S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 1797–1815 (2020) УДК 539.3,517.97
DOI 10.33048/semi.2020.17.122 MSC 35Q74,74M15

A CONTACT OF TWO ELASTIC PLATES CONNECTED ALONG A THIN RIGID INCLUSION

E.V. PYATKINA

Abstract. A contact of two Kirchhoff—Love plates of the same shape and size is considered. The plates are located in parallel without a gap and are clamped at their outer edges. Those plates are connected to each other along a thin rigid inclusion. Three cases are considered. In the first case it is assumed that a force acts at the contact surface. This force is proportional to the difference between displacements of the contact surfaces points of two plates. In the second case a contact of two plates when that force on a contact surface equals zero is considered. The third case corresponds to an equilibrium problem of the two-layer Kirchhoff—Love plate containing thin rigid inclusion. For all three cases a solvability is studied, a variational and differential formulations of the problem are derived and their equivalence is proved. It is shown that the second and the third problems are limit cases of the first one when the value of the force acting on the contact surface tends to zero or to infinity.

Keywords: Kirchhoff—Love plate, contact problem, thin rigid inclusion, nonpenetration condition, variational inequality

1. Introduction

In this paper we consider the equilibrium problem for two Kirchhoff—Love plates. In it, two plates of similar shape and size are located in parallel without a gap. We assume that the plates are connected to each other along a thin rigid inclusion and that their vertical deflections satisfy a non-pertration condition. We consider the case, when a force acts at the contact surface. This force is propotional to a

Pyatkina, E.V., A contact of two elastic plates connected along a thin rigid inclusion.

This research was partially supported by Russian Foundation for Basic Research (grant 18-29-10007).

Received March, 19, 2020, published November, 2, 2020.

difference between displacements of the points of contact surfaces of the plates. We call a positive coefficient of proportionality a "damage parameter". We study the limit cases, as this parameter equals zero or tends to infinity. In the first case we obtain the contact problem for two plates, connected to each other along a thin rigid inclusion, as the elastic force on a contact surface equals zero. In the second case we turn to the equilibrium problem for a two-layer Kirchoff—Love plate, which contains a thin rigid inclusion. We prove an existence and uniqueness for the original problem as well as for both limit cases. Both differential and variational statements of these problems are provided and it is shown that they are equivalent. We also show that the sequence of solutions of the equilibrium problem for two plates with an elastic force acting between them strongly converges to the solution of the contact problem for two plates as the force on the contact surface equals zero, as the damage parameter tends to zero or to the solution of the equilibrium problem for a two-layer plate, which contains a thin rigid inlusion, as the damage parameter tends to infinity.

The equilibrium problem for an elastic plate, which contains a thin rigid inclusion, with and without delamination of the inclusion had been considered in [1]. Studies of the equilibrium of Kirchoff—Love, which contain a delaminated thin rigid inclusion with non-penetration conditions on the edges of a cut can be found in [2,3]. The two-dimensional problem of equilibrium for a two-dimensional elastic body with a thin rigid incluison, with and without delamination of the inclusion had been considered in [4]. In [2,4] shape derivatives of the energy functionals are obtained. In [5], solvability of equilibrium problems for a viscoelastic body with a thin rigid incluison is studied. The problems of equilibrium for a Timoshenko plate and for a two-dimensional body which are containing thin rigid inclusions on their outer borders can be found in [6,7]. In [8] one can find an asymptotic behaviour of a solution in the neighbourhood of the tip of an inclusion for the equilibrium problem of an elastic body with a thin rigid inclusion. A numerical study of the equilibrium problem for a two-dimensional elastic body with a rigid inclusion can be found in [9]. In [10] the equilibrium problem of an elastic body with a thin rigid inclusion is studied both numerically and analytically.

In [11,12,13,14], an asymptotic behaviour of solutions near the tip of rigit line of finitr and semi-finite lenght is studied.

The two-dimensional equilibrium problem for a packet of infinite elastic plates connected along a periodic system of rectilinear segments is studied numerically in [15]. The contact problem for two elastic plates with a non-penetration condition is considered in [16]. In [17,18], one can find two-dimensional equilibrium problems for an elastic body with a defect and for two plates with a defect on the gluing line.

In [19], the two-dimensional equilibrium problem for two elastic half-spaces with an elastic layer between them is considered. Asymptotic modeling of bonding of Kirchhoff—Love plates is conducted in [20,21]. Equations and boundary conditions for an equilibriumproblem of multilayer plates are derived in [22]. In [23], the equilibrium problem of two built-up plates with elastically pliable bracing is considered. In [24,25], equilibrium equations for elastic multilayer plates are derived from three-dimensional theory of elasticity by means of introducing asymptotic expansions with respect to a small parameter defined as a ratio of thickness to characteristic lenght and a variational equationof the problem are obtained. A numerical simulation of equilibrium of a two-layered structure with a through crack is carried out in

[26]. In [27], an equilibrium problem with non-penetration condition for two elastic plates, which are located in parallel and connected along a segment in the contact area is considered.

2. Problem statement

Consider a bounded domain Ω in the Cartesian space $\mathbb{R}^2$ with x_1, x_2 as its axes. Let $\Gamma = \partial\Omega$ be a $C^{1,1}$–class curve. Assume that Ω contains an interval $r = (0,1)\times\{0\}$. By $\overline{r} = [-1,0]\times\{0\}$, denote a closure of r. Let $\nu = (\nu_1, \nu_2)$ be a normal unit vector of r in the plane Ox_1x_2, such that it is directed towards positive values of x_2. Assume axis z is perpendicular to the plane Ox_1x_2. Also let $\Omega_r = \Omega \setminus \overline{r}$.

In this paper we consider the equilibrium problem for two Kirchhoff—Love plates of similar shape and size. Each of them contains thin rigid inclusions, which are positioned similarly and also have the same size. The domain $\Omega\times\{-h/4\} \in \mathbb{R}^3$ is the mid-surface of the lower plate and the domain $\Omega\times\{h/4\}$ is the median plane of the upper plate. The plates occupy the domains in $\mathbb{R}^3$, more specifically $\Omega\times(-h/2, 0)$ and $\Omega\times(0, h/2)$. A thin rigid inclusion occupies the domain $r\times(-h/2, h/2)$ (Fig. 1).

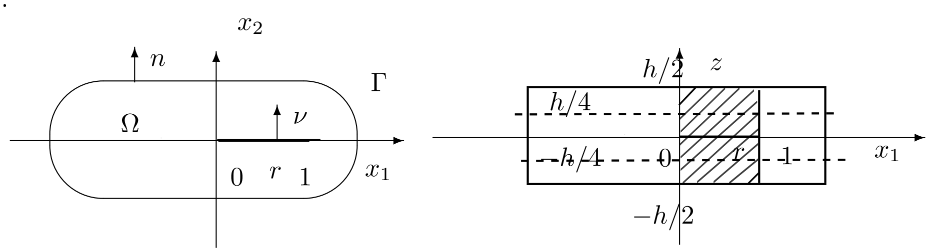

Fig. 1. Domain Ω_r

Denote the displacements of mid-surface' points along the axes x_1, x_2, z by $v_1^I(x), v_2^I(x), w^I(x)$ respectively, where $x = (x_1, x_2)$. Furthermore, from here on out, I can only take values 1 or 2 and we consider all functions with the index $(I = 1)$ to be related to the lower plate and all functions with the index $(I = 2)$ to be related to the upper plate. Let $v^I = (v_1^I, v_2^I)$. As per the Kirchhoff—Love model, we use the following formulae for displacements in the upper and the lower plates respectively:

$$v^{z1} = v^1 - (z + \frac{h}{4})\nabla w^1, \quad w^{z1} = w^1, \qquad x \in \Omega_r,\ z \in (-\frac{h}{2}, 0),$$

$$v^{z2} = v^2 - (z - \frac{h}{4})\nabla w^2, \quad w^{z2} = w^2, \qquad x \in \Omega_r,\ z \in (0, \frac{h}{2}).$$

We will determine the components for a strain tensor by the formula $\varepsilon_{ij}(v^I) = (v_{i,j}^I + v_{j,i}^I)/2$. The components of the stress tensor $\sigma^I(v^I) = \{\sigma_{ij}^I(v^I)\}$, and of the strain tensor $\varepsilon(v^I) = \{\varepsilon_{ij}(v^I)\}$, are connected via linear Hooke's law $\sigma^I(v^I) = A^I\varepsilon(v^I)$. Here $A^I = \{a_{ijkl}^I\}$, $i,j,k,l = 1,2$, is a symmetrical and positive elastic modulus tensor. A case of homogeneous isotropic plates is considered in the paper. The Hooke's law is defined as as follows:

$$\sigma_{11}^I(v^I) = \frac{E_I}{1-\kappa_I^2}\varepsilon_{11}(v^I) + \frac{E_I\kappa_I}{1-\kappa_I^2}\varepsilon_{22}(v^I),$$

$$\sigma_{22}^I(v^I) = \frac{E_I\kappa_I}{1-\kappa_I^2}\varepsilon_{11}(v^I) + \frac{E_I}{1-\kappa_I^2}\varepsilon_{22}(v^I),$$

$$\sigma_{12}^I(v^I) = \frac{E_I}{1+\kappa_I}\varepsilon_{12}(v^I),$$

where E_I is a Young's modulus and κ_I is the Poisson's ratio for the Ith plate.

Let $f^I = (f_1^I, f_2^I) \in L_2(\Omega_r)^2$, $F^I \in L_2(\Omega_r)$, $I = 1,2$, by given external loads, acting on the plates. In this section we will consider the case, when the forces on the contact surface are proportional to the following difference $(v^{z1}(x,0), w^{z1}(x)) - (v^{z2}(x,0), w^{z2}(x))$ of displacements of points of contact surfaces of plates with the ratio coefficient $\mu > 0$, which we will call a damage parameter.

We will first provide the variational statement of a problem. For that, consider the following spaces:

$$H_\Gamma^1(\Omega_\gamma)^2 = \{u = (u_1,u_2) \in H^1(\Omega_\gamma)^2 |\ u = 0 \text{ on } \Gamma\},$$

$$H_\Gamma^2(\Omega_\gamma) = \{w \in H^2(\Omega_\gamma) |\ w = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma\},$$

we also introduce two spaces of infinitesimal rigid displacements on r:

$$R(r) = \{\rho = (\rho_1,\rho_2) |\ \rho(x) = Bx + C, \quad x \in r\},$$

$$L(r) = \{l |\ l(x) = a_0 + a_1x_1 + a_2x_2, \quad x \in r\},$$

where $C = (c^1, c^2)$, $c^1, c^2 = const$, $a_0, a_1, a_2 = const$, and B is a constant skew-symmetric matrix 2×2.

