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APPROXIMATE SOLUTION OF THE SMOOTH TRANSITION EQUATION

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ABSTRACT. The problems of stability and the approximate solution of the integral smooth transition equation first introduced and studied by Yu.I. Chersky are considered. Using the solution of the smooth transition equation under classical assumptions, it is possible to construct the solution of the equation under weaker constraints on the kernels. For the approximate solution, an error estimation and a theorem on the uniqueness and sustainability are provided.

Keywords: smooth transition integral equation, approximate solution, iterative algorithms, stability.

1. INTRODUCTION

The smooth transition equations were first constructed and studied by Yu.I. Chersky [1, 11, 12, 13]. This equation can be represented in the form

$$(1) \quad \begin{aligned} & \varphi(t) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} n_1(t-s)\varphi(s)ds + \\ & + th \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} n_2(t-s)\varphi(s)ds = f(t), t \in \mathbb{R}. \end{aligned}$$

Equation (1) is a special case of a class of equations to which L.S. Rakovshchik referred to as the integral equations with almost difference kernels:

$$(2) \quad \lambda\varphi(t) + \sum_{-\infty}^{\infty} \int_{\mathbb{R}} k_j(t-s)b_j(s)\varphi(s)ds = f(t), t \in \mathbb{R}.$$

In general, there is no constructive method which can be used to solve such equations. Solvability of an equation with an almost difference kernel has been

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studied in works by R.V. Duduchava, N.K. Karapetyants, S.G. Samko, P.V. Kereksha, G.S. Litvinchuk, A.I. Peschansky, V.A. Shevchik, V.A. Lukyanenko, N.Ya. Tikhonenko, L.Ya. Tikhonenko, Fan Tang Da, and others. Special cases of solutions in quadrature are related to the methods of reduction to boundary problems of analytical functions and to the application of analogues of Yu.V. Sokhotsky's formulas for the class of functions $\Phi(z)$ which are analytical in a strip, representable in the integral form

$$\Phi(z) = \frac{1}{2(\beta - \alpha)} \int_{\mathbb{R}} \frac{U(\tau) d\tau}{ch \frac{\pi}{\beta - \alpha} \left(\tau - z + i \frac{\alpha + \beta}{2} \right)}, \quad U(\tau) \in L_2(\mathbb{R})$$

or

$$F(z) = \frac{1}{2(\beta - \alpha)} \int_{\mathbb{R}} \frac{H(\tau) d\tau}{cth \frac{\pi}{\beta - \alpha} \left(\tau - z + i \frac{\alpha + \beta}{2} \right)}, \quad \alpha < y < \beta.$$

Application of the factorization method and the analogues of Yu.V. Sokhotsky's formulas allows us to expand the class of equations of the smooth transition type and multi-element and matrix problems of Carleman type, equivalent to them, which are solved using quadratures or approximately.

In the paper, the problem of finding an approximate solution for the smooth transition equation (of a convolution type) based on a close equation is considered. The closeness is understood with respect to the solution. This approach continues the research described by Yu.I. Chersky and F.D. Gahov in paper [1] and works [2, 3, 4, 5, 6, 14, 15]. It was successfully applied to singular integral equations, equations of the smooth transition kind, Riemann and Carleman boundary problems of theory of analytical functions, and others.

2. APPROXIMATE SOLUTION OF A LINEAR EQUATION. SUSTAINABILITY OF SOLUTIONS

In this section, we provide the necessary information which will be further used, based on results by Yu.I. Chersky [8, 9].

Let K be a given linear operator acting from the linear set X to a linear set Y . With a given element $g \in Y$, the element u to be found is a solution of non-homogeneous linear equation

$$(3) \quad Ku = g.$$

As an approximate solution \tilde{u} of equation (3), a solution of an approximate equation which is more simple in its structure,

$$(4) \quad \tilde{K}\tilde{u} = \tilde{g},$$

is taken. Moreover, it is assumed that \tilde{u} belongs to the same space X , which can be quite wide. The elements $u \in X$ can have singularities, then the approximate solution is constructed using the same singularities. To achieve more closeness of the solutions u, \tilde{u} , a Banach space $X_0 \subset X$ is introduced, and it is assumed that

$$(5) \quad u - \tilde{u} \in X_0.$$

Theorem 1. *Let the following conditions be satisfied:*

1⁰. *Equation (4) has a unique solution \tilde{u} .*

2⁰. *The difference $g - \tilde{g} \in Y_0$, where Y_0 is a linear subset, $Y_0 \subset Y$.*

3⁰. The operator $K - \tilde{K}$ acts from X to Y_0 .

4⁰. On Y_0 , a reverse operator \tilde{K}^{-1} acting from Y_0 to X_0 is defined, moreover, X_0 is a Banach space.

5⁰. The operator $B = \tilde{K}^{-1}(K - \tilde{K})$ is bounded in X_0 with the norm

$$(6) \quad \|B\| = \|\tilde{K}^{-1}(K - \tilde{K})\| < 1.$$

Then equation (3) has a unique solution, equal to

$$(7) \quad u = \tilde{u} + [I + \tilde{K}^{-1}(K - \tilde{K})]^{-1}\tilde{K}^{-1}(g - K\tilde{u}),$$

property (5) holds, and $K\tilde{u} - g \in Y_0$, and also the following error estimation is valid:

$$(8) \quad \|u - \tilde{u}\|_{X_0} \leq \frac{\|\tilde{K}^{-1}(g - K\tilde{u})\|_{X_0}}{1 - \|\tilde{K}^{-1}(K - \tilde{K})\|}.$$

Along with the approximate solution \tilde{u} , it is possible to construct a more accurate approximation \tilde{u}_n using the formula

$$(9) \quad \tilde{u}_n = \tilde{u} + \sum_{k=1}^{n-1} B^{k-1}\tilde{K}^{-1}(g - K\tilde{u}), n > 1.$$

If condition (6) is satisfied, that is, $\|B\| < 1$, then along with (9), we will have that

$$(10) \quad \|u - \tilde{u}_n\|_{X_0} \leq \frac{\|B\|^{n-1}}{1 - \|B\|} \|\tilde{K}^{-1}(g - K\tilde{u})\|_{X_0}.$$

Therefore, if the accuracy of solution (4) with the operator \tilde{K} is not sufficient, then to obtain a solution of a required accuracy it is not necessary to reject the obtained operator \tilde{K}^{-1} , but it suffices to use formula (9).

