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BLASCHKE PRODUCT IN PRIVALOV CLASSES

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ABSTRACT. In terms of zeros, an exact characterization is given of those Blaschke products for which the n th-order derivative belongs to the class of I.I. Privalov in a single circle.

Keywords: Blaschke product, Privalov class, derivative.

1. INTRODUCTION

Let \mathbb{C} be a complex plane and consider a unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ on \mathbb{C} , by $H(D)$ denote the set of all functions that are analytic in D .

Given $0 < q < +\infty$, define a Privalov class as follows:

$$\Pi_q = \{f \in H(D) : \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} (\ln^+ |f(re^{i\varphi})|)^p d\varphi < +\infty\}.$$

The case of $q > 1$ for Π_q was studied by I.I. Privalov in his monograph [1].

It is clear that the class Π_1 is in fact the class N of functions of bounded type on the unit disk, i.e. the Nevalinna class for D (see [2]).

In complex analysis, a well-known Bloch-Nevalinna conjecture, which was proposed by R. Nevalinna in 1929 in [2], states that the class N is invariant with respect to a differential operator. The first example, that confirmed that said conjecture was wrong was presented by Frostman, who constructed the Blaschke product $B(z, z_k)$, such that its derivative does not belong to N , while the closure of its zero-set coincides with the boundary of a circle.

Therefore, a natural question arose: given a Blaschke product $B(z, z_k)$, does its n -th order derivative belong to N ? Interestingly, by considering the problem in terms of limit points of a zero-set on the unit circle T , it is possible to obtain a

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sufficiently complete characterisation of those products, whose derivative belongs not only to N but also to Π_q for all $0 < q < +\infty$,

The following statement holds.

Theorem 1. *Let $\{z_k\}_1^\infty$ be a sequence of points, inside the unit disk D , that satisfies the Blaschke condition, i.e.*

$$\sum_1^\infty (1 - |z_k|) < +\infty, \tag{1}$$

and let E be the set of limit points of the sequence $\{z_k\}_1^\infty$ on the unit circle. By $\{e^{i\alpha_k} e^{i\beta_k}\}_{k=1}^\infty$ denote a sequence of supplementary arcs of a set E on T . Then, if

$$\sum_{k=1}^\infty (\beta_k - \alpha_k) \ln^q \frac{1}{\beta_k - \alpha_k} < +\infty,$$

where $z_n = r_n e^{i\theta_n}$, $n = 1, 2, \dots$ and the amount of points in every interval (α_k, β_k) , $k = 1, 2, \dots$, is uniformly bounded, then for any $n \in N$ the n -th ordered derivative of the Blaschke product belongs to the Privalov class Π_q .

Conversely, there exists a sequence $\{z_k\}_1^\infty \subset D$, satisfying the Blaschke condition, such that if

$$\sum_{k=1}^\infty (\beta_k - \alpha_k) \ln^q \frac{1}{\beta_k - \alpha_k} = +\infty,$$

then $B^{(n)}(z, z_k) \notin \Pi_q$, $n \in N$, $0 < q < +\infty$.

2. PROOF OF SUPPORTING STATEMENTS

Prior to the proof of the theorem, we will prove some supporting statements. By $\rho(F, E)$ denote the distance between sets F and E .

Lemma 1. *Let $Z = \{r_n e^{i\theta_n}\}$ be a sequence of points inside the unit disk D , that satisfies the conditions of Theorem 1, and let E be the set of limit points on the unit circle T . Then the following statements are equivalent:*

$$1) \int_{-\pi}^\pi \ln^q \frac{2\pi}{\rho(e^{i\theta}, E)} d\theta < +\infty, \tag{2}$$

$$2) \int_{-\pi}^\pi \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta < +\infty. \tag{3}$$

Proof. Note that $\rho(e^{i\theta}, Z) \leq \rho(e^{i\theta}, E)$, and since E is a set of limit points on the unit circle, statement 1) immediately follows from 2).

We will now show that 2) follows from 1). For this note that $\frac{2\pi}{\rho(z, E)} \geq 1$ for all $z \in D$, moreover, it follows from 1) that the linear measure of E equals zero, also $\lim_{r \rightarrow 1-0} \rho(re^{i\theta}, E) = \rho(e^{i\theta}, E)$ (see [4]) everywhere on $(-\pi, \pi)$.

