ON PERFECT COLORINGS OF INFINITE MULTIPATH GRAPHS

M.A. LISITSYNA, S.V. AVGUSTINOVICH, O.G. PARSHINA

ABSTRACT. A coloring of vertices of a given graph is called perfect if the color structure of each sphere of radius 1 in the graph depends only on the color of the sphere center. Let n be a positive integer. We consider a lexicographic product of the infinite path graph and a graph G that can be either the complete or empty graph on n vertices. We give a complete description of perfect colorings with an arbitrary number of colors of such graph products.

Keywords: perfect coloring, equitable partition, equivalent colors, infinite multipath graph.

1. INTRODUCTION

Let G be a simple graph and k a positive integer. A *perfect coloring* of the graph G with the parameter matrix $M = (m_{ij})_{i,j=1}^k$ is a map from the vertex set of the graph to the set of integers $\{1, 2, 3, \ldots, k\}$ such that every vertex of color i is adjacent to exactly m_{ij} vertices of color j.

The concept behind the definition of the perfect coloring is quite natural. It arose and developed in connection with the problem of graph isomorphism recognition, and with the coding theory problems. There are several equivalent notions independently introduced in different contexts. For example, the notion of *equitable partition* is used in works of C. Godsil (ex.see [6]). The notion of *partition design* was

introduced by P. Camion, B. Courteau, G. Fournier, S.V. Kanetkar in [4] and is used in combinatorial design theory and in coding theory.

The problem of characterization of perfect colorings is an actual problem of coding theory, because the notion of perfect coloring is closely connected with many known codes such as perfect, completely regular and uniformly packed. For instance, a distance partition of a distance regular graph in accordance to a perfect code is a perfect coloring. It is worth mentioning that the coloring induced by a completely regular code introduced by P. Delsarte [5] is perfect by definition.

The problem of existence of perfect codes in the n-dimensional hypercube graph has been attracting attention of mathematicians for more than half a century. Note that the best upper and lower bounds for a number of various 1-perfect codes differ essentially, what means that the complete description of them is far from being obtained. The perfect coloring with k colors can be interpreted as a generalization of such codes in case of k-ary coding.

Sometimes the graph under consideration can be represented as the product of simpler graphs or graphs of lower dimensions. For example, the hypercube graph E^n is the Cartesian product of graphs E^{n-1} and E. Creation of the constructions that would help to obtain perfect colorings of graph products via colorings of their multipliers is a problem of interest in the areas of coding theory, algebraic combinatorics and graph theory.

Let us note that graphs products are interesting from the point of view of crystallography. Let G^* be a product of graphs G by H. The graph G^* can be interpreted as a graph G with vertices having the structure of kind H, i.e. every its vertex is a "molecule of type H". Perfect colorings of these graphs allow to model structures on crystals having several useful physical and chemical properties.

The lexicographic product of two graphs G and H is the graph $G \cdot H$ such that its vertex set is the Cartesian product $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $\{u_1, u_2\} \in E(G)$ or $u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$. The graph lexicographic product is also known as the graph composition [7].

The *infinite path graph* is the graph whose set of vertices is the set of integers and two vertices u and v are adjacent if $|u - v| = 1$. Hereinafter, we denote the infinite path graph by C_∞ and, for a transitive graph G, we call the graph $C_\infty \cdot G$ the *infinite G-times path*.

Let n be a positive integer. By K_n and $\overline{K_n}$ we denote the complete and empty graphs on n vertices. In this paper we list the perfect colorings of $\overline{K_n}$- and K_n-times paths with an arbitrary finite set of colors. The local structure of the graphs $C_\infty \cdot \overline{K_3}$ and $C_\infty \cdot K_3$ is shown in Figure 1.
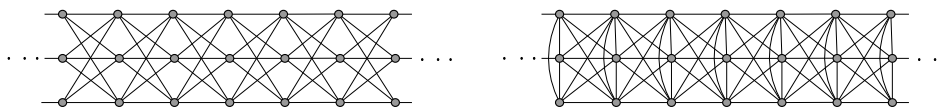

Рис. 1. Local structure of $C_\infty \cdot \overline{K_3}$ (left) and $C_\infty \cdot K_3$ (right)

The graphs under consideration have an extensive structure, in other words, they contain C_∞ as a subgraph. Perfect colorings of graphs with a similar structure, such

as the infinite circulant graphs, infinite transitive grids, infinite prism graph, were studied before.

