

# On metric dimension and fault-tolerant metric dimension of plane graphs

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**Abstract** The problem of characterizing the classes of plane graphs with a bounded fault-tolerant metric dimension or metric dimension is of great interest nowadays. In this article, we obtain the metric dimension and fault-tolerant metric dimension of two interminable classes of rotationally symmetrical plane graphs, that are found to be constant for each of these families of the plane graphs. We show that just 3 vertices properly chosen are adequate to resolve all the vertices of these classes of rotationally symmetrical graphs, and set upper and lower bounds for the fault-tolerant metric dimension of these two classes of the plane graphs. We additionally observe that the upper bound for the fault-tolerant location number for the given family of the plane graph depends upon the distances between the vertices in the taken basis set of the family of the plane graph.

## 1. Introduction

The idea of resolving or locating set and that of metric dimension goes back to the 1950s. They were characterized by L. M. Blumenthal [2] with regards to metric space. Slater [24] presented the idea of a resolving (or locating) set for a connected graph which was likewise autonomously discovered by Melter and Harary [10]. Melter et al. [17] contemplated the location number issue for grid graphs induced by lattice points in the plane when the separations are estimated in the  $L_\infty$  and  $L_1$  metrics. Hernando et al. [5] make an overall investigation of the location number of Cartesian products of graphs. Part of their inspiration for contemplating the location number of Cartesian products are that in two of the applications, to be specific coin gauging (or coin weighing) and Mastermind, the graphs that emerge are Cartesian products. It was noted in [8] that deciding the location number of a graph is an  $NP$ -hard problem.

The invariant metric dimension has been read for trees, heptagonal circular ladder [23], Harary, and circulant graphs [9], etc. Graph theory has applications in numerous zones of figuring, social, and normal sciences and is likewise an affable play area for the investigation of the verification procedure in discrete science. Utilizations of this invariant to issues of picture preparing (or image processing) and design acknowledgment (or pattern recognition) are talked about in [17], to the route of exploring specialist (navigating agent or robots) in systems (or networks) are examined in [15], applications to science are given in [6], application to combinatorial enhancement (or optimization) is yielded in [22], and to issues of check and system revelation (or network discovery) in [1].

Components of metric basis were alluded to as sensors in an application given in [7]. If one of the controls does not work appropriately, we would not have enough data to manage the gatecrasher (fire, intruder, cheat, and so forth). To conquer these sorts of problems, the idea of the fault-tolerant metric dimension was presented by Hernando et al. [11]. A fault-tolerant locating set gives accurate data in any event when one of the controls is not working. Therefore, fault-tolerant locating sets are significant from the relevant point of view.

Javaid et al. demonstrated that the contrast between the fault-tolerant metric dimension and metric dimension can be discretionarily huge [14]. In [20] Raza et al. portrayed some extremal structures of graphs for the fault-tolerant metric dimension. Moreover, they constructed some interminable classes of regular graphs having a steady fault-tolerant metric dimension. Again, Raza et al.

in [21], examined the fault-tolerant resolvability for certain classes of interconnection networks. As of late, in [19] Raza et al. examined the fault-tolerant metric dimension issue for some interminable classes of the graphs of convex polytopes.

A polytope in elementary geometry is a geometric object with flat sides. When polytopes are having an additional property that they are convex sets and are contained in the  $n$ -dimensional space  $\mathbb{R}^n$  (Euclidean space), then they are termed as convex polytopes. Depending upon the problem with which we are dealing, polytopes and convex polytopes are defined accordingly (polytopes and convex polytopes may be defined in several ways by different researchers). Convex polytopes assume a significant job both in different branches of arithmetic and in applied zones, most quite in linear programming. The location number of a few classes of convex polytopes has been considered in [13, 23].

## 2. Preliminaries

In this section, we give some preliminary results and characterize some necessary terminologies. Let  $\Phi = \Phi(\mathbb{V}, \mathbb{E})$  be a basic associated (i.e., simple and connected) and undirected (i.e., all edges are unidirectional, with no direction associated with them) graph (or network) with an edge set  $\mathbb{E}$  and a vertex set  $\mathbb{V}$ . The metric dimension of the network  $\Phi = \Phi(\mathbb{V}, \mathbb{E})$  is the minimal number of nodes (vertices or hubs) in a set with the property that the rundown of good ways from any vertex to those in the set exceptionally distinguishes that vertex. By definition, for an arranged (ordered) subset  $\mathbb{F} = \{\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_z\}$  of nodes in a graph  $\Phi$  and a node  $\varpi$  of  $\Phi$ , the metric code/metric representation of  $\varpi$  concerning  $\mathbb{F}$  is the ordered  $z$ -tuple (or  $z$ -vector)  $\zeta(\varpi|\mathbb{F}) := (d_\Phi(\varpi, \varpi_1), d_\Phi(\varpi, \varpi_2), \dots, d_\Phi(\varpi, \varpi_z))$ .

If each pair of unmistakable vertices of  $\Phi$  have distinctive metric representations then the arranged set  $\mathbb{F}$  is known as a resolving (locating) set of  $\Phi$ . The metric dimension or the location number of  $\Phi$  is the cardinality of the subset  $\mathbb{F}$  i.e., location number (or metric dimension) of the graph  $\Phi$ , indicated by  $dim(\Phi)$  or  $\beta(\Phi)$ , is the smallest size of the locating set on the graph  $\Phi$ ; formally,  $\beta(\Phi) = dim(\Phi) = minimum\{|\mathbb{F}| : \mathbb{F} \text{ is locating set (or resolving set)}\}$ . If a subset  $\mathbb{T}$  of the arrangement of nodes  $\mathbb{V}(\Phi)$  is both resolving and independent, then the set  $\mathbb{T}$  is known as an independent resolving set for the graph  $\Phi$ .

**Example:** Consider the graph  $\Phi$  on 6 nodes as shown in Fig.1. The set  $\mathbb{F}_1 = \{\varpi_1, \varpi_2, \varpi_3\}$  is a locating set for the graph  $\Phi$  since the metric codes for the nodes of the graph  $\Phi$  with respect to  $\mathbb{F}_1$  are:  $\zeta(\varpi_1|\mathbb{F}_1) = (0, 1, 1)$ ;  $\zeta(\varpi_2|\mathbb{F}_1) = (1, 0, 2)$ ;  $\zeta(\varpi_3|\mathbb{F}_1) = (1, 2, 0)$ ;  $\zeta(\varpi_4|\mathbb{F}_1) = (2, 1, 1)$ ;  $\zeta(\varpi_5|\mathbb{F}_1) = (1, 2, 1)$ ;  $\zeta(\varpi_6|\mathbb{F}_1) = (2, 1, 2)$ ;  $\zeta(\varpi_7|\mathbb{F}_1) = (3, 2, 2)$ ;  $\zeta(\varpi_8|\mathbb{F}_1) = (4, 3, 3)$ . However,  $\mathbb{F}_1$  is not a minimum locating set since  $\mathbb{F}_2 = \{\varpi_1, \varpi_3\}$  is likewise a locating set with littler cardinality. Then again, the set  $\mathbb{F}_3 = \{\varpi_1\}$  is not a locating set since  $\zeta(\varpi_2|\mathbb{F}_3) = \zeta(\varpi_3|\mathbb{F}_3) = 1$ . Utilizing a comparable contention it is anything but difficult to watch that none of singleton vertex forms a locating set, and hence  $\beta(\Phi) = 2$ .

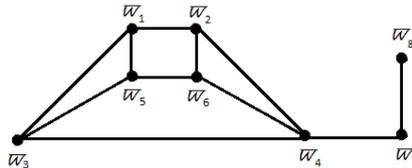


Figure 1: The graph  $\Phi$

For an organized set (or ordered set) of nodes  $\mathbb{F} = \{\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_z\}$  of  $\Phi$ , the  $p$ th component (or distance coordinate) of the code  $\zeta(\varpi|\mathbb{F})$  is zero iff  $\varpi = \varpi_p$ . Subsequently, to watch that the set  $\mathbb{F}$  is a resolving set, it is adequate to confirm that  $\zeta(\varpi|\mathbb{F}) \neq \zeta(\varrho|\mathbb{F})$  for any couple of distinguishable

vertices  $\varrho, \varpi \in \mathbb{V}(\Phi) \setminus \mathbb{F}$ .

In [6] Chartrand et al. demonstrate all the connected graphs with location number\metric dimension one.

