Stability analysis of a prey-predator population model with nonlinear harvesting rate

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Abstract: The present paper discusses a prey-predator fishery model where both species are subjected to nonlinear harvesting effort. The model is a modified version of the classic Lotka-Voltera predator-prey model. The purpose of the work is to offer a mathematical analysis of the model. The dynamical behavior of the proposed model is examined. We investigate the existence and global stability of multiple equilibria, and some characteristic properties of them are mentioned like strong and center manifolds. The existence of a limit cycle is studied. Simulations of the model are performed to illustrate the theoretical results and bifurcation diagrams are analyzed.

Keywords: Stability; Kolmogorov model; Persistent and Permanent; Harvesting; Limit cycle; strong and center manifolds.

Mathematics Subject Classification: 34A34, 34C23, 34C25, 34C45, 34D23, 92d25.

1 Introduction

Many researchers are interested to the dynamic of predator-prey interactions models, and have investigated the processes that affect it [1], [6], [9]-[12], [15]. The interaction between the predator and prey species can be modeled by the classical Lotka-Voltera model [4], which is a special case of Kolmogorov model. By applying the logistic growth on prey population, this model becomes

$$\begin{cases} \dot{x} &= xr\left(1 - \dfrac{x}{k}\right) - axy \\ \dot{y} &= caxy - dy \end{cases} \tag{1}$$

Where x represents the size of prey population and y the size of predator population. Parameters a, r, k, c and d are assumed to be positive values, where:
a presents the rate of predation, such that in the presence of the predator, the prey species decreases at a rate proportional to the functional response ax;
c denotes the factor of the efficiency of predation which divides a maximum per capita birth rate of the predators into a maximum per capita consumption rate;
d is the death rate of the predator which decreases exponentially, in the absence of the preys;
r is the maximum specific growth rate of the prey which grows logistically in the absence of the predator species;
k denotes the environmental carrying capacity with which the prey grows logistically in the absence of the predation.
To enrich the model (1), many researchers modify the nonlinear functional response function and adding some other elements like, pollution, toxicity, harvesting, age of the species, refuge,...etc. [6], [9]-[12], [15].

Because that harvesting is an important and effective prevention and control means of the explosive growth of predators or prey when they are enough, it is reasonable and logic to introduce the harvest of some species of populations into models [6], [11], [12], [15]. Researches on the effects of harvesting in fishery models are becoming more valuable. We then focus in this paper on the predator-prey model with harvest.

Several forms and types of harvesting in prey-predator models are already being studied; researchers have added terms to the prey or predator density like constants [5]; function that is linear if the density of the predator is bellow a switched value and constant otherwise [15]; functions of the form $Q_iF_ix_i/\left(Q_i+F_ix_i\right), i=1,2$ where Q_i and F_i are constants and $x_i, i=1,2$ are the sizes of prey and predator populations respectively [2]; or recently a function constructed by incorporating the Heaviside function [3].

Apparently, the above types of harvesting rates have their own advantages as well as disadvantages in the award of the harvest in the real world. Motivated by these view, in this work, we consider a predator-prey model with another form of harvesting rate. Thus, it is interesting to construct a new kind of harvesting rate that is nonlinear and see what is going on for the density of prey and predator. Taking system (1) as our baseline model, we assume that harvesting takes place and both the prey and the predator are under harvesting. The effect of harvesting on the species is assumed to be different. We introduce harvesting nonlinear functions $H_1(x)$ of the prey and $H_2(y)$ of the predator to model (1) for discussing its dynamical characteristics. We consider then the following differential system

$$\begin{cases} \dot{x} &= xr\left(1-\dfrac{x}{k}\right)-axy-\alpha H_1(x) \\ \dot{y} &= caxy-dy-\beta H_2(y) \end{cases} \tag{2}$$

where α and β are positive parameters related to the harvesting effort. We investigate the existence and stability of multiple equilibria, bifurcations and study the effects of harvest on the dynamics of the predator-prey model (2). The model possesses a varied dynamic.

This paper is organized as follow. In sections 2 we give the formulation of the model which we will study here. Basic results are given in section 3; we present the boundedness and the invariance of the model considered. We then study the existence of multiple equilibria, their types and their local and global stability, the permanence and persistence of the model, bifurcations, and existence of periodic solutions, especially existence of limit cycles for the model. In Section 4, we give numerical simulations to illustrate the established results. Concluding remarks are presented in section 5.

2 Model formulation

Considering prey-predator model (2), we suppose that harvesting functions of the prey and the predator in system (2) are respectively, $H_1(x)=x^2$ and $H_2(y)=y^3$. Model (2) becomes

$$\begin{cases} \dot{x} &= xr\left(1-\dfrac{x}{k}\right)-axy-\alpha x^2 \\ \dot{y} &= caxy-dy-\beta y^3 \end{cases} \tag{3}$$

We can show that solutions of system (3) starting from positive initial conditions are positive and forward bounded. Let

$$x_0>\max\left(\frac{r}{\alpha_1},\frac{d}{ca}\right)$$

Then,

$$D = \left\{(x, y) \in \mathbb{R} \times \mathbb{R};\ 0 < x < x_0, 0 < y < \sqrt{\frac{1}{\beta}\left(cax_0 - d\right)}\right\}$$

is a positive invariant set for system (3).