The perfect colorings of the infinite prism graph with an arbitrary finite number of colors are listed in [12]. Several results on perfect colorings of circulant graphs are obtained by D.B. Khoroshilova in [8, 9]. She showed, in particular, that every perfect coloring listed in [9] yields a perfect coloring of the n-dimensional infinite grid with the same parameter matrix. Perfect 2-colorings for two families of infinite circulant graphs are listed in [14, 15]. The complete description of perfect colorings with an arbitrary finite number of colors is obtained for the infinite circulant graphs with distances 1 and 2 in [13]. Let us note that perfect colorings of circulant graphs can be used in mathematical optimization [10].

A coloring of a graph is called *perfect of radius r* with the parameter matrix $M = (m_{ij})$ if for every vertex x of color i the number of vertices of color j in the sphere of radius r with center x is equal to m_{ij}.

First results on perfect colorings of the infinite rectangular grid graph $G(Z^2)$ were obtained by M. Axenovich [3]. She listed all admissible parameter matrices of perfect 2-colorings of radius 1 for this graph and established several necessary conditions for a matrix to be admissible for the graph in the case $r \geq 2$. Parameters and properties of perfect colorings of $G(Z^2)$ were studied by S.A. Puzynina in her thesis. In [18, 19] she showed that all perfect colorings of the infinite rectangular grid of radius $r > 1$ are periodic and proved their pre-periodicity in the case $r = 1$. A technique of equivalent colors merging is proposed in [18]; we will use this technique to prove the main result. All admissible parameter matrices of order 3 for the graph $G(Z^2)$ were described in [17]. Perfect colorings with up to 9 colors of this graph are listed by D.S. Krotov in [11].

A perfect coloring is called *distance regular* if its parameter matrix can be reduced to the tridiagonal form. The parameters of all distance regular colorings of the infinite rectangular grid were listed by S.V. Avgustinovich, A.Yu. Vasil'eva, and I.V. Sergeeva in [2].

The pre-periodicity of perfect colorings of the hexagonal and triangular grids was proven by S.A. Puzynina in [16]. For the infinite triangular grid, the distance regular colorings were listed by A.Yu. Vasil'eva in [20]; for the hexagonal grid, they were later studied by S.V. Avgustinovich, D.S. Krotov, and A.Yu. Vasil'eva [1].

2. Disjunctive perfect colorings of the graph lexicographic product

Let G and H be simple graphs, where G may be an infinite graph, and $G \cdot H$ their lexicographic product.

Let k be a positive integer. The elements of the finite set $I = \{1, 2, \ldots, k\}$ are called the *colors*. Let $\psi : V(G) \to I$ be a perfect coloring of the graph G and $\Phi = \{\phi_1, \phi_2, \ldots \phi_k\}$ a set of perfect colorings of the graph H with colors from sets $J_1, J_2, \ldots J_k$ respectively, where $J_p \cap J_q = \varnothing$ if $p \neq q$. We define the following coloring for the graph $G \cdot H$:

$$\psi \cdot \Phi : V(G) \times V(H) \to J_1 \cup J_2 \cup \ldots \cup J_k;$$
$$\psi \cdot \Phi(v_1, v_2) = \phi_{\psi(v_1)}(v_2).$$

Such a structure on the graph $G \cdot H$ is called a *disjunctive coloring*. The formula in the definition reflects the fact that the perfect coloring ϕ_i of H with colors from J_i corresponds to the color i in the perfect coloring of G.

Lemma 1. *A disjunctive coloring of the graph $G \cdot H$ is perfect.*

Proof. Consider two vertices (u_1, u_2) and (w_1, w_2) colored with the same color in the structure $\psi \cdot \Phi(v_1, v_2)$:

$$\psi \cdot \Phi(u_1, u_2) = \psi \cdot \Phi(w_1, w_2) \Rightarrow \phi_{\psi(u_1)}(u_2) = \phi_{\psi(w_1)}(w_2).$$