3. SMOOTH TRANSITION EQUATION WITH WEAKER RESTRICTIONS ON KERNELS

Consider a smooth transition equation in the form

$$(11) \quad \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_1(t-s)u(s)ds - g(t) + e^{-t} \left\{ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_2(t-s)u(s)ds - g(t) \right\} = 0, t \in \mathbb{R},$$

and we assume that $g \in L_2(\mathbb{R}), u \in L_2(\mathbb{R})$,

$$(12) \quad 0 < c < |M_j(x)| < C < \infty, j = 1, 2.$$

Equation (11) contains convolutions of the form

$$(13) \quad (A_j u)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_j(t-s)u(s)ds, j = 1, 2,$$

where $m_j(t)$ is a generalized function. Its Fourier transform

$$(14) \quad M_j(x) = \mathcal{F}[m_j](x) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_j(t)e^{itx} dt$$

is assumed to be a locally summable function satisfying (12). Fourier transform (14) is defined in a sense of generalized functions, if the integral does not exist in a

common sense. In the general case, the convolution (13) is a singular integral, which we will understand as an averaged limit [12].

$$\int_{\mathbb{R}} m_j(t-s)u(s)ds = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} m_{j(n)}(t-s)u(s)ds,$$

where

$$m_{j(n)}(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-n}^n M(x)e^{-ixt}dx.$$

Under assumption (12), the operators A_j , defined by equality (13), are linear and bounded in $L_2(\mathbb{R})$, and have bounded reverse operators

$$(A_j^{-1}h_j)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r_j(t-s)h_j(s)ds,$$

where

$$r_j(t) = \mathcal{F}^{-1}[R_j](t), R_j(x) = [M_j(x)]^{-1}, j = 1, 2.$$

The method for reducing equation (11) to Carleman's problem is similar to the one for the case of the smooth transition equation. We represent (11) in the form

$$(15) \quad \begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_1(t-s)u(s)ds - g(t) &= -e^{-t}\varphi(t), \\ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m_2(t-s)u(s)ds - g(t) &= \varphi(t), \end{aligned}$$

where φ is a new unknown function. The left-hand sides of equalities belong to $L_2(\mathbb{R})$ (g and the convolutions $A_j u$ belong to $L_2(\mathbb{R})$), therefore, $\varphi \in L_2(\mathbb{R})$, $e^{-\cdot}\varphi \in L_2(\mathbb{R})$. Following [1], we denote the space of such functions by $\{0, 1\}$.

Theorem 2 ([1, 2, 4, 13]). *Let α, β be real numbers ($\alpha < \beta$). In order for the function $\varphi(t)$ to simultaneously satisfy the conditions*

$$(16) \quad e^{-\alpha t}\varphi(t) \in L_2(\mathbb{R}), e^{-\beta t}\varphi(t) \in L_2(\mathbb{R}),$$

it is necessary and sufficient: 1° for its Fourier integral $\Phi(x)$ to be analytically continuable on the strip $\alpha < \text{Im}z < \beta$, and 2° that there is a constant C such that for every $\alpha \leq y \leq \beta$ the inequality $\int_{\mathbb{R}} |\Phi(x+iy)|^2 dx \leq C < \infty$ is valid. Moreover, the following property for a Fourier integral holds:

$$(17) \quad \mathcal{F}\{e^{-yt}\varphi(t)\} = \Phi(x+iy), \alpha \leq y \leq \beta.$$

The class of functions φ is denoted by $\{\alpha, \beta\}$, and Fourier transforms $\Phi \in \{\{\alpha, \beta\}\}$.

The symbol $\{\{0\}\}$ marks the class of image-functions $U(x)$ belonging to $L_2(\mathbb{R})$ and satisfying Hölder condition on a number line (that is, $U \in L_2(\mathbb{R}) \cap H_\lambda(\mathbb{R})$, $0 < \lambda \leq 1$). By $\{0\}$ we denote a class of original-functions $u(t)$, whose Fourier transforms belong to $\{\{0\}\}$.

Applying Fourier transform to (15) and excluding the function $U(x)$, we arrive to a Carleman's problem in the space of functions $\Phi(x) = \mathcal{F}[\varphi](x) \in \{\{0, 1\}\}$:

$$(18) \quad K\Phi \equiv \Phi(x) + A(x)\Phi(x+i) = H(x), x \in \mathbb{R},$$

where $A(x) = M_2(x)/M_1(x)$, $H(x) = G(x)[M_2(x) - M_1(x)]/M_1(x)$.

The Fourier transform of solution (11) for the case when the Carleman’s problem (18) is solvable is obtained using one of the formulas

$$(19) \quad U(x) = \frac{G(x) + \Phi(x)}{M_2(x)} = \frac{G(x) - \Phi(x+i)}{M_1(x)}.$$

We apply to equation (11) the method of approximate solution, described in the second section. As an approximate solution we consider the smooth transition equation

$$(20) \quad \tilde{u}(t) + (\tilde{k}_1 * \tilde{u})(t) - \tilde{g}(t) + e^{-t}\{\tilde{u}(t) + (\tilde{k}_2 * \tilde{u})(t) - \tilde{g}(t)\} = 0, t \in \mathbb{R}.$$