**Theorem 1.** [6] *A connected graph is the path graph ( $\wp_n, n \geq 2$ ) iff it has metric dimension one.*

The Cartesian product of two graphs  $\Psi_1$  and  $\Psi_2$ , denoted by  $\Phi = \Psi_1 \square \Psi_2$ , is the graph with the vertex set  $\mathbb{V}_1 \times \mathbb{V}_2 = \{(\varpi, \rho) | \varpi \in \mathbb{V}_1, \rho \in \mathbb{V}_2\}$ , then two vertices  $(\varpi_1, \rho_1)$  and  $(\varpi_2, \rho_2)$  are adjacent in  $\Phi$  whenever  $\varpi_1 = \varpi_2$  and  $\rho_1 \rho_2 \in \mathbb{E}_2$ , or  $\rho_1 = \rho_2$  and  $\varpi_1 \varpi_2 \in \mathbb{E}_1$ . By joining of two graphs  $\Psi_1 = \Psi_1(\mathbb{V}_1, \mathbb{E}_1)$  and  $\Psi_2 = \Psi_2(\mathbb{V}_2, \mathbb{E}_2)$ , denoted by  $\Phi = \Psi_1 + \Psi_2$ , we mean a graph  $\Phi = \Phi(\mathbb{V}, \mathbb{E})$  such that  $\mathbb{V} = \mathbb{V}_1 \cup \mathbb{V}_2$  and  $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \{\varrho\varsigma : \varrho \in \mathbb{V}_1 \text{ and } \varsigma \in \mathbb{V}_2\}$ . Then a fan  $F_m$  is characterized as  $F_m = K_1 + \wp_m$  for  $m \geq 1$ , a wheel  $W_m$  is characterized as  $W_m = K_1 + C_m$ , for  $m \geq 3$ , and the Jahangir graph  $J_{2m}$  ( $m \geq 2$ ) is gotten from the wheel graph  $W_{2m}$  by alternately deleting  $m$  spokes of the wheel graph (which is otherwise called as a gear graph).

The graph of Prism  $D_n$  comprises of  $2n$  number of nodes and  $3n$  number of edges. They are cubic, regular graphs. Since the graph of the Prism has symmetries taking every vertex to one another vertex, the prism graphs are vertex-transitive. As polyhedral graphs, they are additionally 3-vertex-connected planar graphs. It has  $n$  3-sided faces and a pair of  $n$ -sided faces. In [4], Caceres et al. proved the following result on the metric dimension of the cartesian product of the path  $\wp_j$  on  $j$  nodes and a cycle  $C_n$  ( $n \geq 3$ ) that

$$\beta(C_n \square \wp_j) = \begin{cases} 3, & \text{if } n \text{ is even natural;} \\ 2, & \text{if } n \text{ is odd natural.} \end{cases}$$

Since the graph of Prism  $D_n$  are the trivalent rotationally symmetrical plane graph acquired by the cross product of the cycle  $C_n$  with a path  $\wp_2$ , this infers

$$\beta(C_n \square \wp_2) = \begin{cases} 3, & \text{if } n \text{ is even natural;} \\ 2, & \text{if } n \text{ is odd natural.} \end{cases}$$

In [5], Caceres et al. decided the location number of the fan graph  $F_m$  ( $m \geq 1$ ) as

**Theorem 2.** [5] *Suppose  $\Phi = F_m$  be the fan graph on  $m$  nodes, then for  $m \notin \{1, 2, 3, 6\}$ , we have*

$$\beta(\Phi = F_m) = \left\lfloor \frac{2m+2}{5} \right\rfloor.$$

In [3] Chartrand et al. decided the location number of the wheel graph  $W_m$  ( $m \geq 3$ ), and Tomescu et al. [26] acquired the location number of the Jahangir graph  $J_{2m}$  ( $m \geq 4$ ) as

**Theorem 3.** [3] *For  $m \geq 3$ , let  $\Phi = W_m$  be the wheel graph on  $m$  nodes, then for  $m \notin \{3, 6\}$ , we have*

$$\beta(\Phi = W_m) = \left\lfloor \frac{2m+2}{5} \right\rfloor.$$

**Theorem 4.** [26] *For  $m \geq 4$ , let  $\Phi = J_{2m}$  be the Jahangir graph on  $2m$  nodes, then we have*

$$\beta(\Phi = J_{2m}) = \left\lfloor \frac{2m}{3} \right\rfloor.$$

Note that the location number of these three families (viz., Fan graph, Wheel graph, and Jahangir graph) of the plane graphs rely on the number of vertices in the graphs and in this way does not comprise the classes of plane graphs with a steady location number (or constant metric dimensions). Now, a property in regards to the metric dimension two of a connected graph  $\Psi = \Psi(\mathbb{V}, \mathbb{E})$  was demonstrated by Khuller et al. in [15] and is

**Theorem 5.** [15] Let  $\mathbb{A} \subseteq \mathbb{V}(\Psi)$  be the basis set of the connected graph  $\Psi = \Psi(\mathbb{V}, \mathbb{E})$  of cardinality two i.e.,  $|\mathbb{A}| = \dim(\Psi) = \beta(\Psi) = 2$ , and say  $\mathbb{A} = \{\varpi, \xi\}$ . Then, the following listed three points are true:

1. Between the vertices  $\varpi$  and  $\xi$ , there exists a unique and shortest path  $\wp$ .
2. The valencies (or degrees) of the nodes  $\varpi$  and  $\xi$  can never exceed 3.
3. The valency of any other node on  $\wp$  can never exceed 5.

Let  $\aleph$  establish a class of connected graphs. We infer that the class  $\aleph$  has steady (or constant) location number if  $\beta(\Psi) = \dim(\Psi)$  is free of the decision of the graph  $\Psi$  in  $\aleph$  and is finite (or limited). In other words, if all the graphs in  $\aleph$  have an indistinguishable location number, at that point  $\aleph$  is known as a family with a steady location number [25]. The cycle  $\mathcal{C}_n$  has location number two for each positive integer  $n$ ;  $n \geq 3$ . Henceforth,  $\mathcal{C}_n$  ( $n \geq 3$ ) and  $\wp_n$  ( $n \geq 2$ ) establish a family of graphs with a steady location number. Additionally,  $P(n, 2)$  (the generalized Petersen graphs) and Harary graphs  $H_{4,n} = C_n(1, 2)$  (or circulant graph), are also the classes of graphs with steady location numbers [13].

A fault-tolerant locating set is a locating set in which the expulsion of a discretionary vertex keeps up the resolvability i.e., a locating set  $\mathfrak{L}^*$  is said to be fault-tolerant, if  $\mathfrak{L}^* \setminus \varpi$  is also a locating set for any  $\varpi \in \mathfrak{L}^*$  [11]. For the sake of simplicity, we can write fault-tolerant locating/resolving set and fault-tolerant metric dimensions as FTRS and FTMD respectively. The fault-tolerant metric basis and fault-tolerant metric dimension (or FTMD) are characterized correspondingly as metric dimensions. We signify the FTMD of the graph  $\Phi$  with  $\beta^*(\Phi)$  or  $f\dim(\Phi)$ . By the definition of FTRS, it is clear that for every graph  $\Phi$ , we have

$$\beta^*(\Phi) \geq \beta(\Phi) + 1 \quad (1)$$

In [20] Raza et al. demonstrate all the connected graphs with FTMD two.

**Theorem 6.** [20] A connected graph is the path graph ( $\wp_n$ ,  $n \geq 2$ ) iff it has FTMD two.

In [11] Hernando et al. indicated for an arbitrary connected graph  $\Phi$ , the following upper bound on FTMD regarding the metric dimension of the graph  $\Phi$  as

**Theorem 7.** [11] The inequality  $\beta^*(\Phi) \leq \beta(\Phi)(1 + 2 \times 5^{\beta(\Phi)-1})$  holds for any arbitrary connected graph  $\Phi$ .

This inequality shows that the FTMD is bounded above by a function of the metric dimension. The accompanying Lemma depicts a connection for a discretionary graph  $\Phi$  among a FTRS and a resolving set. Suppose that  $\mathbb{N}(\varpi)$  represents an open neighborhood of a node  $\varpi \in \mathbb{V}(\Phi)$  where  $\mathbb{N}(\varpi) := \{\vartheta \in \mathbb{V}(\Phi) | \varpi\vartheta \in \mathbb{E}(\Phi)\}$ , and the close neighborhood of a node  $\varpi$  is given as  $\mathbb{N}[\varpi] := \mathbb{N}(\varpi) \cup \{\varpi\}$ .

**Lemma 1.** [11] Let  $\mathfrak{L}$  represents a locating set for the connected graph  $\Phi$ . Then, for any  $\varpi \in \mathfrak{L}$ , let  $\mathbb{T}(\varpi) := \{\rho \in \mathbb{V}(\Phi) : \mathbb{N}(\varpi) \subseteq \mathbb{N}(\rho)\}$ . Then  $\mathfrak{L}^* := \cup_{\varpi \in \mathfrak{L}} (\mathbb{N}[\varpi] \cup \mathbb{T}(\varpi))$  is a FTRS of the graph  $\Phi$ .

Let  $\lambda(\rho, \varrho)$  be the set of common neighbors of the nodes  $\rho$  and  $\varrho$  in the graph  $\Phi$ . For some  $\mathbb{K} \subseteq \mathbb{V}(\Phi)$ , let  $\lambda(\mathbb{K})$  be the set of common neighbors of every node in  $\mathbb{K}$ .

**Lemma 2.** [19] Let  $\mathfrak{L}$  represents a locating set for the connected graph  $\Phi$ . Then  $\mathfrak{L}^* := \cup_{\varpi \in \mathfrak{L}} (\mathbb{N}[\varpi] \cup \lambda(\mathbb{N}(\varpi)))$  is a FTRS of the graph  $\Phi$ .