By the definition of the disjunctive coloring, $\psi(u_1) = \psi(w_1)$. The vertices u_1 and w_1 are colored with the same color in the perfect coloring of G; hence, the color structures of their neighborhoods coincide. Therefore, the copies of the graph H neighboring u_1 and w_1 have equal multisets of colors. The colors of adjacent vertices from the H-copies corresponding to u_1 and w_1 are the same, since the colorings $\phi_{\psi(u_1)}$ and $\phi_{\psi(w_1)}$ are perfect. Thus, the neighborhoods of the vertices (u_1, u_2) and (w_1, w_2) have the same color structure. Consequently, the coloring $\psi \cdot \Phi(v_1, v_2)$ is perfect. □

Herein, we consider the case when $G = C_\infty$ and $H = \overline{K_n}$ or $H = K_n$.

We denote by V_i the copy of $\overline{K_n}$ with number i in the corresponding multipath and call it i-th *block*. The vertices of each block are enumerated with the integers from 1 to n. The j-th vertex of the i-th block in $\overline{K_n}$-times path is denoted by v_{ij}. We use the same enumeration when considering the graph $C_\infty \cdot K_n$.

Note that any coloring of the empty or complete graph is perfect; therefore, it can be used to construct a disjunctive coloring of $C_\infty \cdot \overline{K_n}$ and $C_\infty \cdot K_n$ respectively.

3. Equivalent colors in a perfect coloring

Consider a finite regular graph $G = (V, E)$ and a perfect coloring $\phi : V \to I$ with the parameter matrix M. Two colors i and j in the perfect coloring ϕ are called *equivalent* ($i \sim j$) if the coloring obtained after their identification is perfect. Note that the rows of the parameter matrix corresponding to the colors i and j coincide up to the elements of the columns i and j. This property of the parameter matrix is equivalent to the definition of equivalent colors.

Lemma 2. *The relation " $\sim$ " defined above is an equivalence relation. Moreover, the coloring obtained by identifying colors in equivalent classes is perfect.*

Proof. Reflexivity and symmetry of the relation are obvious. To show transitivity, consider the colors a, b, and c of a perfect coloring ϕ such that $a \sim b$ and $b \sim c$. Without loss of generality, we may suppose that $a = 1$, $b = 2$, and $c = 3$. The fragment of the parameter matrix M corresponding to these colors has the form

$$\begin{pmatrix} & x & p & \dots \\ y & & p & \dots \\ y & q & & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix};$$

the non-identified elements of the first three rows of M coincide by the definition of " $\sim$ ".

Note that $m_{ij} = 0 \Leftrightarrow m_{ji} = 0$ for any two colors i and j. This, in particular, means that if at least one of the numbers x, y, p, and q equals zero, then so do the others. In this case $1 \sim 3$.

Consider the case when none of the elements x, y, p, and q is equal to zero. We denote the number of vertices colored with i by N_i. Consider three subgraphs of G

induced by the sets of vertices colored with the colors 1 and 2, 2 and 3, 1 and 3. Each of the induced subgraphs is a biregular bipartite graph, and the following relations hold:

$$\frac{N_1}{N_2} = \frac{y}{x}; \quad \frac{N_2}{N_3} = \frac{q}{p}; \quad \frac{N_3}{N_1} = \frac{p}{y},$$

what leads to the equalities:

$$\frac{N_1}{N_2} \cdot \frac{N_2}{N_3} \cdot \frac{N_3}{N_1} = \frac{y}{x} \cdot \frac{q}{p} \cdot \frac{p}{y} \Rightarrow \frac{q}{x} = 1 \Rightarrow x = q.$$

The latter means that the colors 1 and 3 are equivalent, which proves transitivity of the relation. Thus " $\sim$ " is an equalence relation.

The set I can be split into disjoint equivalence classes by the relation $\sim$. Let us show that the coloring obtained by identifying colors in equivalent classes is perfect.