In the assumption that the functions $\tilde{k}_1, \tilde{k}_2 \in L_1(\mathbb{R}), \tilde{g}, \tilde{u} \in L_2(\mathbb{R})$ and the conditions of a normal solvability

$$(21) \quad \tilde{M}_j(x) \equiv 1 + \tilde{K}_j(x) \neq 0, x \in \mathbb{R}, j = 1, 2$$

are satisfied, equation (20) is equivalent to Carleman’s problem in the space of functions $\tilde{\Phi}(x) \in \{\{0, 1\}\}$

$$(22) \quad \tilde{K}\tilde{\Phi} \equiv \tilde{\Phi}(x) + \tilde{A}(x)\tilde{\Phi}(x+i) = \tilde{H}(x), x \in \mathbb{R},$$

where

$$\tilde{A}(x) = \tilde{M}_2(x)/\tilde{M}_1(x), \tilde{H}(x) = \tilde{G}(x)[\tilde{M}_2(x) - \tilde{M}_1(x)]/\tilde{M}_1(x).$$

Following [1, 2], we reduce the theorem on solvability of Carleman’s problem (27) to the form convenient for future use.

Theorem 3. *Suppose that*

$$\chi = \text{ind}\tilde{M}_2(x) - \text{ind}\tilde{M}_1(x) \equiv \text{ind}[1 + \tilde{K}_2(x)] - \text{ind}[1 + \tilde{K}_1(x)].$$

Then with $\chi > 0$, a homogeneous Carleman’s problem has in the space $L_2(\mathbb{R})$ exactly χ linearly independent solutions. With $\chi \geq 0$, nonhomogeneous problem (22) is unconditionally solvable for every function $\tilde{H}(x)$, and the solution is defined by the formulas

$$(23) \quad \begin{aligned} \tilde{\Phi}(x) &= F^+ (e^{2\pi x}) e^{\pi x}, \\ \tilde{\Phi}(x+i) &= -F^- (e^{2\pi x}) e^{\pi x}, \end{aligned}$$

where

$$\begin{aligned} F^\pm(\xi) &= X^\pm(\xi) \left[\Psi^\pm(\xi) + \frac{P_{\chi-1}(\xi)}{(\xi+i)^\chi} \right], \\ \Psi^\pm(\xi) &= \pm \frac{\tilde{H}_1(\xi)}{2X^+(\xi)} + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\tilde{H}_1(\tau) d\tau}{X^+(\tau)(\tau-\xi)}, \\ X^+(\xi) &= \exp \left\{ \frac{1}{2} \ln \left[\tilde{A}_1(\xi) \left(\frac{\xi+i}{\xi-i} \right)^\chi \right] + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln \left[\tilde{A}_1(\tau) \left(\frac{\tau+i}{\tau-i} \right)^\chi \right] d\tau}{\tau-\xi} \right\}, \\ X^-(\xi) &= \left(\frac{\xi+i}{\xi-i} \right)^\chi \exp \left\{ -\frac{1}{2} \ln \left[\tilde{A}_1(\xi) \left(\frac{\xi+i}{\xi-i} \right)^\chi \right] + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\ln \left[\tilde{A}_1(\tau) \left(\frac{\tau+i}{\tau-i} \right)^\chi \right] d\tau}{\tau-\xi} \right\}. \end{aligned}$$

The functions $\tilde{A}_1(\xi), \tilde{H}_1(\xi)$ are defined by the formulas

$$\tilde{A}_1(\xi) = \begin{cases} \left[1 + \tilde{K}_2 \left(\frac{\ln \xi}{2\pi}\right)\right] \left[1 + \tilde{K}_1 \left(\frac{\ln \xi}{2\pi}\right)\right]^{-1}, & \xi > 0, \\ 1, & \xi < 0, \end{cases}$$

$$\tilde{H}_1(\xi) = \begin{cases} \tilde{H}(\xi) \left[1 + K_1 \left(\frac{\ln \xi}{2\pi}\right)\right]^{-1}, & \xi > 0, \\ 0, & \xi < 0, \end{cases}$$

$P_{\chi-1}(\xi)$ is an arbitrary polynomial of degree $\chi - 1$. For the case when $\chi \leq 0$, the homogeneous Carleman's problem only has the zero solution. If $\chi < 0$, then to achieve solvability of nonhomogeneous problem (22), it is necessary and sufficient to satisfy the following conditions:

$$\int_{\mathbb{R}} \frac{\tilde{H}_1(\xi)d\xi}{X^+(\xi)(\xi + i)^k} = 0, \quad k = 1, 2, \dots, |\chi|.$$

If the $|\chi|$ conditions for solvability are fulfilled, nonhomogeneous Carleman's problem (22) has a unique solution (23), where we need to put $P_{\chi-1}(\xi) = 0$.

4. THE CASE OF ZERO INDEX

According to Theorem 1, we will define the operators K and \tilde{K} using formulas (18) and (22), and the spaces are as follows: $X = \{\{0, 1\}\}, Y = L_2(\mathbb{R})$.

Suppose that $\chi = \text{ind}A(x) = 0$, which will provide the existence and uniqueness of the solution of Carleman's boundary problem (22). As X_0 we take the space of the functions $\Psi(x) \in X = \{\{0, 1\}\}$, in which the norm

$$\|\Psi\|_{X_0} = \max \left\{ \left(\int_{\mathbb{R}} |\Psi(x)|^2 dx \right)^{1/2}, \left(\int_{\mathbb{R}} |\Psi(x+i)|^2 dx \right)^{1/2} \right\}$$

is introduced, and let Y_0 coincide with $L_2(\mathbb{R})$.

The norm in $\{\{0,1\}\}$ can be also defined in the form

$$\|\Psi\|_{X_0} = \max_{0 \leq y \leq 1} \left(\int_{\mathbb{R}} |\Psi(x+iy)|^2 dx \right)^{1/2}$$

or

$$(24) \quad \|\Psi\|_{X_0} = \left\{ \left(\int_{\mathbb{R}} |\Psi(x)|^2 dx \right)^{1/2} + \left(\int_{\mathbb{R}} |\Psi(x+i)|^2 dx \right)^{1/2} \right\}.$$

From (18) and (22) it follows that the operator $K - \tilde{K}$ has the form

$$(K - \tilde{K}) \Phi(x) = [A(x) - \tilde{A}(x)]\Phi(x+i)$$

and acts from X_0 to Y_0 in a bounded way. Define operator $\tilde{K}^{-1}: Y_0 \rightarrow X_0$ using the formulas from Theorem 3.