The main motivation in characterizing the classes of plane graphs with the bounded metric dimension or FTMD (or with an unbounded metric dimension or unbounded FTMD) is that none of the plane graphs should be left with an unknown metric dimension or FTMD. Throughout this article, all vertex indices are taken to be modulo  $n$ . In this article, we obtain the metric dimension and fault-tolerant metric dimension of two interminable classes of rotationally symmetrical plane graphs when the basis set is taken to be independent. We also compare the upper and lower bounds for the FTMD of these two classes of rotationally plane graphs, for the two cases, when the basis set is independent and when it is not.

Note that Propositions 1 and 2 are also appeared in papers [18, 16] respectively. Be that as it may, for the accommodation of the perusers and for culmination, we give their evidence. In the accompanying section, we acquire the location number of the rotationally symmetrical graph  $\mathfrak{R}_n$ , and also set upper and lower bounds for the FTMD for graph  $\mathfrak{R}_n$  when the basis set is independent.

## 2. Bounds on FTMD for the plane graph $\mathfrak{R}_n$

The plane graph  $\mathfrak{R}_n = R_n$  [18] comprises of  $3n + 2$  number of faces,  $6n$  number of vertices, and  $9n$  number of edges (see figure 2). By  $\mathbb{E}(\mathfrak{R}_n)$  and  $\mathbb{V}(\mathfrak{R}_n)$ , we signify the arrangement of edges and vertices of the plane graph  $\mathfrak{R}_n$  separately. Consequently, we have

$$\mathbb{V}(\mathfrak{R}_n) = \{p_l, q_l, r_l, s_l, t_l, u_l : 1 \leq l \leq n\}$$

and

$$\mathbb{E}(\mathfrak{R}_n) = \{p_l q_l, q_l r_l, r_l s_l, s_l t_l, t_l u_l : 1 \leq l \leq n\} \cup \{p_l p_{l+1}, r_l q_{l+1}, t_l s_{l+1}, u_l u_{l+1} : 1 \leq l \leq n\}$$

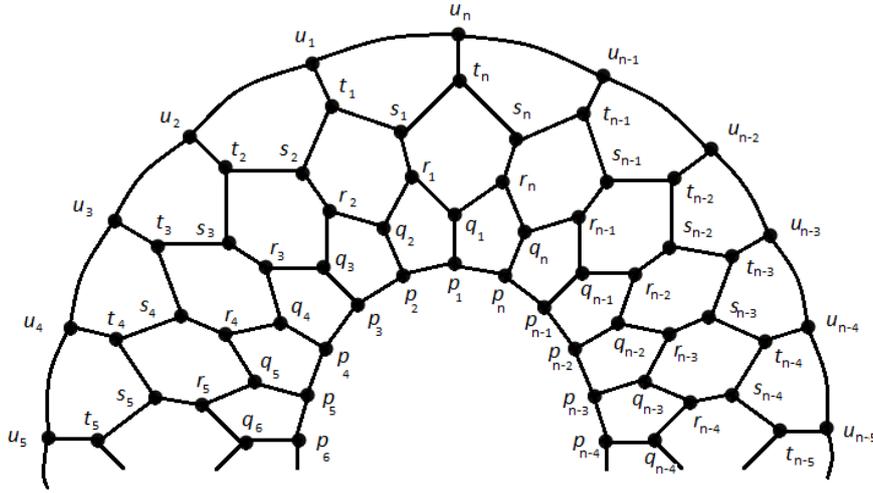


Figure 2: The graph  $\mathfrak{R}_n$

For our simplicity, we refer to the cycle brought forth by the arrangement of vertices  $\{p_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{R}_n$  as the  $p$ -cycle, the cycle brought forth by the arrangement of vertices  $\{q_\eta : 1 \leq \eta \leq n\} \cup \{r_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{R}_n$  as the  $qr$ -cycle, the cycle brought forth by the arrangement of vertices  $\{s_\eta : 1 \leq \eta \leq n\} \cup \{t_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{R}_n$  as the  $st$ -cycle, and the cycle brought forth by the arrangement of vertices  $\{u_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{R}_n$  as the  $u$ -cycle. For our purpose, we consider  $p_1 = p_{n+1}$ ,  $q_1 = q_{n+1}$ ,  $r_1 = r_{n+1}$ ,  $s_1 = s_{n+1}$ ,  $t_1 = t_{n+1}$ , and  $u_1 = u_{n+1}$ . In the accompanying proposition, we show for the rotationally symmetrical plane graph  $\mathfrak{R}_n$  that, there is no locating set with cardinality two in  $\mathfrak{R}_n$ .

**Proposition 1.** *Let  $\mathfrak{R}_n$  be the rotationally symmetrical plane graph on  $6n$  vertices as defined above. Then, for  $n \geq 6$ , we have  $\beta(\mathfrak{R}_n) \geq 3$ .*

*Proof.* Since, for  $n \geq 6$ , we have given that  $\mathfrak{R}_n$  be the rotationally symmetrical plane graph on  $6n$  nodes. We have to show that  $\beta(\mathfrak{R}_n) \geq 3$  i.e., there exists a locating (or resolving) set, say  $\mathcal{L}$  with cardinality greater than or equals to 3. Suppose on the contrary that  $\beta(\mathfrak{R}_n) \leq 2$ . Now, as the graph is not a path [6], so it means that there is a locating set of cardinality 2 i.e.,  $|\mathcal{L}| = 2$ . To demonstrate this, we eagerly consider the resulting two cases relying on the positive integer  $n$  i.e.,

when the positive whole number  $n$  is even and when it is odd.

**Case(I)** When the integer  $n$  is even.

For this situation, the integer  $n$  can be written as  $n = 2\varrho$ , where  $\varrho \in \mathbb{N}$  and  $\varrho \geq 3$ . At that point, we have the accompanying prospects to be talked about.