Such a coloring, denoted by $\hat{\phi}$, is obtained in the following way: every two vertices colored in equivalent colors in ϕ get the same color in $\hat{\phi}$. The color corresponding to the equivalence class $[j]$ is denoted by j^*. The rows of the parameter matrix corresponding to the elements x and y from the class $[j]$ coincide up to the elements of the columns x and y. By regularity of G, the sums $m_{xx} + m_{yx}$ and $m_{xy} + m_{yy}$ are equal, where the sums stand for the number of vertices of color j^* adjacent to a vertex colored in j^*. Thus, the number of vertices of each color in every unit sphere with the center of color j^* is the same and the coloring $\hat{\phi}$ is perfect. $\qquad\square$

The identifying of colors in each class of equivalence from the factor-set $I/\sim$ is referred to as the *gluing operation*. The coloring $\hat{\phi}$ obtained by gluing the colors of some perfect coloring ϕ is called a *reduced* coloring. Let ϕ be a reduced coloring. Every perfect coloring ψ such that $\phi = \hat{\psi}$ is called the *splitting* of ϕ. Observe that a reduced coloring ϕ can have several different splittings. Thus, to enumerate all perfect colorings of the graph G, it suffices, first, to obtain a complete description of its reduced colorings and, then, to consider all admissible splittings of the latter.

4. REDUCED COLORINGS OF $\overline{K_n}$- AND K_n-TIMES PATHS

Any perfect coloring of the graph $C_\infty \cdot \overline{K_n}$ (or $C_\infty \cdot K_n$) is described by the sequence of color sets of it blocks up to equivalence. It is easy to be shown (using Dirichlet principle) that any such sequence is periodic, as if parameters of the perfect coloring are known and coloring of two neighbour blocks is fixed, then the whole coloring is uniquelly restored. Length of period of perfect coloring in this case is equal to number of blocks in period of such sequence of color sets.

To describe a perfect coloring of such graph, it suffices to indicate its least period, which means that the perfect colorings of $\overline{K_n}$- and K_n-times paths are the homomorphic inverse images of the perfect colorings of the corresponding finite graphs. The latter allows us to apply methods and results of Section 3 to infinite multipath graphs under consideration.

A perfect coloring of the graph $C_\infty \cdot \overline{K_n}$ (or $C_\infty \cdot K_n$) is called *block-monochrome* if the vertices have the same color in each block. The period of the block-monochrome perfect coloring of the graph $C_\infty \cdot \overline{K_n}$ (or $C_\infty \cdot K_n$) will be denoted by a row in square brackets whose elements are the colors of the blocks.

Lemma 3. *The reduced colorings of the graphs $C_\infty \cdot \overline{K_n}$ and $C_\infty \cdot K_n$ are block-monochrome.*

Proof. Consider the perfect colorings of the graphs from $\overline{K_n}$- and K_n-times paths. The neighborhoods of the vertices of a $\overline{K_n}$-copy in $C_\infty \cdot \overline{K_n}$ coincide; therefore, the colors assigned to such vertices are equivalent. The neighborhoods of two vertices from the same block of $C_\infty \cdot K_n$ differ in exactly these two vertices. Thus, the colors corresponding to such vertices are also equivalent.

Hence, gluing the perfect colorings of such graphs leads to the vertices of each copy to be colored with one color, i.e., a block-monochrome perfect coloring is obtained. $\square$

There is a one-to-one correspondence between the perfect colorings of an infinite path graph and the block-monochrome perfect colorings of $C_\infty \cdot \overline{K_n}$ and $C_\infty \cdot K_n$; the latter are obtained by $\overline{K_n}$-times or K_n-times copying of the former.

The perfect colorings of infinite path graph, in their turn, are well studied. We recall the terminology and well-known facts that can be found in [12].

Every perfect coloring of C_∞ is periodic. We also denote its period by a row in square brackets. The same notation is chosen due to the fact that the infinite path graph is a particular case of the graph $C_\infty \cdot \overline{K_n}$. The colorings of the infinite path graph with the periods $S_{11}(k) = [k{-}1 \; k{-}2 \; \ldots \; 1 \; 0 \; 1 \; \ldots \; k{-}3 \; k{-}2]$, $S_{12}(k) = [k{-}1 \; k{-}2 \; \ldots \; 1 \; 0 \; 1 \; \ldots \; k{-}2 \; k{-}1]$, and $S_{22}(k) = [k{-}1 \; k{-}2 \; \ldots \; 1 \; 0 \; 0 \; 1 \ldots \; k{-}2 \; k{-}1]$ are called the *mirror colorings of types* $(1,1)$, $(1,2)$ *and* $(2,2)$; with the periods $S(k) = [0 \; 1 \; 2 \; \ldots \; k{-}2 \; k{-}1]$, they are the *cyclic* colorings. The type of a mirror coloring is determined by the number of vertices of the first and last colors (0 and $k{-}1$) in the period.