We will estimate the norm of the operator $B = \tilde{K}^{-1}(K - \tilde{K})$

$$\|B\Phi\|_{X_0} = \max \left\{ \left(\int_{\mathbb{R}} \left| \tilde{X}^+(e^{2\pi x}) \left[\frac{(K - \tilde{K})\Phi(x)}{2\tilde{X}^+(e^{2\pi x})} + \frac{1}{2i} \int_{\mathbb{R}} \frac{(K - \tilde{K})\Phi(s)ds}{\tilde{X}^+(e^{2\pi s})sh\pi(s-x)} \right] \right|^2 dx \right)^{1/2}, \dots \right\}$$

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left| \tilde{X}^-(e^{2\pi x}) \left[\frac{(K - \tilde{K})\Phi(x)}{2\tilde{X}^+(e^{2\pi x})} - \frac{1}{2i} \int_{\mathbb{R}} \frac{(K - \tilde{K})\Phi(s)ds}{\tilde{X}^+(e^{2\pi s})sh\pi(s-x)} \right] \right|^2 dx \right)^{1/2} \leq \\ & \leq \max \left\{ \max_x |\tilde{X}^+(e^{2\pi x})|, \max_x |\tilde{X}^-(e^{2\pi x})| \right\} \cdot \max_x |\tilde{X}^+(e^{2\pi x})|^{-1} \times \\ & \times \max \left\{ \left(\int_{\mathbb{R}} |(K - \tilde{K})\Phi(x)|^2 dx \right)^{1/2}, \left(\int_{\mathbb{R}} |(K - \tilde{K})\Phi(x)|^2 dx \right)^{1/2} \right\} \leq \\ & \leq L \cdot \max_x |A(x) - \tilde{A}(x)| \|\Phi\|_{X_0} \leq L\varepsilon \|\Phi\|_{X_0} \leq \delta \|\Phi\|_{X_0}, \\ & \varepsilon = \max_x |A(x) - \tilde{A}(x)|, L\varepsilon \leq \delta, \end{aligned}$$

where

$$(25) \quad L \equiv N_1 \cdot N_2 = \max \left\{ \max_x \tilde{X}^+(e^{2\pi x}), \max_x \tilde{X}^-(e^{2\pi x}) \right\} \max_x |\tilde{X}^+(e^{2\pi x})|^{-1}.$$

We choose the functions $\tilde{M}_1(x) = 1 + \tilde{K}_1(x)$, $\tilde{M}_2(x) = 1 + \tilde{K}_2(x)$ close to the functions $M_1(x)$ and $M_2(x)$ in such a way that the value of δ is less than one, hence, $\|B\| < 1$. All the conditions of Theorem 1 are satisfied, therefore, equation (18) has a unique solution which is obtained using formula (7):

$$(26) \quad \Phi(x) = \tilde{\Phi}(x) + [1 + B]^{-1} \tilde{K}^{-1}(H(x) - K\tilde{\Phi}(x))$$

and we have error estimation (8):

$$(27) \quad \|\Phi - \tilde{\Phi}\|_{X_0} \leq \frac{\|\tilde{K}^{-1}(H(x) - K\tilde{\Phi}(x))\|_{X_0}}{1 - \delta} \leq \frac{L}{1 - \delta} \|H(x) - K\tilde{\Phi}(x)\|_{Y_0},$$

where L, δ are defined by inequality (25), and the approximate solution $\tilde{\Phi}(x)$ is found by the formulas from Theorem 3 with $\chi = 0$. The Fourier transform of the solution $\tilde{u}(t)$ of equation (20) is defined by the solution of problem (22) using the formulas

$$(28) \quad \tilde{U}(x) = \frac{\tilde{G}(x) + \tilde{\Phi}(x)}{\tilde{M}_2(x)} = \frac{\tilde{G}(x) - \tilde{\Phi}(x+i)}{\tilde{M}_1(x)}, x \in \mathbb{R}.$$

Due to the equivalence of problems (18), (22) and equations (11), (20), we come to the following result:

Theorem 4. *Suppose that in a smooth transition equation (11) the Fourier integrals $M_1(x)$ and $M_2(x)$ are locally summable functions satisfying (12); the inequality $L\varepsilon \leq \delta < 1$ (25) is fulfilled, conditions (21) and the index of coefficient in problem (22) is as follows: $\chi = \text{ind}\tilde{A}(x) = 0$. Then with any right-hand side $g \in L_2(\mathbb{R})$, equation (11) has a unique solution (19) and (26). The function $\tilde{u}(t) = \mathcal{F}^{-1}[\tilde{U}](t)$, where $U(x)$ is defined by (28) and the formulas from Theorem 3, is an approximate solution of equation (11), and the average quadratic error estimation*

$$(29) \quad \|u - \tilde{u}\|_{L_2} \leq M(\|g - \tilde{g}\|_{L_2} + \|\Phi - \tilde{\Phi}\|_{X_0}) + \delta_1(\|\tilde{g}\|_{L_2} + \|\tilde{\Phi}\|_{X_0}),$$

is valid, where

$$M = \max_x \frac{1}{|M_2(x)|} \leq \frac{1}{c}, \delta_1 = \max_x \left| \frac{\tilde{M}_2(x) - M_2(x)}{\tilde{M}_2(x)M_2(x)} \right|.$$

M is a restricted value, δ_1 is a small one, and for the norm $\|\Phi - \tilde{\Phi}\|_{X_0}$, inequality (27) holds.

