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ON THE GENERIC EXISTENTIAL THEORY OF FINITE GRAPHS

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Abstract. Finite graphs are the most important mathematical objects that are used for solving many practical problems of optimization, computer science, modeling. Many such problems can be formulated as problems related with solving systems of equations over graphs, which lead to the need for the development of algebraic geometry. Algebraic geometry over such objects is closely related to properties of existential theories. From a practical point of view, the most important questions concern decidability and computational complexity of these theories. Generic (existential) theory consists of all (existential) statements which are true for almost all graphs. Classical 0-1 law for graphs implies that generic theory of finite graphs is decidable, while the classical elementary theory of graphs is undecidable. In this article we study the generic existential theory of finite graphs. We describe this theory as the set of all existential statements that are consistent with the theory of graphs. We prove that this theory is NP-complete. This means that there are no polynomial algorithms that recognize this theory, provided the inequality of classes P and NP.

Keywords: graphs, generic theory.

1. Introduction

Due to the rapid development of computer technology and applied mathematics in the 20th century, studies concerning finite combinatoric and algebraic structures moved to the center of the stage. One of the most important classes of such objects is the class of finite graphs. Graph theory is widely used while solving practical problems connected with networks, routes, object classification, etc. While

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algebraic and combinatorial methods of studying finite graphs can be considered classical, a new method, connected with logic and model theory, had been created as a part of so-called universal algebraic geometry [3]. Many important problems related to finite graphs can be stated as problems of solving systems of equations on graphs, providing a need for development of algebraic geometry. Moreover, algebraic geometry over finite graphs is heavily connected with properties of existential theories.

From a practical point of view, the most important questions concern decidability and computational complexity of these theories. In [4], it had been shown that the elementary theory of the class of connected graphs is undecidable. However, the existential theory of this class is decidable in polynomial time (since it is equal to the existential theory of a graph with only one vertex). In [5], a notion of generic theory of the class of algebraic systems as a set of all sentences, which are true for "almost all" algebraic systems of a given signature, where the notion of "almost all" can be formalized by introducing a canonical measure on the set of all models. A generic theory, in a sense, approximates the original elementary theory and is usually simpler. For instance, the classical 0-1 law for graphs (see [2]) implies that the generic theory of a class of finite graphs is decidable.

In this paper we consider a generic existential theory of the class of finite graphs. We describe this theory as a set of all existential sentences which are true in the theory of finite graphs. We also show that the recognition problem of this theory is NP-complete. This means that, provided that P $\neq$ NP, there exists no polynomial algorithm, which recognizes this theory.

2. Preliminaries

Recall the following notions from graph theory. A graph G is a pair (V, E), where V is a set of vertices and $E \subseteq V \times V$ is a set of edges of G. We will consider directed graphs with loops, however results, similar to the ones in our paper can be easily obtained for undirected graphs. For every set A by $|A|$ denote its cardinality. Two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are called *isomorphic* if $|V_1| = |V_2|$, $|E_1| = |E_2|$ and there exists a bijection $f : V_1 \to V_2$, such that $(u, v) \in E_1$ iff $(f(u), f(v)) \in E_2$. In that case, we write $G \cong H$. The subgraph isomorphism problem can be stated as follows: Given two graphs $G = (V, E)$ and $H = (V', E')$, it is required to determine whether there exists a subgraph $G_0 = (V_0, E_0)$, where $V_0 \subseteq V$ and $E_0 = (V_0 \times V_0) \cap E$, such that $G_0 \cong H$. This problem is known to be NP-complete [1].

For every finite graph $G = (V, E)$, consider a finite algebraic structure $\mathfrak{A}_G = \langle V, \sigma_{\mathcal{G}} \rangle$, with a set V of vertices of G as its domain and such that its signature $\sigma_{\mathcal{G}} = \{E^{(2)}, =\}$ consists of two binary relation symbols: of a vertex adjacency relation $E(x, y)$ and equality relation. The symbol $E^{(2)}$ in the algebraic structure $\mathfrak{A}_G$ is interpreted as follows: for all $v_1, v_2 \in V$ it is true that $E(v_1, v_2) \Leftrightarrow (v_1, v_2) \in E$. From now on, we will identify a finite graph G and the corresponding algebraic structure $\mathfrak{A}_G$. It is easy to see, that for two isomorphic graphs G and H, the structures $\mathfrak{A}_G$ and $\mathfrak{A}_H$ are also isomorphic. Moreover, if H is a subgraph of G, then the structure $\mathfrak{A}_H$ is a substructure of $\mathfrak{A}_G$.