Henceforth, by the cyclic and mirror colorings of $\overline{K_n}$- and K_n-times paths we understand the block-monochrome colorings of these graphs corresponding to the cyclic and mirror colorings of C_∞.

The following lemma describes all perfect colorings of the infinite path graph [12]:

Lemma 4. *The perfect colorings of the graph C_∞ are exhausted by the following four infinite series: three series of mirror and one series of cyclic colorings.*

Corollary 1. *The reduced colorings of the graphs $C_\infty \cdot \overline{K_n}$ and $C_\infty \cdot K_n$ are exhausted by four infinite series corresponding to the perfect colorings of the graph C_∞.*

Thus, we obtained a complete description of the reduced perfect colorings of the infinite $\overline{K_n}$- and K_n-times path graphs.

Let V_i be a block of one of the multipath graphs under consideration, ϕ the reduced coloring of that multipath graph, and ψ its splitting. Thus, the multiset of colors corresponding to the vertices of this block in the coloring ψ is $\psi(V_i)$.

Lemma 5. *Let V_i and V_j be the blocks of $C_\infty \cdot \overline{K_n}$ or $C_\infty \cdot K_n$ the colors of which in the reduced coloring ϕ do not coincide and are equal to a and b respectively. Then $\psi(V_i) \cap \psi(V_j) = \varnothing$ for every splitting ψ of ϕ.*

Proof. Otherwise, by applying the gluing operation to the perfect coloring ψ, we obtain $a = b$. $\square$

Note that if there are blocks V_p and V_q in the graph $C_\infty \cdot G$ such that $\psi(V_p) \neq \psi(V_q)$ and $\psi(V_p) \cap \psi(V_q) \neq \varnothing$, then the coloring ψ is not disjunctive.

Thus, to complete the characterization of the perfect colorings of the $\overline{K_n}$- and K_n-times path graphs, it remains to describe all disjunctive and non-disjunctive splittings of the reduced colorings of these graphs.

5. Perfect colorings of the $\overline{K_n}$-times path

The graph $C_\infty \cdot \overline{K_n}$ being bipartite is one of its important structural properties. This allows us to use the ideas on colorings of bipartite graphs presented in [12]. Let us give some necessary definitions.

Let $G(V_1, V_2)$ be a bipartite graph with the parts V_1 and V_2. A coloring of one of the graph parts is a *semicoloring* of G. A semicoloring is called *admissible* if it is a part of the perfect coloring of G. If a semicoloring belongs to the reduced coloring of G it is *reduced* semicoloring. Two admissible semicolorings of a graph are *conjugate*, if they complement each other to make a perfect coloring of the graph.

Admissible semicolorings of $C_\infty \cdot \overline{K_n}$ are periodic, because perfect colorings of the whole graph are periodic. Length of period of admissible semicoloring is equal to number of blocks in such period.

A perfect coloring of a bipartite graph is *bipartite* if the color sets of its semicolorings are disjoint; otherwise, the coloring is *non-bipartite*. Note that the color sets of semicolorings coincide in non-bipartite case if G is connected.

In non-disjunctive perfect colorings, the number of vertices of the given color in different blocks can be different. Denote by $N_j(i)$ the number of j-colored vertices in the block V_i of $C_\infty \cdot \overline{K_n}$ or $C_\infty \cdot K_n$.

Let us describe a construction for the $\overline{K_n}$-times path. We consider a coloring ψ of $C_\infty \cdot \overline{K_n}$ with period of length 4. If equality $N_j(i-1)+N_j(i+1) = N_j(i)+N_j(i+2)$ holds for every color j of ψ and every i then semicolorings of ψ are called *matched*. It is easy to be shown that the validaty of latter condition for all colors of ψ implies its perfectness.

To make the structure of the further arguments clear, let us formulate the main result of this section:

Theorem 1. *The perfect colorings of the graph $C_\infty \cdot \overline{K_n}$ are exhausted by the following list:*

1 Disjunctive perfect colorings;

2 Non-disjunctive bipartite colorings obtained by conjugation of 2-periodic semicolorings with disjoint sets of colors;

3 Non-disjunctive non-bipartite colorings obtained by conjugation of two matched 2-periodic semicolorings.

We start the study of the perfect colorings of the $\overline{K_n}$-times path graph with a description of its admissible reduced semicolorings.