Based on (19), (28), we obtain that

$$\|u - \tilde{u}\|_{L_2} = \left\| \frac{G(x)}{M_2(x)} - \frac{\tilde{G}(x)}{\tilde{M}_2(x)} + \frac{\Phi(x)}{M_2(x)} - \frac{\tilde{\Phi}(x)}{\tilde{M}_2(x)} \right\|_{L_2}$$

which implies inequality (29). If the approximate solution $\tilde{u}(t)$ is obtained with an insufficient accuracy, then the solution $u_n(t)$ can be acquired with a required accuracy using formula (9) with estimation (10).

5. THE CASE OF POSITIVE INDEX

Requirement 1⁰ of Theorem 1 on uniqueness of the solution \tilde{u} of smooth transition equation (20) was provided by the condition that the coefficient $\tilde{A}(x)$ in the problem (22) had to equal zero. In the case

$$(30) \quad \chi = \text{ind} \tilde{A}(x) \geq 0,$$

the requirement on uniqueness of a solution of equation (11) or (20) can be satisfied by a corresponding choice of spaces, or replaced with other conditions [7]. To study the sustainability of the solution (approximate solution) of these equations, we will use the corresponding results for Carleman’s problem (18), (22). We keep the assumptions from Section 3 regarding the kernels of equations (11), (20).

Suppose that $X = \{\{0, 1\}\}$, and Y is a space of vector-columns $P = (H(x), g_1, g_2, \dots, g_\chi)^T$, where $H \in L_2(\mathbb{R})$ is a known function, and g_1, g_2, \dots, g_χ are the given numbers, the superscript T denotes the transpose of the row. Let a system of linearly independent functionals in $X : Q_i, \tilde{Q}_i, i = 1, 2, \dots, \chi$ be given. We define the original operators A and \tilde{A} in the form:

$$(31) \quad \begin{aligned} A\Phi &\equiv (K\Phi, (Q_1, \Phi), \dots, (Q_\chi, \Phi))^T = (H, g, \dots, g_\chi)^T, \\ \tilde{A}\tilde{\Phi} &\equiv (\tilde{K}\tilde{\Phi}, (\tilde{Q}_1, \tilde{\Phi}), \dots, (\tilde{Q}_\chi, \tilde{\Phi}))^T = (\tilde{H}, \tilde{g}, \dots, \tilde{g}_\chi)^T, \end{aligned}$$

where the operators K, \tilde{K} are defined respectively by formulas (18) and (22). If (30) is satisfied, a solution of equation (22) $\tilde{K}\tilde{\Phi} = \tilde{H}$, according to the formulas from Theorem 3, has the form $\tilde{\Phi}(x) = \tilde{\Phi}_1(x) + \tilde{\Phi}_0(x)$, where $\tilde{\Phi}_1(x)$ is a special solution, and $\tilde{\Phi}_0(x)$ is a general solution of the homogeneous equation

$$(32) \quad \tilde{\Phi}_0(x) = \sum_{k=1}^{\chi} \tilde{c}_k \tilde{\Phi}_{0k}(x), \tilde{\Phi}_{0k}(x) = \tilde{X}^+(e^{2\pi x}) \frac{e^{\pi x} e^{2\pi(k-1)x}}{(e^{2\pi x} + i)^\chi},$$

where \tilde{c}_k are arbitrary constants, $\tilde{X}^+(e^{2\pi x})$ are defined by the formulas from Theorem 3, and $\tilde{\Phi}_1(x)$ is a particular solution of a non-homogeneous equation.

We will assume that the determinant is as follows:

$$(33) \quad \tilde{\Delta} = \begin{vmatrix} \tilde{\lambda}_{11} & \dots & \tilde{\lambda}_{1\chi} \\ \dots & \dots & \dots \\ \tilde{\lambda}_{\chi 1} & \dots & \tilde{\lambda}_{\chi\chi} \end{vmatrix} \equiv |(\tilde{Q}_j, \tilde{\Phi}_{0k})| \neq 0, j, k = 1, 2, \dots, \chi, \tilde{\lambda}_{jk} = (\tilde{Q}_j, \tilde{\Phi}_{0k}).$$

The fulfillment of these conditions helps to determine the unique solution of problem (31): $\tilde{A}\tilde{\Phi} = \tilde{P}$ (and of equation (20)).

Following Theorem 1, we take as Y_0 the space of vectors $P \subset Y$, in which the norm

$$(34) \quad \|P\|_{Y_0} = \left(\|H(x)\|_{L_2}^2 + \sum_{k=1}^{\chi} |g_k|^2 \right)^{1/2}$$

is introduced. As \tilde{P} , a vector close to P is chosen, so that $P - \tilde{P} \in Y_0$. The norm in $X_0 = \{\{0, 1\}\}$ is introduced similarly as in (24). Approximate problem (31) on the space X_0 has a unique solution that is defined by the operator $\tilde{A}^{-1} : Y_0 \rightarrow X_0$.

Indeed, the solution of Carleman’s problem (22) contains χ arbitrary constants $\tilde{c}_k, k = 1, \dots, \chi$, belonging to $\tilde{\Phi}(x)$ (32). To define them, we use χ equations (31) $(\tilde{a}_j, \tilde{\Phi}) = \tilde{g}_j$ and with respect to \tilde{c}_k obtain the system of linear algebraic equations with χ variables

$$(35) \quad \sum_{k=1}^{\chi} \tilde{\lambda}_{jk} \tilde{c}_k = \tilde{\gamma}_j, j = 1, 2, \dots, \chi,$$

$$\tilde{\gamma}_j = \tilde{g}_j - \int_{\mathbb{R}} \tilde{X}^+(e^{2\pi x}) \left[\frac{\tilde{H}(x)}{2\tilde{X}^+(e^{2\pi x})} + \frac{1}{2i} \int_{\mathbb{R}} \frac{\tilde{H}(s)ds}{\tilde{X}^+(e^{2\pi x})sh\pi(s-x)} \right] \tilde{Q}_j(x)dx$$

