ALGEBRAIC LIMIT CYCLES OF PLANAR CUBIC SYSTEMS

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ABSTRACT. We study algebraic limit cycles of differential systems of the form $\dot{x} = x + P_3(x,y)$, $\dot{y} = y + Q_3(x,y)$ where $P_3(x,y)$ and $Q_3(x,y)$ are homogeneous cubic polynomials.

Keywords: polynomial systems, algebraic limit cycles, non-algebraic limit cycles, phase portraits.

INTRODUCTION

The problem of researching limit cycles is one of the central ones in the qualitative theory of ordinary differential equations. A notable subsection of studies in this perspective is devoted to limit cycles of autonomous polynomial systems of the form

$$(1) \qquad \dot{x} = P(x,y), \ \dot{y} = Q(x,y).$$

Here $P(x,y)$, $Q(x,y)$ are real polynomials of the variables x, y, and $t \in \mathbb{R}$ is taken as an independent variable $t \in \mathbb{R}$. The term degree of a system is used to refer to the maximum of degrees of the polynomials P, Q. If $n = 3$, system (1) is called a cubic system.

A limit cycle of system (1) refers to a periodic solution whose trajectory is isolated among the trajectories of all the periodic solutions. The limit cycle of system (1) is called an algebraic one of degree m if it is a real oval of the irreducible algebraic curve $H(x,y) = 0$ of degree m.

The problem of searching for algebraic solutions for polynomial systems, in particular, algebraic limit cycles, goes back to J. G. Darboux and A. Poincare [1, 2] and is currently actively developing (see [3] and the literature cited there).

In this paper, we study this problem with regard to a differential system of the form

$$\dot{x} = x + P_3(x, y), \ \dot{y} = y + Q_3(x, y), \tag{2}$$

which we call a cubic system of a Darboux type. Here $P_3(x, y)$, $Q_3(x, y)$ are homogeneous real polynomials of degree 3 of the variables x, y. The systems of such kind emerged in Darboux's works on geometry.

We have obtained necessary and sufficient conditions for the existence of hyperbolic limit cycle for system (2), moreover, this cycle turns to be unique. The rest of the trajectories (except from the quiescent point in the origin) have a limit cycle as an α- or ω-limit set, and cannot be algebraic curves. Only a limit cycle can be an algebraic curve. Therefore, the number of algebraic curves (in this case, ovals) does not exceed 1. This statement provides a solution of Poincare's problem for the case of system (2), in which we need to specify for a polynomial system of degree n the maximal possible number $N(n)$ of algebraic solutions.

We proved that the degree of the algebraic limit cycle of system (2) equals 2 and obtained necessary and sufficient conditions for the existence of an algebraic quadratic limit cycle for system (2).

The considered examples demonstrate applications of our results.

1. The main part

Consider a system of Darboux type of a more general form compared to (2),

$$\dot{x} = sx + P_n(x, y), \ \dot{y} = sy + Q_n(x, y), \ s \neq 0, \ n > 1. \tag{3}$$

We define the following functions:

$$f(\vartheta) = \cos \vartheta P_n(\cos \vartheta, \sin \vartheta) + \sin \vartheta Q_n(\cos \vartheta, \sin \vartheta),$$
$$g(\vartheta) = \cos \vartheta Q_n(\cos \vartheta, \sin \vartheta) - \sin \vartheta P_n(\cos \vartheta, \sin \vartheta).$$

From the results obtained in [4, 5], follows

Theorem 1. *1) If n is even, system (3) does not have any periodic solutions.*

2) If n is odd and $g(\vartheta)$ has zeros on the segment $[0, 2\pi]$, system (3) does not have any periodic solutions.

3) System (3) has at most one limit cycle.

4) In order for the unique limit cycle of system (3), which surrounds the origin, to exist, it is necessary and sufficient that the following conditions be satisfied:

$$g(\vartheta) \neq 0, \ \vartheta \in [0, 2\pi]; \ sg(0) \int_0^{2\pi} \frac{f(\vartheta)}{g(\vartheta)} d\vartheta < 0.$$

This cycle is a hyperbolic one.

Consider a cubic Darboux system

$$\dot{x} = x + P_3(x, y), \ \dot{y} = y + Q_3(x, y), \tag{2}$$

which is obtained from system (3) when $s = 1$, $n = 3$. Let system (2) have a unique hyperbolic limit cycle (see Theorem 1).