3 Preliminary results

3.1 Existence of Equilibria

In this section, we inspect the existence of all equilibria of model (3). Their existence and number depend principally on the harvesting rates. We know that only the positive equilibria are of a biological interest, but in this paper, for a mathematical objective, we will study and explore all equilibria. We denote $\alpha_1 = \alpha + \frac{r}{k}$, $d_1 = d - \frac{car}{\alpha_1}$ and $\Delta = \frac{c^2a^4}{\alpha_1^2} - 4\beta d_1$. We present our results of the existence of equilibria in system (3) as follows:

Theorem 3.1. *According to the values of the harvesting parameter β and the mortality rate d of the predator species in the absence of the prey, system (3) has the following coexistence equilibria: In the case $\beta < \frac{c^2a^4}{4d_1\alpha_1^2}$ and $d_1 > 0$, or in the case $d_1 < 0$, system (3) admits four equilibria: $(0, 0)$, $\left(\frac{r}{\alpha_1}, 0\right)$, (x_1, y_1) and (x_2, y_2) where*

$$x_1 = \frac{r}{\alpha_1} - \frac{a}{\alpha_1}y_1,\ y_1 = \frac{\dfrac{-ca^2}{\alpha_1} - \sqrt{\Delta}}{2\beta},\ x_2 = \frac{r}{\alpha_1} - \frac{a}{\alpha_1}y_2 \text{ and } y_2 = \frac{\dfrac{-ca^2}{\alpha_1} + \sqrt{\Delta}}{2\beta}$$

If $\beta = \frac{c^2a^4}{4d_1\alpha_1^2}$, system (3) has three equilibria: $(0, 0)$, $\left(\frac{r}{\alpha_1}, 0\right)$ and $\left(x_1, \frac{-ca^2}{2\alpha_1\beta}\right)$. When $\beta > \frac{c^2a^4}{4d_1\alpha_1^2}$ and $d_1 > 0$ there exist two equilibria: $(0, 0)$ and $\left(\frac{r}{\alpha_1}, 0\right)$. If $d_1 = 0$, system (3) admits three equilibria: $(0, 0)$, $\left(\frac{r}{\alpha_1}, 0\right)$ and (x_1, y_1).

Proof. An equilibrium point of system (3) satisfies

$$\begin{cases} xr\left(1 - \dfrac{x}{k}\right) - axy - \alpha x^2 = 0 \\ caxy - dy - \beta y^3 = 0 \end{cases} \tag{4}$$

As d is positive, we get by equations $x = 0$ or $y = 0$, two equilibria $(0, 0)$ and $\left(\frac{r}{\alpha_1}, 0\right)$. On the other hand, equation

$$r\left(1 - \frac{x}{k}\right) - ay - \alpha x = 0 \tag{5}$$

gives $x = \frac{r}{\alpha_1} - \frac{a}{\alpha_1}y$. Solutions of equation

$$cax - d - \beta y^2 = 0 \tag{6}$$

are given based on values of d_1 and β compared to that of $\frac{c^2a^4}{4d_1\alpha_1^2}$. More precisely, it follows from equations (5) and (6): two equilibria (x_1, y_1) and (x_2, y_2) either if $\beta < \frac{c^2a^4}{4d_1\alpha_1^2}$ and $d_1 > 0$, or if $d_1 < 0$; one equilibrium point (x_1, y_1) if $d_1 = 0$; one equilibrium point $\left(x_1, \frac{-ca^2}{2\alpha_1\beta}\right)$ if $\beta = \frac{c^2a^4}{4d_1\alpha_1^2}$; and none if $\beta > \frac{c^2a^4}{4d_1\alpha_1^2}$ and $d_1 > 0$.

Remark 3.1. *The origin represents an equilibrium when both the prey and predator population die out and extinct, and the point* $\left(\frac{r}{\alpha_1}, 0\right)$ *represents the equilibrium when the prey population survives in the absence of the predator population. The nonlinear harvest strategy applied to model (3) leads to the predator extinct when the harvesting rate conditions satisfy* $\beta \geq \frac{c^2 a^4}{4 d_1 \alpha_1^2}$ *and* $d_1 \geq 0$*. In all other cases, the prey and predator coexist.*

3.2 Nature and stability of equilibria

In this section,we give the nature of equilibria of system (3) and their stability will be discussed.

First, we can show that the equilibrium point $\left(\frac{r}{\alpha_1}, 0\right)$ is a saddle, locally unstable, if $d_1 < 0$, and a locally stable node when $d_1 > 0$. However, if $d_1 = 0$, then $\left(\frac{r}{\alpha_1}, 0\right)$ is a degenerate equilibrium point, and further study is needed. In this case, in the following theorem we will study the stability of the two equilibria, origin and $\left(\frac{r}{\alpha_1}, 0\right)$ and describe the behavior of the solutions of (3) near this last equilibrium point.

Theorem 3.2. *Equilibrium* $(0, 0)$ *is unstable for system (3). If* $d_1 = 0$*, equilibrium point* $\left(\frac{r}{\alpha_1}, 0\right)$ *is a saddle-node, with two hyperbolic sectors and a parabolic one.*
There exists an invariant strong unstable manifold W_u *tangent at point* $\left(\frac{r}{\alpha_1}, 0\right)$ *to the* x*-axis, on which the behavior of system (3) is repulsive.*
On one side of W_u*, for each point* (x, y)*, there exists a center manifold* $W_c\,(x, y)$ *tangent at point* $\left(\frac{r}{\alpha_1}, 0\right)$ *to the line* $y = -\frac{\alpha_1}{a}x + \frac{r}{a}$*, and on the other side, all the center manifolds coincide and are tangent at point* $\left(\frac{r}{\alpha_1}, 0\right)$ *to the same line.*

Proof. The eigenvalues of the Jacobian matrix associated to system (3) at the equilibrium point $(0, 0)$ are $r > 0$ and $(-d) < 0$, then origin is an unstable saddle.

Now we suppose that $d_1 = 0$. First we translate the equilibrium point $\left(\frac{r}{\alpha_1}, 0\right)$ to the origin by the change of variables $X = x - \frac{r}{\alpha_1}$ and $Y = y$. System (3) becomes

$$\begin{cases} \dot{X} &= -rX - \dfrac{ar}{\alpha_1}Y - aXY - \alpha_1 X^2 \\ \dot{Y} &= caXY - \beta Y^3 \end{cases}$$

By using the transvection $u = X + \frac{r}{\alpha_1}Y$, $v = Y$ and by reversing the time $\tau = -t$, we get

$$\begin{cases} u' &= ru - \left(a - \dfrac{ca^2}{\alpha_1}\right)uv + \alpha_1 u^2 - \dfrac{ca^3}{\alpha_1^2}v^2 - \dfrac{\beta a}{\alpha_1}v^3 \\ v' &= -cauv + \dfrac{ca^2}{\alpha_1}v^2 + \beta v^3 \end{cases} \tag{7}$$

where $(')$ denotes $\left(\frac{d}{dt}\right)$. By exchanging the roles of u and v in system (7) by setting $\bar{x} = v$,