The displacements of points of a thin rigid inclusion is $\rho(x) - z\nabla l(x)$, and the vertical displacement is $l(x)$. Here $z \in (-h/2, h/2)$ and $\rho(x) \in R(r)$, $l(x) \in L(r)$ are some functions. For convenience we will denote the aforementioned horizontal displacements of points on $r \times \{-h/2\}$ by $\rho^1(x)$, and by $\rho^2(x)$ for points on $r \times \{h/2\}$, where $\rho^I(x) \in R(r)$. It then follows from above that on $r \times \{0\}$

$$\rho^1(x) - \frac{h}{4}\nabla l(x) = \rho^2(x) + \frac{h}{4}\nabla l(x).$$

This equality states the relation between ρ^1 and ρ^2, while equalities $v^I = \rho^I$, $w^I = l(x)$ and $\nabla w^I = \nabla l(x)$ on r represent the conditions of gluing of displacements of plates and those of a thin rigid inclusion. Set $\nabla l(x) = a$, for $a = (a_1, a_2)$. We now introduce the set of admissible displacements

$$K = \{(\bar v^1, \bar w^1, \bar v^2, \bar w^2) \in H_\Gamma^1(\Omega_r)^2 \times H_\Gamma^2(\Omega_r) \times H_\Gamma^1(\Omega_r)^2 \times H_\Gamma^2(\Omega_r) |\ \bar w^2 - \bar w^1 \geq 0 \text{ in } \Omega_r,$$

$$\bar v^I = \bar\rho^I, \ \ \bar w^I = \bar l, \ \ \nabla \bar w^I = \bar a, \ \ \bar\rho^1 - \frac{1}{2}\bar a = \bar\rho^2 + \frac{1}{2}\bar a \ \text{ on } r, \ \bar\rho^I \in R(r), \ \bar l \in L(r)\}. \quad (1)$$

Let $(\cdot,\cdot)$ be a scalar product in $L_2(\Omega_r)$. We define the potential energy functional as follows:

$$\Pi_\mu(\bar v^1, \bar w^1, \bar v^2, \bar w^2) = \sum_{I=1}^{2}\{(\sigma^I(\bar v^I), \varepsilon(\bar v^I)) + B_I(\bar w^I, \bar w^I) - (f^I, \bar v^I) - (F^I, \bar w^I)\} +$$

$$+\mu(\bar g, \bar g) + \mu(\bar w^2 - \bar w^1, \bar w^2 - \bar w^1). \quad (2)$$

For simplicity, in (1), (2) and later in this paper, we consider flexural rigidity $D_I = E_I h^3/12 \cdot 2^3(1-\kappa_I^2)$ and thickness $h/2$ of both plates to be equal to 1. We introduce bilinear forms B_I:

$$B_I(w, \bar w) = \int_{\Omega_r} (w_{,11}\bar w_{,11} + w_{,22}\bar w_{,22} + \kappa_I(w_{,11}\bar w_{,22} + w_{,22}\bar w_{,11}) + 2(1-\kappa_I)w_{,12}\bar w_{,12}).$$

For reducing the size of formulae we introduced a function

$$\bar{g} = \bar{v}^1 - \bar{v}^2 - \frac{1}{2}\nabla\bar{w}^1 - \frac{1}{2}\nabla\bar{w}^2$$

which is a difference of horizontal displacements of points of the upper surface of the lower plate and the lower surface of the upper plate. It can be seen that $\bar{g}|_\Gamma = \bar{g}|_r = 0$. We denote $\sigma^I_{ij}(\bar{v}^I)\varepsilon_{ij}(\bar{v}^I)$ by $\sigma^I(\bar{v}^I)\varepsilon(\bar{v}^I)$ (the usual summation convention over repeted indices is employed). A product of two vectors $f^I v^I$ is a sum $f^I_i v^I_i$. In the potential energy formula (2) the last two terms are obtained as a work of elastic forces, which are connecting the plates and acting perpendicular and along the contact surfaces.

A solution to the equilibrium problem for two plates, connected along a thin rigid rectilinear inclusion is a solution to the problem of minimising a functional Π_μ on the set of admissible displacements

$$\Pi_\mu(v^1, w^1, v^2, w^2) = \inf_{(\bar{v}^1,\bar{w}^1,\bar{v}^2,\bar{w}^2)\in K} \Pi_\mu(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2). \tag{3}$$

K is a closed convex set and is therefore weakly bounded in the space $H^1_\Gamma(\Omega_r)^2 \times H^2_\Gamma(\Omega_r) \times H^1_\Gamma(\Omega_r)^2 \times H^2_\Gamma(\Omega_r)$. The functional $\Pi_\mu(v^1, w^1, v^2, w^2)$ is weakly semicontinuous and is coercive on K if $\mu > 0$. Hence (see [28, pages 30-32], a solution for (3) exists and satisfies the variational inequality

$$(v^1, w^1, v^2, w^2) \in K, \tag{4}$$

$$\sum_{I=1}^{2}\{(\sigma^I(v^I), \varepsilon(\bar{v}^I - v^I)) + B_I(w^I, \bar{w}^I - w^I) - (f^I, \bar{v}^I - v^I) - (F^I, \bar{w}^I - w^I)\}+$$
$$+\mu(g, \bar{g} - g) + \mu(w^2 - w^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) \geq 0 \quad \forall(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) \in K. \tag{5}$$

Since $\Pi_\mu(v^1, w^1, v^2, w^2)$ is a strictly convex functional, then the solution of the problem $(4) - (5)$ is unique.

Then we consider the following boundary value problem: for given $f^I \in L_2(\Omega_r)^2$, $F^I \in L_2(\Omega_r)$ find $(v^I(x), w^I(x))$:

$$-\text{div } \sigma^1(v^1) = f^1 - \mu g, \quad \sigma^1(v^1) = A^1\varepsilon(v^1) \quad \text{in } \Omega_r, \tag{6}$$

$$-\text{div } \sigma^2(v^2) = f^2 + \mu g, \quad \sigma^2(v^2) = A^2\varepsilon(v^2) \quad \text{in } \Omega_r, \tag{7}$$

$$\Delta^2 w^1 + \Delta^2 w^2 - F^1 - F^2 = -\mu \text{div} g, \ \Delta^2 w^1 - F^1 + \frac{1}{2}\mu \text{div} g \leq \mu(w^2 - w^1) \ \text{ in } \Omega_r, \tag{8}$$

$$w^2 - w^1 \geq 0, \quad (\Delta^2 w^1 - F^1 + \frac{1}{2}\mu \text{ div} g - \mu(w^2 - w^1))(w^2 - w^1) = 0 \ \text{ in } \Omega_r, \tag{9}$$

$$v^I = 0, \quad w^1 = \frac{\partial w^I}{\partial n} = 0 \qquad \text{on } \Gamma, \tag{10}$$

$$v^I = \rho^I(x), \quad w^I = l(x), \quad \nabla w^I = a, \quad \rho^1 - \frac{1}{2}a = \rho^2 + \frac{1}{2}a \ \text{ on } r, \tag{11}$$

$$\int_r ([\sigma^1(v^1)]\nu + [\sigma^2(v^2)]\nu)\tilde{\rho} = 0, \quad \forall \ \tilde{\rho} \in R(r), \tag{12}$$

$$-\int_r [\sigma^1_{12}(v^1)]\tilde{l}_{,1} + \int_r ([t^1(w^1)] + [t^2(w^2)])\tilde{l} = 0, \tag{13}$$

$$\int_r ([m^1(w^1)] + [m^2(w^2)] + [\sigma^1_{22}(v^1)])\tilde{l}_{,2} = 0, \quad \forall \ \ \tilde{l} \in L(r). \tag{14}$$

The components of tangential unit vector to some curve can be rewritten via components of the curve's normal vector using the formula $\tau = (\tau_1, \tau_2) = (-\nu_2, \nu_1)$. A vector $\sigma^I \nu = (\sigma^I_{1j}\nu_j, \sigma^I_{2j}\nu_j)$ is a product of the stress tensor and the normal vector. Moreover, $\sigma^I_\nu = \sigma^I_{ij}\nu_j\nu_i$, $\sigma^I_\tau = \sigma^I\nu - \sigma^I_\nu \nu$, $\sigma^I_\tau = (\sigma^I_{\tau 1}, \sigma^I_{\tau 2})$. A divergence of the stress tensor σ^I is a vector $\sigma^I_{ij,j}$. Bending moment and transverse force on a curve can be defined as [29]

$$m^I(w^I) = \kappa_I \Delta w^I + (1-\kappa_I)\frac{\partial^2 w^I}{\partial \nu^2}, \quad t^I(w^I) = \frac{\partial}{\partial \nu}(\Delta w^I + (1-\kappa_I)\frac{\partial^2 w^I}{\partial \tau^2}).$$

Denote by Δ the Laplace operator, by $[\cdot]$ the jump of function at γ.

The equalities in (6) and (7) are the equilibrium equations for plates in Ω_r, while the stresses $\sigma^I_{i3} = -\mu g$ act on upper and lower surfaces of the lower and upper plate respectively. Furthermore, normal load $(\Delta^2 w^1 + \mu \mathrm{div} g/2 - F^1)$ acts at the upper surface of the lower plate, while normal load $(-\Delta^2 w^2 - \mu \mathrm{div} g/2 + F^2)$ acts at the contact surface of the upper plate. The first equality in (8) represents the equality of forces at the contact surfaces. The first inequality (9) is a non-penetration condition for the plates. From the second relation of (9) we derive that for points, where $w^2 > w^1$, the surface force, acting at the contact surfaces is equal to $\mu(w^2 - w^1)$. However, for the points, where $w^2 = w^1$, the second inequality of (8) holds. Conditions (10) represent the fact that the plates are clamped on Γ.