Resolving sets	Contradictions
$\{p_1, p_g\}, p_g (2 \leq g \leq n)$	For $2 \leq g \leq \varrho$ , we have $\zeta(p_n \{p_1, p_g\}) = \zeta(q_1 \{p_1, p_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(p_2 \{p_1, p_{\varrho+1}\}) = \zeta(p_n \{p_1, p_{\varrho+1}\})$ , a contradiction.
$\{q_1, q_g\}, q_g (2 \leq g \leq n)$	For $g = 2$ , we have $\zeta(t_n \{q_1, q_2\}) = \zeta(t_1 \{q_1, q_2\})$ , when $g = 3$ , we have $\zeta(s_2 \{q_1, q_3\}) = \zeta(p_4 \{q_1, q_3\})$ , when $4 \leq g \leq \varrho$ , we have $\zeta(q_n \{q_1, q_g\}) = \zeta(s_1 \{q_1, q_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(p_2 \{q_1, q_{\varrho+1}\}) = \zeta(p_n \{q_1, q_{\varrho+1}\})$ , a contradiction.
$\{r_1, r_g\}, r_g (2 \leq g \leq n)$	For $g = 2$ , we have $\zeta(t_1 \{r_1, r_2\}) = \zeta(p_2 \{r_1, r_2\})$ , when $g = 3$ , we have $\zeta(t_2 \{r_1, r_3\}) = \zeta(p_4 \{r_1, r_3\})$ , when $g = 4$ , we have $\zeta(p_3 \{r_1, r_4\}) = \zeta(q_3 \{r_1, r_4\})$ , when $5 \leq g \leq \varrho$ , we have $\zeta(p_1 \{r_1, r_g\}) = \zeta(r_2 \{r_1, r_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(p_2 \{r_1, r_{\varrho+1}\}) = \zeta(p_1 \{r_1, r_{\varrho+1}\})$ , a contradiction.
$\{s_1, s_g\}, s_g (2 \leq g \leq n)$	For $2 \leq g \leq 3$ , we have $\zeta(r_1 \{s_1, s_g\}) = \zeta(t_n \{s_1, s_g\})$ , when $4 \leq g \leq \varrho$ , we have $\zeta(s_2 \{s_1, s_g\}) = \zeta(u_n \{s_1, s_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(p_1 \{s_1, s_{\varrho+1}\}) = \zeta(p_2 \{s_1, s_{\varrho+1}\})$ , a contradiction.
$\{t_1, t_g\}, t_g (2 \leq g \leq n)$	For $g = 2$ , we have $\zeta(r_1 \{t_1, t_2\}) = \zeta(t_n \{t_1, t_2\})$ , when $g = 3$ , we have $\zeta(u_3 \{t_1, t_3\}) = \zeta(q_3 \{t_1, t_3\})$ , when $4 \leq g \leq \varrho$ , we have $\zeta(r_2 \{t_1, t_g\}) = \zeta(t_n \{t_1, t_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(u_2 \{t_1, t_{\varrho+1}\}) = \zeta(u_n \{t_1, t_{\varrho+1}\})$ , a contradiction.
$\{u_1, u_g\}, u_g (2 \leq g \leq n)$	For $2 \leq g \leq \varrho$ , we have $\zeta(u_n \{u_1, u_g\}) = \zeta(t_1 \{u_1, u_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(u_2 \{u_1, u_{\varrho+1}\}) = \zeta(u_n \{u_1, u_{\varrho+1}\})$ , a contradiction.
$\{p_1, q_g\}, q_g (1 \leq g \leq n)$	For $g = 1, \varrho + 1$ , we have $\zeta(p_n \{p_1, q_g\}) = \zeta(p_2 \{p_1, q_g\})$ , when $g = 2$ , we have $\zeta(q_3 \{p_1, q_2\}) = \zeta(s_1 \{p_1, q_2\})$ , and when $3 \leq g \leq \varrho$ , we have $\zeta(q_1 \{p_1, q_g\}) = \zeta(p_n \{p_1, q_g\})$ , a contradiction.
$\{p_1, r_g\}, r_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(t_n \{p_1, r_1\}) = \zeta(t_1 \{p_1, r_1\})$ , when $g = 2$ , we have $\zeta(p_3 \{p_1, r_2\}) = \zeta(r_1 \{p_1, r_2\})$ , when $3 \leq g \leq \varrho$ , we have $\zeta(q_n \{p_1, r_g\}) = \zeta(r_n \{p_1, r_g\})$ , when $g = \varrho$ , we have $\zeta(r_1 \{p_1, r_{\varrho}\}) = \zeta(q_n \{p_1, r_{\varrho}\})$ , and when $g = \varrho + 1$ , we have $\zeta(q_2 \{p_1, r_{\varrho+1}\}) = \zeta(r_n \{p_1, r_{\varrho+1}\})$ , a contradiction.
$\{p_1, s_g\}, s_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(t_n \{p_1, s_1\}) = \zeta(t_1 \{p_1, s_1\})$ , when $g = 2$ , we have $\zeta(q_3 \{p_1, s_2\}) = \zeta(s_1 \{p_1, s_2\})$ , when $3 \leq g \leq \varrho - 1$ , we have $\zeta(q_n \{p_1, s_g\}) = \zeta(r_n \{p_1, s_g\})$ , when $g = \varrho$ , we have $\zeta(r_1 \{p_1, s_{\varrho}\}) = \zeta(q_n \{p_1, s_{\varrho}\})$ , and when $g = \varrho + 1$ , we have $\zeta(r_n \{p_1, s_{\varrho+1}\}) = \zeta(q_2 \{p_1, s_{\varrho+1}\})$ , a contradiction.
$\{p_1, t_g\}, t_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(r_n \{p_1, t_1\}) = \zeta(p_3 \{p_1, t_1\})$ , when $g = 2$ , we have $\zeta(q_3 \{p_1, t_2\}) = \zeta(s_1 \{p_1, t_2\})$ , when $g = 3$ , we have $\zeta(q_4 \{p_1, t_3\}) = \zeta(s_2 \{p_1, t_3\})$ , when $4 \leq g \leq \varrho$ , we have $\zeta(r_1 \{p_1, t_g\}) = \zeta(q_2 \{p_1, t_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(r_n \{p_1, t_{\varrho+1}\}) = \zeta(q_n \{p_1, t_{\varrho+1}\})$ , a contradiction.
$\{p_1, u_g\}, u_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(t_n \{p_1, u_1\}) = \zeta(s_2 \{p_1, u_1\})$ , when $2 \leq g \leq \varrho$ , we have $\zeta(r_1 \{p_1, u_g\}) = \zeta(q_2 \{p_1, u_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(r_n \{p_1, u_{\varrho+1}\}) = \zeta(q_n \{p_1, u_{\varrho+1}\})$ , a contradiction.

Resolving sets	Contradictions
$\{q_1, r_g\}, r_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(r_n q_1, r_1) = \zeta(p_1 q_1, r_1)$ , when $g = 2$ , we have $\zeta(p_2 q_1, r_2) = \zeta(s_1 q_1, r_2)$ , when $g = 3$ , we have $\zeta(r_2 q_1, r_3) = \zeta(p_3 q_1, r_3)$ , when $4 \leq g \leq \varrho - 1$ , we have $\zeta(q_n q_1, r_g) = \zeta(s_1 q_1, r_g)$ , when $g = \varrho$ , we have $\zeta(p_n q_1, r_\varrho) = \zeta(q_2 q_1, r_\varrho)$ , and when $g = \varrho + 1$ , we have $\zeta(p_2 q_1, r_{\varrho+1}) = \zeta(q_n q_1, r_{\varrho+1})$ , a contradiction.
$\{q_1, s_g\}, s_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(t_n q_1, s_1) = \zeta(t_1 q_1, s_1)$ , when $g = 2$ , we have $\zeta(q_2 q_1, s_2) = \zeta(s_1 q_1, s_2)$ , when $3 \leq g \leq 4$ , we have $\zeta(s_2 q_1, s_g) = \zeta(q_3 q_1, s_g)$ , when $5 \leq g \leq \varrho$ , we have $\zeta(t_n q_1, s_g) = \zeta(r_2 q_1, s_g)$ , and when $g = \varrho + 1$ , we have $\zeta(q_n q_1, s_{\varrho+1}) = \zeta(p_2 q_1, s_{\varrho+1})$ , a contradiction.
$\{q_1, t_g\}, t_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(u_1 q_1, t_1) = \zeta(s_2 q_1, t_1)$ , when $2 \leq g \leq 3$ , we have $\zeta(s_1 q_1, t_g) = \zeta(q_2 q_1, t_g)$ , when $4 \leq g \leq \varrho$ , we have $\zeta(t_n q_1, t_g) = \zeta(r_2 q_1, t_g)$ , and when $g = \varrho + 1$ , we have $\zeta(s_1 q_1, t_{\varrho+1}) = \zeta(q_n q_1, t_{\varrho+1})$ , a contradiction.
$\{q_1, u_g\}, u_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(q_n q_1, u_1) = \zeta(p_2 q_1, u_1)$ , when $2 \leq g \leq \varrho$ , we have $\zeta(t_n q_1, u_g) = \zeta(r_2 q_1, u_g)$ , and when $g = \varrho + 1$ , we have $\zeta(t_n q_1, u_{\varrho+1}) = \zeta(r_{n-1} q_1, u_{\varrho+1})$ , a contradiction.
$\{r_1, s_g\}, s_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(t_1 r_1, s_1) = \zeta(t_n r_1, s_1)$ , when $2 \leq g \leq 4$ , we have $\zeta(r_2 r_1, s_g) = \zeta(t_1 r_1, s_g)$ , when $5 \leq g \leq \varrho$ , we have $\zeta(t_n r_1, s_g) = \zeta(r_2 r_1, s_g)$ , and when $g = \varrho + 1$ , we have $\zeta(u_1 r_1, s_{\varrho+1}) = \zeta(u_n r_1, s_{\varrho+1})$ , a contradiction.
$\{r_1, t_g\}, t_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(r_2 r_1, t_1) = \zeta(t_n r_1, t_1)$ , when $2 \leq g \leq 3$ , we have $\zeta(r_2 r_1, t_g) = \zeta(t_1 r_1, t_g)$ , when $4 \leq g \leq \varrho$ , we have $\zeta(t_n r_1, t_g) = \zeta(r_2 r_1, t_g)$ , and when $g = \varrho + 1$ , we have $\zeta(t_1 r_1, t_{\varrho+1}) = \zeta(r_n r_1, t_{\varrho+1})$ , a contradiction.
$\{r_1, u_g\}, u_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(q_1 r_1, u_1) = \zeta(q_2 r_1, u_1)$ , when $2 \leq g \leq \varrho$ , we have $\zeta(t_n r_1, u_g) = \zeta(r_2 r_1, u_g)$ , and when $g = \varrho + 1$ , we have $\zeta(t_1 r_1, u_{\varrho+1}) = \zeta(r_n r_1, u_{\varrho+1})$ , a contradiction.
$\{s_1, t_g\}, t_g (1 \leq g \leq n)$	For $1 \leq g \leq 2$ , we have $\zeta(r_1 s_1, t_g) = \zeta(t_n s_1, t_g)$ , when $g = 3$ , we have $\zeta(u_1 s_1, t_3) = \zeta(s_2 s_1, t_3)$ , when $4 \leq g \leq \varrho$ , we have $\zeta(s_2 s_1, t_g) = \zeta(u_n s_1, t_g)$ , and when $g = \varrho + 1$ , we have $\zeta(s_n s_1, t_{\varrho+1}) = \zeta(u_1 s_1, t_{\varrho+1})$ , a contradiction.
$\{s_1, u_g\}, u_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(q_1 s_1, u_1) = \zeta(q_2 s_1, u_1)$ , when $2 \leq g \leq \varrho$ , we have $\zeta(s_2 s_1, u_g) = \zeta(u_n s_1, u_g)$ , and when $g = \varrho + 1$ , we have $\zeta(s_n s_1, u_{\varrho+1}) = \zeta(u_1 s_1, u_{\varrho+1})$ , a contradiction.
$\{t_1, u_g\}, u_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(s_1 t_1, u_1) = \zeta(s_2 t_1, u_1)$ , when $2 \leq g \leq \varrho$ , we have $\zeta(t_n t_1, u_g) = \zeta(r_2 t_1, u_g)$ , and when $g = \varrho + 1$ , we have $\zeta(u_n t_1, u_{\varrho+1}) = \zeta(u_2 t_1, u_{\varrho+1})$ , a contradiction.