Lemma 6. *The admissible reduced semicolorings of the graph $C_\infty \cdot \overline{K_n}$ are exhausted by the following four infinite series: three series of mirror and one series of cyclic semicolorings.*

Proof. Partitioning any reduced perfect coloring of the graph $C_\infty \cdot \overline{K_n}$ (see Corollary 1) into semicolorings produces two color sequences. Each of them belongs to cyclic or one of three mirror series. For example, the reduced coloring of $\overline{K_n}$-times path with period $S_{22}(3) = [2\ 1\ 0\ 0\ 1\ 2]$ is obtained by conjugation of two cyclic semicolorings – $[0\ 1\ 2]$ and $[0\ 2\ 1]$. $\qquad\square$

In Lemma 7 and Corollary 2 we characterize the reduced colorings of the $\overline{K_n}$-times path graph admitting a non-disjunctive splitting.

Lemma 7. *If a perfect coloring ψ of the graph $C_\infty \cdot \overline{K_n}$ is obtained by non-disjunctive splitting of a reduced coloring ϕ, then ϕ is a conjugation of one-color semicolorings (either bipartite or non-bipartite).*

Proof. Let V_i and V_j be two a-colored copies of the empty graph admitting a non-disjunctive splitting, i.e., $\psi(V_i) \neq \psi(V_j)$. Consider the following two cases: the blocks V_i and V_j belong to one part of the graph $C_\infty \cdot \overline{K_n}$ in the first case and to different parts in the second.

Study the first case. Assume that the reduced semicoloring of the part which V_i and V_j belong to contains more than one color. By Lemma 6, at least one of the blocks V_{i-2} or V_{i+2} in the coloring ϕ is colored with a color b different from a. Without loss of generality, let V_{i+2} be such a block. Let the block V_{i+1} be colored with a color x, which can coincide with a or b. By Corollary 1, the right or left neighbor of the copy of V_j is also of color x. For definiteness, let V_{j+1} be such a copy; then V_{j+2} is colored with b.

The color sets of the neighborhoods of the x-colored vertices in the coloring ψ coincide; therefore,

$$\psi(V_i) \cup \psi(V_{i+2}) = \psi(V_j) \cup \psi(V_{j+2}).$$

By Lemma 5, the latter equality is valid only in the case when $\psi(V_i) = \psi(V_j)$ and $\psi(V_{i+2}) = \psi(V_{j+2})$; a contradiction. Hence, the part which V_i and V_j belong to is monochrome colored in ϕ.

Describe all reduced perfect colorings of the graph under consideration to which belong that semicoloring. It is easy to see that these are the colorings $S(1)$, $S(2)$, and $S_{11}(3)$. Note that $S_{11}(3)$ cannot be obtained by gluing the colors of the perfect coloring: the corresponding 4-periodic perfect colorings get glued immediately in $S(2)$.

For the second case, the proof is similar, with the only difference that we suppose that at least one semicoloring in ϕ is not one-colored; a contradiction.

Thus, the reduced coloring admitting a non-disjunctive splitting is a conjugation of one-colored semicolorings (bipartite or non-bipartite). □

Corollary 2. *The only reduced colorings of the graph $C_\infty \cdot \overline{K_n}$ admitting non-disjunctive splittings are $S(1)$ (non-bipartite case) and $S(2)$ (bipartite case).*

All non-disjunctive colorings of the $\overline{K_n}$-times path graph are described in Lemma 8.

Lemma 8. *The non-disjunctive perfect colorings of the graph $C_\infty \cdot \overline{K_n}$ are exhausted by the following list:*

1 Non-disjunctive bipartite colorings obtained by conjugation of arbitrary 2-periodic semicolorings;

2 Non-disjunctive non-bipartite colorings obtained by conjugation of two matched 2-periodic semicolorings.

Proof. By Corollary 2, we shall describe the non-disjunctive splittings of the colorings $S(1)$ and $S(2)$. Consider the sequence of blocks V_{i-1}, V_i, V_{i+1}, V_{i+2}, and V_{i+3}. Due to the fact that V_{i+1} and V_{i+3} have the same color in $S(1)$ and $S(2)$ and the vertices of V_{i+2} are in the 1-neighborhoods of both V_{i+1} and V_{i+3}, we obtain $\phi(V_i) = \phi(V_{i+4})$ for every splitting of such block-monochrome colorings. Since i may be arbitrary, the perfect coloring $\phi(v)$ is 4-periodic and its semicolorings have the period of length 2.