We will find out when this limit cycle is algebraic.

The slope field of system (2) is symmetric with respect to the origin. In this case, the trajectories of this system and the formulas generating them should also have the symmetry property. In particular, closed algebraic curves will be defined by polynomials of the form $H(x, y) = h_0 + h_2(x, y) + h_4(x, y) + ...h_{2k}(x, y)$, where $h_0 = \text{const} \neq 0$, $h_2(x, y)$, $h_4(x, y), ...h_{2k}(x, y)$ are homogeneous polynomials of even degrees. Without loss of generality, we can put that $h_0 = 1$.

After conversion to polar coordinates, system (2) turns into the system

$$\dot{r} = r + r^3 f(\vartheta), \ \dot{\vartheta} = r^2 g(\vartheta),$$

which we substitute with the equation

(4) $$\frac{dR}{d\vartheta} = \frac{2f(\vartheta)}{g(\vartheta)} R + \frac{2}{g(\vartheta)}, \ R = r^2,$$

where now we have

$$f(\vartheta) = P_3(\cos\vartheta, \sin\vartheta)\cos\vartheta + Q_3(\cos\vartheta, \sin\vartheta)\sin\vartheta,$$
$$g(\vartheta) = Q_3(\cos\vartheta, \sin\vartheta)\cos\vartheta - P_3(\cos\vartheta, \sin\vartheta)\sin\vartheta.$$

Periodic solutions of this equation (with period $T = 2\pi$) form periodic solutions of system (2).

We introduce the function $F(\vartheta) = 2f(\vartheta)/g(\vartheta)$.

We designate the solution of linear equation (4) with condition $R(0) = R_0$ by $R(\vartheta; R_0)$:

(5) $$R(\vartheta; R_0) = \left(R_0 + \int_0^\vartheta \frac{2}{g(\tau)} \exp(-\int_0^\tau F(\varsigma)d\varsigma)d\tau\right) \exp(\int_0^\vartheta F(\varsigma)d\varsigma).$$

To obtain a periodic solution, we need to choose $R_0 = R_0^*$ from the condition $R(2\pi; R_0) = R_0$. In the case under consideration, there is a unique solution of this kind:

(6) $$R_0^* = \frac{\exp(\int_0^{2\pi} F(\varsigma)d\varsigma)}{1 - \exp(\int_0^{2\pi} F(\varsigma)d\varsigma)} \int_0^{2\pi} \frac{2}{g(\tau)} \exp(-\int_0^\tau F(\varsigma)d\varsigma)d\tau.$$

The orbits of system (2) have a parametric definition

(7) $$x = \sqrt{R(\vartheta; R_0)}\cos\vartheta, \ y = \sqrt{R(\vartheta; R_0)}\sin\vartheta, \ \vartheta \in [0, 2\pi],$$

where $R(\vartheta; R_0)$ is taken from (5).

For a periodic orbit we need to take $R_0 = R_0^*$ from (5).

Let the limit cycle of system (2) be algebraic.

We will use the approach proposed in [6].

Consider an irreducible polynomial $H(x, y) = 1 + h_2(x, y) + ...h_{2k}(x, y)$, such that $H(x, y) = 0$ contains an oval defined by (7) with $R_0 = R_0^*$; $H(x, y)$ is an invariant algebraic curve of system (2). After substituting (7) into $H(x, y)$, we obtain with respect to the variable R the polynomial $\tilde{H}(R, \vartheta)$ whose coefficients are trigonometric polynomials $h_i(\cos\vartheta, \sin\vartheta)$. Moreover, $R = R(\vartheta; R_0^*)$ will constitute its R-root. In [6] it has been proven that only periodic solutions of equation (4) can be the R-roots of the polynomial $\tilde{H}(R, \vartheta)$, and such solution is unique: $R = R(\vartheta; R_0^*)$. We obtain that the polynomial $\tilde{H}(R, \vartheta)$ has one and only one R-root, hence, it has the form $\tilde{H}(R, \vartheta) = 1 + h_2(\cos\vartheta, \sin\vartheta)R$. This means that in Cartesian

coordinate system, the algebraic limit cycle of system (2) is defined by a polynomial of second degree.