In this manner, the above conversation explains that there is no resolving set comprising of two vertices for  $\mathbb{V}(\mathfrak{R}_n)$  inferring that  $\beta(\mathfrak{R}_n) \geq 3$  in this case.

**Case(II)** When the integer  $n$  is odd.

For this situation, the integer  $n$  can be written as  $n = 2\varrho + 1$ , where  $\varrho \in \mathbb{N}$  and  $\varrho \geq 3$ . Now, on expecting that  $\beta(\mathfrak{R}_n) = 2$ , we consider that to be are parallel prospects as talked about in Case(I) and logical inconsistency can be inferred correspondingly. Consequently,  $\beta(\mathfrak{R}_n) \geq 3$  for this situation too, which concludes the proposition.  $\square$

In the accompanying result, we demonstrate some lower and upper bounds on the FTMD of the rotationally symmetrical plane graph  $\mathfrak{R}_n$ .

**Theorem 8.** *Let  $\mathfrak{R}_n$  be the rotationally symmetrical plane graph on  $6n$  vertices as defined above. Then, for  $n \geq 6$ , we have  $\beta^*(\mathfrak{R}_n) \geq 4$  and*

$$\beta^*(\mathfrak{R}_n) \leq \begin{cases} 9, & \text{if } n = 6; \\ 10, & \text{if } n = 7; \\ 11, & \text{if } n \geq 8. \end{cases}$$

*Proof. Claim 1:* The rotationally symmetrical graph  $\mathfrak{R}_n$  has a locating set with cardinality 3.

To demonstrate this, we eagerly consider the resulting two cases relying on the positive integer  $n$  i.e., when the positive whole number  $n$  is even and when it is odd.

**Case(I)** When the integer  $n$  is even.

For this situation, the integer  $n$  can be written as  $n = 2\varrho$ , where  $\varrho \in \mathbb{Z}^+$  and  $\varrho \geq 3$ . Let  $\mathfrak{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{R}_n)$ . Now, in order to unveil that  $\mathfrak{L}$  is a locating or resolving set for the rotationally symmetrical graph  $\mathfrak{R}_n$ , we consign the metric codes for each vertex of  $\mathbb{V}(\mathfrak{R}_n) \setminus \mathfrak{L}$  regarding the set  $\mathfrak{L}$ .

Presently, the metric codes for the vertices of  $p$ -cycle  $\{p_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(p_\eta|\mathfrak{L}) = \begin{cases} (1, \varrho, 1), & \eta = 1; \\ (\eta - 2, \varrho - \eta + 1, \eta), & 3 \leq \eta \leq \varrho \\ (2\varrho - \eta + 2, \eta - \varrho - 1, 2\varrho - \eta), & \varrho + 2 \leq \eta \leq 2\varrho - 1. \end{cases}$$

The metric codes for the nodes of  $qr$ -cycle  $\{q_\eta : 1 \leq \eta \leq n\} \cup \{r_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(q_\eta|\mathfrak{L}) = \begin{cases} (2, \varrho + 1, 2), & \eta = 1; \\ (\eta - 1, \varrho - \eta + 2, \eta + 1), & 2 \leq \eta \leq \varrho; \\ (\varrho, 1, \varrho), & \eta = \varrho + 1; \\ (2\varrho - \eta + 3, \eta - \varrho, 2\varrho - \eta + 1), & \varrho + 2 \leq \eta \leq 2\varrho. \end{cases}$$

and

$$\zeta(r_\eta|\mathfrak{L}) = \begin{cases} (2, \varrho + 1, 3), & \eta = 1; \\ (\eta, \varrho - \eta + 2, \eta + 2), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho, 2, \varrho + 1), & \eta = \varrho; \\ (\varrho + 1, 2, \varrho), & \eta = \varrho + 1; \\ (2\varrho - \eta + 3, \eta - \varrho + 1, 2\varrho - \eta + 1), & \varrho + 2 \leq \eta \leq 2\varrho - 1; \\ (2\varrho - \eta + 3, \eta - \varrho + 1, 2), & \eta = 2\varrho. \end{cases}$$

The metric codes for the nodes of  $st$ -cycle  $\{s_\eta : 1 \leq \eta \leq n\} \cup \{t_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(s_\eta|\mathfrak{L}) = \zeta(r_\eta|\mathfrak{L}) + (1, 1, 1).$$

and

$$\zeta(t_\eta|\mathfrak{L}) = \begin{cases} (4, \varrho + 2, 4), & \eta = 1; \\ (\eta + 2, \varrho - \eta + 3, \eta + 4), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho + 2, 4, \varrho + 2), & \eta = \varrho; \\ (2\varrho - \eta + 4, \eta - \varrho + 3, 2\varrho - \eta + 2), & \varrho + 1 \leq \eta \leq 2\varrho - 2; \\ (2\varrho - \eta + 4, \eta - \varrho + 3, 4), & \eta = 2\varrho - 1; \\ (2\varrho - \eta + 4, \eta - \varrho + 3, 4), & \eta = 2\varrho. \end{cases}$$

The metric codes for the vertices of  $u$ -cycle  $\{u_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(u_\eta|\mathfrak{L}) = \zeta(t_\eta|\mathfrak{L}) + (1, 1, 1).$$

We notice that no two vertices are having indistinguishable metric codes, inferring that  $\mathfrak{L}$  is a locating set for the rotationally symmetrical plane graph  $\mathfrak{R}_n$  with cardinality 3.

**Case(II)** When the integer  $n$  is odd.

For this situation, the integer  $n$  can be written as  $n = 2\varrho + 1$ , where  $\varrho \in \mathbb{Z}^+$  and  $\varrho \geq 3$ . Let  $\mathfrak{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{R}_n)$ . Now, in order to unveil that  $\mathfrak{L}$  is a locating or resolving set for the rotationally symmetrical graph  $\mathfrak{R}_n$ , we conisgn the metric codes for each vertex of  $\mathbb{V}(\mathfrak{R}_n) \setminus \mathfrak{L}$  regarding the set  $\mathfrak{L}$ .

Presently, the metric codes for the vertices of  $p$ -cycle  $\{p_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(p_\eta|\mathfrak{L}) = \begin{cases} (1, \varrho, 1), & \eta = 1; \\ (\eta - 2, \varrho - \eta + 1, \eta), & 3 \leq \eta \leq \varrho \\ (\varrho, 1, \varrho - 1), & \eta = \varrho + 2; \\ (2\varrho - \eta + 3, \eta - \varrho - 1, 2\varrho - \eta + 1), & \varrho + 3 \leq \eta \leq 2\varrho. \end{cases}$$

The metric codes for the nodes of  $qr$ -cycle  $\{q_\eta : 1 \leq \eta \leq n\} \cup \{r_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(q_\eta|\mathfrak{L}) = \begin{cases} (2, \varrho + 1, 2), & \eta = 1; \\ (\eta - 1, \varrho - \eta + 2, \eta + 1), & 3 \leq \eta \leq \varrho \\ (\varrho + 1, 2, \varrho), & \eta = \varrho + 2; \\ (2\varrho - \eta + 4, \eta - \varrho, 2\varrho - \eta + 2), & \varrho + 3 \leq \eta \leq 2\varrho + 1. \end{cases}$$

and

$$\zeta(r_\eta|\mathfrak{L}) = \begin{cases} (2, \varrho + 1, 3), & \eta = 1; \\ (\eta, \varrho - \eta + 2, \eta + 2), & 2 \leq \eta \leq \varrho; \\ (\varrho + 1, 2, \varrho + 1), & \eta = \varrho + 1; \\ (2\varrho - \eta + 4, \eta - \varrho + 1, 2\varrho - \eta + 2), & \varrho + 2 \leq \eta \leq 2\varrho; \\ (2\varrho - \eta + 4, \eta - \varrho + 1, 2), & \eta = 2\varrho + 1. \end{cases}$$

The metric codes for the nodes of  $st$ -cycle  $\{s_\eta : 1 \leq \eta \leq n\} \cup \{t_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(s_\eta|\mathfrak{L}) = \zeta(r_\eta|\mathfrak{L}) + (1, 1, 1).$$

and

$$\zeta(t_\eta|\mathfrak{L}) = \begin{cases} (4, \varrho + 2, 4), & \eta = 1; \\ (\eta + 2, \varrho - \eta + 3, \eta + 4), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho + 2, 4, \varrho + 3), & \eta = \varrho; \\ (\varrho + 3, 4, \varrho + 2), & \eta = \varrho + 1; \\ (2\varrho - \eta + 5, \eta - \varrho + 3, 2\varrho - \eta + 3), & \varrho + 2 \leq \eta \leq 2\varrho - 1; \\ (2\varrho - \eta + 5, \eta - \varrho + 3, 4), & \eta = 2\varrho; \\ (2\varrho - \eta + 5, \varrho + 3, 4), & \eta = 2\varrho + 1. \end{cases}$$

The metric codes for the vertices of  $u$ -cycle  $\{u_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(u_\eta|\mathfrak{L}) = \zeta(t_\eta|\mathfrak{L}) + (1, 1, 1).$$

We notice that no two vertices are having indistinguishable metric codes, inferring that  $\mathcal{L}$  is a locating set for the rotationally symmetrical plane graph  $\mathfrak{R}_n$  with cardinality 3.