We will show that, if condition (1) holds, then

$$I = \int_{-\pi}^\pi \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta < +\infty.$$

Let $z_n = r_n e^{i\theta_n}$, $n = 1, 2, \dots$. It is clear that

$$I = \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta.$$

Moreover,

$$\rho(e^{i\theta}, Z)^2 = \inf_{n \geq 1} |e^{i\theta} - r_n e^{i\theta_n}|^2 = \inf_{n \geq 1} \left((1 - r_n)^2 + 4r_n \sin^2 \frac{\theta - \theta_n}{2} \right).$$

It can be seen that

$$|z_n - e^{i\theta}|^2 \geq c((1 - r_n)^2 + (\theta - \theta_n)^2) > c|\theta - \theta_n|^2. \quad (4)$$

Therefore,

$$\rho(e^{i\theta}, Z) \geq c \inf_{n \geq 1} |\theta_n - \theta|.$$

From this we derive that

$$\int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta \leq \int_{\alpha_k}^{\beta_k} \sup_{n \geq 1} \ln^q \frac{2\pi}{|\theta - \theta_n|} d\theta. \quad (5)$$

Now we fix k and get the boundary on

$$I_k = \int_{\alpha_k}^{\beta_k} \sup_{n \geq 1} \ln^q \frac{2\pi}{|\theta - \theta_n|} d\theta.$$

Let $\gamma_k = \frac{\alpha_k + \beta_k}{2}$ be the center of an interval (α_k, β_k) ,

$$\Delta_k = \{\theta : |\theta - \gamma_k| < (\beta_k - \alpha_k)\}.$$

If $\theta_n \notin \Delta_k$, then, given $\theta \in [\alpha_k, \beta_k]$ $|\theta_n - \theta| \geq \frac{\beta_k - \alpha_k}{2}$, we have

$$\inf_{\theta_n \notin \Delta_k} |\theta - \theta_n| \geq \frac{(\beta_k - \alpha_k)}{2}.$$

Therefore,

$$\int_{\alpha_k}^{\beta_k} \sup_{n \geq 1, \theta_n \notin \Delta_k} \ln^q \frac{2\pi}{|\theta - \theta_n|} d\theta \leq (\beta_k - \alpha_k) \ln^q \frac{4\pi}{\beta_k - \alpha_k}.$$

Now we will obtain bounds on the same integral, when $\theta_n \in \Delta_k$. Then the following cases are possible: $\theta_n \notin [\alpha_k, \beta_k]$ and $\theta_n \in [\alpha_k, \beta_k]$.

Given the conditions of this Lemma,

$$\begin{aligned} I_k &= \int_{\alpha_k}^{\beta_k} \sup_{\theta_n \in \Delta_k} \ln^q \frac{2\pi}{|\theta_n - \theta|} d\theta = \\ &= \int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{|\theta_{n_0} - \theta|} d\theta \end{aligned}$$

for some n_0 (it is assumed that θ_{n_0} is positioned to the left of the interval (α_k, β_k)). Therefore,

$$I_k = \int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{|\theta_{n_0} - \theta|} d(\theta - \theta_{n_0}).$$

Using integration by parts, we obtain

$$\begin{aligned} I_k &= (\theta - \theta_{n_0}) \ln^q \frac{2\pi}{(\theta - \theta_{n_0})} \Big|_{\alpha_k}^{\beta_k} + \int_{\alpha_k}^{\beta_k} (\theta - \theta_{n_0}) q \ln^{q-1} \frac{2\pi}{(\theta - \theta_{n_0})} \frac{2\pi}{(\theta - \theta_{n_0})} d\theta = \\ &= (\beta_k - \theta_{n_0}) \ln^q \frac{2\pi}{\beta_k - \theta_{n_0}} - (\alpha_k - \theta_{n_0}) \ln^q \frac{2\pi}{\alpha_k - \theta_{n_0}} + q \int_{\alpha_k}^{\beta_k} \ln^{q-1} \frac{2\pi}{\theta - \theta_{n_0}} d\theta. \end{aligned}$$

From this we get

$$\int_{\alpha_k}^{\beta_k} \ln^q \frac{2\pi}{\theta - \theta_n} \left(1 - \frac{2\pi q}{\ln \frac{2\pi}{\theta - \theta_n}} \right) d\theta \leq c(\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k}.$$

Since $\beta_k - \alpha_k \rightarrow 0$ as $k \rightarrow +\infty$, one can use the last obtained bound to show that

$$I_k \leq c(\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k}. \tag{6}$$

The case $\theta_{n_0} \in [\alpha_k, \beta_k]$ is checked in a similar way.

Finally, from (5) and (6) we derive

$$\int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, Z)} d\theta \leq c \int_{-\pi}^{\pi} \ln^q \frac{2\pi}{\rho(e^{i\theta}, E)} d\theta \leq c \sum_{k=1}^{\infty} (\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k},$$

where c is some positive constant.

The lemma is proved. □

One may be familiar with the statement of the following lemma, but we will give it for the sake of completeness.