$\bar{y} = u$, we obtain

$$\begin{cases} \bar{x}' &= -ca\bar{x}\bar{y} + \dfrac{ca^2}{\alpha_1}\bar{x}^2 + \beta\bar{x}^3 \\ \bar{y}' &= r\bar{y} - \left(a - \dfrac{ca^2}{\alpha_1}\right)\bar{x}\bar{y} + \alpha_1\bar{y}^2 - \dfrac{ca^3}{\alpha_1^2}\bar{x}^2 - \dfrac{\beta a}{\alpha_1}\bar{x}^3 \end{cases} \tag{8}$$

We denote $A(x,y) = -caxy + \frac{ca^2}{\alpha_1}x^2 + \beta x^3$. Equation $\bar{y}' = 0$ gives

$$\alpha_1\bar{y}^2 + \left(r - \left(a - \frac{ca^2}{\alpha_1}\right)\bar{x}\right)\bar{y} - \frac{ca^3}{\alpha_1^2}\bar{x}^2 - \frac{\beta a}{\alpha_1}\bar{x}^3 = 0 \tag{9}$$

The solutions of equation (9) depend on the sign of

$$\bar{\Delta} = r^2 + \left(a + \frac{ca^2}{\alpha_1}\right)^2\bar{x}^2 - 2r\left(a - \frac{ca^2}{\alpha_1}\right)\bar{x} + 4a\beta\bar{x}^3$$

Because $(\bar{x},\bar{y})$ is supposed to be in the neighborhood of the origin, then since $\bar{x}$ is smallest than r^2, we obtain $\bar{\Delta} > 0$ and equation (9) has two solutions

$$\bar{y}_1(\bar{x}) = \frac{1}{2\alpha_1}\left(-\left(r - \left(a - \frac{ca}{\alpha_1}\right)\bar{x}\right) - \sqrt{\Delta_3}\right)$$

and

$$\bar{y}_2(\bar{x}) = \frac{1}{2\alpha_1}\left(-\left(r - \left(a - \frac{ca}{\alpha_1}\right)\bar{x}\right) + \sqrt{\Delta_3}\right)$$

We can show that $\bar{y}_1$ is excluded and $\bar{y} = \bar{y}_2$. By the third order Taylor development, we get

$$A(\bar{x},\bar{y}_2(\bar{x})) = \frac{ca^2}{\alpha_1}\bar{x}^2 + \left(\beta - \frac{c^2a^4}{r\alpha_1^2}\right)\bar{x}^3 + o\left(\bar{x}^3\right)$$

Since $\frac{ca^2}{\alpha_1} > 0$, following the similar arguments in [7] which is based on removing of flat terms in order to get the $\mathcal{C}^\infty$ normal forms by using the homotopic method, we conclude that origin is a saddle-node for system (8). It follows from the stable manifold theorem ([8], [16]) that:

- there exists an invariant strong unstable manifold W_u tangent at origin to the $\bar{y}$-axis such that on W_u, system (8) is analytically conjugate to $\bar{y}' = r\bar{y}$, and the behavior is repulsive;

- for any $(\bar{x}_0,\bar{y}_0)$, if $\bar{x}_0 > \bar{x}_{W_u}$, where $\bar{x}_{W_u}$ is the abscissa of W_u at the point whose ordinate is $\bar{y}_0$, it passes a center manifold W_c tangentially at origin to the $\bar{x}$-axis. On W_c, system (8) is $\mathcal{C}^\infty$-conjugate to $\bar{x}' = \bar{x}^2 + \left(\beta - \frac{c^2a^4}{r\alpha_1^2}\right)\bar{x}^3$. If $\bar{x}_0 < \bar{x}_{W_u}$, all the center manifolds coincide and are tangent at origin to $\bar{x}$-axis;

- system (8) is $\mathcal{C}^\infty$-conjugate to

$$\begin{cases} \bar{x}' &= \bar{x}^2 + \left(\beta - \dfrac{c^2a^4}{r\alpha_1^2}\right)\bar{x}^3 \\ \bar{y}' &= r\bar{y} \end{cases}$$

and $\mathcal{C}^0$-conjugate to

$$\begin{cases} \bar{x}' &= \bar{x}^2 \\ \bar{y}' &= \bar{y} \end{cases}$$

Going back to the coordinates (x,y) by: inverting transvection; exchanging the roles of u and v; translation; and by change time, we obtain the result.

Remark 3.2. *In the proof of theorem 3.2, more details are given about the behavior of solutions near the equilibrium point* $\left(\frac{r}{\alpha_1}, 0\right)$.

Because that the coordinates y_2 is positive if and only if $d_1 \leq 0$, and $y_1 < 0$, equilibrium point (x_1, y_1) is not biological meaningful and it is the same for (x_2, y_2) if $d_1 > 0$, but we will study them mathematically, because of their mathematical interest; we note that x_1 and x_2 remain positive. We will show in the following theorem that stability and nature of equilibria depend essentially upon of the values of the mortality rate d of the predator and of the harvesting parameters α and β.

Theorem 3.3. *The equilibrium point* (x_1, y_1) *is locally stable if* $\beta \neq \frac{c^2a^4}{4d_1\alpha_1^2}$ *and locally unstable if* $\beta = \frac{c^2a^4}{4d_1\alpha_1^2}$. *The equilibrium point* (x_2, y_2) *is locally unstable if* $d_1 \geq 0$, *and it is either a linear center, a focus or a node if* $d_1 < 0$; *it is locally stable if* $d_1 < 0$ *and* $\left(1 - \frac{ca}{\alpha_1}\right) \geq 0$.