Theorem 1. *Assume that* $(v^1, w^1, v^2, w^2) \in K$ *and div* $\sigma^I(v^I) \in L_2(\Omega_r)^2$, $\Delta^2 w^I \in L_2(\Omega_r)$. *Then the variational* (4)–(5) *and differential* (6)–(14) *problems are equivalent.*

Proof. We will show that (4)–(5) and (6)–(14) are equivalent if the solution to the equilibrium problem (3) and test functions belong to the set K, and define, the exact meaning of (6)–(14). For that we will use the results of [28,29]. Let us extend r until it becomes a closed $C^{1,1}$–class curve Σ, which lies in Ω. A unit normal vector ν is defined on Σ, moreover on r equals to the normal chosen earlier. We introduce the spaces $H^{N-\frac{1}{2}}(\Sigma)$, where $N = 1, 2$, with the norm

$$\|\varphi\|^2_{H^{N-\frac{1}{2}}(\Sigma)} = \|\varphi\|^2_{H^{N-1}(\Sigma)} + \int\limits_\Sigma \int\limits_\Sigma \frac{|D^{N-1}\varphi(x) - D^{N-1}\varphi(y)|^2}{|x-y|^2}.$$

The following inclusions take place

$$v^I \in H^{1/2}(\Sigma)^2, \qquad w^I \in H^{3/2}(\Sigma), \qquad \nabla w^I \in H^{1/2}(\Sigma)^2.$$

Also consider the spaces $H^{N-\frac{1}{2}}_{00}(\Sigma \setminus \bar{r})$ which we define as follows:

$$H^{N-\frac{1}{2}}_{00}(\Sigma \setminus \bar{r}) = \{\varphi \in H^{N-\frac{1}{2}}_0(\Sigma \setminus \bar{r}) |\ \rho^{-1/2} D^{N-1}\varphi \in L_2(\Sigma \setminus \bar{r})\},$$

with the norm

$$\|\varphi\|^2_{H^{N-\frac{1}{2}}_{00}(\Sigma\setminus\bar{r})} = \|\varphi\|^2_{H^{N-1}(\Sigma\setminus\bar{r})} + \|\rho^{-1/2}D^{N-1}\varphi\|^2_{L_2(\Sigma\setminus\bar{r})},$$

where ρ is a distance between x and ∂r. We use the following property of spaces $H^{N-\frac{1}{2}}_{00}(\Sigma \setminus \bar{r})$. Assume function φ is defined on $\Sigma \setminus \bar{r}$. Denote by $\bar{\varphi}$ its extension by zero on r. Note that $\bar{\varphi} \in H^{N/2}(\Sigma)$ iff $\varphi \in H^{N/2}_{00}(\Sigma \setminus \bar{r})$. This means that

$$[w^I] \in H^{3/2}_{00}(\Sigma \setminus \bar{r}), \quad [\nabla w^I] \in H^{1/2}_{00}(\Sigma \setminus \bar{r})^2, \quad [g] \in H^{1/2}_{00}(\Sigma \setminus \bar{r})^2.$$

Denote by $H^{-N+\frac{1}{2}}(\Sigma)$ and $H_{00}^{-N+\frac{1}{2}}(\Sigma\backslash\bar{r})$ the spaces, which are dual of $H^{N-\frac{1}{2}}(\Sigma)$ and of $H_{00}^{N-\frac{1}{2}}(\Sigma\setminus\bar{r})$ respectively. Let the brackets $<\ \cdot,\cdot\ >_{N-\frac{1}{2},\Sigma}$ denote the duality pairing between the spaces $H^{N-\frac{1}{2}}(\Sigma)$ and $H^{-N+\frac{1}{2}}(\Sigma)$, while the brackets $<\cdot,\cdot>_{N-\frac{1}{2},\Sigma\setminus\bar{r}}^{00}$ denote the same for spaces $H_{00}^{N-\frac{1}{2}}(\Sigma\setminus\bar{r})$ and $H_{00}^{-N+\frac{1}{2}}(\Sigma\setminus\bar{r})$.

First we derive (6)–(14) from the variational inequality. To obtain the equilibrium equations (6), (7) we use $\bar{v}^I = v^I \pm \phi$, $\phi \in C_0^\infty(\Omega_r)^2$, $\bar{w}^I = w^I$ as test functions. Then the equations are true in the following sense

$$(\sigma^I(v^I), \varepsilon(\phi)) - (f^I, \phi) + (-1)^{I+1}\mu(g, \phi) = 0 \quad \forall\phi \in C_0^\infty(\Omega_r)^2. \tag{15}$$

We now derive the first equality of (8). Let $\bar{v}^I = v^I$, $\bar{w}^I = w^I \pm \psi$, where $\psi \in C_0^\infty(\Omega_r)$. From there we obtain that the equation

$$\sum_{I=1}^{2}\{B_I(w^I,\psi) - (F^I,\psi)\} + \mu(g,\nabla\psi) = 0 \quad \forall\psi \in C_0^\infty(\Omega_r) \tag{16}$$

holds in the sense of generalised functions.

Then, to obtain the inequality in (8) we will use $\bar{v}^I = v^I$, $\bar{w}^I = w^I + \tilde{w}^I$, $\tilde{w} = \tilde{w}^2 - \tilde{w}^1 \geq 0$, where $\tilde{w}^I \in H_0^2(\Omega)$, $\tilde{w} \in H_0^2(\Omega_r)$, as test functions. This leads to the inequality

$$\sum_{I=1}^{2}\{B_I(w^I,\tilde{w}^I) - (F^I,\tilde{w}^I)\} - \frac{1}{2}\mu(g,\nabla\tilde{w}^1 + \nabla\tilde{w}^2) + \mu(w^2 - w^1, \tilde{w}) \geq 0.$$

Using Green's theorem [26] and (16), we get

$$(\Delta^2 w^1 - F^1, \tilde{w}) - \frac{1}{2}\mu(g,\nabla\tilde{w}) - \mu(w^2 - w^1,\tilde{w}) \leq 0 \ \ \forall\tilde{w} \in H_0^2(\Omega_r),\ \tilde{w} \geq 0. \tag{17}$$

We simplify inequality (4)–(5) using Green's theorem and (15), (16):

$$\begin{gathered}-(\Delta^2 w^1 - F^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) + \frac{1}{2}\mu(g, \nabla\bar{w}^2 - \nabla\bar{w}^1 - \nabla w^2 + \nabla w^1) + \\ +\mu(w^2 - w^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) + \\ +\sum_{I=1}^{2}\{-[<\sigma_\nu^I(v^I), \bar{v}_\nu^I - v_\nu^I>_{1/2,\Sigma}] - [<\sigma_\tau^I(v^I), \bar{v}_\tau^I - v_\tau^I>_{1/2,\Sigma}] + \\ +[<m^I(w^I), \frac{\partial\bar{w}^I}{\partial\nu} - \frac{\partial w^I}{\partial\nu}>_{1/2,\Sigma}] + [<t^I(w^I), \bar{w}^I - w^I>_{3/2,\Sigma}]\} \geq 0.\end{gathered} \tag{18}$$

Here $\phi = (\phi_1,\phi_2)$, $\phi_\nu = \phi\cdot\nu$, $\phi_\tau = \phi - \phi_\nu\cdot\nu$.

We now proceed to derivation of (12)–(14). We choose $\bar{v}^I = v^I \pm \tilde{v}^I$, $\bar{w}^I = w^I$, where $\tilde{v}^I \in H_0^1(\Omega)^2$, $\tilde{v}^1 = \tilde{v}^2 \in R(r)$ on r, as test functions. It then follows from (18) that

$$\begin{gathered}\sum_{I=1}^{2}\{<[\sigma_\nu^I(v^I)], \tilde{v}_\nu^I>_{1/2,\Sigma} + <[\sigma_\tau^I(v^I)], \tilde{v}_\tau^I>_{1/2,\Sigma}\} = \\ =<[\sigma_\nu^1(v^1)], \tilde{v}_\nu^1 - \tilde{v}_\nu^2>_{1/2,\Sigma} + <[\sigma_\tau^1(v^1)], \tilde{v}_\tau^1 - \tilde{v}_\tau^2>_{1/2,\Sigma} + \\ + <[\sigma_\nu^1(v^1)] + [\sigma_\nu^2(v^2)], \tilde{v}_\nu^2>_{1/2,\Sigma} + <[\sigma_\tau^1(v^1)] + [\sigma_\tau^2(v^2)], \tilde{v}_\tau^2>_{1/2,\Sigma} = 0.\end{gathered}$$

We define functionals $[\sigma_\nu^I(v^I)] \in H_{00}^{-1/2}(\Sigma\setminus\bar{r})$ using the formula $<[\sigma_\nu^I(v^I)],\varphi>_{1/2,\Sigma\setminus\bar{r}}^{00} = <[\sigma_\nu^I(v^I)],\bar{\varphi}>_{1/2,\Sigma}$ $\forall\ \bar{\varphi} = \varphi$ on $\Sigma\setminus\bar{r}$, $\bar{\varphi} = 0$ on r, $\bar{\varphi} \in H^{1/2}(\Sigma)$. Functionals $[\sigma_\tau^I(v^I)] \in H_{00}^{-1/2}(\Sigma\setminus\bar{r})^2$ are defined similarly.