Lemma 2. *Let $Z = \{z_k\}_1^{\infty}$ be the sequence of points inside the unit disk D , that satisfies the Blaschke condition. Also let $B(z, z_k) = \prod_{k=1}^{\infty} \frac{z_k - z}{1 - \bar{z}_k z} \frac{\bar{z}_k}{|z_k|}$ be the Blaschke product with Z as its zero-set. Then there exists a positive number C , depending only on n , such that*

$$|B^{(n)}(z, z_k)| \leq \frac{C(n)}{\rho^{2n}(z, Z)}, \quad z \in D.$$

Proof. We will prove this lemma by induction.

First consider $n = 1$.

We have that

$$\frac{B^{(1)}(z, z_k)}{B(z, z_k)} = \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{(z - z_k)(1 - \bar{z}_k z)}, \quad z \in D.$$

Let $b_k(z) = \frac{z_k - z}{1 - \bar{z}_k z} \frac{\bar{z}_k}{|z_k|}$, $B_k = B/b_k$. Therefore,

$$B^{(1)}(z) = B(z) \cdot \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{(z - z_k)(1 - \bar{z}_k z)}.$$

Since $B(z, z_k) = B_k(z, z_m) \frac{z - z_k}{1 - \bar{z}_k z}$, we can use the last equality to derive that

$$B^{(1)}(z, z_k) = \sum_{k=1}^{\infty} \frac{B_k(z, z_k)(z - |z_k|^2)}{(1 - \bar{z}_k z)^2}, \quad (7)$$

i.e. $|B^{(1)}(z, z_k)| \leq \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)}{|1 - \bar{z}_k z|^2}$. But since $|1 - \bar{z}_k z| \geq |z_k - z|$, $z, z_k \in D$, we can use the last bound to show that

$$|B^{(1)}(e^{i\theta}, z_k)| \leq \frac{\sum_{k=1}^{\infty} (1 - |z_k|)}{\rho(e^{i\theta}, Z)^2}.$$

Now, by considering (7) we get that

$$B^{(m)}(z, z_k) = \sum_{k=1}^{\infty} (1 - |z_k|^2) (B_k(z, z_k) \times \frac{1}{(1 - \bar{z}_k z)^2})^{(m-1)}.$$

By using the Leibniz formula we obtain the following:

$$\begin{aligned} B^{(m)}(z, z_k) &= \sum_{k=1}^{\infty} (1 + |z_k|) \sum_{s=0}^{m-1} C_{m-1}^s \times B_k^{(s)}(z) \cdot \frac{\bar{z}_k^{m-1-s} (1 + m - s)!}{(1 - \bar{z}_k z)(m - 1 - s)!} = \\ &= \sum_{k=1}^{\infty} (1 - |z_k|) \sum_{s=0}^{m-1} C_{m-1}^s B_k^{(s)}(z) \cdot z_k^{m-1-s} \times \frac{(1 + m - s)!}{(1 - \bar{z}_k z)^{1+m-s}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |B^{(m)}(z, z_k)| &\leq A(m) \sum_{k=1}^{\infty} \left(\sum_{s=0}^{m-1} \frac{|B_k^{(s)}(z_k)|}{|z - z_k|^m} \right) \times (1 - |z_k|) \leq \\ &\leq A_1(m) \sum_{s=0}^{m-1} \frac{1}{\rho(z, Z)^{2s}} \cdot \frac{1}{\rho(z, Z)^{m+1-s}}. \end{aligned}$$

From this, one can immediately derive the bound

$$|B^{(m)}(z, z_k)| \leq \frac{C(m)}{\rho(z, Z)^{2m}}, \quad z \in D, \quad m \in N.$$

The lemma is proved. \square

Note. By using the proof of Lemma 2, one can also obtain the following bound:

$$|B^{(n)}(re^{i\varphi}, z_k)| \leq \frac{C(n)}{\rho(e^{i\varphi}, Z)^{2n}}, \quad r \in [0, 1], \quad (8).$$

3. PROOF OF THE MAIN RESULT

Proof of theorem. The first part of the theorem follows directly from Lemma 1, Lemma 2 and (8).

We will now prove the second part, i.e. if

$$\sum_{k=1}^{+\infty} (\beta_k - \alpha_k) \ln^q \frac{2\pi}{\beta_k - \alpha_k} = +\infty,$$

then there exists a sequence $\{z_k\}_1^\infty \subset D$, such that

$$\sum_{k=1}^{+\infty} (1 - |z_k|) < +\infty,$$

at the same time $B^{(n)}(z, z_k) \notin \Pi_q$ for all $0 < q < +\infty$.