Summarising the above considerations, we formulate

Proposition 1. *If a cubic Darboux system (2) has an algebraic limit cycle, then it has degree 2 and is defined by the algebraic curve*

$$1 + ax^2 + 2bxy + cy^2 = 0.$$

In [7], we have proved Theorem 3.2, which conducts a classification of planar homogeneous cubic vector fields. The corollary of this theorem in the considered case is

Proposition 2. *The system*

$$\dot{x} = x + P_3(x, y), \quad \dot{y} = y + Q_3(x, y) \tag{2}$$

has a limit cycle only in the case when there exists a linear transformation $\sigma \in GL(2; \mathbb{R})$ and time scaling, which turn system (2) into a system of the form

$$
\begin{aligned}
\dot{x} &= sx + p_1 x^3 + (p_2 - \alpha)x^2 y + p_3 xy^2 - \alpha y^3 \equiv sx + \tilde{P}_3(x, y), \\
\dot{y} &= sy + \alpha x^3 + p_1 x^2 y + (p_2 + \alpha)xy^2 + p_3 y^3 \equiv sy + \tilde{Q}_3(x, y), \\
&\quad p_1,\ p_2,\ p_3,\ s \in \mathbb{R},\ \alpha = \pm 1.
\end{aligned} \tag{8}
$$

For system (8),

$$f(\vartheta) = \frac{1}{2}(p_1 + p_3 + (p_1 - p_3)\cos 2\vartheta + p_2 \sin 2\vartheta),\ g(\vartheta) \equiv \alpha,$$

$$sg(0) \int_0^{2\pi} \frac{f(\vartheta)}{g(\vartheta)} d\vartheta = \pi s(p_1 + p_3).$$

Due to Theorem 1, we have that in order for the unique hyperbolic limit cycle of system (8) to exist, it is necessary and sufficient that the condition

$$s(p_1 + p_3) < 0 \tag{9}$$

is satisfied.

Consider the question on the existence of an algebraic limit cycle for system (8).

Recall that if the trajectory of a planar polynomial system of differential equations

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1}$$

lies on the irreducible algebraic curve $H(x, y) = 0$, when there is a polynomial $k(x, y)$ (co-factor), such that

$$\frac{\partial H(x, y)}{\partial x} P(x, y) + \frac{\partial H(x, y)}{\partial y} Q(x, y) = k(x, y) H(x, y). \tag{10}$$

Obviously, the degree of the co-factor does not exceed $n - 1$, if n is the maximum of degrees of the polynomials $P(x, y),\ Q(x, y)$,

As it follows from Proposition 1, for systems (2), (8) the closed algebraic curve (algebraic limit cycle) will be defined by a polynomial of the form $H(x, y) = 1 + h_2(x, y)$, where $h_2(x, y)$ is a homogeneous polynomial of second degree; $k(x, y)$ in this case is a quadratic form $k_2(x, y)$ of variables $x,\ y$.

Let system (8) have a quadratic limit cycle $H = 0$, where $H = 1 + h_2 \equiv 1 + ax^2 + 2bxy + cy^2$.

Condition (10) takes the form

$$\frac{\partial H}{\partial x}(sx + \tilde{P}_3) + \frac{\partial H}{\partial y}(sy + \tilde{Q}_3) = k_2 H.$$

Applying Euler's formula for the homogeneous polynomial $h_2(x, y)$, we get that

$$2sh_2 = k_2, \ \tilde{P}_3\frac{\partial h_2}{\partial x} + \tilde{Q}_3\frac{\partial h_2}{\partial y} = k_2 h_2.$$

Finally, we obtain

$$\tilde{P}_3\frac{\partial h_2}{\partial x} + \tilde{Q}_3\frac{\partial h_2}{\partial y} = 2sh_2^2.$$

If we equate the coefficients of equal powers of the variables x, y on the left-hand and right-hand sides, after trivial transformations we arrive to the system of equalities

$$-a^2 s + ap_1 + \alpha b = 0,$$
$$-4abs + ap_2 - \alpha a + 2bp_1 + \alpha c = 0,$$
$$-2acs + ap_3 - 4b^2 s + 2bp_2 + cp_1 = 0,$$
$$-\alpha a - 4bcs + 2bp_3 + cp_2 + \alpha c = 0,$$
$$-\alpha b - c^2 s + cp_3 = 0.$$

Here and after, to conduct symbolic and number calculations we use the system *Mathematica*[1].