Proof. We start the proof with the case $d_1 > 0$. We assume that $\Delta \neq 0$. The Jacobian matrix of system (3) at a point (x, y) such that $x = \frac{r}{\alpha_1} - \frac{a}{\alpha_1}y$, is given by

$$J_{(x,y)} = \begin{pmatrix} -r + ay & -a\left(\frac{r}{\alpha_1} - \frac{a}{\alpha_1}y\right) \\ cay & ca\left(\frac{r}{\alpha_1} - \frac{a}{\alpha_1}y\right) - d - 3\beta y^2 \end{pmatrix}$$

We denote $A = -r + ay$, $B = -a\left(\frac{r}{\alpha_1} - \frac{a}{\alpha_1}y\right)$, $C = cay$ and $D = ca\left(\frac{r}{\alpha_1} - \frac{a}{\alpha_1}y\right) - d - 3\beta y^2$. The characteristic equation is $\lambda^2 - (A + D)\lambda + (AD - BC) = 0$ and the eigenvalues of the matrix are given according to the sign of $\Delta_1 = (A - D)^2 + 4BC$. If $y = y_1$ or $y = y_2$, then y is negative, and we get $BC > 0$, and $\Delta_1 > 0$. Hence, matrix $J_{(x,y)}$ has two simple real eigenvalues, and (x, y) is a saddle or a node. In the other hand, $\det\left(J_{(x,y)}\right) = (r - ay)\left(-2\frac{ca^2}{\alpha_1}y + d_1 + 3\beta y^2\right)$. The sign of $I_1(x, y) := 2\frac{ca^2}{\alpha_1}y + d_1 + 3\beta y^2$ depends on that of $\Delta_2 := \frac{c^2a^4}{\alpha_1^2} - 3\beta d_1$. We can write $\Delta_2 = \Delta + \beta d_1$ which is positive. This means that $I_1(x, y)$ has two roots,

$$\overline{y} = \frac{-\frac{c^2a^4}{\alpha_1^2} - \sqrt{\Delta_2}}{3\beta} \text{ and } \overline{\overline{y}} = \frac{-\frac{c^2a^4}{\alpha_1^2} + \sqrt{\Delta_2}}{3\beta}$$

First, we will compare y_2 with respect to $\overline{\overline{y}}$. To doing this, suppose that $y_2 > \overline{\overline{y}}$. Then

$$-\frac{ca^2}{\alpha_1} + \sqrt{\Delta} > \frac{2}{3}\left(-\frac{c^2a^4}{\alpha_1^2} + \sqrt{\Delta_2}\right)$$

It follows that $\frac{ca^2}{\alpha_1}\sqrt{\Delta_2} < \frac{c^2a^4}{\alpha_1^2} - 6\beta d_1$. It leads after some calculus to $\frac{c^2a^4}{\alpha_1^2} - 4\beta d_1 < 0$, which is absurd. By the same way, we suppose that $y_2 < \overline{y}$. Then, $\sqrt{\Delta} < \frac{ca^2}{3\alpha_1} - \frac{2}{3}\sqrt{\Delta_2}$. Suppose that $\frac{ca^2}{3\alpha_1} - \frac{2}{3}\sqrt{\Delta_2} > 0$. By squaring and after simplifications, we obtain the same absurdity as above. We deduce that, $I_1(x_2, y_2) < 0$ and (x_2, y_2) is a saddle.
We now suppose that $y_1 > \overline{y}$. Then

$$\sqrt{\Delta} < -\frac{ca^2}{3\alpha_1} + \frac{2}{3}\sqrt{\Delta_2}$$

But we have just shown that $-\frac{ca^2}{3\alpha_1}+\frac{2}{3}\sqrt{\Delta_2}>0$. It follows that

$$\frac{ca^2}{\alpha_1}\sqrt{\Delta_2}<-\frac{c^2a^4}{\alpha_1^2}+6\beta d_1 \tag{10}$$

Suppose that $-\frac{c^2a^4}{\alpha_1^2}+6\beta d_1>0$ and by squaring in (10), we get $\Delta<0$ which is absurd. Then $I_1(x_1,y_1)>0$, and (x_1,y_1) is a node.
On the other hand, $tr\left(J_{(x_1,y_1)}\right)=(-d_1-r)+\left(a\left(1-\frac{ca}{\alpha_1}-3\beta y_1\right)\right)y_1$. We show that $\left(1-\frac{ca}{\alpha_1}-3\beta y_1\right)>0$, and then $A+D<0$. We deduce that (x_1,y_1) is a stable node.

Now we assume that $d_1<0$. We begin with the equilibrium point (x_2,y_2). We have

$$\det\left(J_{(x,y)}\right)=(r-ay)\left(-\frac{ca}{\alpha_1}(r-ay)y+d+3\beta y^2+\frac{ca^2}{\alpha_1}y\right)$$

We denote $I_2(x,y)=2\frac{ca^2}{\alpha_1}y+d_1+3\beta y^2$. Its sign depends on that of $\Delta_3:=\frac{c^2a^4}{\alpha_1^2}-3\beta d_1$. We have $\Delta_3=\Delta+\beta d_1$, it is positive. This means that $I_2(x,y)$ has two roots,

$$\overline{y}=\frac{-\frac{c^2a^4}{\alpha_1^2}-\sqrt{\Delta_2}}{3\beta}\text{ and }\overline{\overline{y}}=\frac{-\frac{c^2a^4}{\alpha_1^2}+\sqrt{\Delta_2}}{3\beta}$$

We will compare y_2 with respect to $\overline{\overline{y}}$. Suppose that $y_2<\overline{\overline{y}}$. Then

$$2\frac{c^2a^4}{\alpha_1^2}+3\left(-\frac{ca^2}{\alpha_1}+\sqrt{\Delta}\right)<\sqrt{\Delta_2} \tag{11}$$

By squaring, since $-\frac{ca^2}{\alpha_1}+\sqrt{\Delta}=2\beta y_2$, the term in the left of (11) is positive and we get $\Delta<\frac{ca^2}{\alpha_1}\sqrt{\Delta_2}$. It follows by some calculus that $-4\beta d_1<0$ which is absurd, and then $I_2(x_2,y_2)>0$. Moreover, we can show that $r-ay_2>0$; otherwise $\sqrt{\Delta}<\frac{ca^2}{\alpha_1}+\frac{2r\beta}{a}$, and by squaring and simplifications, we get $d<-\frac{r^2\beta}{a^2}<0$ which contradicts the assumption. Hence (x_2,y_2) is a node, a center or a focus.
Since $y_1<0$, we have $\Delta_1>0$ at the point (x_1,y_1). Therefore, equilibrium (x_1,y_1) is a saddle or a node. To complete the study of the stability of (x_1,y_1), we will look at the sign of $I_2(x_1,y_1)$. Suppose that $y_1>\overline{y}$. Then,

$$\sqrt{\Delta}<-\frac{ca^2}{3\alpha_1}+\frac{2}{3}\sqrt{\Delta_2}$$

But $-\frac{ca^2}{3\alpha_1}+\frac{2}{3}\sqrt{\Delta_2}>0$. It follows, by squaring and after simplifications, that

$$\frac{ca^2}{\alpha_1}\sqrt{\Delta_2}<-\frac{c^2a^4}{\alpha_1^2}+6\beta d_1 \tag{12}$$