which, according to (33), has a determinant $\tilde{\Delta} \neq 0$. By Cramer’s rule, solution (35) can be written in the form

$$\tilde{c}_k = \frac{\tilde{\Delta}_k}{\tilde{\Delta}} = \frac{1}{\tilde{\Delta}} \sum_{k=1}^{\chi} \tilde{A}_{jk} \tilde{\gamma}_j, k = 1, 2, \dots, \chi,$$

where the determinant $\tilde{\Delta}_k$ differs from $\tilde{\Delta}$ by its k th column which equals $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_\chi)^T$, and \tilde{A}_{jk} are the minors are of $\chi - 1$ th order $\tilde{\Delta}_k$ with respect to the k th column. Placing \tilde{c}_k into the solution of problem (22), we obtain the unique solution of equation (31):

$$\tilde{\Phi}(x) = (\tilde{A}^{-1}\tilde{P})(x) = \tilde{X}^+(e^{2\pi x}) \left[\frac{\tilde{H}(x)}{2\tilde{X}^+(e^{2\pi x})} + \frac{1}{2i} \int_{\mathbb{R}} \frac{\tilde{H}(s)ds}{\tilde{X}(e^{2\pi x})sh\pi(s-x)} \right] + \frac{1}{\tilde{\Delta}} \sum_{j=1}^{\chi} \tilde{A}_{jk} \tilde{\gamma}_j \tilde{\Phi}_{0k}(x).$$

Taking into account that

$$(A - \tilde{A})\Phi(x) = ([A(x) - \tilde{A}(x)]\Phi(x + i), (Q_1 - \tilde{Q}_1, \Phi), \dots, (Q_\chi - \tilde{Q}_\chi, \Phi))^T,$$

and also applying the estimations for the norm of the operator $\tilde{K}^{-1}(K - \tilde{K})$ from Section 4 to the norm of the operator $B = \tilde{A}^{-1}(A - \tilde{A})$ in X_0 , we obtain

$$\begin{aligned} \|\tilde{A}^{-1}(A - \tilde{A})\Phi\|_{X_0} &\leq L\varepsilon\|\Phi\|_{X_0} + \frac{N_1}{2} \sum_{k=1}^{\chi} |c_k| \left(\left| \int_{\mathbb{R}} \frac{e^{\pi x} e^{2\pi(k-1)x}}{(e^{2\pi x} + i)^k} dx \right| \right)^{1/2} \leq \\ &\leq L\varepsilon\|\Phi\|_{X_0} + \frac{N_1}{2} \sum_{k=1}^{\chi} |c_k|. \end{aligned}$$

In the estimations it is clarified that

$$\int_{\mathbb{R}} \frac{e^{2\pi x} e^{4\pi(k-1)x} dx}{(e^{4\pi x} + i)^k} \leq \int_{\mathbb{R}} \frac{e^{2\pi x} e^{4\pi(k-1)x} dx}{(e^{4\pi x} + i)^k} = \frac{1}{4}.$$

From the estimation for the coefficients, it follows that

$$\begin{aligned}
 |c_k| &\leq \frac{1}{|\Delta|} \sum_{j=1}^{\chi} |\tilde{A}_{jk}| |Q_j - \tilde{Q}_j, \Phi| + \\
 &+ \frac{1}{\tilde{\Delta}} \sum_{j=1}^{\chi} |\tilde{A}_{jk}| \int_{\mathbb{R}} \tilde{X}^+(e^{2\pi x}) \left[\frac{(K - \tilde{K})\Phi(x)}{2\tilde{X}^+(e^{2\pi x})} + \frac{1}{2i} \int_{\mathbb{R}} \frac{(K - \tilde{K})\Phi(s)ds}{\tilde{X}^+(e^{2\pi s})sh\pi(s-x)} \right] \times \\
 &\times (Q_j - \tilde{Q}_j) dx \leq \frac{1}{|\Delta|} \left(1 + \max_x |\tilde{X}^+(e^{2\pi k})| N_2 \varepsilon \right) \sum_{j=1}^{\chi} |\tilde{A}_{jk}| |Q_j - \tilde{Q}_j, \Phi| \leq \\
 &\leq \frac{1}{|\Delta|} (1 + L\varepsilon) \sum_{j=1}^{\chi} |\tilde{A}_{jk}| \|Q_j - \tilde{Q}_j\|_{L_2} \cdot \|\Phi\|_{X_0}, k = 1, 2, \dots, \chi.
 \end{aligned}$$

Finally, we obtain that $\|B\Phi\|_{X_0} \leq \delta \|\Phi\|_{X_0}$, where

$$(36) \quad \delta = \left[L\varepsilon + \frac{N_2(1 + L\varepsilon)}{2|\tilde{\Delta}|} \sum_{k,j=1}^{\chi} |A_{jk}| \|Q_j - \tilde{Q}_j\| \right]$$

here the denotations for L, N_2, ε remain the same (25).

Let the functions $M_j(x), \tilde{M}_j(x), \tilde{G}(x), G(x), Q_j$ and \tilde{Q}_j be chosen in a way that $\delta < 1$, then $\|B\| < 1$ and the conclusion of Theorem 1 is valid, that is, the changes in Theorem 2 are obvious.

Remark 1. From expression (36) for δ , it follows that the accuracy of the solution will be better if $Q_j \equiv \tilde{Q}_j$, and $g_j = \tilde{g}_j, j = 1, 2, \dots, \chi$, that is, $(\tilde{Q}_j, \tilde{\Phi}_{0k}) = (Q_j, \Phi_{0k})$, in this case, the estimations remain similar to the ones for $\chi = 0$.