System (11) can be considered a nonhomogeneous system of linear equations $AX = B$ with respect to the variables p_1, p_2, p_3 with parameters a, b, c, s, α

$$A = \begin{pmatrix} a & 0 & 0 \\ 2b & a & 0 \\ c & 2b & a \\ 0 & c & 2b \\ 0 & 0 & c \end{pmatrix}, \quad B = \begin{pmatrix} a^2 s - \alpha b \\ \alpha(a - c) + 4abs \\ 2acs + 4b^2 s \\ \alpha(a - c) + 4bcs \\ \alpha b + c^2 s \end{pmatrix}, \quad X = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The system has a solution if and only if the rank of the matrix A equals the rank of the extended matrix $(A|B)$.

Obviously, in (11), we have that $a \neq 0$. Hence, rank $A = 3$. In this case, all the minors of 4th order of the matrix $(A|B)$ turn into zero. These minors are

$$(A|B)_1 = \alpha(a^2 c^2 - 8ab^2 c - 2ac^3 + 8b^4 + 4b^2 c^2 + c^4),$$
$$(A|B)_2 = \alpha(-3a^2 bc + 4ab^3 + 2abc^2 + bc^3),$$
$$(A|B)_3 = \alpha(-a^3 c + 2a^2 b^2 + 2a^2 c^2 - ac^3 + 2b^2 c^2),$$
$$(A|B)_4 = \alpha(a^3 b + 2a^2 bc - 3abc^2 + 4b^3 c),$$
$$(A|B)_5 = \alpha(a^4 - 2a^3 c + 4a^2 b^2 + a^2 c^2 - 8ab^2 c + 8b^4),$$

where $(A|B)_i$ designates the minor obtained from the matrix $(A|B)$ by crossing out the i-th line. Note that the obtained expressions do not depend on s and contain α as their factor.

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[1]Wolfram Research, Inc., *Mathematica. Version 8.0.0.0.* (Licence L332-8225 Academic 8.0.0.0)

As $\alpha = \pm 1$, it suffices to consider the system of homogeneous equations

$$a^2c^2 - 8ab^2c - 2ac^3 + 8b^4 + 4b^2c^2 + c^4 = 0,$$
$$-3a^2bc + 4ab^3 + 2abc^2 + bc^3 = 0,$$
(11) $$-a^3c + 2a^2b^2 + 2a^2c^2 - ac^3 + 2b^2c^2 = 0,$$
$$a^3b + 2a^2bc - 3abc^2 + 4b^3c = 0,$$
$$a^4 - 2a^3c + 4a^2b^2 + a^2c^2 - 8ab^2c + 8b^4 = 0.$$

The second and the fourth equations have the form

(12) $$b(-3a^2c + 4ab^2 + 2ac^2 + c^3) = 0, \;\; b(a^3 + 2a^2c - 3ac^2 + 4b^2c) = 0.$$

Case 1. $b = 0$.
In this case, system (12) is reduced to the system

$$c^2(a-c)^2 = ac(a-c)^2 = a^2(a-c)^2 = 0,$$

which implies that we have a nonzero solution of system (12), $a = c \neq 0, \; b = 0$.

Case 2. If $b \neq 0$, then $-3a^2c + 4ab^2 + 2ac^2 + c^3 = a^3 + 2a^2c - 3ac^2 + 4b^2c = 0$, which implies that $(a-c)(a+c)^3 = 0$. If $a = c$, then from the fourth equation of system (12) we obtain $c = 0$, and then from the fifth equation we have that $b = 0$, which contradicts the assumption that $b \neq 0$. In the case when $a = -c$, the unique real solution of system (13) will be $a = b = c = 0$, which again leads to a contradiction.

System (11), given the obtained values of the parameters $a, \; b, \; c$, is reduced to the form

$$ap_1 = a^2s, \;\; ap_2 = 0, \;\; ap_1 + ap_3 = 2a^2s, \;\; ap_2 = 0, \;\; ap_3 = a^2s, \;\; a \neq 0,$$

and has a nonzero solution $p_1 = p_3 \neq 0, \; p_2 = 0, \; a = c = p_1/s, \; b = 0$.