Since $-\frac{c^2a^4}{\alpha_1^2}+6\beta d_1<0$, then $y_1<\overline{y}$ and $I_2(x_1,y_1)>0$. Since $(r-ay_1)>0$, we deduce that the equilibrium point (x_1,y_1) is a node.

To study the stability of the equilibria (x_1,y_1) and (x_2,y_2), we look at

$$tr\left(J_{(x,y)}\right)=(r-ay)\left(-1+\frac{ca}{\alpha_1}\right)-d-3\beta y^2$$

and it is easy to see that if $1-\frac{ca}{\alpha_1}\geq 0$, then the two equilibria are stable. A complementary study of the vector field (3) shows that (x_1,y_1) is a stable if $d_1<0$.

If $d_1=0$, then $(x_2,y_2)=\left(\frac{r}{\alpha_1},0\right)$ and is, by theorem 3.2, a saddle-node.

If $\beta=\frac{c^2a^4}{4d_1\alpha_1^2}$, then $(x_1,y_1)=(x_2,y_2)=\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)$. The Jacobian matrix associated to system (3) at this equilibrium point is

$$J_{\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)}=\begin{pmatrix} -r-\frac{ca^3}{2\alpha_1\beta} & -\frac{c^2a^3}{2\alpha_1\beta}\\ \frac{-a}{\alpha_1}\left(r+\frac{ca^3}{2\alpha_1\beta}\right) & cax_1-d-3\beta\left(\frac{ca^2}{2\alpha_1\beta}\right)^2\end{pmatrix}$$

We have

$$\det\left(J_{\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)}\right)=-\left(r+\frac{ca^3}{2\alpha_1\beta}\right)\left(-d+\frac{car}{\alpha_1}-\frac{c^2a^4}{4\alpha_1\beta}\right)-\frac{c^2a^4}{2\alpha_1^2\beta}\left(r+\frac{ca^3}{2\alpha_1\beta}\right)$$

Then, $\det\left(J_{\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)}\right)=\left(r+\frac{ca^3}{2\alpha_1\beta}\right)\left(d_1-\frac{c^2a^4}{4\alpha_1\beta}\right)$. Since $d_1-\frac{c^2a^4}{4\alpha_1\beta}=0$, then $\det\left(J_{\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)}\right)=0$. On the other hand,

$$tr\left(J_{\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)}\right)=-r-\frac{ca^3}{2\alpha_1\beta}+\frac{ca}{\alpha_1}\left(r+\frac{ca^3}{2\alpha_1\beta}\right)-d-\frac{3c^2a^4}{4\alpha_1^2\beta}$$

It follows that

$$\begin{aligned} tr\left(J_{\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)}\right) &= -r-\frac{ca^3}{2\alpha_1\beta}+\frac{1}{4\beta}\left(\frac{c^2a^4}{\alpha_1^2}-4\beta d\left(d-\frac{ca}{\alpha_1}\right)\right)-\frac{c^2a^4}{2\alpha_1^2\beta}\\ &= -r-\frac{ca^3}{2\alpha_1\beta}-\frac{c^2a^4}{2\alpha_1^2\beta}<0\end{aligned}$$

Then, $\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)$ is a semi-hyperbolic equilibrium point; it is either a node, a saddle or a saddle-node. We get by an additional study of the vector field (3), that $\left(x_1,\frac{-ca^2}{2\alpha_1\beta}\right)$ is a saddle-node; then it is unstable.

In the following result, we use Dulac's criterion [14] to show the nonexistence of limit cycles in system (3).

Theorem 3.4. *System (3) does not have a periodic solution and then it does not have a limit cycle in*

$$D^+=\{(x,y)\in\mathbb{R}\times\mathbb{R};\, x>0,y>0\}.$$

Proof. Consider a real-valued continuously differentiable Dulac function

$$h(x,y)=\frac{1}{xy}$$

for $(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$ and denote

$$\begin{cases} f(x,y)=xr\left(1-\frac{x}{k}\right)-axy-\alpha x^2\\ g(x,y)=caxy-dy-\beta y^3\end{cases}$$

We have

$$\frac{\partial(hf)}{\partial x}(x,y)+\frac{\partial(hg)}{\partial y}(x,y)=-\frac{r}{ky}-\frac{\alpha}{y}-\frac{2\beta y}{x}$$

Since $x>0$ and $y>0$, then $\frac{\partial(hf)}{\partial x}(x,y)+\frac{\partial(hg)}{\partial y}(x,y)$ is not identically zero and is of one sign. Therefore, system (3) does not have a closed orbit in D^+.

Remark 3.3. *The proof of theorem 3.4 remains true if we replace D^+ by the simply connected region*

$$D^- := \{(x,y)\in\mathbb{R}\times\mathbb{R};\, x>0, y<0\}.$$

Since x-axis is invariant under system (3) and because that (3) has no equilibria in the region

$$D' := \{(x,y)\in\mathbb{R}\times\mathbb{R};\, x<0, y\in\mathbb{R}\}$$

we can deduce that system (3) has no closed orbits in all the plan.

We can now state results about the global stability of equilibria of model (3).