Therefore, we can rewrite the previous equality as follows:

$$< [\sigma^1_\nu(v^1)], \tilde v^1_\nu - \tilde v^2_\nu >^{00}_{1/2,\Sigma\setminus\bar r} + < [\sigma^1_\nu(v^1)] + [\sigma^2_\nu(v^2)], \tilde v^2_\nu >_{1/2,\Sigma} = 0,$$

$$< [\sigma^1_\tau(v^1)], \tilde v^1_\tau - \tilde v^2_\tau >^{00}_{1/2,\Sigma\setminus\bar r} + < [\sigma^1_\tau(v^1)] + [\sigma^2_\tau(v^2)], \tilde v^2_\tau >_{1/2,\Sigma} = 0.$$

Since $\tilde v^1$ и $\tilde v^2$ are chosen independently on Σ, we obtain the following conditions:

$$< [\sigma^I_\nu(v^I)], \varphi_\nu >^{00}_{1/2,\Sigma\setminus\bar r} = 0, \quad < [\sigma^I_\tau(v^I)], \varphi_\tau >^{00}_{1/2,\Sigma\setminus\bar r} = 0 \ \ \forall \varphi \in H^{1/2}_{00}(\Sigma\setminus\bar r)^2, \tag{19}$$

$$< [\sigma^1_\nu(v^1)] + [\sigma^2_\nu(v^2)], \phi_\nu >_{1/2,\Sigma} = 0, \ < [\sigma^1_\tau(v^1)] + [\sigma^2_\tau(v^2)], \phi_\tau >_{1/2,\Sigma} = 0$$
$$\forall \phi \in H^1_0(\Omega)^2, \ \phi \in R(r). \tag{20}$$

Now choose $\bar v^I = v^I \pm \tilde v^I$, $\bar w^I = w^I \pm \tilde w$ from K. Then, taking into account the relation between ρ^1 and ρ^2, we obtain

$$- < [m^1(w^1)] + [m^2(w^2)] + [\sigma^1_\nu(v^1)], \frac{\partial \tilde w^I}{\partial \nu} >_{1/2,\Sigma} +$$

$$+ < [t^1(w^1)] + [t^2(w^2)], \tilde w^I >_{3/2,\Sigma} - < [\sigma^1_\tau(v^1)], \frac{\partial \tilde w^I}{\partial \tau} >_{1/2,\Sigma} = 0,$$

where $\partial\tilde w^I/\partial\tau = \nabla\tilde w^I - \nu\cdot\partial\tilde w^I/\partial\nu$, on r we have $\partial\tilde w^I/\partial\tau = (\tilde w^I_{,1}, 0)$. From this equality we get

$$< [m^1(w^1)] + [m^2(w^2)] + [\sigma^1_\nu(v^1)], \frac{\partial \tilde w}{\partial \nu} >_{1/2,\Sigma} = 0, \tag{21}$$

$$< [t^1(w^1)] + [t^2(w^2)], \tilde w >_{3/2,\Sigma} -$$
$$- < [\sigma^1_\tau(v^1)], \frac{\partial \tilde w}{\partial \tau} >_{1/2,\Sigma} = 0 \ \ \forall\, \tilde w \in H^2_0(\Omega), \ \tilde w \in L(r). \tag{22}$$

Considering (19)–(22) we rewrite (18) as follows

$$-(\Delta^2 w^1 - F^1, \bar w^2 - \bar w^1 - w^2 + w^1) + \frac{1}{2}\mu(g, \nabla\bar w^2 - \nabla\bar w^1 - \nabla w^2 + \nabla w^1) +$$

$$+\mu(w^2 - w^1, \bar w^2 - \bar w^1 - w^2 + w^1) - < [m^1(w^1)], \frac{\partial \bar w^1}{\partial\nu} - \frac{\partial \bar w^2}{\partial\nu} - \frac{\partial w^1}{\partial\nu} + \frac{\partial w^2}{\partial\nu} >^{00}_{1/2,\Sigma\setminus\bar r} +$$
$$+ < [t^1(w^1)], \bar w^1 - \bar w^2 - w^1 + w^2 >^{00}_{3/2,\Sigma\setminus\bar r} \geq 0. \tag{23}$$

Here the functional $[t^1(w^1)] \in H^{-3/2}_{00}(\Sigma\setminus\bar r)$ is defined using the formula $< [t^1(w^1)], \psi >^{00}_{3/2,\Sigma\setminus\bar r} = < [t^1(w^1)], \bar\psi >_{3/2,\Sigma}$ $\forall\ \bar\psi = \psi$ on $\Sigma\setminus\bar r$, $\bar\psi = 0$ on r, $\bar\psi \in H^{3/2}(\Sigma)$.

Substitute test functions $\bar w^I = w^I \pm \tilde w^I$, where $\tilde w = \tilde w^2 - \tilde w^1 \geq 0$ in Ω_r, into (23). Since $\tilde w$ and its tangental derivative $\partial\tilde w/\partial\tau$ from $\partial\tilde w/\partial\nu$ on $\Sigma\setminus\bar r$ are chosen independently, we obtain that the fourth term on the left side of (23) is equal to zero

$$< [m^1(w^1)], \varphi >^{00}_{1/2,\Sigma\setminus\bar r} = 0 \quad \forall\ \varphi \in H^{1/2}_{00}(\Sigma\setminus\bar r). \tag{24}$$

From (17), we know that the sum of the first three terms of (23) is non-negative. Now we only have to estimate the fifth term of the left-hand side of the inequality. For that we use the following property. Function $\tilde w$ can be constructed in such a way, that for a given small $\delta > 0$ the domain of a constracted function lies in the δ-neighbourhood of a domain $\Sigma\setminus\bar r$ of its trace. From that, we get that

$$< [t^1(w^1)], \tilde w >^{00}_{3/2,\Sigma\setminus\bar r} \leq 0 \ \ \forall\ \tilde w \in H^{3/2}_{00}(\Sigma\setminus\bar r), \ \tilde w \geq 0. \tag{25}$$

Now we substitute two sets of test functions $(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = (0,0,0,0)$ and $(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = 2(v^1, w^1, v^2, w^2)$ into (23) and from there we get the equality

$$-(\Delta^2 w^1 - F^1, w^2 - w^1) + \frac{1}{2}\mu(g, \nabla w^2 - \nabla w^1) +$$
$$+\mu(w^2 - w^1, w^2 - w^1) + < [t^1(w^1)], w^1 - w^2 >^{00}_{3/2,\Sigma\setminus\bar{r}} = 0.$$

On the left-hand side we have a non-negative sum of the first three terms and a non-negative fourht term. Therefore,

$$-(\Delta^2 w^1 - F^1, w^2 - w^1) + \frac{1}{2}\mu(g, \nabla w^2 - \nabla w^1) + \mu(w^2 - w^1, w^2 - w^1) = 0, \quad (26)$$

$$< [t^1(w^1)], w^2 - w^1 >^{00}_{3/2,\Sigma\setminus\bar{r}} = 0. \quad (27)$$

We obtained that the second relation of (8) holds in the sense of (17) and the equality from (9) holds in the sense of (26). Relations (19), (20) specify a precise way to state (12). Condition (13) holds in the sense (22), (25), (27), and the equality (14) holds in the sense (21), (24). Therefore we have obtained all relations (6)–(14).

Now we show the opposite, i.e. derive (4)–(5) from (6)–(14). By addition of (17) and (26), we obtain the inequality

$$-(\Delta^2 w^1 - F^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) + \mu(w^2 - w^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) +$$
$$+\frac{1}{2}\mu(g, \nabla\bar{w}^2 - \nabla\bar{w}^1 - \nabla w^2 + \nabla w^1) \geq 0, \quad (28)$$

where the solution (v^1, w^1, v^2, w^2) and test functions $(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2)$ belong to the set K of possible displacements.

We will rewrite the first term of the left-hand side of (28) using the equilibrium equation (16) and Green's theorem, then we get that

$$\sum_{I=1}^{2}(\Delta^2 w^I - F^I, \bar{w}^I - w^I) - \mu(g, \nabla\bar{w}^2 - \nabla w^2) +$$
$$+\mu(w^2 - w^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) + \frac{1}{2}\mu(g, \nabla\bar{w}^2 - \nabla\bar{w}^1 - \nabla w^2 + \nabla w^1) \geq 0.$$

Using the definition of the function $\bar{g}$, we obtain an inequality

$$\sum_{I=1}^{2}(\Delta^2 w^I - F^I, \bar{w}^I - w^I) + \mu(w^2 - w^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) +$$
$$+\frac{1}{2}\mu(g, \bar{g} - g) - \frac{1}{2}\mu(g, \bar{v}^1 - \bar{v}^2 - v^1 + v^2) \geq 0. \quad (29)$$

By using Green's theorem on the first term of the left-hand side of (29), we get

$$\sum_{I=1}^{2}\{B_I(w^I, \bar{w}^I - w^I) - (F^I, \bar{w}^I - w^I) + < [m^I(w^I)], \frac{\partial\bar{w}^I}{\partial\nu} - \frac{\partial w^I}{\partial\nu} >_{1/2,\Sigma} -$$
$$- < [t^I(w^I)], \bar{w}^I - w^I >_{3/2,\Sigma}\} + \mu(w^2 - w^1, \bar{w}^2 - \bar{w}^1 - w^2 + w^1) +$$
$$+\frac{1}{2}\mu(g, \bar{g} - g) - \mu(g, \bar{v}^1 - \bar{v}^2 - v^1 + v^2) \geq 0. \quad (30)$$

We introduce the following notation

$$J_1 = \sum_{I=1}^{2} < [m^I(w^I)], \frac{\partial\bar{w}^I}{\partial\nu} - \frac{\partial w^I}{\partial\nu} >_{1/2,\Sigma},$$

$$J_2 = -\sum_{I=1}^{2} < [t^I(w^I)], \bar{w}^I - w^I >_{3/2,\Sigma}$$

and estimate each term separately.

For rewriting J_1, we will use (21) and (24), therefore

$$J_1 = - < [\sigma^1_\nu(v^1)], \frac{\partial \bar{w}^2}{\partial \nu} - \frac{\partial w^2}{\partial \nu} >_{1/2,\Sigma} .$$

We will use (22) for rewriting J_2:

$$J_2 = - < [t^1(w^1)], \bar{w}^1 - \bar{w}^2 - w^1 + w^2 >_{3/2,\Sigma} - < [t^1(w^1)] + [t^2(w^2)], \bar{w}^2 - w^2 >_{3/2,\Sigma} =$$

$$= - < [t^1(w^1)], \bar{w}^1 - \bar{w}^2 - w^1 + w^2 >^{00}_{3/2,\Sigma\setminus\bar{\Gamma}} - < [\sigma^1_\tau(v^1)], \frac{\partial \bar{w}^2}{\partial \tau} - \frac{\partial w^2}{\partial \tau} >_{1/2,\Sigma} .$$

From here, by using (27) and (25), we obtain an estimate

$$J_2 \le - < [\sigma^1_\tau(v^1)], \frac{\partial \bar{w}^2}{\partial \tau} - \frac{\partial w^2}{\partial \tau} >_{1/2,\Sigma} .$$

Therefore, we have an estimate of the sum

$$J_1 + J_2 \le - < [\sigma^1_\nu(v^1)], \frac{\partial \bar{w}^2}{\partial \nu} - \frac{\partial w^2}{\partial \nu} >_{1/2,\Sigma} - < [\sigma^1_\tau(v^1)], \frac{\partial \bar{w}^2}{\partial \tau} - \frac{\partial w^2}{\partial \tau} >_{1/2,\Sigma} .$$