By using claim 1 and proposition 1, for rotationally symmetrical plane graph  $\mathfrak{R}_n$ , we find that  $\beta(\mathfrak{R}_n) = 3$ . Then by using equation (1), we obtain the lower bound for FTMD for graph  $\mathfrak{R}_n$  as  $\beta^*(\mathfrak{R}_n) \geq 4$ . One can easily see that, for  $n = 6$  and  $n = 7$ , the upper bound for the FTMD of the graph  $\mathfrak{R}_n$  are 9 and 10 respectively (see graph  $\mathfrak{R}_n$  and use Lemma 2). Now, to complete the proof, we have to obtain the upper bound for the FTMD of the plane graph  $\mathfrak{R}_n$ , for  $n \geq 10$ .

**Claim 2:** The rotationally symmetrical plane graph  $\mathfrak{R}_n$  has a FTRS of cardinality 11.

**Case(I)** When the integer  $n$  is even.

Note that for this situation,  $\mathcal{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{R}_n)$  is a locating set for the plane graph  $\mathfrak{R}_n$  by claim 1. We shall show that the plane graph  $\mathfrak{R}_n$  has a FTRS of cardinality 11. From Fig. 2, one can clearly see that  $N[p_2] = \{p_1, p_2, p_3, q_2\}$ ,  $N[p_{\varrho+1}] = \{p_{\varrho}, p_{\varrho+1}, p_{\varrho+2}, q_{\varrho+1}\}$ , and  $N[p_n] = \{p_1, p_n, p_{n-1}, q_n\}$ . Besides, we find that  $\lambda(N(p_2)) = \lambda(N(p_{\varrho+1})) = \lambda(N(p_n)) = \phi$ . Along these lines by utilizing Lemma 2, we find that  $\mathcal{L}^* = \{p_1, p_2, p_3, p_{\varrho}, p_{\varrho+1}, p_{\varrho+2}, p_{n-1}, p_n, q_2, q_{\varrho+1}, q_n\}$  is a FTRS of  $\mathfrak{R}_n$ . Thus, we find that there exists a FTRS for the rotationally symmetrical plane graph  $\mathfrak{R}_n$  of cardinality 11, if the positive integer  $n$  is even.

**Case(II)** When the integer  $n$  is odd.

Note that for this situation,  $\mathcal{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{R}_n)$  is a locating set for the plane graph  $\mathfrak{R}_n$  by claim 1. In this case, we also show that the plane graph  $\mathfrak{R}_n$  has a FTRS of cardinality 11. From Fig. 2, one can clearly see that  $N[p_2] = \{p_1, p_2, p_3, q_2\}$ ,  $N[p_{\varrho+1}] = \{p_{\varrho}, p_{\varrho+1}, p_{\varrho+2}, q_{\varrho+1}\}$ , and  $N[p_n] = \{p_1, p_n, p_{n-1}, q_n\}$ . Besides, we find that  $\lambda(N(p_2)) = \lambda(N(p_{\varrho+1})) = \lambda(N(p_n)) = \phi$ . Along these lines by utilizing Lemma 2, we find that  $\mathcal{L}^* = \{p_1, p_2, p_3, p_{\varrho}, p_{\varrho+1}, p_{\varrho+2}, p_{n-1}, p_n, q_2, q_{\varrho+1}, q_n\}$  is a FTRS of  $\mathfrak{R}_n$ . Thus, we find that there exists a FTRS for the rotationally symmetrical plane graph  $\mathfrak{R}_n$  of cardinality 11, if the positive integer  $n$  is odd.

Hence, from these two cases, we can obtained that there exists a FTRS for the rotationally symmetrical plane graph  $\mathfrak{R}_n$  of cardinality 11, and thus, the claim 2.  $\square$

Again, by utilizing these same arguments, we obtain the following result regarding FTMD of the rotationally symmetrical plane graph  $\mathfrak{R}_n$ , when the taken basis set is not independent i.e.,  $\mathcal{L} = \{p_1, p_2, p_{\varrho+1}\} \subset \mathbb{V}(\mathfrak{R}_n)$  [12]:

**Theorem 9.** *Let  $\mathfrak{R}_n$  be the rotationally symmetrical plane graph on  $6n$  vertices as defined above. Then, for  $n \geq 6$ , we have  $\beta^*(\mathfrak{R}_n) \geq 4$  and*

$$\beta^*(\mathfrak{R}_n) \leq \begin{cases} 9, & \text{if } 6 \leq n \leq 7; \\ 10, & \text{if } n \geq 8. \end{cases}$$

The immediate conclusion of the Theorems 8 and 9 is the following corollary

**Corollary 1.** *The FTMD for the family of plane graph  $\mathfrak{R}_n$  is constant.*

In the accompanying section, we acquire the location number of the rotationally symmetrical plane graph  $\mathfrak{S}_n$ , and also set lower and upper bounds for the FTMD for the graph  $\mathfrak{S}_n$ .

### 3. Bounds on FTMD for the plane graph $\mathfrak{S}_n$

The plane graph  $\mathfrak{S}_n = R_n$  [16] comprises of  $4n + 2$  number of faces,  $5n$  number of vertices, and  $9n$  number of edges (see figure 3). By  $\mathbb{E}(\mathfrak{S}_n)$  and  $\mathbb{V}(\mathfrak{S}_n)$ , we signify the arrangement of edges and vertices of the plane graph  $\mathfrak{S}_n$  separately. Consequently, we have

$$\mathbb{V}(\mathfrak{S}_n) = \{p_l, q_l, r_l, s_l, t_l : 1 \leq l \leq n\}$$

and

$$\mathbb{E}(\mathfrak{S}_n) = \{p_l q_l, q_l r_l, r_l s_l, s_l t_l : 1 \leq l \leq n\} \cup \{p_l p_{l+1}, s_l s_{l+1}, t_l t_{l+1}, s_l r_{l+1}, q_l p_{l+1} : 1 \leq l \leq n\}$$

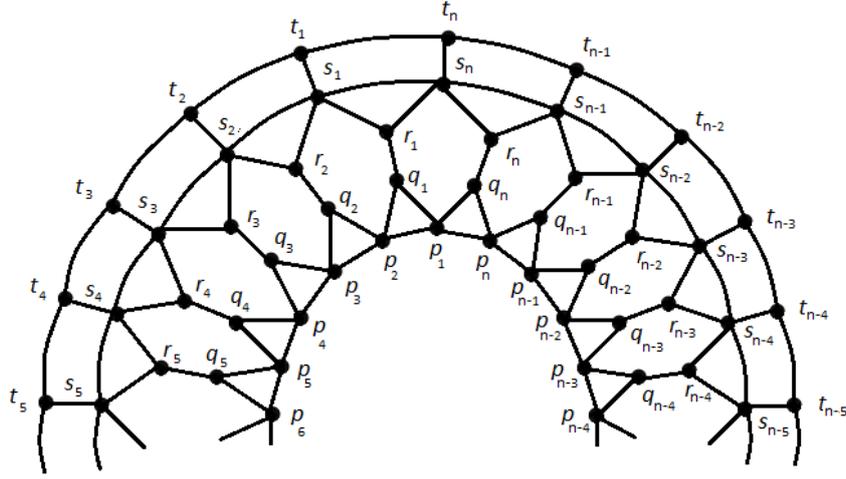


Figure 3: The graph  $\mathfrak{G}_n$

For our simplicity, we refer to the cycle brought forth by the arrangement of vertices  $\{p_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{G}_n$  as the  $p$ -cycle, the arrangement of vertices  $\{q_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{G}_n$  as the set of inward vertices, the arrangement of vertices  $\{r_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{G}_n$  as the set of outward vertices, the cycle brought forth by the arrangement of vertices  $\{s_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{G}_n$  as the  $s$ -cycle, the cycle brought forth by the arrangement of vertices  $\{t_\eta : 1 \leq \eta \leq n\}$  in the graph,  $\mathfrak{G}_n$  as the  $t$ -cycle. For our purpose, we consider  $p_1 = p_{n+1}$ ,  $q_1 = q_{n+1}$ ,  $r_1 = r_{n+1}$ ,  $s_1 = s_{n+1}$ , and  $t_1 = t_{n+1}$ . In the accompanying proposition, we show for the rotationally symmetrical plane graph  $\mathfrak{G}_n$  that, there is no locating set with cardinality two in  $\mathfrak{G}_n$ .

