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UNIQUE DETERMINATION OF CONFORMAL TYPE FOR DOMAINS. III

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ABSTRACT. The article is the third (final) part of a review series entitled "Unique determination of conformal type for domains," initiated by the author's eponymous paper, published in *Sib. Èlektron. Mat. Izv.*, **16**, 692–708 (2019). The main result of the present article is that for $n \le 3$, any n-connected plane domain U is uniquely determined by the relative conformal moduli of pairs of boundary components.

Keywords: finitely connected plane domain, relative conformal modulus of pairs of boundary components, Teichmüller and Grötzsch extremal domains.

1. Introduction

In the first and second parts of the series of review articles (see [1] and [2]), we discussed problems of unique determination of spatial domains by the relative conformal moduli of their boundary condensers. The present article is devoted to solving the analogous problem for plane domains. We study the unique determination of n-connected plane domains by the relative conformal moduli of pair of their boundary components in the case of $n \le 3$.

In this article, we keep to the notations and use the notions and assertions of [1] and [2]. For this reason, we refer the reader for them to [1] and [2].

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2. On Unique Detemination of 3-Connected Plane Domain by the Relative Conformal Moduli of Pairs of Boundary Components

Let U be a plane domain. Consider a family Γ of continuous nonconstant paths $\gamma : \langle 0,1\rangle \to U$, where $\langle 0,1\rangle$ is an open, half-open, or closed interval such that $\operatorname{Im}(\gamma) = \gamma(\langle 0,1\rangle) \subset U$. The conformal modulus of the family Γ is the quantity

$$M_2(\Gamma) = \inf_{\rho\in\mathcal{R}(\Gamma)} \int_{\mathbb{R}^2} \rho^2 dxdy.$$

Here $\mathcal{R}(\Gamma)$ is the family of all nonnegative Borel measurable functions ρ such that

$$\int_\gamma \rho dl \geq 1$$

for each locally rectifiable path $\gamma \in \Gamma$ (the set of all such functions ρ is called the set of admissible functions for the family Γ and denoted by $\mathcal{R}(\Gamma)$; more details on the definition of moduli of families of curves and their properties can be found, e.g., in [3], [4]).

Let E_0 and E_1 be two boundary components of a finitely connected plane domain U, and let $\gamma : \langle 0,1\rangle \to U$ be a path such that $\operatorname{cl}(\operatorname{Im}(\gamma)) \cap E_j \neq \varnothing$ (here, $j = 0,1$ and $\operatorname{cl} A$ denotes the closure of a set A). In this case, we say that the path γ joins E_0 and E_1 in U.

Suppose that U_1 and U_2 are two plane n-connected domains. Recall that a domain U is said to be simply connected if its boundary consists of a continuum or a single point (considering the boundary of a plane domain, we identify the field of complex numbers with the Riemann 2-sphere). Otherwise, the domain is considered multiply connected. A domain is said to be 2-connected, 3-connected, n-connected if its boundary consists of two, three, n pairwise disjoint continua (some of which may degenerate into a point), respectively. These continua are called the boundary components of U. The relative conformal modulus $M^U(F)$ of a pair $F = \{F_1, F_2\}$ of boundary components F_1 and F_2 of a domain U is defined as the conformal modulus $M_2(\Gamma(F))$ of the family $\Gamma(F)$ of paths joining the components F_1 and F_2 in U. Pairs $F = \{F_1, F_2\}$ with $F_1 = F_2$ are not excluded. If there exists a conformal mapping $f : U_1 \to U_2$ between domains U_1 and U_2, then it induces a natural bijection G between the sets of boundary components of these domains such that $M^{U_1}(F) = M^{U_2}(G(F))$ for each pair $F = \{F_1, F_2\}$ of boundary components of U_1 (so that $G(F) = \{G(F_1), G(F_2)\}$ is the pair of boundary components of U_2 corresponding to F_1 and F_2). This gives rise to the question of whether the converse is true. In this paper, we give a partial answer to this question in the case of finitely connected domains. In the course of study, it has turned out that the problem is very difficult and undoubtedly interesting. Our main result in this section is the following theorem ([5]).