We add a term equal to zero (from (19)) to both parts of the last inequality

$$-\frac{1}{2} < [\sigma^1_\nu(v^1)], \frac{\partial \bar{w}^1}{\partial \nu} - \frac{\partial \bar{w}^2}{\partial \nu} - \frac{\partial w^1}{\partial \nu} + \frac{\partial w^2}{\partial \nu} >^{00}_{1/2,\Sigma\setminus\bar{\Gamma}} -$$
$$-\frac{1}{2} < [\sigma^1_\tau(v^1)], \frac{\partial \bar{w}^1}{\partial \tau} - \frac{\partial \bar{w}^2}{\partial \tau} - \frac{\partial w^1}{\partial \tau} + \frac{\partial w^2}{\partial \tau} >^{00}_{1/2,\Sigma\setminus\bar{\Gamma}} = 0.$$

Then we obtain that

$$J_1 + J_2 \le -\frac{1}{2} < [\sigma^1_\nu(v^1)], \frac{\partial \bar{w}^1}{\partial \nu} + \frac{\partial \bar{w}^2}{\partial \nu} - \frac{\partial w^1}{\partial \nu} - \frac{\partial w^2}{\partial \nu} >_{1/2,\Sigma} -$$
$$-\frac{1}{2} < [\sigma^1_\tau(v^1)], \frac{\partial \bar{w}^1}{\partial \tau} + \frac{\partial \bar{w}^2}{\partial \tau} - \frac{\partial w^1}{\partial \tau} - \frac{\partial w^2}{\partial \tau} >_{1/2,\Sigma} .$$

For further rewriting of the right-hand side of the last inequality, we will use the definition of the function $\bar{g}$ and the condition (20). At the right-hand side of the inequality

$$J_1 + J_2 \le \frac{1}{2} < [\sigma^1_\nu(v^1)], \bar{g}_\nu - g_\nu >^{00}_{1/2,\Sigma\setminus\bar{\Gamma}} + \frac{1}{2} < [\sigma^1_\tau(v^1)], \bar{g}_\tau - g_\tau >^{00}_{1/2,\Sigma\setminus\bar{\Gamma}} -$$
$$-\sum_{I=1}^{2} \{ < [\sigma^I_\nu(v^I)], \bar{v}^I_\nu - v^I_\nu >_{1/2,\Sigma} + < [\sigma^I_\tau(v^I)], \bar{v}^I_\tau - v^I_\tau >_{1/2,\Sigma} \}$$

we use (19), to show that the first two terms are equal to zero. We can apply Green's theorem to other summands to obtain that

$$J_1 + J_2 \le \sum_{I=1}^{2} \{ (\sigma^I(v^I), \varepsilon(\bar{v}^I - v^I)) + (\mathrm{div}\sigma^I(v^I), \bar{v}^I - v^I) \}.$$

From here, using the equilibrium equations (15), we get the estimate

$$J_1 + J_2 \le \sum_{I=1}^{2} \{ (\sigma^I(v^I), \varepsilon(\bar{v}^I - v^I)) - (f^I, \bar{v}^I - v^I) \} + \mu(g, \bar{v}^1 - \bar{v}^2 - v^1 + v^2). \quad (31)$$

By substituting (31) into (30), we get (4), which is precisely what we needed. □

3. The limiting case as $\mu \to 0$.

In this section, we consider the problem (4)–(5) as μ tends to zero. For this, we will put test functions $(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = (0,0,0,0)$ and

$$(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = 2(v^1, w^1, v^2, w^2)$$

into the variational inequality (5). From the equality

$$\sum_{I=1}^{2}\{(\sigma^I(v^I), \varepsilon(v^I)) + B_I(w^I, w^I) - (f^I, v^I) - (F^I, w^I)\}+$$
$$+\mu(g,g) + \mu(w^2 - w^1, w^2 - w^1) = 0 \tag{32}$$

we obtain the estimates

$$\|v^I\|_{H^1_\Gamma(\Omega_r)^2} \le C_1, \quad \|w^I\|_{H^2_\Gamma(\Omega_r)} \le C_1, \tag{33}$$

$$\|g\|_{L_2(\Omega_r)^2} \le C_2, \qquad \|w^2 - w^1\|_{L_2(\Omega_r)} \le C_2. \tag{34}$$

Here, the constants C_1, C_2 are independent of μ.

It follows from (33), that there exist subsequences v^I, w^I, such that

$$v^I \to v^{0I} \ \text{ weakly in } H^1_\Gamma(\Omega_r)^2, \quad w^I \to w^{0I} \ \text{ weakly in } H^2_\Gamma(\Omega_r),$$

as $\mu \to 0$, $(v^{01}, w^{01}, v^{02}, w^{02}) \in K$.

Passing to the lower limit as $\mu \to 0$ in (5), we obtain the variational inequality for the contact problem for two plates, which are connected along a thin rigid inclusion and there is no any elastic force acting at the contact surface:

$$(v^{01}, w^{01}, v^{02}, w^{02}) \in K, \tag{35}$$

$$\sum_{I=1}^{2}\{(\sigma^I(v^{0I}), \varepsilon(\bar{v}^I - v^{0I})) + B_I(w^{0I}, \bar{w}^I - w^{0I}) - (f^I, \bar{v}^I - v^{0I}) - (F^I, \bar{w}^I - w^{0I})\} \ge 0$$
$$\forall (\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) \in K. \tag{36}$$

This inequality means that the limit functions v^{0I} and w^{0I} are the solutions to the problem, corresponding to the case when $\mu = 0$. A set of admissible displacements if defined in (1). The inequality (35)–(36) is variational for a minimization problem on K

$$\Pi_0(v^{01}, w^{01}, v^{02}, w^{02}) = \inf_{(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) \in K} \Pi_(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2)$$

of a functional

$$\Pi_0(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = \sum_{I=1}^{2}\{(\sigma^I(\bar{v}^I), \varepsilon(\bar{v}^I)) + B_I(\bar{w}^I, \bar{w}^I) - (f^I, \bar{v}^I) - (F^I, \bar{w}^I)\}.$$

The solution of minimization problem for a functional $\Pi_0(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2)$ on the set K exists, is unique and satisfies the variational inequality (35) – (36) (see [28, pp. 30–32]).

We obtain the boundary value problem for given $f^I \in L_2(\Omega_r)^2$, $F^I \in L_2(\Omega_r)$ find $(v^{0I}(x), w^{0I}(x))$:

$$-\text{div } \sigma^1(v^{01}) = f^1, \quad \sigma^1(v^{01}) = A^1\varepsilon(v^{01}) \quad \text{in } \Omega_r, \tag{37}$$

$$-\text{div } \sigma^2(v^{02}) = f^2, \quad \sigma^2(v^{02}) = A^2\varepsilon(v^{02}) \quad \text{in } \Omega_r, \tag{38}$$

$$w^{02} - w^{01} \geq 0, \quad \Delta^2 w^{01} + \Delta^2 w^{02} = F^1 + F^2, \quad \Delta^2 w^{01} - F^1 \leq 0 \ \text{ in } \Omega_r, \tag{39}$$

$$(\Delta^2 w^{01} - F^1)(w^{02} - w^{01}) = 0 \ \text{ in } \Omega_r, \tag{40}$$

$$v^{0I} = 0, \quad w^{0I} = \frac{\partial w^{0I}}{\partial n} = 0 \quad \text{on } \Gamma, \tag{41}$$

$$w^{0I} = l_0(x), \quad \nabla w^{0I} = a, \quad v^{0I} = \rho_0^I(x) \quad \rho_0^1 - \frac{1}{2}a = \rho_0^2 + \frac{1}{2}a \ \text{ on } r, \tag{42}$$

$$\int\limits_r ([\sigma^1(v^{01})]\nu + [\sigma^2(v^{02})]\nu)\tilde{\rho} = 0 \ \ \forall \ \tilde{\rho} \in R(r), \tag{43}$$

$$-\int\limits_r [\sigma^1_{12}(v^{01})]\tilde{l}_{,1} + \int\limits_r ([t^1(w^{01})] + [t^2(w^{02})])\tilde{l} = 0, \tag{44}$$

$$\int\limits_r ([m^1(w^{01})] + [m^2(w^{02})] + [\sigma^1_{22}(v^{01})])\tilde{l}_{,2} = 0 \ \ \forall \ \ \tilde{l} \in L(r). \tag{45}$$

In that case the following theorem holds:

Theorem 2. *Assume that* $(v^{01}, w^{01}, v^{02}, w^{02}) \in K$ *and div* $\sigma^I(v^{0I}) \in L_2(\Omega_r)^2$, $\Delta^2 w^{0I} \in L_2(\Omega_r)$. *Then the variational* (35) – (36) *and differential* (37) – (45) *problems are equivalent.*

The proof of this statement can be obtained from the proof of Theorem 1, if one assumes $\mu = 0$. We show the exact sence of the relations of a boundary value problem (37) – (45).