For convenience, we consider finite graphs with $\{1, 2, \dots, n\}$, $n \in \mathbb{N}$ as their sets of vertices. By $\mathcal{G}$ denote a class of algebraic systems $\mathfrak{A}_G$ corresponding to all such finite graphs G. We will call the number of vertices in a graph G the size of G

(and also the size of the corresponding algebraic structure $\mathfrak{A}_G$). By $\mathcal{G}_n$ denote the set of graphs of size n, i.e. the set of all finite graphs with $\{1, 2, \ldots, n\}$ as a set of their vertices. Since every directed finite graph can be represented by an adjacency matrix, we obtain that $|\mathcal{G}_n| = 2^{n^2}$. For every $S \subseteq \mathcal{G}$ define the following sequence

$$\rho_n(S) = \frac{|S_n|}{|\mathcal{G}_n|}, \ n = 1, 2, 3, \ldots,$$

where $S_n = S \cap \mathcal{G}_n$ is a set of graphs from S of size n. By *asymptotic density* of S we mean a limit

$$\rho(S) = \lim_{n\to\infty} \rho_n(S),$$

given such limit exists. A set S is said to be *generic*, if $\rho(S) = 1$ and *negligible* if $\rho(S) = 0$. It is easy to see that S is generic iff $\mathcal{G} \setminus S$ is neglectable.

A *generic theory* for the class of finite graphs $GTh(\mathcal{G})$ is the set of all sentences (closed formulae) Φ in a signature $\sigma_{\mathcal{G}}$, such that

$$\{\mathfrak{A}_G : \mathfrak{A}_G \models \Phi\}$$

is a generic set. A *generic existential theory* of the class of finite graphs $GTh_\exists(\mathcal{G})$ is the set of all $\exists$-sentences of $GTh(\mathcal{G})$.

3. Description of the theory

Lemma 1. *Consider a finite graph G. Then the set of finite graphs containing a subgraph isomorphic to G is generic.*

Proof. We will show that the set $\mathcal{A}$ of all finite graphs which do not contain a subgraph isomorphic to G is negligible. Assume that G has m vertices $v_1, \ldots, v_m$. Consider a set B whose elements are graphs H with n vertices $w_1, \ldots, w_n$, such that a mapping φ_k, defined by $\varphi_k(w_{km+j}) = v_j$, $j = 1, \ldots, m$, for every induced subgraph of H with $\{w_{km+1}, \ldots, w_{km+m}\}$, $k = 0, \ldots, [n/m] - 1$ being the vertex set of that subgraph, is not an isomorpishm. It is obvious that $\mathcal{A} \subseteq \mathcal{B}$.

It can be calculated that

$$|\mathcal{B}_n| = 2^{n^2 - [\frac{n}{m}]m^2}(2^{m^2} - 1)^{[\frac{n}{m}]}.$$

This equality follows from the fact hat there are $[n/m]$ "prohibited" matrices equal to the adjacency matrix of G which one can never encounter on the diagonals of adjacency matrices of graphs from B.

Therefore,

$$\rho(\mathcal{B}) = \lim_{n\to\infty} \frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} = \lim_{n\to\infty} \frac{2^{n^2 - [\frac{n}{m}]m^2}(2^{m^2} - 1)^{[\frac{n}{m}]}}{2^{n^2}} =$$

$$= \lim_{n\to\infty} \frac{(2^{m^2} - 1)^{[n/m]}}{2^{[n/m]m^2}} = \lim_{n\to\infty} \Big(1 - \frac{1}{2^{m^2}}\Big)^{[n/m]} = 0.$$

This shows that $\mathcal{B}$, hence its subset $\mathcal{A}$ is also negligible. □

Theorem 1. *The generic existential theory of finite graphs consists of existnetial sentences, which are true in the theory of graphs, i.e.*

$$GTh_\exists(\mathcal{G}) = \{\Phi : \Phi \textit{ is an existential sentence and } \exists G \in \mathcal{G}\ \mathfrak{A}_G \models \Phi\}.$$

Proof. Let $\Phi \in GTh_\exists(\mathcal{G})$. Then, since $\{\mathfrak{A}_G : \mathfrak{A}_G \models \Phi\}$ is generic, there exists a graph G, such that $\mathfrak{A}_G \models \Phi$.