Theorem 3.5. *1) If $d_1<0$, then the positive equilibrium point (x_2,y_2) is asymptotically stable; it is globally asymptotically stable in $D\cap D^+$.*
2) If $d_1>0$, then the equilibrium point $\left(\frac{r}{\alpha_1},0\right)$ is globally asymptotically stable in $D\cap(D^+\cup(\mathbb{R}_+\times\{0\}))$; if moreover $\beta>\frac{c^2a^4}{4d_1\alpha_1^2}$, it is globally asymptotically stable in $D\cap(D^+\cup D^-\cup(\mathbb{R}_+\times\{0\}))$.

Proof. It follows from theorem 3.4 that, when $d_1<0$, the equilibrium point (x_2,y_2) can not be a center for system (3). By using in addition theorem 3.3, we deduce that (x_2,y_2) is locally asymptotically stable.
Moreover, we know by theorem 3.1 that there is not any other equilibrium of system (3) than (x_2,y_2) in D^+ which is positively invariant for system (3). It follows from theorem 3.4 that any trajectory of system (3) in $D\cap D^+$ tends to (x_2,y_2) when t tends to positive infinity, and the first part of the theorem is shown.

We know that if $d_1>0$, then equilibrium point $\left(\frac{r}{\alpha_1},0\right)$ is a stable node. It is, except the origin which is instable, the unique equilibrium point of system (3) in $D^+\cup(\mathbb{R}_+\times\{0\})$ which is positively invariant for system (3). On the other hand, if moreover $\Delta<0$, then $\left(\frac{r}{\alpha_1},0\right)$ is the unique equilibrium point in $D^+\cup D^-\cup(\mathbb{R}_+\times\{0\})$; the direction of vector fields allows us to conclude the second part of the theorem.

Using the same arguments as in theorem 3.5, we show the following result.

Theorem 3.6. *If $d_1<0$, then the equilibrium point (x_1,y_1) is locally asymptotically stable; it is globally asymptotically stable in D^-.*

Theorem 3.7. *System (3) is permanent and persistent if and only if $d_1<0$.*

Proof. We consider in the case $d_1<0$, the points:

$A:=(x_0,y_0)$ a point of the curve $y=\sqrt{\frac{1}{\beta}(cax-d)}$ such that $\frac{d}{ca}<x_0<x_2$;
B the point of intersection of the horizontal line passing through A with the line $y=(\frac{-r}{ak}-\frac{\alpha}{a})x+\frac{r}{a}$;

E the point of intersection of the vertical line through B with the curve $y = \sqrt{\frac{1}{\beta}(cax - d)}$;
M the point of intersection of the horizontal line through E with the line $y = (\frac{-r}{ak} - \frac{\alpha}{a})x + \frac{r}{a}$;
N the point of intersection of the vertical line through M with the curve $y = \sqrt{\frac{1}{\beta}(cax - d)}$;
C the part of the curve $y = \sqrt{\frac{1}{\beta}(cax - d)}$ connecting the points M and A.
Let D_1 be the region of the plan delimited by the straight segments AB, BE, EM and MN, and by the curved line C. We show that D_1 is positively invariant compact for system (3). Since in the case $d_1 < 0$, the point (x_2, y_2) is the unique equilibrium in the first quadrant of the plan and using, according to theorem 3.5, the fact that (x_2, y_2) is globally asymptotically stable, we can then conclude.

3.3 Bifurcation analysis

We present in this part the different bifurcation diagrams of multiple equilibria of system (3), for various parameter values. Note that model (3) exhibits a forward and a backward bifurcation.

Example 3.1. *Choosing $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$ and $\beta = 0.1$, we get $d_1 \leq 0$ for $\alpha \leq 0.25$. The two equilibria (x_2, y_2) and $(\dfrac{r}{\alpha_1}, 0)$ coincide at the bifurcation value $\alpha = 0.25$ as shown in Figure 1.*

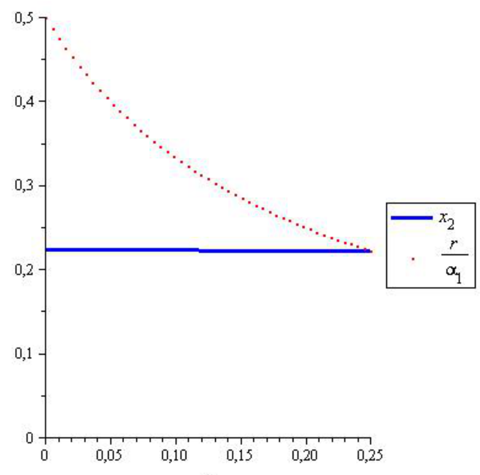

Figure 1: A part of the bifurcation diagram for model (3) with x_2 and $\dfrac{r}{\alpha_1}$ versus α. The continued line presents the curve of the prey with coexistence equilibrium (x_2, y_2), and the dotted line indicates the curve of the prey with coexistence equilibrium $(\dfrac{r}{\alpha_1}, 0)$.

Example 3.2. *Using the following parameter values $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$ and $\beta = 0.1$, we get $d_1 \geq 0$ and $\beta > \frac{c^2a^4}{4d_1\alpha_1^2}$ for $0.25 \leq \alpha \leq 1.6318$. When α varies, the two equilibria (x_1, y_1) and (x_2, y_2) coincide at the forward bifurcation value $\alpha = 1.6318$ as shown in Figure 2.*