Note that the result of conjugation of arbitrary 2-periodic semicolorings of $C_\infty \cdot \overline{K_n}$ with disjoint sets of colors is a perfect coloring. The set of non-disjunctive bipartite perfect colorings of the $\overline{K_n}$-times path graph consists of such conjugations.

In a non-disjunctive non-bipartite perfect coloring ψ of $C_\infty \cdot \overline{K_n}$, there is a pair of the same-colored vertices in adjacent blocks V_i and V_{i+1}. The sets of colors of their neighborhoods coincide; therefore, $N_j(i-1) + N_j(i+1) = N_j(i) + N_j(i+2)$ for each color j of ψ. Thus, perfect coloring ψ is obtained by conjugation of two matched semicolorings.

The characterization of non-disjunctive perfect colorings of a $\overline{K_n}$-times path graph is completed. $\qquad\square$

6. Perfect colorings of the K_n-times path graph

Theorem 2. *The perfect colorings of the graph $C_\infty \cdot K_n$ are exhausted by the following list:*
1 Disjunctive perfect colorings;
2 Non-disjunctive 3-periodic colorings.

Proof. The proof can be reduced to description of non-disjunctive splittings of the reduced perfect colorings of the K_n-times path graph. Let ϕ be a reduced coloring of the graph under consideration and ψ one of its non-disjunctive splittings. Consider the following two cases: with the coloring ϕ either being one-colored or consisting of more than one color.

Studying the first case, we consider the blocks following one another, starting with V_i. Show that $\psi(V_i) = \psi(V_{i+3})(*)$. If there exists a pair of vertices of the same color in V_{i+1} and V_{i+2}, then equality $(*)$ holds. Suppose that there are no same-colored vertices in these blocks and $\psi(v_{(i+1)j}) = p$ and $\psi(v_{(i+2)l}) = q$ for some j and l. Since p and q are equivalent, the number of vertices of color s $(s \neq p,\ s \neq q)$ adjacent to $v_{(i+1)j}$ is equal to the number of such neighbors of $v_{(i+2)l}$:

$$N_s(i) + N_s(i+1) + N_s(i+2) = N_s(i+1) + N_s(i+2) + N_s(i+3) \Rightarrow N_s(i) = N_s(i+3).$$

If there is a vertex of color t different from p and q in V_{i+1} or V_{i+2}, then $t \sim p$ and $t \sim q$. For these colors, by writing down an equality similar to the latter, we obtain $N_q(i) = N_q(i+3)$ and $N_p(i) = N_p(i+3)$; consequently, $\psi(V_i) = \psi(V_{i+3})$.

The lack of vertices of color t means that the blocks V_{i+1} and V_{i+2} are monochrome-colored with the colors p and q respectively. Show that all admissible extensions of such a fragment are 3-periodic perfect colorings.

If there is a color r in the set $\psi(V_{i+3})$ such that $r \neq p$ and $r \neq q$, then by equivalence of q and r the block V_{i+4} is monochrome-colored with color p. The multiset $\psi(V_{i+5})$ consists only of the elements q, since $p \sim r$. Such a fragment is uniquely extended to a 3-periodic perfect coloring.

Let only p and q be the elements of the set $\psi(V_{i+3})$; hence, the coloring ψ is two-colored. In the case $q \in \phi(V_{i+3})$, the color set of the neighborhoods of vertices of color q is defined uniquely. This allows us to extend the coloring $\phi(v)$ to the right by a p-colored block. The admissible extensions of this fragment are exhausted by 3-periodic non-disjunctive and two disjunctive colorings $S_{12}(2)$ and $S_{22}(2)$.

In the case of the p-monochrome block V_{i+3}, the perfect extensions of such a structure are exhausted by the disjunctive colorings $S_{12}(2)$ and $S(2)$. The characterization of the non-disjunctive splittings of a one-colored reduced coloring is thus completed.