The equilibrium equations (37), (38) and the second relation of (39) are fulfilled in the sense of generalised functions

$$(\sigma^I(v^I), \varepsilon(\phi)) - (f^I, \phi) = 0 \quad \forall \phi \in C_0^\infty(\Omega_r)^2,$$

$$\sum_{I=1}^{2} \{B_I(w^I, \psi) - (F^I, \psi)\} = 0 \quad \forall \psi \in C_0^\infty(\Omega_r).$$

For the third relation of (39) and equality (40) we have that

$$(\Delta^2 w^1 - F^1, \tilde{w}) \leq 0 \quad \forall \tilde{w} \in H_0^2(\Omega_r), \ \tilde{w} \geq 0,$$

$$(\Delta^2 w^1 - F^1, w^2 - w^1) = 0.$$

The following equalities specify an exact way to state (43):

$$< [\sigma^I(v^{0I})\nu], \varphi >^{00}_{1/2, \Sigma \setminus \bar{r}} = 0 \ \ \forall \varphi \in H_{00}^{1/2}(\Sigma \setminus \bar{r})^2,$$

$$< [\sigma^1(v^{01})\nu] + [\sigma^2(v^{02})\nu], \phi >_{1/2, \Sigma} = 0 \ \ \forall \phi \in H_0^1(\Omega)^2, \ \phi \in R(r).$$

Condition (44) is true in the sense of

$$< [t^1(w^{01})] + [t^2(w^{02})], \tilde{w} >_{3/2, \Sigma} -$$

$$- < [\sigma^1_\tau(v^{01})], \frac{\partial \tilde{w}}{\partial \tau} >_{1/2, \Sigma} = 0 \ \ \forall \ \tilde{w} \in H_0^2(\Omega), \ \tilde{w} = \tilde{l} \in L(r),$$

$$< [t^1(w^{01})], \tilde{w} >^{00}_{3/2, \Sigma \setminus \bar{r}} \leq 0 \ \ \forall \ \tilde{w} \in H_{00}^{3/2}(\Sigma \setminus \bar{r}), \ \tilde{w} \geq 0,$$

$$< [t^1(w^{01})], w^2 - w^1 >^{00}_{3/2, \Sigma \setminus \bar{r}} = 0.$$

For (45), we have

$$< [m^1(w^{01})] + [m^2(w^{02})] +$$

$$+[\sigma^1_\nu(v^{01})], \frac{\partial \tilde{w}}{\partial \nu} >_{1/2,\Sigma}= 0 \ \ \forall\, \tilde{w} \in H^2_0(\Omega),\ \ \tilde{w} = \tilde{l} \in L(r),$$
$$< [m^1(w^{01})], \psi >^{00}_{1/2,\Sigma\setminus\bar{r}}= 0 \quad \forall\, \psi \in H^{1/2}_{00}(\Sigma \setminus \bar{r}).$$

4. The limiting case as $\mu \to \infty$.

Now we consider the problem (4)–(5) as μ tends to infinity.

From the equality (32), we get estimates (which are uniform by μ) for large values of the parameter:

$$\|v^I\|_{H^1_\Gamma(\Omega_r)^2} \le C_3, \quad \|w^I\|_{H^2_\Gamma(\Omega_r)} \le C_3, \tag{46}$$

$$\mu\|g\|_{L_2(\Omega_r)^2} \le C_2, \qquad \mu\|w^2 - w^1\|_{L_2(\Omega_r)} \le C_2. \tag{47}$$

According to (46) we can choose weakly converging subsequences v^I, w^I, such that, as $\mu \to \infty$

$$v^I \to v^{*I} \ \text{ weakly in } \ H^1_\Gamma(\Omega_r)^2, \qquad w^I \to w^{*I} \ \text{ weakly in } \ H^2_\Gamma(\Omega_r),$$

where $(v^{*1}, w^{*1}, v^{*2}, w^{*2}) \in K$. The inequalities (47) mean that $w^{*2} - w^{*1} = 0$ and $v^{*1} - v^{*2} - \nabla w^{*1}/2 - \nabla w^{*2}/2 = 0$ almost everywhere in Ω_r. That means that the limit functions satisfy the displacement gluing condition in Ω_r. Let $w^* = w^{*I}$.

For the test functions, we choose those that satisfy the displacement gluing condition at the entire contact surface $\bar{w} = \bar{w}^1 = \bar{w}^2$, $\bar{v}^1 - \nabla\bar{w}^1/2 = \bar{v}^2 + \nabla\bar{w}^2/2$ and send μ to infinity in (5). As a result, we obtain the variational equality

$$(v^{*1}, v^{*2}, w^*) \in K^*, \tag{48}$$

$$\sum_{I=1}^{2}\{(\sigma^I(v^{*I}), \varepsilon(\bar{v}^I)) + B_I(w^*, \bar{w}) - (f^I, \bar{v}^I) - (F^I, \bar{w})\} = 0 \quad \forall (\bar{v}^1, \bar{v}^2, \bar{w}) \in K^*. \tag{49}$$

In that case, the set of admissible displacements becomes:

$$K^* = \{(\bar{v}^1, \bar{v}^2, \bar{w}) \in H^1_\Gamma(\Omega_r)^2 \times H^1_\Gamma(\Omega_r)^2 \times H^2_\Gamma(\Omega_r) | \bar{v}^1 - \bar{v}^2 - \nabla\bar{w} = 0 \in \Omega_r,$$
$$\bar{w} = \bar{l}, \quad \nabla\bar{w} = \bar{a}, \quad \bar{v}^I = \bar{\rho}^I, \quad \bar{\rho}^1 - \bar{\rho}^2 = \bar{a} \ \text{ on } r, \ \bar{\rho}^I \in R(r), \ \bar{l} \in L(r)\}.$$

The equation (48)–(49) corresponds to the minimization problem

$$\Pi^*(v^{*1}, v^{*2}, w^*) = \inf_{(\bar{v}^1, \bar{v}^2, \bar{w}) \in K^*} \Pi^*(\bar{v}^1, \bar{v}^2, \bar{w}), \tag{50}$$

where

$$\Pi^*(\bar{v}^1, \bar{v}^2, \bar{w}) = \sum_{I=1}^{2}\{(\sigma^I(\bar{v}^I), \varepsilon(\bar{v}^I)) + B_I(\bar{w}, \bar{w}) - (f^I, \bar{v}^I) - (F^I, \bar{w})\}.$$

It can be seen that K^* is a closed convex set and Π^* is a coercive and weakly lower semi-continuous functional. Therefore the problem (50) is solvable and since Π^* is strictly convex the solution is unique. The set K^* is a linear subspace, hence the solution of (50) satisfies the variatonal inequality (48)–(49) see [28, pages 30-32].

We get the boundary value problem for given $f^I \in H^1(\Omega_r)^2$, $F^I \in L_2(\Omega_r)$ find v^{*1}, v^{*2}, w^*:

$$-\text{div}\sigma^1(v^{*1}) - \text{div}\sigma^2(v^{*2}) = f^1 + f^2, \qquad \sigma^I(v^{*I}) = A^I\varepsilon(v^{*I}) \qquad \text{in } \Omega_r, \tag{51}$$

$$(D_1+D_2)\Delta^2 w^* + \frac{h^2}{8}(\sigma^1_{ij,ij}(v^{*1}) + f^1_{i,i} - \sigma^2_{ij,ij}(v^{*2}) - f^2_{i,i}) = F^1 + F^2 \quad \text{in } \Omega_r, \tag{52}$$

$$v^{*1} - \frac{h}{4}\nabla w^* = v^{*2} + \frac{h}{4}\nabla w^* \quad \text{in } \Omega_r, \tag{53}$$

$$v^{*I} = 0, \quad w^* = \frac{\partial w^*}{\partial n} = 0 \quad \text{on } \Gamma, \tag{54}$$

$$\int_r ([\sigma^1(v^{*1})]\nu + [\sigma^2(v^{*2})]\nu)\tilde{\rho} = 0 \ \ \forall\, \tilde{\rho} \in R(r), \tag{55}$$

$$-\frac{h^2}{4}\int_r [\sigma^1_{12}(v^{*1})]\tilde{l}_{,1} + \int_r (D_1[t^1(w^*)] + D_2[t^2(w^*)] + \frac{h^2}{8}([\sigma^1_{2j,j}(v^{*1})] - [\sigma^2_{2j,j}(v^{*2})]))\tilde{l} = 0, \tag{56}$$

$$\int_r (D_1[m^1(w^*)] + D_2[m^2(w^*)] + \frac{h^2}{4}[\sigma^1_{22}(v^{*1})])\tilde{l}_2 = 0, \ \ \forall\ \ \tilde{l} \in L(r). \tag{57}$$

In (51)–(57) the coefficinets D_I and $h/2$, previously said to be equal to 1 are restored.

Theorem 3. *Assume that* $(v^{*1}, v^{*2}, w^*) \in K^*$ *and*

$$div\ \sigma^I(v^{*I}) \in L_2(\Omega_r)^2,\ \sigma^I_{ij,ij}(v^{*I}) \in L_2(\Omega_r),\ \Delta^2 w^* \in L_2(\Omega_r).$$

Let $f^I_{i,i} \in L_2(\Omega_r)$. *Then the variational* (48) – (49) *and differential* (51) – (57) *problems are equivalent.*

Proof. First we consider the problem (51)−(57) and derive equations and boundary conditions for r from the variational equality (48) – (49).

Substitute test functions $\bar{v}^I = \phi$, $\phi \in C_0^\infty(\Omega_r)^2$, $\bar{w} = 0$ into (49). Then (51) is true in case of generalised functions

$$\sum_{I=1}^{2}\{\frac{h}{2}(\sigma^I(v^{*I}), \varepsilon(\phi)) - \frac{h}{2}(f^I, \phi)\} = 0 \quad \forall \phi \in C_0^\infty(\Omega_r)^2. \tag{58}$$

To obtain (52), we take $\bar{w} = \psi$, where $\psi \in C_0^\infty(\Omega_r)$ and $\bar{v}^1 = \frac{h}{4}\nabla\psi$, $\bar{v}^2 = -\frac{h}{4}\nabla\psi$. From there we get

$$\frac{h^2}{8}\sum_{I=1}^{2}(-1)^{I+1}\{(\sigma^I(v^{*I}), \varepsilon(\nabla\psi)) - (f^I, \nabla\psi)\} + \sum_{I=1}^{2}\{D_I B_I(w^*, \psi) - (F^I, \psi)\} = 0. \tag{59}$$

Hence (52) is true in the sense of generalised functions.

We now have to obtain conditions (55) − (57) on r. For that we rewrite (49). We introduce a new set of test functions

$$\bar{u} = \bar{v}^I - (-1)^{I+1}\frac{h}{4}\nabla\bar{w}, \quad \bar{w} = \bar{w}.$$

It is clear that $\bar{u} \in H^1_\Gamma(\Omega_r)^2$, $\bar{u} \in R(r)$ on r. A function $\bar{u}$ sets possible horizontal displacements of points of the plane $z = 0$ by using horizontal displacements $\bar{v}^I$ of mid-surfaces of the plates, which form a monolithic plate of thickness h. Admissible vertical displacements of points of the plane $z = 0$ are similar to vertical displacements $\bar{w}$ of mid-surfaces $z = \pm h/2$ of two layers.