Conversely, assume that there exists G, such that $\mathfrak{A}_G \models \Phi$. Then consider all graphs K, which contain a subgraph isomorphic to G. Since $\mathfrak{A}_G$ is a substructure $\mathfrak{A}_K$ for all K and Φ is an existential sentence, it is true that $\mathfrak{A}_K \models \Phi$. Therefore, the set of all such graphs is generic. Hence $\Phi \in GTh_\exists(\mathcal{G})$. □

4. NP-completeness of the theory

For every finite graph $G = (V, E)$ construct a quantifier-free formula φ_G in variables $x_1, \ldots, x_n$, where $n = |V|$, as follows

$$\varphi_G(x_1, \ldots, x_n) = \bigwedge_{0 \le i < j \le n} (x_i \neq x_j) \wedge \bigwedge_{(i,j) \in E} E(x_i, x_j) \wedge \bigwedge_{(i,j) \notin E} \neg E(x_j, x_j).$$

Let G, H be finite graphs with n and m vertices respectively. Define the following existential sentence:

$$\Psi_{G,H} = \exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_m \ \big(\varphi_G(x_1, \ldots, x_n) \wedge \varphi_H(y_1, \ldots, y_m) \wedge$$
$$\wedge \bigwedge_{i=1,\ldots,m} (y_i = x_1 \vee \ldots \vee y_i = x_n)\big).$$

Lemma 2. *Consider two finite graphs G and H. Then H is isomorphic to some subgraph of G iff there exists a graph K, such that $\mathfrak{A}_K \models \Psi_{G,H}$.*

Proof. Assume that $\mathfrak{A}_K \models \Psi_{G,H}$ for some graph K. This means that the formula $\varphi_G(v_1, \ldots, v_n)$ is true for some vertices $v_1, \ldots, v_n$ of K and the formula $\varphi_H(w_1, \ldots, w_m)$ is true for some vertices $w_1, \ldots, w_m$ of K, therein $\{w_1, \ldots, w_m\} \subseteq \{v_1, \ldots, v_n\}$. It is now obvious that the mapping between vertices $w_1, \ldots, w_m$ and corresponding vertices from $v_1, \ldots, v_n$ is an isomorphism between H and a subgraph in G.

Now assume that H is isomorphic to some subgraph of G. Then we can take G as the aforementioned graph K. Indeed, one can substitute variables $y_1, \ldots, y_m$ with the corresponding variables from the set $x_1, \ldots, x_n$ in such a way that a quantifier-free part of the sentence $\Psi_{G,H}$ only contains the formula $\varphi_G(x_1, \ldots, x_n)$, while other subformulas will be consumed by it. As a result, we obtain the sentence

$$\Phi_G = \exists x_1 \ldots \exists x_n \varphi_G(x_1, \ldots, x_n).$$

From the definition of φ_G, it follows that $\mathfrak{A}_G \models \Phi_G$. Hence $\mathfrak{A}_G \models \Psi_{G,H}$. □

Theorem 2. *The generic existential theory of finite graphs is NP-complete.*

Proof. By Theorem 1, a theory $GTh_\exists(\mathcal{G})$ is equal to the set

$$T = \{\Phi : \Phi \text{ is an existential sentence and } \exists G \in \mathcal{G}\ \mathfrak{A}_G \models \Phi\}.$$

Hence we only need to show that T is NP-complete.

We will first show, that $T \in$ NP. We need to find a predicate $A(x, y)$ such that $A(x, y)$ is computable in polynomial time and for every existential sentence $\Phi = \exists x_1 \ldots \exists x_n \varphi(x_1, \ldots, x_n)$ in signature σ_G, it is true that

$$\Phi \in T \Leftrightarrow \exists y :\ |y| < |\Phi|^2,\ A(\Phi, y).$$

Some graph G, represented via its adjacency matrix, will serve as a hint (verificator??) for y and $A(x, y)$ will check, whether $\mathfrak{A}_G \models \varphi(v_1, \ldots, v_n)$.

Indeed, if $\Phi \in T$, then it is true that $\mathfrak{A}_K \models \Phi$ for some graph K, i.e. there exist vertices $v_1, \ldots, v_n$ in K, such that $\mathfrak{A}_K \models \varphi(v_1, \ldots, v_n)$. Get rid of all vertices in

K apart from $v_1, \dots, v_n$ and all edges apart from those which are not incident to any vertex of the set $\{v_1, \dots, v_n\}$. We have obtained G. It is obvious that $\mathfrak{A}_G \models \varphi(v_1, \dots, v_n)$ and that $A(\Phi, G)$.

Converesely, consider a graph G with n vertices, such that $A(\Phi, G)$, i.e. $\mathfrak{A}_G \models \varphi(v_1, \dots, v_n)$, hence $\mathfrak{A}_G \models \Phi$. Then $\Phi \in T$.

Now we will prove polynomial reducibilty of an NP-complete subgraph isomorphism problem to T. We will define a reduction f between these two problems as follows. If (G, H) is a pair of graphs, represented by their adjacency matrices, we set $f(G, H) = \Psi_{G,H}$. By Lemma 2, H is isomorphic to some subgraph of G iff $f(G, H) = \Psi_{G,H} \in T$. Since the size of $\Psi_{G,H}$ is polynomially bounded by the number of vertices and edges of graphs G and H, it follows that f is computable in polynomial time. $\square$

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