Consider the case of a reduced coloring ϕ with two or more colors. Let V_i and V_j be copies of a complete graph admitting non-disjunctive splitting; i.e., the elements of such blocks in ϕ are colored with the same color a, but $\psi(V_i) \neq \psi(V_j)$.

At least one of the blocks adjacent to V_i is colored differently in the reduced coloring, with a color b ($b \neq a$). This is also true for the neighbors of the copy of V_j. Without loss of generality, we assume that the blocks V_{i+1} and V_{j+1} are b-colored. Since $\psi(V_i) \neq \psi(V_j)$ and the color sets of the neighborhoods of vertices from V_i and V_j in the coloring $\psi(v)$ coincide, we have $\psi(V_{i-1}) \cap \psi(V_j) \neq \varnothing$ and $\psi(V_i) \cap \psi(V_{j-1}) \neq \varnothing$. Consequently, the color a corresponds to the elements of the copies V_{i-1} and V_{j-1} in ϕ. Arguing similarly for V_{i+1} and V_{j+1}, we find that the vertices of V_{i+2} and V_{j+2} are also a-colored in this coloring. Thus, the period of the reduced coloring ϕ has the form $[aba]$.

Prove that every non-disjunctive splitting ψ of such a reduced coloring is 3-periodic. Consider the sequence of blocks V_i, V_{i+1}, V_{i+2}, and V_{i+3}. Let their colors in the coloring ϕ be equal to a, a, b, and a respectively and $\psi(V_i) \neq \psi(V_{i+1})$. Show that $\psi(V_i) = \psi(V_{i+3})$.

Suppose that this is not true. Hence, there is a color c such that the numbers of vertices of this color in the blocks V_i, V_{i+1}, and V_{i+3} are equal to x, y, and z respectively, while $x \neq z$. Without loss of generality, we can assume that $z < x$, i.e., $x - z > 0$. The number of neighbors of color c in the neighborhood of each vertex is thus defined. For the vertices to which the color a corresponds in the reduced coloring, this number is equal to $x+y$, while for the elements of the b-colored copies it is $y + z$. Calculating the number of vertices of color c in the blocks V_{i+4}, V_{i+5}, V_{i+6}, etc., we obtain $N_c(i+3p) = z - (p-1)(x-z)$. This implies that $N_c(i+3p)$ is monotone decreasing with growth of p, which contradicts the infinity of the graph under consideration; therefore, our assumption is false and $x = z$. Hence, we obtain the 3-periodicity of the coloring ψ.

Consequently, all non-disjunctive perfect colorings of a K_n-times path graph have the period of length 3.   □

Thus, the set of the perfect colorings of the graph $C_\infty \cdot K_n$ consists of two infinite series: disjunctive colorings and non-disjunctive 3-periodic colorings.

Conclusion

Creating the constructions that make it possible to obtain perfect colorings of different types of graph products from the perfect colorings of their factors is an important problem of graph theory. In particular, the n-dimensional binary cube E^n can be presented as the product of hypercubes of smaller dimension. The complete description of its perfect colorings is not known yet even in the case of two colors.

In this article, we study the lexicographic product of graphs. We show that the set of perfect colorings of the graph $G \cdot H$ splits up into two subsets – disjunctive and non-disjunctive colorings. Exploring constructions of non-disjunctive colorings for different pairs (G, H) is a natural and interesting problem.

The simplest graph of such a type is the lexicographic product of the infinite path graph and an arbitrary transitive graph G, i.e. the G-times path graph. We described all perfect colorings of the $\overline{K_n}$- and K_n-times path graphs with an arbitrary finite number of colors.

The multipath graphs can be viewed as extensions of the infinite path graph and, consequently, the structures built on them may find applications in the group theory and crystallography.

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Mariya Aleksandrovna Lisitsyna
Budyonny Military Academy of the Signal Corps,
3, pr. Tikhoretsky ave.,
St Petersburg, 194064, Russia
Email address: lisitsyna.mariya.mathematician@gmail.com

Sergey Vladimirovich Avgustinovich
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: avgust@math.nsc.ru

Olga Gennad'evna Parshina
Czech Technical University in Prague,
13, Trojanova str.,
Prague, 120 00, Czech Republic
Email address: parolja@gmail.com