Theorem 2.1. *For $n \leq 3$, any n-connected plane domain U is uniquely determined by the relative conformal moduli of pairs of boundary components, i.e., if V is another n-connected plane domain and there exists a bijection G between the sets of boundary components of U and V such that $M^U(F) = M^V(G(F))$ for any pair $F = \{F_1, F_2\}$ of boundary components F_1 and F_2 of U and the pair $G(F) = \{G(F_1), G(F_2)\}$ of the boundary components of V that are their images under G then V is conformally equivalent to U.*

Lemmas 4.1-4.3 mentioned in the proof of Theorem 2.1 are formulated and proved in Appendix.

Proof of Theorem 2.1. By Lemma 4.1, we can assume that all boundary components of the domains under consideration do not degenerate into a point. If $n = 1$ and the boundary of U consists of more than one point then the required assertion immediately follows from the Riemann theorem on a conformal mapping of a connected plane domain onto a disk.

Let $n = 2$. By [6, Chapter V, Section 1, Theorem 1], each two-connected plane domain can be conformally mapped onto a circular ring (with concentric boundary circles). Consequently, taking into account Lemma 4.1 and the fact that the conformal modulus of a family of curves (paths) is conformlally invariant, it suffices to prove that if $U = \{z \in \mathbb{C} : r < |z| < 1\}$, $V = \{z \in \mathbb{C} : R < |z| < 1\}$, and there exists a bijection G of the set of boundary components of these domains that preserves the relative conformal moduli of pairs of boundary components, then the relative conformal modulus of the pair $F = \{F_1, F_2\}$ of boundary components $F_1 = \{z \in \mathbb{C} : |z| = r\}$ and $F_2 = \{z \in \mathbb{C} : |z| = 1\}$ of the circular ring U coincides with the relative conformal modulus of the pair $F^* = \{F_1^*, F_2^*\}$ of boundary components $F_1^* = \{z \in \mathbb{C} : |z| = R\}$ and $F_2^* = F_2$ of the circular ring V. Since these moduli are equal to $2\pi/\log(r^{-1})$ and $2\pi/\log(R^{-1})$ respectively, we have $V = U$, which implies the required assertion in the case $n = 2$.

The case $n = 3$ is the most complicated among the cases covered by this theorem. Considering a three-connected plane domain U and assuming that U has no isolated degenerate components, we can use [6, Chapter V, Theorem 1 in Section 1 and Theorem 4 in Section 3] to assert that U is conformally equivalent to the domain

$$\begin{aligned}\widetilde{U} = \{z \in \mathbb{C} : |z| < 1\} \setminus (\{z \in \mathbb{C} : |z| \le r < 1\} \\ \cup \{z \in \mathbb{C} : r < x_1 \le \operatorname{Re} z \le x_2 < 1, \operatorname{Im} z = 0\}).\end{aligned}$$

We make some conformal transformations of this domain. First, by a suitable linear fractional transformation, we map $\widetilde{U}$ onto the domain

$$\begin{aligned}\widetilde{\widetilde{U}} = \mathbb{C} \setminus (\{z \in \mathbb{C} : |z - z_1| \le r_1, \operatorname{Im} z_1 = 0\} \cup \{z \in \mathbb{C} : |z - z_2| \le r_2, \operatorname{Im} z_2 = 0\} \\ \cup \{z \in \mathbb{C} : z_2 + r_2 < a \le \operatorname{Re} z \le b < \infty, \operatorname{Im} z = 0\}),\end{aligned}$$

where $\operatorname{Im} a = \operatorname{Im} b = 0$, $0 < r_j < \infty$, $-\infty < z_j < \infty$ $(j = 1, 2)$, and $z_1 + r_1 < z_2 - r_2 < z_2 + r_2 < a < b$. Further, the so-obtained domain $\widetilde{\widetilde{U}}$ is conformally mapped onto the complement of the union of three disjoint segments, which can be done by mapping conformally the part of the domain $\widetilde{\widetilde{U}}$ located in the upper half-plane onto the upper half-plane itself in such a way that the upper half-circles of two boundary circles of the domain $\widetilde{\widetilde{U}}$ and the segment $\{z \in \mathbb{C} : z_2 + r_2 < a \le \operatorname{Re} z \le b, \operatorname{Im} z = 0\}$ are transformed into the segments