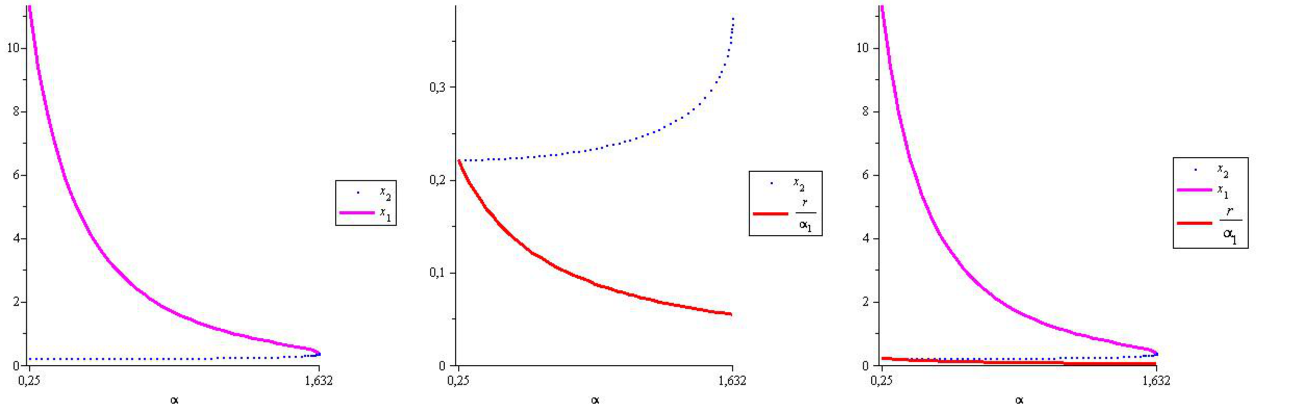

Figure 2: The bifurcation diagram for model (3) with x_1, x_2, and $\dfrac{r}{\alpha_1}$ versus α, between the forward and the backward bifurcations. The two continued lines present the curves of the prey with coexistence equilibrium (x_1, y_1) and $(\dfrac{r}{\alpha_1}, 0)$, and the dotted line indicates the curve of the prey with coexistence equilibrium (x_2, y_2).

Example 3.3. *Selecting the following parameter values $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$ and $\beta = 0.1$, we get $d_1 \geq 0$ for $\alpha \geq 0.25$. When α varies, the two equilibria (x_1, y_1) and (x_2, y_2) extinct after the backward bifurcation value $\alpha = 1.6318$ as shown in Figure 3.*

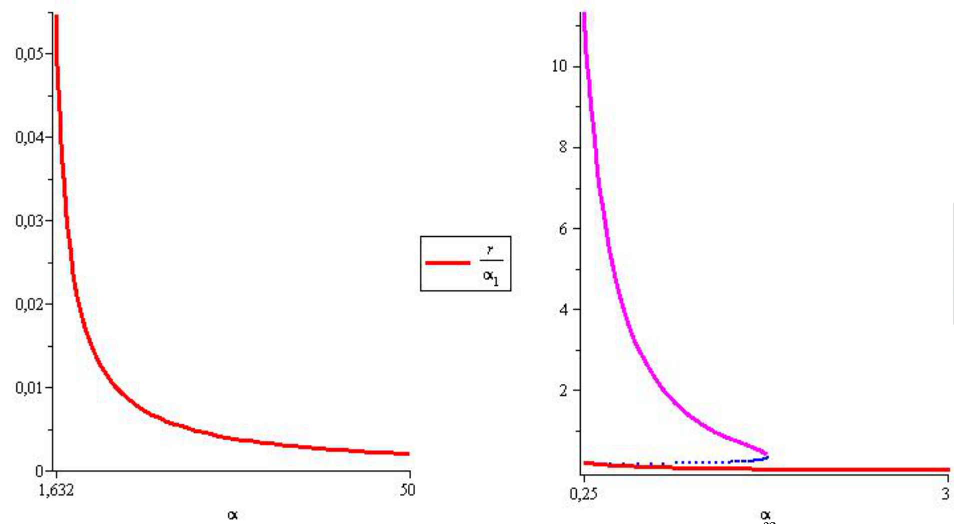

Figure 3: The bifurcation diagram for model (3) with x_1, x_2, and $\dfrac{r}{\alpha_1}$ versus α, after the backward bifurcation for the right figure, and after the forward bifurcation for the left one. The two continued lines present the curves of the prey with coexistence equilibrium (x_1, y_1) and $(\dfrac{r}{\alpha_1}, 0)$, and the dotted line indicates the curve of the prey with coexistence equilibrium (x_2, y_2).

Remark 3.4. *If the harvesting rate parameter α exceeds the value giving $d_1 = 0$, system (3) does not have positive equilibria (Figures 1-3), and there is a predator extinction. If α is less than this value, an unstable equilibrium and a stable one exist in model (3), and the prey and the predator coexist.*

4 Numerical simulation

We give in this section some interesting numerical examples for system (3).

Example 4.1. *Setting parameter values $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$ and $\alpha = 0.1$, we get $d_1 = -0.1 < 0$. The equilibrium point (x_2, y_2) exists and is positive and globally asymptotically stable (Figure 4). In this case, $(\frac{r}{\alpha_1}, 0)$ is an unstable equilibrium point.*

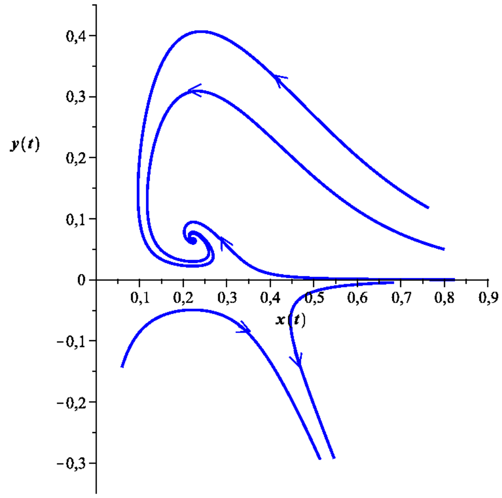

Figure 4: The phase portrait of system (3) when (x_2, y_2) is globally asymptotically stable and $(\frac{r}{\alpha_1}, 0)$ is unstable.

Example 4.2. *Setting $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$ and $\alpha = 0.25$, we get $d_1 = 0$. The equilibrium point (x_2, y_2) coincides with the equilibrium $(\frac{r}{\alpha_1}, 0)$ giving a saddle-node point and a forward (a transcritical) bifurcation occurs (Figure 5); in this case there is no other positive equilibria. A forward bifurcation diagram is illustrated in Figure 1 and Figure 2 . The center manifold of the saddle-node point divides the plan into two regions. The region above is the basin of attraction for $(\frac{r}{\alpha_1}, 0)$ and below is the attracting basin for the non-positive equilibrium point (x_1, y_1).*

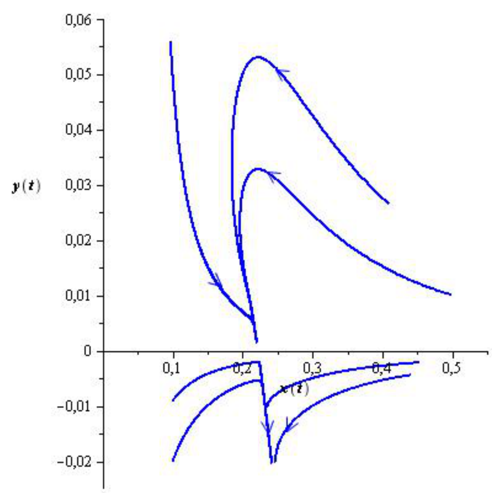

Figure 5: The phase portrait of system (3) when (x_2, y_2) meets $(\frac{r}{\alpha_1}, 0)$ at a saddle-node equilibrium.