In this case, $H(x,y) = 1 + \frac{p_1}{s}(x^2 + y^2)$, and the algebraic curve $H(x,y) = 0$ defines a real oval (circle) if the condition $p_1/s < 0$ is satisfied (compare to (9)).

Therefore, we have proved

Theorem 2. *System (8) has a hyperbolic algebraic limit cycle if and only if*

(13) $$p_1 = p_3, \;\; p_2 = 0, \;\; p_1s < 0.$$

Moreover, this cycle is defined by the algebraic curve

$$H \equiv 1 + \frac{p_1}{s}(x^2 + y^2) = 0.$$

System (8), for which conditions (14) are fulfilled, has the form

(14) $$\dot{x} = sx + px^3 - \alpha x^2 y + pxy^2 - \alpha y^3, \quad \dot{y} = sy + \alpha x^3 + px^2 y + \alpha xy^2 + py^3.$$

We designate $\delta = \sqrt{-s/p}$.
We can directly verify that

$$x(t) = \delta \cos \delta^2 t, \; y(t) = \alpha \delta \sin \delta^2 t$$

constitutes a periodic solution of system (15). This solution is a suitable parameterization of the circle $H = 0$, mentioned in Theorem 2. The period of the obtained cycle equals $T = 2\pi/\delta^2$.

The cycle will be stable if $s > 0$, and unstable if $s < 0$.

We conduct in (15) a substitution of the variables

$$u = ax + by, \ v = cx + dy, \ \tau = t/s,$$

where $a, \ b, \ c, \ d$ are arbitrary real numbers, such that $ad - bc \neq 0$.

We obtain the system

$$
(15) \qquad
\begin{aligned}
\dot{u} &= u + p_{30}u^3 + p_{21}u^2v + p_{12}uv^2 + p_{03}v^3 \equiv \mathcal{P}_3(u, v), \\
\dot{v} &= v + q_{30}u^3 + q_{21}u^2v + q_{12}uv^2 + q_{03}v^3 \equiv \mathcal{Q}_3(u, v),
\end{aligned}
$$

in which

$$p_{30} = -s\frac{\left(c^2 + d^2\right)\left(\alpha(ac + bd) + p(ad - bc)\right)}{(bc - ad)^3},$$

$$p_{21} = s\frac{a^2(\alpha(3c^2 + d^2) + 2cdp) + 2ab(2\alpha cd + p(d^2 - c^2)) + b^2(\alpha(c^2 + 3d^2) - 2cdp)}{(bc - ad)^3},$$

$$p_{12} = -s\frac{(a^2 + b^2)(3\alpha ac + adp + 3\alpha bd - bcp)}{(bc - ad)^3},$$

$$
(16) \qquad p_{03} = s\frac{\alpha(a^2 + b^2)^2}{(bc - ad)^3}, \qquad q_{30} = -s\frac{\alpha\left(c^2 + d^2\right)^2}{(bc - ad)^3},
$$

$$q_{21} = s\frac{(c^2 + d^2)(3\alpha(ac + bd) - p(ad - bc))}{(bc - ad)^3},$$

$$q_{12} = -s\frac{a^2(\alpha(3c^2 + d^2) - 2cdp) + 2ab(2\alpha cd + p(c^2 - d^2)) + b^2(\alpha(c^2 + 3d^2) + 2cdp)}{(bc - ad)^3},$$

$$q_{03} = s\frac{(a^2 + b^2)(\alpha(ac + bd) - p(ad - bc))}{(bc - ad)^3}.$$

The dot designates the differentiation with respect to the variable τ.

Theorem 3. *System (16) has an algebraic limit cycle if and only if the coefficients* $p_{ij}, \ q_{ij}, \ i, j = 0, 1, 2, 3, \ i + j = 3$ *can be represented in the form (17), where* $a, b, c, d, p, s \in \mathbb{R}, \ ad - bc \neq 0, \ ps < 0, \ \alpha = \pm 1.$

Доказательство. Sufficiency. Directly calculations show that if relations (17) between the coefficients of system (16) are fulfilled, for the expression

$$H(u, v) = 1 + \frac{ps}{(ad - bc)^2}((av - cu)^2 + (bv - du)^2)$$

the equality

$$\frac{\partial H}{\partial u}\mathcal{P}_3(u, v) + \frac{\partial H}{\partial v}\mathcal{Q}_3(u, v) = \frac{2ps((av - cu)^2 + (bv - du)^2)}{(ad - bc)^2}H(u, v).$$

holds, that is, $H(u, v) = 0$ is a partial integral of system (16) . The corresponding level line is an ellipse and depicts a hyperbolic limit cycle (stable if $s > 0$, and unstable if $s < 0$). The period of the cycle equals $T = 2\pi|p|$.