$$\begin{gathered}\{z \in \mathbb{C} : -\infty < \alpha_1 \le \operatorname{Re} z \le \beta_1, \operatorname{Im} z = 0\}, \\ \{z \in \mathbb{C} : \beta_1 < \alpha_2 \le \operatorname{Re} z \le \beta_2, \operatorname{Im} z = 0\}, \\ \{z \in \mathbb{C} : \beta_2 < \alpha_3 \le \operatorname{Re} z \le \beta_3 < \infty, \operatorname{Im} z = 0\}\end{gathered} \tag{2.1}$$

respectively (by [6, Chapter II, Section 3, Theorem 4′], the last mapping can be chosen so that it can be extended to a homeomorphic mapping of the closure relative to the Riemann sphere of these domains). Applying the symmetry principle, we extend this mapping to the whole domain $\tilde{\tilde{U}}$. Moreover, without loss of generality, we can assume that $\alpha_1 = -1$ and $\beta_1 = 0$.

Taking into account Lemma 4.1, considering three-connected plane domains U and V, and assuming that there exists a bijection G of the set of boundary components of these domains preserving the relative conformal moduli of pairs of boundary components, we make a number of such transformations of U and also of V (the corresponding parameters for V are defined as in (2) and denoted by α_1^*, β_1^*, α_2^*, β_2^*, α_3^*, β_3^*). Making the additional linear fractional transformations $w = (1+\beta_3)z/(\beta_3 - z)$ and $w^* = (1+\beta_3^*)z/(\beta_3^* - z)$ of U and V, we reduce the study to the case where these domains take the form

$$\begin{aligned} U = \mathbb{C} \setminus (F_1 \cup F_2 \cup F_3) = \mathbb{C} \setminus (&\{z \in \mathbb{C} : -1 \le \operatorname{Re} z \le 0, \operatorname{Im} z = 0\} \\ &\cup \{z \in \mathbb{C} : \alpha \le \operatorname{Re} z \le \beta, \operatorname{Im} z = 0\} \cup \{z \in \mathbb{C} : u \le \operatorname{Re} z < \infty, \operatorname{Im} z = 0\}), \end{aligned} \tag{2.2}$$

$$\begin{aligned} V = \mathbb{C} \setminus (F_1^* \cup F_2^* \cup F_3^*) = \mathbb{C} \setminus (F_1 \cup \{z \in \mathbb{C} : \alpha^* \le \operatorname{Re} z \le \beta^*, \operatorname{Im} z = 0\} \\ \cup \{z \in \mathbb{C} : u^* \le \operatorname{Re} z < \infty, \operatorname{Im} z = 0\}), \end{aligned} \tag{2.3}$$

where $F_1 = \{z \in \mathbb{C} : -1 \le \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}$, $F_2 = \{z \in \mathbb{C} : \alpha \le \operatorname{Re} z \le \beta, \operatorname{Im} z = 0\}$, $F_3 = \{z \in \mathbb{C} : u \le \operatorname{Re} z < \infty, \operatorname{Im} z = 0\}$, $F_1^* = F_1$, $F_2^* = \{z \in \mathbb{C} : \alpha^* \le \operatorname{Re} z \le \beta^*, \operatorname{Im} z = 0\}$, $F_3^* = \{z \in \mathbb{C} : u^* \le \operatorname{Re} z < \infty, \operatorname{Im} z = 0\}$, $0 < \alpha < \beta < u$ and $0 < \alpha^* < \beta^* < u^*$,

$$u = \frac{(\beta_3 + 1)\alpha_3}{\beta_3 - \alpha_3}, \quad u^* = \frac{(\beta_3^* + 1)\alpha_3^*}{\beta_3^* - \alpha_3^*}.$$