**Proposition 2.** *Let  $\mathfrak{G}_n$  be the rotationally symmetrical plane graph on  $5n$  vertices as defined above. Then, for  $n \geq 6$ , we have  $\beta(\mathfrak{G}_n) \geq 3$ .*

*Proof.* Since, for  $n \geq 6$ , we have given that  $\mathfrak{G}_n$  be the rotationally symmetrical plane graph on  $5n$  nodes. We have to show that  $\beta(\mathfrak{G}_n) \geq 3$  i.e., there exists a locating (or resolving) set, say  $\mathfrak{L}$  with cardinality greater than or equals to 3. Suppose on the contrary that  $\beta(\mathfrak{G}_n) \leq 2$ . Now, as the graph is not a path [6], so it means that there is a locating set of cardinality 2 i.e.,  $|\mathfrak{L}| = 2$ . At that point by Theorem 5, we find that the valency of basis nodes can never exceed 3. But except the vertices of the set  $\{p_\eta, s_\eta : 1 \leq \eta \leq n\}$ , all other nodes of the rotationally symmetrical plane graph  $\mathfrak{G}_n$  have a valency less than or equals to 3. To demonstrate this, we eagerly consider the resulting two cases relying on the positive integer  $n$  i.e., when the positive whole number  $n$  is even and when it is odd.

**Case(I)** When the integer  $n$  is even.

For this situation, the integer  $n$  can be written as  $n = 2\varrho$ , where  $\varrho \in \mathbb{N}$  and  $\varrho \geq 3$ . At that point, we have the accompanying prospects to be talked about.

Resolving sets	Contradictions
$\{q_1, q_g\}, q_g (2 \leq g \leq n)$	For $2 \leq g \leq \varrho - 1$ , we have $\zeta(q_n   \{q_1, q_g\}) = \zeta(p_n   \{q_1, q_g\})$ , when $g = \varrho$ , we have $\zeta(q_{n-1}   \{q_1, q_\varrho\}) = \zeta(r_2   \{q_1, q_\varrho\})$ , and when $g = \varrho + 1$ , we have $\zeta(p_1   \{q_1, q_{\varrho+1}\}) = \zeta(p_2   \{q_1, q_{\varrho+1}\})$ , a contradiction.
$\{r_1, r_g\}, r_g (2 \leq g \leq n)$	For $2 \leq g \leq \varrho$ , we have $\zeta(r_n   \{r_1, r_g\}) = \zeta(t_n   \{r_1, r_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(s_1   \{r_1, r_{\varrho+1}\}) = \zeta(s_n   \{r_1, r_{\varrho+1}\})$ , a contradiction.
$\{t_1, t_g\}, t_g (2 \leq g \leq n)$	For $2 \leq g \leq \varrho$ , we have $\zeta(t_n   \{t_1, t_g\}) = \zeta(s_1   \{t_1, t_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(t_2   \{t_1, t_{\varrho+1}\}) = \zeta(t_n   \{t_1, t_{\varrho+1}\})$ , a contradiction.
$\{q_1, r_g\}, r_g (1 \leq g \leq n)$	For $1 \leq g \leq \varrho$ , we have $\zeta(r_n   \{q_1, r_g\}) = \zeta(t_n   \{q_1, r_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(s_1   \{q_1, r_{\varrho+1}\}) = \zeta(s_n   \{q_1, r_{\varrho+1}\})$ , a contradiction.

Resolving sets	Contradictions
$\{q_1, t_g\}, t_g (1 \leq g \leq n)$	For $1 \leq g \leq \varrho - 1$ , we have $\zeta(r_n \{q_1, t_g\}) = \zeta(s_{n-1} \{q_1, t_g\})$ , when $g = \varrho$ , we have $\zeta(s_n \{q_1, t_\varrho\}) = \zeta(q_2 \{q_1, t_\varrho\})$ , and when $g = \varrho + 1$ , we have $\zeta(s_1 \{q_1, t_{\varrho+1}\}) = \zeta(q_n \{q_1, t_{\varrho+1}\})$ , a contradiction.
$\{r_1, t_g\}, t_g (1 \leq g \leq n)$	For $g = 1$ , we have $\zeta(p_1 \{r_1, t_1\}) = \zeta(p_2 \{r_1, t_1\})$ , when $2 \leq g \leq \varrho$ , we have $\zeta(r_2 \{r_1, t_g\}) = \zeta(t_n \{r_1, t_g\})$ , and when $g = \varrho + 1$ , we have $\zeta(t_1 \{r_1, t_{\varrho+1}\}) = \zeta(r_n \{r_1, t_{\varrho+1}\})$ , a contradiction.

In this manner, the above conversation explains that there is no resolving set comprising of two vertices for  $\mathbb{V}(\mathfrak{S}_n)$  inferring that  $\beta(\mathfrak{S}_n) \geq 3$  in this case.

**Case(II)** When the integer  $n$  is odd.

For this situation, the integer  $n$  can be written as  $n = 2\varrho + 1$ , where  $\varrho \in \mathbb{N}$  and  $\varrho \geq 3$ . Now, on expecting that  $\beta(\mathfrak{S}_n) = 2$ , we consider that to be are parallel prospects as talked about in Case(I) and logical inconsistency can be inferred correspondingly. Consequently,  $\beta(\mathfrak{S}_n) \geq 3$  for this situation too, which concludes the proposition.  $\square$

In the accompanying result, we demonstrate some lower and upper bounds on the FTMD of the rotationally symmetrical plane graph  $\mathfrak{S}_n$ .

**Theorem 10.** *Let  $\mathfrak{S}_n$  be the rotationally symmetrical plane graph on  $5n$  vertices as defined above. Then, for  $n \geq 6$ , we have  $\beta^*(\mathfrak{S}_n) \geq 4$  and*

$$\beta^*(\mathfrak{S}_n) \leq \begin{cases} 12, & \text{if } n = 6; \\ 13, & \text{if } n = 7; \\ 14, & \text{if } n \geq 8. \end{cases}$$

*Proof.* **Claim 1:** The rotationally symmetrical graph  $\mathfrak{S}_n$  has a locating set with cardinality 3.

To demonstrate this, we eagerly consider the resulting two cases relying on the positive integer  $n$  i.e., when the positive whole number  $n$  is even and when it is odd.

**Case(I)** When the integer  $n$  is even.

For this situation, the integer  $n$  can be written as  $n = 2\varrho$ , where  $\varrho \in \mathbb{Z}^+$  and  $\varrho \geq 3$ . Let  $\mathfrak{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{S}_n)$ . Now, in order to unveil that  $\mathfrak{L}$  is a locating or resolving set for the rotationally symmetrical graph  $\mathfrak{S}_n$ , we consign the metric codes for each vertex of  $\mathbb{V}(\mathfrak{S}_n) \setminus \mathfrak{L}$  regarding the set  $\mathfrak{L}$ .

Presently, the metric codes for the vertices of  $p$ -cycle  $\{p_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(p_\eta|\mathfrak{L}) = \begin{cases} (1, \varrho, 1), & \eta = 1; \\ (\eta - 2, \varrho - \eta + 1, \eta), & 3 \leq \eta \leq \varrho; \\ (2\varrho - \eta + 2, \eta - \varrho - 1, 2\varrho - \eta), & \varrho + 2 \leq \eta \leq 2\varrho - 1. \end{cases}$$

The metric codes for the set of inward nodes  $\{q_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(q_\eta|\mathfrak{L}) = \begin{cases} (1, \varrho, 2), & \eta = 1; \\ (\eta - 1, \varrho - \eta + 1, \eta + 1), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho - 1, 1, \varrho), & \eta = \varrho; \\ (\varrho, 1, \varrho - 1), & \eta = \varrho + 1; \\ (2\varrho - \eta + 2, \eta - \varrho, 2\varrho - \eta), & \varrho + 2 \leq \eta \leq 2\varrho - 1; \\ (2\varrho - \eta + 2, \eta - \varrho, 1), & \eta = 2\varrho. \end{cases}$$

The metric codes for the set of outward nodes  $\{r_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(r_\eta|\mathfrak{L}) = \zeta(q_\eta|\mathfrak{L}) + (1, 1, 1).$$

The metric codes for the nodes of  $s$ -cycle  $\{s_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(s_\eta|\mathfrak{L}) = \begin{cases} (3, \varrho + 1, 4), & \eta = 1; \\ (\eta + 1, \varrho - \eta + 3, \eta + 3), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho + 1, 3, \varrho + 1), & \eta = \varrho; \\ (2\varrho - \eta + 3, \eta - \varrho + 2, 2\varrho - \eta + 1), & \varrho + 1 \leq \eta \leq 2\varrho - 2; \\ (2\varrho - \eta + 3, \eta - \varrho + 3, 3), & 2\varrho - 1 \leq \eta \leq 2\varrho. \end{cases}$$

The metric codes for the nodes of  $t$ -cycle  $\{t_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(t_\eta|\mathfrak{L}) = \zeta(s_\eta|\mathfrak{L}) + (1, 1, 1).$$

We notice that no two vertices are having indistinguishable metric codes, inferring that  $\mathfrak{L}$  is a locating set for the rotationally symmetrical plane graph  $\mathfrak{S}_n$  with cardinality 3.