Necessity. Let system (16) have a quadratic limit cycle. According to [5], there exists a linear transformation $\sigma \in GL(2, \mathbb{R})$ and time scaling $\tau = t/s$, which turn system (16) into system (8), which has a quadratic limit cycle. Due to Theorem 3, the obtained system has the form (15). The inverse transformation σ^{-1} returns system (15) back to system (16), where the coefficients are represented in the form (17). $\qquad\square$

Applying the embedded command `Eliminate[expr, vars]` to the modified set of relations (17), we can formulate

Proposition 3. *If system (16) has an algebraic limit cycle, it is necessary that the coefficients p_{ij}, q_{ij} satisfy the relations*

$$p_{03}(4p_{21} + 4q_{12}) = p_{12}^2 + 2p_{12}q_{03} - 3q_{03}^2,$$

$$p_{03}q_{21}(2p_{12} + 6q_{03}) = -2p_{12}p_{21}q_{03} + 6p_{21}q_{03}^2 + p_{12}^2q_{12} - 2p_{12}q_{03}q_{12} - 3q_{03}^2q_{12},$$

$$2p_{03}q_{21}^2 = -3p_{21}p_{30}q_{03} + 6p_{30}q_{03}q_{12} - 5p_{21}q_{03}q_{21} + p_{12}q_{12}q_{21} + q_{03}q_{12}q_{21} - 18q_{03}^2q_{30},$$

$$2p_{12}q_{21}^2 = 2p_{21}^2p_{30} - 9p_{30}^2q_{03} - 2p_{21}p_{30}q_{12} - 4p_{30}q_{12}^2 + 2p_{21}^2q_{21} +$$

$$+ 12p_{30}q_{03}q_{21} + 2p_{21}q_{12}q_{21} - q_{03}q_{21}^2,$$

$$p_{12}(p_{30} + q_{21}) = q_{03}(3p_{30} - q_{21}),$$

$$p_{03}(3p_{30} + q_{21}) = 2p_{21}q_{03} - p_{12}q_{12} - q_{03}q_{12},$$

$$p_{03}q_{12}(-2p_{12} + 6q_{03}) + 4p_{03}^2q_{21} = q_{03}(-p_{12}^2 + 4p_{12}q_{03} - 3q_{03}^2),$$

$$p_{21}(p_{30}^2 + 2p_{30}q_{21} + q_{21}^2) = 2p_{30}^2q_{12} + 2p_{30}q_{12}q_{21} - 6p_{30}q_{03}q_{30} - 2q_{03}q_{21}q_{30},$$

$$p_{12}^2q_{21} + p_{12}(-2p_{21}q_{12} - 2q_{12}^2 + 8q_{03}q_{21}) =$$

$$= q_{03}(-4p_{21}^2 + 18p_{30}q_{03} - 2p_{21}q_{12} + 2q_{12}^2 - 9q_{03}q_{21}),$$

$$p_{12}q_{30} = -p_{21}p_{30} + 2p_{30}q_{12} - p_{21}q_{21} - 3q_{03}q_{30},$$

$$4p_{21}q_{30} = -3p_{30}^2 + 2p_{30}q_{21} + q_{21}^2 - 4q_{12}q_{30},$$

$$2p_{03}q_{21}q_{30} = -p_{21}p_{30}q_{12} + 2p_{30}q_{12}^2 - p_{30}q_{03}q_{21} - p_{21}q_{12}q_{21} + q_{03}q_{21}^2 - 6q_{03}q_{12}q_{30},$$

$$4p_{03}q_{30}^2 = -p_{30}^2q_{12} + q_{12}q_{21}^2 - 2p_{30}q_{03}q_{30} - 4q_{12}^2q_{30} + 2q_{03}q_{21}q_{30},$$