Let G be a bijection from the set of boundary components $\{F_1, F_2, F_3\}$ of U onto the set of boundary components $\{F_1^*, F_2^*, F_3^*\}$ of V (cf. (2.2) and (2.3)). Moreover, without loss of generality, we may assume that $M^V(\{F_1^*, F_3^*\}) = M^U(\{F_1, F_3\})$, $M^V(\{F_1^*, F_2^*\}) = M^U(\{F_1, F_2\})$, and $M^V(\{F_2^*, F_3^*\}) = M^U(\{F_2, F_3\})$. By what was said above, it suffices to show that $u^* = u$, $\alpha^* = \alpha$, and $\beta^* = \beta$.

For this purpose, we need to make some additional observations. First of all, we show that the conformal modulus $\psi(u)$ of the family Γ_u of paths joining the boundary components of the Teichmüller extremal domain

$$\begin{aligned} T(u) = \mathbb{C} \setminus (\{z \in \mathbb{C} : -1 \le \operatorname{Re} z \le 0, \operatorname{Im} z = 0\} \\ \cup \{z \in \mathbb{C} : u \le \operatorname{Re} z < \infty, \operatorname{Im} z = 0\}), \end{aligned} \tag{2.4}$$

where $u > 0$, in $\mathbb{C}+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is a strictly monotone function of u; moreover,

$$0 < \psi(u) < \infty. \tag{2.5}$$

Indeed, the right inequality in (2.5) follows from the fact that the family of paths joining the boundary circles of the circular ring $\{z \in \mathbb{C} : 1/2 < |z + 1/2| < u + 1/2\}$ within this ring minorizes the family Γ_u

$$\psi(u) \le \frac{2\pi}{\log \frac{u+1/2}{1/2}} = \frac{2\pi}{\log(2u+1)}$$

(for the notion of minorization for families of curves, see [4]).

To prove the left inequality, we consider the half-ring

$$\Big\{z \in \mathbb{C} : \frac{u}{2} < |z - \frac{u}{2}| < 1 + \frac{u}{2}, \operatorname{Im} z > 0\Big\}.$$

The family Γ of all paths joining $\{z \in \mathbb{C} : -1 \le \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}$ and $\{z \in \mathbb{C} : u \le \operatorname{Re} z \le u+1, \operatorname{Im} z = 0\}$ in this half-ring is a subfamily of the above family Γ_u of paths joining boundary components of the Teichmüller extremal domains $T(u)$ (in $\mathbb{C}_+$). Therefore,

$$\psi(u) \ge M_2(\Gamma) = \frac{1}{\pi} \log \frac{u+2}{u} > 0.$$

Further, following [7], consider the functions

$$\Phi(u) = \exp\left(\frac{2\pi}{M_2(\Gamma_{G(u)})}\right), \quad u \in]1, \infty[,$$

$$\Psi(u) = \exp\left(\frac{2\pi}{M_2(\Gamma_{T(u)})}\right), \quad u \in]0, \infty[,$$

where $M_2(\Gamma_{G(u)})$ and $M_2(\Gamma_{T(u)})$ are the relative conformal moduli of the families $\Gamma_{G(u)}$ and $\Gamma_{T(u)}$ of paths joining boundary components of the plane Grötzsch

$$G(u) = \mathbb{C} \setminus (\{z \in \mathbb{C} : |z| \le 1\} \cup \{z \in \mathbb{C} : u \le \operatorname{Re} z < \infty, \operatorname{Im} z = 0\}), \quad u > 1,$$

and Teichmüller $T(u)$ extremal domains in these domains (in [7], the functions Φ and Ψ are introduced in some other manner, but the method in [7] is equivalent to the method used in this paper). By what was said above Lemma 4.3, we have

$$\psi(u) = \pi(\log \Psi(u))^{-1}. \tag{2.6}$$

From the first part of Lemma 4.2 we conclude that Φ is strictly increasing, whereas the second part of Lemma 4.2 and the relation (2.6) imply that ψ is strictly decreasing.