Then (48)–(49) becomes

$$\sum_{I=1}^{2}\{\frac{h}{2}(\sigma^I(v^{*I}),\varepsilon(\bar{u}))+(-1)^{I+1}\frac{h^2}{8}(\sigma^I(v^{*I}),\varepsilon(\nabla\bar{w}))+D_IB_I(w^*,\bar{w})-$$

$$-\frac{h}{2}(f^I,\bar{u})-(-1)^{I+1}\frac{h^2}{8}(f^I,\nabla\bar{w})-(F^I,\bar{w})\}=0. \tag{60}$$

We simplify (60) using Green's theorem and (58), (50):

$$\sum_{I=1}^{2}\{-\frac{h}{2}<[\sigma^I_\nu(v^{*I})],\bar{u}_\nu>_{1/2,\Sigma}-\frac{h}{2}<[\sigma^I_\tau(v^{*I})],\bar{u}_\tau>_{1/2,\Sigma}-$$

$$-D_I<[m^I(w^*)],\frac{\partial\bar{w}}{\partial\nu}>_{1/2,\Sigma}+$$

$$+D_I<[t^I(w^*)],\bar{w}>_{3/2,\Sigma}+(-1)^{I+1}\frac{h^2}{8}(<[\sigma^I_{ij,j}(v^{*I})\nu_i],\bar{w}>_{3/2,\Sigma}-$$

$$-<[\sigma^I_\nu(v^{*I})],\frac{\partial\bar{w}}{\partial\nu}>_{1/2,\Sigma}-<[\sigma^I_\tau(v^{*I})],\frac{\partial\bar{w}}{\partial\tau}>_{1/2,\Sigma})\}=0. \tag{61}$$

We choose $\bar{w}=0$. By substituting it into (61), we obtain

$$\sum_{I=1}^{2}\{<[\sigma^I_\nu(v^{*I})],\bar{u}_\nu>_{1/2,\Sigma}+<[\sigma^I_\tau(v^{*I})],\bar{u}_\tau>_{1/2,\Sigma}\}=0\quad\forall\bar{u}\in H^1_\Gamma(\Omega_r)^2,\ \bar{u}\in R(r).$$

From this, we get that (55) is true in the following sense:

$$<[\sigma^1_\nu(v^{*1})]+[\sigma^2_\nu(v^{*2})],\phi_\nu>_{1/2,\Sigma}=0, \tag{62}$$

$$<[\sigma^1_\tau(v^{*1})]+[\sigma^2_\tau(v^{*2})],\phi_\tau>_{1/2,\Sigma}=0\quad\forall\phi\in H^1_0(\Omega)^2,\ \phi\in R(r). \tag{63}$$

Now, we choose test functions such that $\bar{u}=0$. Then, from (61), we obtain

$$\sum_{I=1}^{2}\{-D_I<[m^I(w^*)],\frac{\partial\bar{w}}{\partial\nu}>_{1/2,\Sigma}+D_I<[t^I(w^*)],\bar{w}>_{3/2,\Sigma}+$$

$$+(-1)^{I+1}\frac{h^2}{8}(<[\sigma^I_{ij,j}(v^{*I})\nu_i],\bar{w}>_{3/2,\Sigma}-$$

$$-<[\sigma^I_\nu(v^{*I})],\frac{\partial\bar{w}}{\partial\nu}>_{1/2,\Sigma}-[\sigma^I_\tau(v^{*I})],\frac{\partial\bar{w}}{\partial\tau}>_{1/2,\Sigma})\}=0.$$

Note that $\partial\bar{w}/\partial\nu$ is independent of $\bar{w}$ and $\partial\bar{w}/\partial\tau$ on Σ. From that and (62), (63) we get that (56) and (57) are true in the sense of

$$\sum_{I=1}^{2}D_I<[m^I(w^*)],\phi_\nu>_{1/2,\Sigma}+$$

$$+\frac{h^2}{4}<[\sigma^1_\nu(v^{*1})],\phi_\nu>_{1/2,\Sigma}=0\ \forall\phi\in H^1_0(\Omega)^2,\ \phi_\nu\in\mathbb{R}\text{ on }r \tag{64}$$

$$-\frac{h^2}{4}<[\sigma^1_\tau(v^{*1})],\frac{\partial\bar{w}}{\partial\tau}>_{1/2,\Sigma}+\sum_{I=1}^{2}\{D_I<[t^I(w^*)],\bar{w}>_{3/2,\Sigma}+$$

$$+(-1)^{I+1}\frac{h^2}{8}<[\sigma^I_{ij,j}(v^{*I})\nu_i],\bar{w}>_{3/2,\Sigma}\}=0\ \forall\bar{w}\in H^2_0(\Omega),\ \bar{w}\geq0,\bar{w}\in L(r). \tag{65}$$

We now show that the opposite is true, i.e. one can derive (48)–(49) from (51)–(57). Let (v^{*1}, v^{*2}, w^*), $(\bar{v}^1, \bar{v}^2, \bar{w}) \in K^*$. Consider the sum of scalar products

$$\sum_{I=1}^{2}\{-\frac{h}{2}(\mathrm{div}\sigma^I(v^{*I}), \bar{v}^1 - \frac{h}{4}\nabla\bar{w}) - \frac{h}{2}(f^I, \bar{v}^1 - \frac{h}{4}\nabla\bar{w})\} = 0,$$

which is equal to zero because of (58). Applying Green's theorem, as well as (53), the first boundary condition (54), (62) and (63) to the terms on the left-hand side, we obtain

$$\sum_{I=1}^{2}\{\frac{h}{2}(\sigma^I(v^{*I}), \varepsilon(\bar{v}^I)) - \frac{h}{2}(f^I, \bar{v}^I)\} -$$

$$-\sum_{I=1}^{2}\frac{h^2}{8}\{< [\sigma_\nu^I(v^{*I})], \frac{\partial\bar{w}}{\partial\nu} >_{1/2,\Sigma} + < [\sigma_\tau^I(v^{*I})], \frac{\partial\bar{w}}{\partial\tau} >_{1/2,\Sigma}\} -$$

$$-\sum_{I=1}^{2}(-1)^{I+1}\frac{h^2}{8}\{(\sigma^I_{ij,ij}(v^{*I}) + f^I_{i,i}, \bar{w}) + < [\sigma^I_{ij,j}(v^{*I})\nu_i], \bar{w} >_{3/2,\Sigma}\} = 0. \quad (66)$$

We use the equilibrium equation (59) as well as (54):

$$\sum_{I=1}^{2}(-1)^{I+1}\frac{h^2}{8}\{(\sigma^I_{ij,ij}(v^{*I}) + f^I_{i,i}, \bar{w})\} = -\sum_{I=1}^{2}\{D_I B_I(w^*, \bar{w}) - (F^I, \bar{w}) +$$

$$+D_I < [m^I(w^*)], \frac{\partial\bar{w}}{\partial\nu} >_{1/2,\Sigma} - D_I < [t^I(w^*)], \bar{w} >_{3/2,\Sigma}\} = 0.$$

Considering the boundary conditions (64) and (65), we obtain

$$\sum_{I=1}^{2}(-1)^{I+1}\frac{h^2}{8}\{(\sigma^I_{ij,ij}(v^{*I}) + f^I_{i,i}, \bar{w})\} = -\sum_{I=1}^{2}\{D_I B_I(w^*, \bar{w}) - (F^I, \bar{w})\} +$$

$$+\sum_{I=1}^{2}(-1)^{I+1}\frac{h^2}{8} < [\sigma^I_{ij,j}(v^{*I})\nu_i], \bar{w} >_{3/2,\Sigma} -$$

$$-\sum_{I=1}^{2}\frac{h^2}{8}\{< [\sigma_\nu^I(v^{*I})], \frac{\partial\bar{w}}{\partial\nu} >_{1/2,\Sigma} + < [\sigma_\tau^I(v^{*I})], \frac{\partial\bar{w}}{\partial\tau} >_{1/2,\Sigma}\}.$$

Substitute the last equality into (66) and assume flexural rigidness D_I and thickness $h/2$ of both plates to be equal to 1. Thus we obtain the variational equality (48)–(49). □

The system (51) – (57) states the equilibrium equations for a two-layered Kirchhoff—Love plate (see, for instance, [22, page 50;22]), which contains a thin rigid inclusion. Each layer has a thickness of $h/2$ with the thickness of the whole plate being equal to h. A thin rigid inclusion occupies the domain $r \times (-h/2, h/2)$. Therefore, a weak limit of the sequence of solutions of (4) – (5), as $\mu \to \infty$, is the solution to the equilibrium problem for a two-layered Kirchhoff—Love plate, which contains a thin rigid inclusion.

5. Strong convergence of the solution to the problem (4) – (5) to the solution of limiting cases as μ tends to zero or infinity

In sections 2 and 3, we proved the weak convergence of the solution to (4)–(5) to the solution to (35)–(36) as $\mu \to 0$, and to the solution to (48)–(49) as $\mu \to \infty$. We will show that, in fact, the strong convergence of sulution to (35)–(36) to the aforementioned solutions also takes place.

A The case of $\mu \to 0$. By substituting test functions $(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = (0,0,0,0)$ and $(\bar{v}^1, \bar{w}^1, \bar{v}^2, \bar{w}^2) = 2(v^{01}, w^{01}, v^{02}, w^{02})$ into (35)–(36), we derive the equality

$$\sum_{I=1}^{2} \{(\sigma^I(v^{0I}), \varepsilon(v^{0I})) + B_I(w^{0I}, w^{0I}) - (f^I, v^{0I}) - (F^I, w^{0I})\} = 0. \tag{67}$$

Now, we use the weak convergence of $v^I \to v^{0I}$ in $H^1_\Gamma(\Omega_r)^2$, $w^I \to w^{0I}$ in $H^2_\Gamma(\Omega_r)$ as $\mu \to 0$, which we showed in section 2 of this paper. From (32) and (67) and while considering (34), we obtain that, as $\mu \to 0$,

$$\sum_{I=1}^{2} \{(\sigma^I(v^I), \varepsilon(v^I)) + B_I(w^I, w^I)\} = \sum_{I=1}^{2} \{(f^I, v^I) + (F^I, w^I)\} -$$

$$-\mu \|g\|^2_{L_2(\Omega_r)} - \mu \|w^2 - w^1\|^2_{L_2(\Omega_r)} \to \sum_{I=1}^{2} \{(f^I, v^{0I}) + (F^I, w^{0I})\} =$$

$$= \sum_{I=1}^{2} \{(\sigma^I(v^{0I}), \varepsilon(v^{0I})) + B_I(w^{0I}, w^{0I})\}.$$

From here, considering the conditions

$$\liminf (\sigma^I(v^I), \varepsilon(v^I)) \geq (\sigma^I(v^{0I}), \varepsilon(v^{0I})), \qquad \liminf B_I(w^I, w^I) \geq B_I(w^{0I}, w^{0I}),$$

we obtain that

$$(\sigma^I(v^I), \varepsilon(v^I)) \to (\sigma^I(v^{0I}), \varepsilon(v^{0I})), \qquad B_I(w^I, w^I) \to B_I(w^{0I}, w^{0I}).$$

Together with the weak convergence of the sequence of functions, it shows that $v^I \to v^{0I}$ strongly in $H^1_\Gamma(\Omega_r)^2$ and $w^I \to w^{0I}$ strongly in $H^2_\Gamma(\Omega_r)$.