Remark 2. The results, similar to the ones for the case $\chi \geq 0$, can be obtained also for $\chi < 0$. As X we take the space of vectors $(\Phi(x), c, \dots, c_{|\chi|})^T, \Phi \in \{\{0, 1\}\}, c_k \in \mathbb{C}$, and the system of functionals $\tilde{Q}_j, j = 1, \dots, |\chi|$ is introduced with the condition

$$\left(\tilde{Q}_j(x), \frac{e^{\pi x}}{\tilde{X}^+(e^{2\pi x}(e^{2\pi x} + i)^k)} \right) \neq 0, j, k = 1, 2, \dots, |\chi|,$$

which allows us to find \tilde{c}_k in such way that the fulfillment of the necessary and sufficient conditions for solvability of Carleman's problem is ensured. The operator \tilde{A} is defined as follows,

$$\tilde{A}(\Phi, \tilde{c}_1, \dots, \tilde{c}_{|\chi|})^T = (\tilde{K}\tilde{\Phi})(x) - \sum_{k=1}^{\chi} \tilde{c}_k \tilde{Q}_k(x),$$

and the operator A is defined in a similar way.

Remark 3. The solution of smooth transition equation (20) is stable with respect to small disturbance of the kernels and the right-hand sides ($\|B\| \approx \delta < 1$), which allows us to obtain a solution of a more general equation (11) with less strict conditions for the kernels (12).

On the other hand, if the kernels of approximate and exact equation of the form (20) satisfy similar conditions, then the kernels and the right-hand side of the approximate equation have to be chosen in such a way to make the calculation of the function with tilde much easier. That can be achieved, for example, by an

approximate factorization of the coefficient $A(x)$ from the Carleman's problem in an explicit form (without transition to the equivalent Riemann's problem) $A(x) \approx \tilde{A}(x) = \tilde{X}(x)[\tilde{X}(x+i)]^{-1}$. Such approach will be applied in the next section for the smooth transition equation of the «first kind». Note that if $\tilde{A}(x)(\tilde{A}(\pm\infty) = 1)$ is a rational function of the form [1]

$$\prod_{k=1}^n \frac{x^2 + a_k^2}{x^2 + (a_k + 1)^2}, n \geq 2, a_k \neq 0, a_k \neq -1,$$

then

$$\tilde{X}(z) = \prod_{k=1}^n \frac{z + ia_k}{z - i(a_k + 1)}.$$

6. APPROXIMATE SOLUTION OF SMOOTH TRANSITION EQUATION FOR ONE EXCEPTIONAL CASE

To illustrate the specifics of application of the approximate factorization method, we consider the following equation of the «first kind» [3]

$$(37) \quad (k_1 * u)(t) - g(t) + e^{-t}\{u(t) + (k_2 * u)(t) - g(t)\} = 0, t \in \mathbb{R}.$$

Suppose that the functions $K_j(x) = \mathcal{F}[k_j](x)$, $j = 1, 2$, which are the Fourier transforms of the kernels of the equations, satisfy the conditions

$$(38) \quad |1 + K_2(x)| \geq \varepsilon > 0, K(x)\sqrt{x^2 + 1} \neq 0, \\ \lim_{x \rightarrow \infty} K(x)\sqrt{x^2 + 1} = 1, ind[K(x)\sqrt{x^2 + 1}] = 0,$$

where $K(x) = K_1(x)[1 + K_2(x)]^{-1}$. We choose the classes for the solution $u(t)$ and the absolute term $g(t)$ according to the class of the solution $\Phi(x)$ and the constant term $H(x)$ of the equivalent (37) Carleman's problem:

$$(39) \quad K\Phi \equiv \Phi(x+i) + K(x)\Phi(x) = H(x), x \in \mathbb{R},$$

where $H(x) = [1 - K(x)]G(x)$, then

$$u(t) \equiv \mathcal{F}^{-1}[U](t) = \mathcal{F}^{-1}\{[G(x) + \Phi(x)][1 + K_2(x)]^{-1}\}(t).$$

The approximate solution $\tilde{u}(t)$ of equation (37) is an exact solution of an approximate equation which is more simple in its structure (close in the sense used in Theorem 1 to the original one, which is ensured by the choice of the kernels $\tilde{k}_j(t)$ and the constant term $\tilde{g}(t)$):

$$(\tilde{k}_1 * \tilde{u})(t) - \tilde{g}(t) + e^{-t}\{\tilde{u}(t) + (\tilde{k}_2 * \tilde{u})(t) - \tilde{g}(t)\} = 0, t \in \mathbb{R}.$$

The solution $\tilde{u}(t)$ is defined with respect to the solution of the following Carleman's problem

$$(40) \quad \tilde{K}\tilde{\Phi} \equiv \tilde{\Phi}(x+i) + \tilde{K}(x)\tilde{\Phi}(x) = \tilde{H}(x), x \in \mathbb{R}, \\ \tilde{H}(x) = [1 - \tilde{K}(x)]\tilde{G}(x),$$

in which the coefficient $\tilde{K}(x)$ satisfies all the conditions (38), and, moreover, the Hölder condition, and permits the factorization

$$(41) \quad \tilde{K}(x)\sqrt{x^2 + 1} = \tilde{X}(x)[\tilde{X}(x+i)]^{-1},$$

where $\tilde{X}(z)$ is a continuous, bounded function which has no zeros in the strip $0 \leq Imz \leq 1$ and is analytical within this strip. To achieve representation (41),

in addition to conditions (38), it suffices to assume also that the original of the function $\tilde{D}(x) = \tilde{K}(x)\sqrt{x^2 + 1} - 1$ satisfies the conditions

$$(42) \quad \tilde{d} \in L_1(\mathbb{R}), \tilde{d}' \in L_2(\mathbb{R}),$$

which, when fulfilled, allows $\tilde{X}(x), \tilde{X}(x+i)$ to be obtained using the formulas from Theorem 3, where we need to put $H(x) = \ln[\tilde{K}(x)\sqrt{x^2 + 1}]$.