Let us show that from the equality $M^V(\{F_1^*, F_3^*\}) = M^U(\{F_1, F_3\})$ of the conformal moduli of the pairs of boundary components $\{F_1, F_3\}$ and

$$\{F_1^*, F_3^*\} = \{G(F_1), G(F_3)\}$$

we obtain the equality of the conformal moduli of the plane Teichmüller extremal domains

$$U_2(u) = \mathbb{C} \setminus (E_0 \cup E_1) = \mathbb{C} \setminus (F_1 \cup F_3),$$
$$U_2(u^*) = \mathbb{C} \setminus (E_0 \cup E_1^*) = \mathbb{C} \setminus (F_1^* \cup F_3^*),$$

where $E_0 = F_1 = F_1^*$, $E_1 = F_3$, and $E_1^* = F_3^*$. For this purpose we use Lemmas 4.2 and 4.3. As a result, we first find $u^* = u$. Further, considering the pair of boundary components E_0 and $E_2 = F_2$ of U and the pair of the boundary components E_0 and $E_2^* = F_2^*$ of V corresponding to them under G and performing suitable linear fractional transformations of U and V, we again get into the situation where each of the domains U and V satisfies the assumptions of Lemma 4.3 (more exactly, Remark 4.1), where, at present,

$$u = \frac{(1+\beta)\alpha}{\beta - \alpha}, \quad u^* = \frac{(1+\beta^*)\alpha^*}{\alpha^* - \beta^*}.$$

The above-mentioned linear fractional mappings have the form

$$w(z) = \frac{(1+\beta)z}{\beta - z}, \quad w^*(z) = \frac{(1+\beta^*)z}{\beta^* - z}.$$

Therefore, repeating the argument of the first case, we obtain the identity

$$\frac{(1+\beta^*)\alpha^*}{\beta^* - \alpha^*} = \frac{(1+\beta)\alpha}{\beta - \alpha}.$$

Finally, considering the pair E_2, E_1 of boundary components of U and the pair E_2^*, E_1^* of boundary components of V corresponding to them under G and arguing as above, we obtain the identity

$$\frac{u^* - \beta^*}{\beta^* - \alpha^*} = \frac{u - \beta}{\beta - \alpha},$$

where we used the auxiliary mappings

$$w(z) = \frac{z - \beta}{\beta - \alpha}, \quad w^*(z) = \frac{z - \beta^*}{\beta^* - \alpha^*}).$$

Thus, we obtain the system of algebraic equations

$$\begin{gathered} u^* = u, \\ \frac{(1+\beta^*)\alpha^*}{\beta^* - \alpha^*} = \frac{(1+\beta)\alpha}{\beta - \alpha}, \\ \frac{u^* - \beta^*}{\beta^* - \alpha^*} = \frac{u - \beta}{\beta - \alpha}. \end{gathered}$$

Assume that the parameters u, α, and β in this system are fixed ($0 < \alpha < \beta < u < \infty$). Then for α^* and β^* ($0 < \alpha^* < \beta^* < u$) we obtain the system of equations

$$\begin{gathered} (\alpha^* - \alpha) - \alpha^*\alpha\Big(\frac{1}{\beta^*} - \frac{1}{\beta}\Big) + \Big(\frac{\alpha^*}{\beta^*} - \frac{\alpha}{\beta}\Big) = 0, \\ u[(\beta^* - \alpha^*) - (\beta - \alpha)] = \beta^*\alpha - \alpha^*\beta. \end{gathered}$$

From the second equation of this system we have $\beta^* = \frac{\alpha^*(\beta-u)-u(\beta-\alpha)}{\alpha-u}$. Substituting this expression for β^* in the first equation, we obtain an equation for α^*:

$$(\alpha^* - \alpha) + \alpha^*\alpha\left[\frac{\alpha - u}{\alpha^*(\beta - u) - u(\beta - \alpha)} - \frac{1}{\beta}\right] - \left[\frac{\alpha}{\beta} - \frac{\alpha^*(\alpha - u)}{\alpha^*(\beta - u) - u(\beta - \alpha)}\right] = 0.$$