$$p_{03}(3q_{12}^2 - 2q_{03}q_{21}) + 3p_{03}^2q_{30} = q_{03}(-p_{21}q_{03} + 2p_{12}q_{12} - q_{03}q_{12}),$$

$$p_{03}(-2q_{12}q_{21} + 6q_{03}q_{30}) = -3p_{30}q_{03}^2 + 2p_{21}q_{03}q_{12} - p_{12}q_{12}^2 - q_{03}q_{12}^2 + 3q_{03}^2q_{21},$$

$$p_{30}^3 + p_{30}^2q_{21} + p_{30}(-q_{21}^2 + 4q_{12}q_{30}) = q_{21}^3 - 4q_{12}q_{21}q_{30} + 8q_{03}q_{30}^2.$$

The question on the existence of a quadratic limit cycle in different cubic systems of a special form (which differs from system (2)) was considered in papers [8–10].

Example 1. Using Theorems 2, 3, we can build systems of the form (2), which have as their algebraic limit cycle any ellipse symmetric with respect to the origin.

Suppose that the following ellipse is given:

(17)
$$\frac{5}{16}x^2 - \frac{3}{8}xy + \frac{5}{8}y^2 = 1.$$

The linear substitution $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$

$$x \to 2x - y, \; y \to 2x + y$$

tuns this ellipse into the circle

(18)
$$x^2 + y^2 = 1.$$

Now we take a system of the form (8) in which $p_1 = p_3 = -1, p_2 = 0, \alpha = 1, s = 1$,

(19) $$\dot{x} = x - x^3 - x^2y - xy^2 - y^3, \quad \dot{y} = y + x^3 - x^2y + xy^2 - y^3.$$

By Theorem 2, circle (19) will be a stable cycle of system (20).

The inverse substitution σ^{-1} has the form

$$x \to \frac{1}{4}x + \frac{1}{4}y, \; y \to -\frac{1}{2}x + \frac{1}{2}y.$$

and transforms the unit circle into an ellipse (18) .

Substituting in (17)

$$a = 2,\ b = -1,\ c = 2,\ d = 1,\ p = -1,\ s = 1, \alpha = 1,$$

we obtain the system

$$(20)\ \ \dot{x} = x - \frac{5}{64}x^3 - \frac{19}{64}x^2y + \frac{25}{64}xy^2 - \frac{25}{64}y^3,\quad \dot{y} = y + \frac{25}{64}x^3 - \frac{65}{64}x^2y + \frac{67}{64}xy^2 - \frac{35}{64}y^3,$$

which has an oval (17) as its hyperbolic stable cycle.

The phase portrait of system (21) is provided on Picture 1 (the cycle is shown with a bold line).
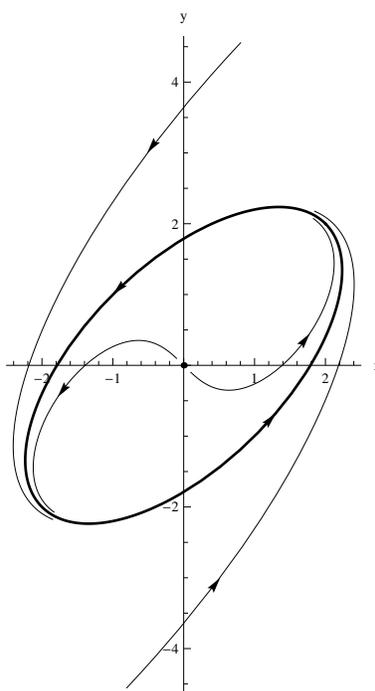

Fɪɢ. 1. Phase portrait of system (21)

Example 2. As a second example, consider the system

$$(21)\qquad \dot{x} = x + (y - x)(x^2 - xy + y^2),\ \dot{y} = y - (x - y)(x^2 - xy + y^2),$$

studied in [11]. In this paper, it has been shown that system (22) has a stable hyperbolic limit cycle, surrounding the origin, and also an explicit description of the cycle in polar and Cartesian coordinates has been obtained.

It is easy to see that the coefficients of system (22) do not satisfy the conditions of Proposition 3 (even the first of the required correspondences is violated). Therefore, the mentioned cycle is not algebraic.

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