**Case(II)** When the integer  $n$  is odd.

For this situation, the integer  $n$  can be written as  $n = 2\varrho + 1$ , where  $\varrho \in \mathbb{Z}^+$  and  $\varrho \geq 3$ . Let  $\mathfrak{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{S}_n)$ . Now, in order to unveil that  $\mathfrak{L}$  is a locating or resolving set for the rotationally symmetrical graph  $\mathfrak{S}_n$ , we consign the metric codes for each vertex of  $\mathbb{V}(\mathfrak{S}_n) \setminus \mathfrak{L}$  regarding the set  $\mathfrak{L}$ .

Presently, the metric codes for the vertices of  $p$ -cycle  $\{p_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(p_\eta|\mathfrak{L}) = \begin{cases} (1, \varrho, 1), & \eta = 1; \\ (\eta - 2, \varrho - \eta + 1, \eta), & 3 \leq \eta \leq \varrho; \\ (\varrho, 1, \varrho - 1), & \eta = \varrho + 2; \\ (2\varrho - \eta + 3, \eta - \varrho - 1, 2\varrho - \eta + 1), & \varrho + 3 \leq \eta \leq 2\varrho + 1. \end{cases}$$

The metric codes for the set of inward nodes  $\{q_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(q_\eta|\mathfrak{L}) = \begin{cases} (1, \varrho, 2), & \eta = 1; \\ (\eta - 1, \varrho - \eta + 1, \eta + 1), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho - 1, 1, \varrho + 1), & \eta = \varrho; \\ (\varrho, 1, \varrho), & \eta = \varrho + 1; \\ (2\varrho - \eta + 3, \eta - \varrho, 2\varrho - \eta + 1), & \varrho + 2 \leq \eta \leq 2\varrho; \\ (2\varrho - \eta + 3, \eta - \varrho, 1), & \eta = 2\varrho + 1. \end{cases}$$

The metric codes for the set of outward nodes  $\{r_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(r_\eta|\mathfrak{L}) = \zeta(q_\eta|\mathfrak{L}) + (1, 1, 1).$$

The metric codes for the nodes of  $s$ -cycle  $\{s_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(s_\eta|\mathfrak{L}) = \begin{cases} (3, \varrho + 1, 4), & \eta = 1; \\ (\eta + 1, \varrho - \eta + 3, \eta + 3), & 2 \leq \eta \leq \varrho - 1; \\ (\varrho + 1, 3, \varrho + 2), & \eta = \varrho; \\ (\varrho + 2, 3, \varrho + 1), & \eta = \varrho + 1; \\ (2\varrho - \eta + 4, \eta - \varrho + 2, 2\varrho - \eta + 2), & \varrho + 2 \leq \eta \leq 2\varrho - 1; \\ (2\varrho - \eta + 4, \eta - \varrho + 2, 3), & \eta = 2\varrho; \\ (2\varrho - \eta + 4, \varrho + 2, 3), & \eta = 2\varrho + 1. \end{cases}$$

The metric representation of the nodes of  $t$ -cycle  $\{t_\eta : 1 \leq \eta \leq n\}$  are

$$\zeta(t_\eta|\mathfrak{L}) = \zeta(s_\eta|\mathfrak{L}) + (1, 1, 1).$$

We notice that no two vertices are having indistinguishable metric codes, inferring that  $\mathfrak{L}$  is a locating set for the rotationally symmetrical plane graph  $\mathfrak{S}_n$  with cardinality 3.

By using claim 1 and proposition 2, for rotationally symmetrical plane graph  $\mathfrak{S}_n$ , we find that  $\beta(\mathfrak{S}_n) = 3$ . Then by using equation (1), we obtain the lower bound for FTMD for graph  $\mathfrak{S}_n$  as  $\beta^*(\mathfrak{S}_n) \geq 4$ . One can easily see that, for  $n = 6$  and  $n = 7$ , the upper bound for the FTMD of the graph  $\mathfrak{S}_n$  are 12 and 13 respectively (see graph  $\mathfrak{S}_n$  and use Lemma 2). Now, to complete the proof, we have to obtain the upper bound for the FTMD of the plane graph  $\mathfrak{S}_n$ , for  $n \geq 8$ .

**Claim 2:** The rotationally symmetrical plane graph  $\mathfrak{S}_n$  has a FTRS of cardinality 14.

**Case(I)** When the integer  $n$  is even.

Note that for this situation,  $\mathfrak{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{S}_n)$  is a locating set for the plane graph  $\mathfrak{S}_n$  by claim 1. We shall show that the plane graph  $\mathfrak{S}_n$  has a FTRS of cardinality 14. From Fig. 3, one can clearly see that  $N[p_2] = \{p_1, p_2, p_3, q_1, q_2\}$ ,  $N[p_{\varrho+1}] = \{p_\varrho, p_{\varrho+1}, p_{\varrho+2}, q_\varrho, q_{\varrho+1}\}$ , and  $N[p_n] = \{p_1, p_n, p_{n-1}, q_{n-1}, q_n\}$ . Besides, we find that  $\lambda(N(p_2)) = \lambda(N(p_{\varrho+1})) = \lambda(N(p_n)) = \phi$ . Along these lines by utilizing Lemma 2, we find that  $\mathfrak{L}^* = \{p_1, p_2, p_3, p_\varrho, p_{\varrho+1}, p_{\varrho+2}, p_{n-1}, p_n, q_1, q_2, q_\varrho, q_{\varrho+1}, q_{n-1}, q_n\}$  is a FTRS of  $\mathfrak{S}_n$ . Thus, we find that there exists a FTRS for the rotationally symmetrical plane graph  $\mathfrak{S}_n$  of cardinality 14, if the positive integer  $n$  is even.

**Case(II)** When the integer  $n$  is odd.

Note that for this situation,  $\mathfrak{L} = \{p_2, p_{\varrho+1}, p_n\} \subset \mathbb{V}(\mathfrak{S}_n)$  is a locating set for the plane graph  $\mathfrak{S}_n$  by claim 1. In this case, we also show that the plane graph  $\mathfrak{S}_n$  has a FTRS of cardinality 14. From Fig. 3, one can clearly see that  $N[p_2] = \{p_1, p_2, p_3, q_1, q_2\}$ ,  $N[p_{\varrho+1}] = \{p_\varrho, p_{\varrho+1}, p_{\varrho+2}, q_\varrho, q_{\varrho+1}\}$ , and  $N[p_n] = \{p_1, p_n, p_{n-1}, q_{n-1}, q_n\}$ . Besides, we find that  $\lambda(N(p_2)) = \lambda(N(p_{\varrho+1})) = \lambda(N(p_n)) = \phi$ . Along these lines by utilizing Lemma 2, we find that  $\mathfrak{L}^* = \{p_1, p_2, p_3, p_\varrho, p_{\varrho+1}, p_{\varrho+2}, p_{n-1}, p_n, q_1, q_2, q_\varrho, q_{\varrho+1}, q_{n-1}, q_n\}$  is a FTRS of  $\mathfrak{S}_n$ . Thus, we find that there exists a FTRS for the rotationally symmetrical plane graph  $\mathfrak{S}_n$  of cardinality 14, if the positive integer  $n$  is odd.

Hence, from these two cases, we can obtained that there exists a FTRS for the rotationally symmetrical plane graph  $\mathfrak{S}_n$  of cardinality 14, and thus, the claim 2.  $\square$

Again, by utilizing these same arguments, we obtain the following result regarding FTMD of the rotationally symmetrical plane graph  $\mathfrak{S}_n$ , when the taken basis set is not independent i.e.,  $\mathfrak{L} = \{p_1, p_2, p_{\varrho+1}\} \subset \mathbb{V}(\mathfrak{S}_n)$  [16]:

**Theorem 11.** *Let  $\mathfrak{S}_n$  be the rotationally symmetrical plane graph on  $5n$  vertices as defined above. Then, for  $n \geq 6$ , we have  $\beta^*(\mathfrak{S}_n) \geq 4$  and*

$$\beta^*(\mathfrak{S}_n) \leq \begin{cases} 11, & \text{if } 6 \leq n \leq 7; \\ 12, & \text{if } n \geq 8. \end{cases}$$

The immediate conclusion of the Theorems 10 and 11 is the following corollary

**Corollary 2.** *The FTMD for the family of plane graph  $\mathfrak{S}_n$  is constant.*

## Conclusion

In this article, we obtain the metric dimension and fault-tolerant metric dimension of two infinite classes of rotationally symmetrical plane graphs, that are found to be constant for each of these two families of the plane graphs. We additionally find that the upper bound for the FTMD for the given family of a plane graph depends upon the distances between the vertices in the taken basis set of the family of a plane graph. We besides saw that the basis set  $\mathfrak{L}$  is self-governing (or independent) for these two rotationally symmetrical families of plane graphs viz.,  $\mathfrak{R}_n$ , and  $\mathfrak{S}_n$ . We show that just 3 vertices properly chosen are adequate to determine all the vertices of these classes of rotationally symmetrical graphs, and set lower and upper bounds for the FTMD of these classes of the rotationally symmetrical plane graphs when the basis set is taken to be independent.

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