We factorize the function $\sqrt{x^2 + 1}$ in the form

$$(43) \quad \sqrt{x^2 + 1} = T(x+i)[T(x)]^{-1},$$

$$T(z) = e^{\frac{\pi z}{2}} \sqrt{\Gamma(1-iz)\Gamma(-1-iz)} = \pi e^{\frac{\pi z}{2}} [\sqrt{z(z-1)} \operatorname{sh} \pi z \Gamma(iz)]^{-1},$$

where $\Gamma(z)$ are gamma-functions. From the properties of $\Gamma(z)$ it follows that $|T(x)|$ behaves as $|x|^{-\frac{1}{2}}$, and $|T(x+i)|$ – as $|x|^{\frac{1}{2}}$ with $x \rightarrow \infty$. We represent boundary condition (40), taking into account (38), (41), and (43), in the form

$$(44) \quad \tilde{\Phi}(x+i) + \frac{\tilde{X}(x)T(x)}{\tilde{X}(x+i)T(x+i)} \tilde{\Phi}(x) = \tilde{H}(x), x \in \mathbb{R}.$$

According to Theorem 1, we need to take as Y_0 the class of functions, such that

$$(45) \quad T(x+i)\tilde{H}(x) \in L_2(\mathbb{R}), \quad (|x|^{\frac{1}{2}}\tilde{H}(x) \in L_2(\mathbb{R})).$$

We search for the solution $\tilde{\Phi}(x)$ in the space of the functions X such that $T(x)\tilde{\Phi}(x) \in \{0, 1\}$, that is,

$$(46) \quad T(x)\tilde{\Phi}(x), T(x+i)\tilde{\Phi}(x+i) \in L_2(\mathbb{R})$$

(or $|x|^{-\frac{1}{2}}\tilde{\Phi}(x) \in L_2(\mathbb{R}), |x|^{\frac{1}{2}}\tilde{\Phi}(x+i) \in L_2(\mathbb{R})$).

Based on the results from Section 3, the solution of equation (44) can be written using the operators \mathcal{F} and \mathcal{F}^{-1} in the form

$$(47) \quad \begin{aligned} \tilde{\Phi}(x) &= (\tilde{K}^{-1}\tilde{H})(x) = \\ &= \frac{1}{T(x)\tilde{X}(x)} \mathcal{F} \left\{ \frac{1}{1+e^{-t}} \mathcal{F}^{-1} \{T(x+i)\tilde{X}(x+i)\tilde{H}(x)\} \right\} (x), \end{aligned}$$

$$\tilde{\Phi}(x+i) = \frac{1}{T(x+i)\tilde{X}(x+i)} \mathcal{F} \left\{ \frac{e^{-t}}{1+e^{-t}} \mathcal{F}^{-1} \{T(x+i)\tilde{X}(x+i)\tilde{H}(x)\} \right\} (x).$$

Let X_0 be the space of functions $\Phi \in X$ such that $T(z)\Phi(z) \in \{0, 1\}$ (that is, they satisfy Theorem 2 with $\alpha = 0, \beta = 1$), with the norm

$$(48) \quad \begin{aligned} \|\Phi\|_{X_0} &= \\ &= \max \left\{ \left(\int_{\mathbb{R}} |T(x)\Phi(x)|^2 dx \right)^{\frac{1}{2}}, \left(\int_{\mathbb{R}} |T(x+i)\Phi(x+i)|^2 dx \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Then, based on the reasoning similar to the one performed above, the following theorem is valid.

Theorem 5. *Suppose that there exists a function $\tilde{K}(x)$ continuous in Hölder's sense, which satisfies conditions (38) and (42), and the inequality*

$$(49) \quad \|B\| \equiv \|\tilde{K}^{-1}(K - \tilde{K})\| \leq L \sup_x |[K(x) - \tilde{K}(x)]\sqrt{x^2 + 1}| = \rho < 1,$$

holds, where $L = N_1 N_2$,

$$N_1 = \max \left\{ \sup_x |\tilde{X}^{-1}(x)|, \sup_x |\tilde{X}^{-1}(x+i)| \right\}, \quad N_2 = \sup_x |\tilde{X}(x+i)|.$$

Then for every function $H(x) \in Y_0$ (45), Carleman's problem (39) has a unique solution in the class of functions X_0 with finite norm (48). That solution is given by formula (26)

$$\Phi(x) = \tilde{\Phi}(x) + [1 + \tilde{K}^{-1}(K - \tilde{K})]^{-1} \tilde{K}^{-1}(H(x) - K\tilde{\Phi}(x)),$$

in which the function $\tilde{\Phi}(x) = (\tilde{K}^{-1}\tilde{H})(x)$ and the operators \tilde{K} , \tilde{K}^{-1} are defined respectively by formulas (47), (39), (40). The error of the approximate solution is estimated by inequality (27), (49).

These results, taking into account the equivalence of (38) and (39), are transferred into the original equation (37). The error estimation for $\tilde{u}(t)$ is defined by inequality (29). The classes for the solution $u(t)$ and the constant term $g(t)$ are chosen according to (45) and (46). Due to (49), the approximate solution can be specified by formula (9) with estimation (10).

As an example of the considered equation we can take equation (37), if we use as the kernel $k_1(t)$ the McDonald function (cylindrical function of imaginary argument)

$$k_1(t) = \sqrt{\frac{z}{\pi}} K_0(|t|), \quad \mathcal{F}[k_1](x) = K_1(x) = \frac{1}{\sqrt{x^2 + 1}}$$

which can be quite often encountered in applications.

CONCLUSION

The scheme, proposed for constructing an approximate solution of a smooth transition equation based on a close equation which has a simpler structure or a simpler solution algorithm, can be transferred on a wide class of equations of convolution type or boundary problems from analytical functions theory, equivalent to them. The most simple algorithms correspond to the method of factorization or incomplete factorization. Together with the regularization method, the solution scheme can be expanded to equations of the first kind, extremal problems, discrete equations of convolution type.

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