B The case of $\mu \to \infty$. Here we use (32), (49) and the weak convergence $v^I \to v^{*I}$ in $H^1_\Gamma(\Omega_r)^2$, $w^I \to w^{*I}$ in $H^2_\Gamma(\Omega_r)$ and derive a sequence of inequalities, which hold, as $\mu \to \infty$:

$$0 \leq \liminf \{\mu \|g\|^2_{L_2(\Omega_r)} + \mu \|w^2 - w^1\|^2_{L_2(\Omega_r)}\} \leq$$

$$\leq \limsup \{\mu \|g\|^2_{L_2(\Omega_r)} + \mu \|w^2 - w^1\|^2_{L_2(\Omega_r)}\} =$$

$$= \limsup \sum_{I=1}^{2} \{(-\sigma^I(v^I), \varepsilon(v^I)) - B_I(w^I, w^I)\} + (f^I, v^I) + (F^I, w^I)\} \leq$$

$$\leq \limsup \sum_{I=1}^{2} \{(-\sigma^I(v^{*I}), \varepsilon(v^{*I})) - B_I(w^*, w^*) + (f^I, v^{*I}) + (F^I, w^*)\} = 0.$$

Therefore, we showed that

$$\mu \|g\|^2_{L_2(\Omega_r)} + \mu \|w^2 - w^1\|^2_{L_2(\Omega_r)} \to 0.$$

Hence, from (32) and (49) we obtain the convergence

$$\sum_{I=1}^{2}\{(\sigma^I(v^I),\varepsilon(v^I))+B_I(w^I,w^I)\}\to\sum_{I=1}^{2}\{(\sigma^I(v^{*I}),\varepsilon(v^{*I}))+B_I(w^*,w^*)\}.$$

From the estimates

$$\liminf(\sigma^I(v^I),\varepsilon(v^I))\geq(\sigma^I(v^{*I}),\varepsilon(v^{*I})),\qquad \liminf B_I(w^I,w^I)\geq B_I(w^*,w^*),$$

we have that $v^I\to v^{*I}$ strongly in $H^1_\Gamma(\Omega_r)^2$ and that $w^I\to w^{*I}$ strongly in $H^2_\Gamma(\Omega_r)$.

6. Conclusion

1 In the paper we considered the equilibrium of two plates, which are connected along a rectilinear thin rigid inclusion. The case of elastic force acting at the contact surface both in the contact plane and perpendicularly to it had been studied. The value of that force is characterised by the damage parameter. Non-penetration condition for the contact deflections of the plates is taken into consideration. We showed that this problem is solvable and has a unique solution, both variational and differential statements of that problem are presented and it is proved that they are equivalent.

2 The limit cases $\mu=0$ и $\mu=\infty$ of the damage parameter were considered. Case $\mu=0$ corresponds to the contact problem for two Kirchhoff—Love plates, with no elastic force acting at their contact surfaces. Case $\mu=\infty$ corresponds to the equilibrium problem for a two-layered Kirchhoff—Love plate, which contains a thin rigid inclusion.

3 We have shown strong convergence of the sequence of solutions to the contact problem for two plates with elastic force acting between them to either the solution of the contact problem for two plates with no elastic force acting between them, or the solution to the equilibrium problem for a two-layered plate as μ tends to zero or infinity respectively.

References

[1] A.M. Khludnev, *On bending an elastic plate with a delaminated thin rigid inclusion*, J. Appl. Ind. Math., **5**:4 (2011), 582–594. Zbl 1230.74111

[2] A.M. Khludnev, *Shape control of thin rigid inclusions and cracks in elastic bodies*, Arch. Appl. Mech., **83**:10 (2013), 1493–1509. Zbl 1293.74136

[3] V.V. Shcherbakov, *Existence of an optimal shape for thin rigid inclusions in the Kirchhoff–Love plate*, J. Appl. Ind. Math., **8**:1 (2014), 97–105. Zbl 1340.74087

[4] E.M. Rudoy, *Shape sensitivity analysis of an equilibrium problem for bodies with a thin rigid inclusion*, J. Math. Sci., **211**:6 (2015), 847–862. Zbl 1349.74150

[5] T.S. Popova, *The equilibrium problem for a viscoelastic body with a thin rigid inclusion*, Mat. Zamet. SVFU, **21**:1 (2014), 47–55. Zbl 1340.49012

[6] N.P. Lazarev, S. Das, M.P. Grigoryev, *Optimal control of a thin rigid stiffener for a model describing equilibrium of a Timoshenko plate with a crack*, Sib. Electr. Math. Izv., **15** (2018), 1485–1497. Zbl 1414.49010

[7] N. Lazarev, G. Semenova, *An optimal size of a rigid thin stiffener reinforcing an elastic two-dimensional body on the outer edge*, J. Optim. Theory Appl., **178**:2 (2018), 614–626. Zbl 1401.49012

[8] H. Itou, A.M. Khludnev, E.M. Rudoy, A. Tani, *Asymptotic behaviour at a tip of a rigid line inclusion in linearized elasticity*, Z. Angew. Math. Mech., **92**:9 (2012), 716–730. Zbl 1348.74041

[9] E.M. Rudoy, *On numerical solving a rigid inclusions problem in 2D elasticity*, Z. Angew. Math. Phys., **68**:1, Paper No. 19 (2017), Zbl 06735700

[10] N.A. Kazarinov, E.M. Rudoy, V.Yu. Slesarenko, V.V. Shcherbakov, *Mathematical and numerical simulation of an elastic body reinforced by a thin elastic inclusion*, Comp. Math. Math. Phys., **58**:5 (2018), 761–774. Zbl 06920540

[11] G.Ya. Popov, *Concentration of elastic stresses near punches, cuts, thin inclusions and stiffeners*, Nauka, Moscow, 1982. [in Russian]

[12] L.T. Berezhnitskiy, V.V. Panasyuk, N.G. Stashchuk, *Interaction of hard linear inclusions and cracks in a deformable body*, Naukova dumka, Kiyev, 1983.

[13] I.I. Il'ina, V.V. Sil'vestrov, *The problem of a thin interfacial inclusion detached from the medium along one side*, Mech. Solids, **40**:3 (2005), 123–133.

[14] V.V. Sil'vestrov, Yu.O. Vasil'eva, *Problem of a thin rigid inclusion inserted in an interfacial crack in the vicinity of its tip*, J. Appl. Mech. Tech. Phys., **56**:3 (2015), 510–518. Zbl 1336.74062

[15] V.V. Sil'vestrov, I.A. Ivanov, *A packet of elastic plates joined along a periodic system of segments*, J. Appl. Mech. Tech. Phys., **42**:4 (2001), 731–736. Zbl 1001.74072

[16] A.M. Khludnev, *Problem on the contact of two elastic plates*, J. Appl. Math. Mech., **47**:1 (1984), 110–115. Zbl 0535.73091

[17] A.M. Khludnev, *On modelling elastic bodies with defects*, Sib. Elektron. Mat. Izv., **15** (2018), 153–166. Zbl 1390.35354

[18] I.V. Fankina, *On equilibrium problem for a two-layer structure in the presence of a defect*, Sib. Elektron. Mat. Izv., **16** (2019), 959–974. Zbl 1428.35574

[19] Y. Beneveniste, T. Miloh, *Imperfect soft and stiff interfaces in two-dimensional elasticity*, Mech. of Materials., **33** 2001, 309–323.

[20] E.M. Rudoy, *Asymptotic modelling of bonded plates by soft thin adhesive layer*, Sib. Elektron. Mat. Izv., **17** (2020), 615–625. Zbl 1434.74078

[21] A.I. Furtsev, E.M. Rudoy, *Variational approach to modelling soft and stiff interfaces in the Kirchhoff — Love theory of plates*, I. J. Sol. Str., **202** (2020), 562–572.

[22] V.V. Bolotin, Yu.N. Novichkov, *Mechanics of Multilayered Structures*, Mashinostroenie, Moscow, 1980.

[23] A.R. Rzhanitsyn, *Built-up Bars and Plates*, Stroiizdat, Moscow, 1986.

[24] Yu.I. Dimitrienko, *Asymptotic theory of multylayer thin plates*, Vestnik Moskov.Gos. Univ. im. Baumana Ser. Estestv Nauki, **3** (2012), 86–99.

[25] Yu.I. Dimitrienko, E.A. Gubareva, Yu.V. Yurin, *Variational equations of asymptotic theory of multylayer thin plates*, Vestnik Moskov.Gos. Univ. im. Baumana Ser. Estestv Nauki, **4** (2015), 67–87.

[26] E.M. Rudoy, N.A. Kazarinov, V.Yu. Slesarenko, *Numerical simulation of equilibrium of an elastic two-layer structure with a through crack*, Numer. Analys. Appl., **10**:1 (2017), 63–73.

[27] E.V. Pyatkina, *A problem of glueing of two Kirchhoff—Love plate*, Sib. Elektron. Mat. Izv., **16** (2019), 1351–1374. Zbl 1428.35580

[28] A.M. Khludnev, *Elasticity Theory Problems in Nonsmooth Domains*, Fizmatlit, Moscow, 2010.

[29] A.M. Khludnev, V.A. Kovtunenko, *Analysis of Cracks in Solids*, WIT-Press, Boston, Southampton, 2000.

Evdokia Vladimirovna Pyatkina
Lavrentyev Institute of Hydrodynamics,
15, acad. Lavrentyeva ave.,
Novosibirsk, 630090, Russia
Email address: dusya_pyatkina@mail.ru