This equation has two solutions $\alpha^* = \alpha$ and $\alpha^* = -\frac{u(\beta-\alpha)(1+\beta)}{(u-\beta)(\alpha+\beta)}$ (< 0). Since the second solution is impossible, the first one is true. It is easy to verify that $\beta^* = \beta$. Consequently, the desired relations between α^* and α, β^* and β, and u^* and u (i.e., the identities $\alpha^* = \alpha$, $\beta^* = \beta$, and $u^* = u$) are established. The theorem is proved. □

3. Unsolved problems

In relation to results considered in [1] and this paper, the following questions appear to be important and interesting.

1. Do Theorem 2.1 and Corollary 3.1 from [1] remain valid for $n = 3$?

2. Bearing the results of [8] in mind, can we hope to remove the condition that the mapping f preserves the relative n-moduli of boundary ring-shaped condensers from the statement of Corollary 3.1 from [1]?

3. Is it possible to generalize the main results of [1] to the case of polyhedral domains with boundary of a more complicated geometric structure (e.g., not necessarily connected)?

4. Does Theorem 2.1 of the present paper remain valid for n-connected plane domains, where $n \geq 4$?

4. Appendix

Lemma 4.1. *Assume that U and V are finitely connected plane domains such that there exists a bijection G between the sets of boundary components of these domains that preserves the relative conformal moduli of pairs of boundary components of U. Then U and V have the same connection and the isolated boundary components of U which degenerate into a point are transformed into sets of the same kind (for V) under G.*

Proof. If E is an isolated boundary component of U that degenerates into a point then, considering the pair of boundary components $F = \{E, E\}$ of U and the pair $G(F) = \{G(E), G(E)\}$ of boundary components of V, under the assumption that $G(E)$ does not degenerate штto a point, we can use the properties of the moduli of families of curves (cf. [4]) to obtain $M^U(F) = 0$, whereas $M^V(G(F)) \neq 0$. □

Lemma 4.2. *The function $u^{-1}\Phi(u)$, $1 < u < \infty$, is nonincreasing and*

$$\Psi(u) = \{\Phi((u+1)^{1/2})\}^2$$

for $0 < u < \infty$.

The second part of this assertion is proved in [7, Subsection 2.7]. We can prove the first part in the same way as the first part of Lemma 6 in [9] with natural modifications caused by the fact that the dimension of $\mathbb{R}^2$ is equal to 2.

Lemma 4.3. *The following relation holds:*

$$M_2(\Gamma) = \frac{1}{2} M_2(\Gamma_T) = \frac{1}{2} M_2(\Gamma_U), \tag{4.1}$$

where the domain U is defined by (2.2), Γ is the family of paths joining E_0 $(= F_1)$ and E_1 $(= F_3)$ in $\mathbb{C}_+$, and Γ_T and Γ_U are the families of paths joining E_0 and E_1 in $T = T(u)$ and in U respectively, where $T(u)$ is the Teichmüller domain (2.4).

Proof. We argue as in [10, Lemma 3.3], taking into account the specificity in the case of dimension $n = 2$. We outline our arguments.

First, following the proof of Lemma 3.3 in [10], we establish the equality

$$M_2(\Gamma) = \frac{1}{2} M_2(\Gamma_T).$$

Then we prove the series of relations

$$2M_2(\Gamma) = M_2(\Gamma) + M_2(\widetilde{\Gamma}) = M_2(\Gamma \cup \widetilde{\Gamma}) \leq M_2(\Gamma_U), \tag{4.2}$$

where $\widetilde{\Gamma}$ is the family of paths joining E_0 and E_1 in $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, and then, taking into account that $\Gamma_U \subset \Gamma_T$, we find the final series

$$M_2(\Gamma_U) \leq M_2(\Gamma_T) = 2M_2(\Gamma).$$

□

Remark 4.1. *Under the assumptions of Lemma 4.3, the segment $[\alpha_2, \beta_2]$ can be replaced with any closed set $A \subset \operatorname{cl} \mathbb{R} \setminus \{E_0 \cup E_1\}$.*

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