Example 4.3. *Using the following parameter values $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$ and $\alpha = 0.7$, we obtain $d_1 = 0.1 > 0$. The equilibrium point (x_2, y_2) is not positive and* *it is unstable and there is no positive equilibria other than* $(\frac{r}{\alpha_1}, 0)$*; this last equilibrium is stable. The equilibrium point* (x_1, y_1) *is stable. The phase portrait of model (3) is shown in Figure 6.*

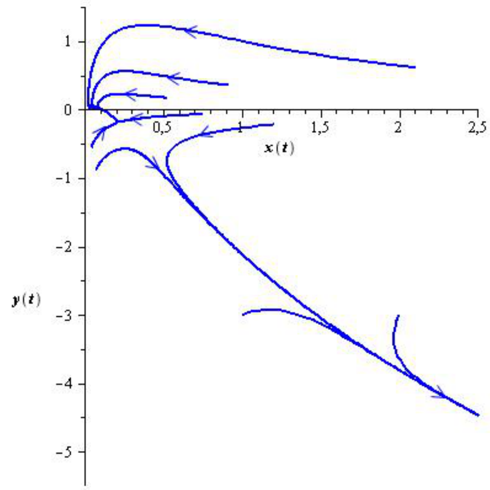

Figure 6: The phase portrait of system (3) with stable equilibria (x_1, y_1) and $(\frac{r}{\alpha_1}, 0)$ and an unstable equilibrium point (x_1, y_1).

Example 4.4. *Taking* $d = 0.2$*,* $r = 0.1$*,* $k = 0.5$*,* $c = 1.8$*,* $a = 0.5$*,* $\beta = 0.1$ *and* $\alpha = 1.6318$ *we get* $d_1 = 0.150868 > 0$*. The equilibrium point* (x_2, y_2) *meets the equilibrium* (x_1, y_1) *giving an unstable equilibrium. A backward bifurcation (a saddle-node bifurcation) occurs as illustrated in Figure 3. There is no positive equilibria other than* $(\frac{r}{\alpha_1}, 0)$ *which is stable. The phase portrait of model (3) is shown in Figure 7.*

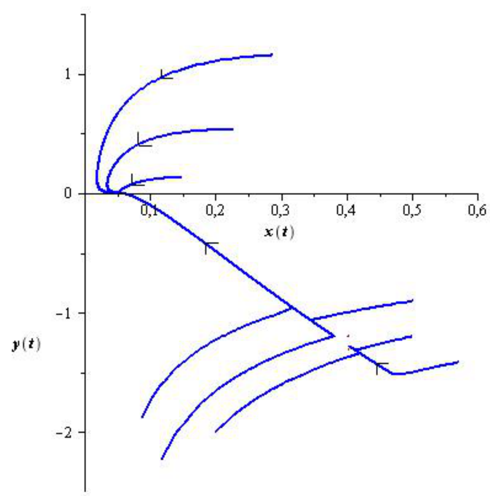

Figure 7: The phase portrait of system (3) when the unstable equilibrium (x_2, y_2) coincides with the stable equilibrium point (x_1, y_1).

Example 4.5. *Taking* $d = 0.2$*,* $r = 0.1$*,* $k = 0.5$*,* $c = 1.8$*,* $a = 0.5$*,* $\beta = 0.1$ *and* $\alpha = 1.8$ *we obtain* $d_1 = 0.045 > 0$*. Equilibria* (x_1, y_1) *and* (x_2, y_2) *disappear. In this case, model (3) admits only two equilibria, origin and a stable equilibrium* $(\frac{r}{\alpha_1}, 0)$*. The phase portrait of model (3) is shown in Figure 8.*

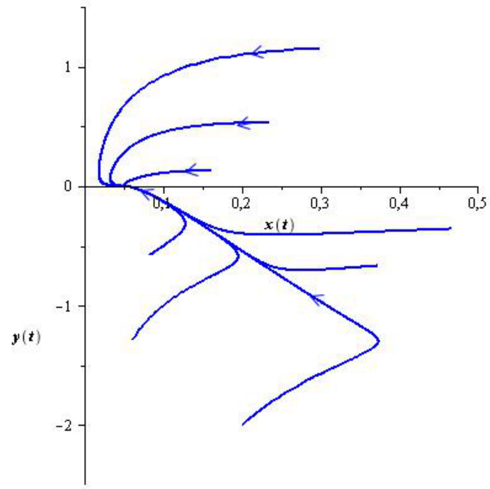

Figure 8: The phase portrait of system (3) when the two equilibria (x_1, y_1) and (x_2, y_2) no longer exist. The equilibrium $(\dfrac{r}{\alpha 1}, 0)$ is globally asymptotically stable.

5 Conclusion

In this model, we discussed the effects of nonlinear harvesting in a predator-prey system in which both the species are harvested. The equilibria of model (3) are examined and the stability is discussed. We showed that it occurs two interesting bifurcations which are a backward and a forward bifurcations (a saddle-node and a transcritical bifurcation). Moreover, we showed that model (3) exhibits for some values of the parameters, a varied dynamic, like the permanence and the persistence of the model; a unique positive globally asymptotically stable coexistence equilibrium, connecting with a coexistence of positive saddle equilibrium point and changing its stability; saddle-node equilibria; extinction of two equilibria. We showed by the Bendixson-Dulac criterion that model (3) have no periodic solutions and than nonexistence of limit cycle is proved. Finally a numerical simulation is taken to verify the results we obtained.

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