We will first prove this statement for $n = 1$. From (7), we can derive that

$$B^{(1)}(e^{i\theta}, z_k) = B(e^{i\theta}, z_k) e^{-i\theta} \sum_{k=1}^{+\infty} \frac{(1 - |z_k|^2)}{|e^{i\theta} - z_k|^2}.$$

Hence

$$|B^{(1)}(e^{i\theta}, z_k)| = \sum_{k=1}^{+\infty} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2}$$

almost everywhere on the segment $[-\pi, \pi]$.

Now we construct the sequence $\{z_k\}_1^\infty$. Set $z_n = (1 - \varepsilon_n)e^{i\alpha_n}$, $\tilde{z}_n = (1 - \varepsilon_n)e^{i\beta_n}$, $\varepsilon_n = \beta_n - \alpha_n$, $n = 1, 2, \dots$. It is clear that $1 - |\tilde{z}_n| = 1 - |z_n| = \varepsilon_n$, hence the sequences $\{z_n\}_1^\infty$ and $\{\tilde{z}_n\}_1^\infty$ satisfy the Blaschke condition. Let $w_{2k-1} = z_k$, $w_{2k} = \tilde{z}_k$. It can be seen that the sequence $\{w_k\}_{k=1}^{+\infty}$ also satisfies the Blaschke condition.

Let $B(z, w_k) = \prod_1^\infty \frac{w_k - z}{1 - \bar{w}_k z} \frac{\bar{w}_k}{|w_k|}$, $z \in D$. Then, by considering (7), we obtain that

$$B^{(1)}(e^{i\theta}, w_k) = e^{-i\theta} B(e^{i\theta}, w_k) \sum_{k=1}^{+\infty} \frac{1 - |w_k|^2}{|e^{i\theta} - w_k|^2},$$

for almost all $\theta \in [-\pi, \pi]$. Therefore,

$$|B^{(1)}(e^{i\theta}, w_k)| = \sum_{k=1}^{+\infty} \frac{1 - |w_k|^2}{|e^{i\theta} - w_k|^2}.$$

Considering the definition of the sequence $\{w_k\}_1^\infty$, we can use this equality to derive that

$$|B^{(1)}(e^{i\theta}, w_k)| \geq \sum_{k=1}^{+\infty} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2}$$

$$\theta \in [-\pi, \pi]$$

Therefore,

$$\int_{-\pi}^{\pi} \left(\ln^+ |B^{(1)}(e^{i\theta}, w_k)| \right)^q d\theta \geq \int_{-\pi}^{\pi} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{(1 - |z_k|^2)}{|e^{i\theta} - z_k|^2} \right) \right)^q d\theta, \tag{9}$$

Now note that $z_k = (1 - \varepsilon_k)e^{i\alpha_k}$,

$$|e^{i\theta} - z_k|^2 = (1 - r_k)^2 + 4r_k \sin^2 \frac{\theta - \alpha_k}{2} = \varepsilon_k^2 + 4(1 - \varepsilon_k) \sin^2 \frac{\theta - \alpha_k}{2}.$$

By the arguments similar to those used in the proof of Lemma 1, we get that $|e^{i\theta} - z_k|^2 \geq \delta(\varepsilon_k^2 + (\theta - \alpha_k)^2)$ for some positive $\delta > 0$.

Now, turning back to the inequality (9), we obtain

$$\int_{-\pi}^{\pi} \left(\ln^+ |B^{(1)}(e^{i\theta}, w_k)| \right)^q d\theta \geq \int_{-\pi}^{\pi} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta + c. \quad (10)$$

We will now present the lower bound on the last integral. Set

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta = \\ &= \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta \end{aligned} \quad (11).$$

Let

$$I_n = \int_{\alpha_n}^{\beta_n} \left(\ln^+ \left(\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{(\theta - \alpha_k)^2 + \varepsilon_k^2} \right) \right)^q d\theta.$$

Set $\gamma_n = \frac{\beta_n + \alpha_n}{2}$, $n = 1, 2, \dots$. Then

$$I_n \geq \int_{\alpha_n}^{\gamma_n} \left(\ln^+ \frac{\varepsilon_n}{(\theta - \alpha_n)^2 + \varepsilon_n^2} \right)^q d\theta \geq \left(\ln^+ \frac{\varepsilon_n}{\frac{\varepsilon_n^2}{4} + \varepsilon_n^2} \right)^q \cdot \frac{\varepsilon_n}{4},$$

therefore,

$$I_n \geq \varepsilon_n \left(\ln \frac{4}{5\varepsilon_n} \right)^q, \quad n = 1, 2, \dots$$

Now, considering (11) and the final bound, we get that

$$\int_{-\pi}^{\pi} \left(\ln^+ |B^{(1)}(e^{i\theta}, w_k)| \right)^q + c_1 = +\infty.$$

The theorem is proved.

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