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A CLASS OF PLANAR DIFFERENTIAL SYSTEMS WITH EXPLICIT EXPRESSION FOR TWO LIMIT CYCLES

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Abstract. The existence of limit cycles is interesting and very important in applications. It is a key to understand the dynamic of polynomial differential systems. The aim of this paper is to investigate a class of a multi-parameter planar polynomial differential systems. Under some suitable conditions, the existence of two limit cycles, these limit cycles are explicitly given. Some examples are presented in order to illustrate the applicability of our results. algebras.

Keywords: limit cycle, Riccati equation, invariant algebraic curve, first integral.

1. Introduction

One of the main problems in the qualitative theory of differential equations [8, 14] is the study of the limit cycles of planar differential systems and specially of the planar polynomial differential systems of the form

$$(1)\qquad \begin{cases} x' = \dfrac{dx}{dt} = P(x,y), \\[2ex] y' = \dfrac{dy}{dt} = Q(x,y), \end{cases}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y . Let n be the maximum degree of the polynomials P and Q, we say that system (1) is of

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degree n. In the literature equivalent mathematical objects to refer to these planar differential systems appear as a vector field

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

A limit cycle of system (1) is an isolated periodic solution in the set of all its periodic solutions of system (1), and it is said to be algebraic if it is contained in the zero level set of a polynomial function [1, 12] . In 1900 Hilbert [11] in the second part of his 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane $\mathbb{R}^2$, even more difficult problem is to give an explicit expression of them [2, 4, 16]. This was one of the main problems in the qualitative theory of planar differential equations in the 20th century. In [3, 10] examples of explicit limit cycles which are not algebraic are given. To distinguish when a limit cycle is algebraic or not, usually, it is not easy. Thus, the well known limit cycle of the van der Pol differential system exhibited in 1926, was not proved until 1995 by Odani [13] that it was not algebraic. The van der Pol system can be written as a polynomial system (1) of degree 3, but its limit cycle is not known explicitly.

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x}P(x,y) + \frac{\partial H(x,y)}{\partial y}Q(x,y) \equiv 0 \ \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant.

Since for such vector fields the notion of integrability is based on the existence of a first integral [5, 6, 9], the following question arises: Given the polynomial differential system (1), how to recognize if this polynomial differential system has a first integral? and how to compute it when it exists?

Given a system (1) we define an invariant algebraic curve as an algebraic curve $U(x,y) = 0$, where $U(x,y)$ is a polynomial with real coefficients, such that:

$$\frac{\partial U(x,y)}{\partial x}P(x,y) + \frac{\partial U(x,y)}{\partial y}Q(x,y) = K(x,y)U(x,y). \tag{2}$$

where $K(x,y)$ is a polynomial called the cofactor of $U(x,y)$. It is easy to see that if P and Q are polynomials of degree at most n, then the cofactor is of degree at most $n-1$, if the cofactor is identically zero, then $U(x,y)$ is a polynomial first integral for system (1). The corresponding cofactor of $U(x,y)$ is always polynomial whether $U(x,y)$ is algebraic or non-algebraic. If $U(x,y)$ is real, the curve $U(x,y) = 0$ is an invariant under the flow of differential system (1) and the set

$$\left\{(x,y) \in \mathbb{R}^2, U(x,y) = 0\right\}$$

is formed by orbits of system (1). Several authors have remarked the importance of an invariant algebraic curve to understand the dynamics of a system; we refer to [7, 15] an exhaustive survey on this topic. There are strong relationships between the integrability of the system (1) and its number of invariant algebraic curves. The importance of the invariant algebraic curve to understand the dynamics of a system (1) .

In this paper we give an explicit expression of invariant algebraic curves,then we prove that these systems are integrable and we introduce an explicit expression of a first integral of a multi-parameter planar polynomial differential system of degree nine of the form

$$\left\{\begin{aligned} x' &= \frac{dx}{dt} = xS_4(x,y) + P_7(x,y) + xR_8(x,y), \\ y' &= \frac{dy}{dt} = yS_4(x,y) + Q_7(x,y) + yR_8(x,y), \end{aligned}\right. \tag{3}$$

where

$$\begin{aligned} P_7(x,y) &= \frac{1}{3}\left(x^2+y^2\right)^2\left((2a-b)\,x^3 + (15d-6c)\,x^2y + (2b-a)\,xy^2 + (6d-3c)\,y^3\right), \\ Q_7(x,y) &= -\frac{1}{3}\left(x^2+y^2\right)^2\left((6d-3c)\,x^3 + (b-2a)\,x^2y - 3dxy^2 + (a-2b)\,y^3\right), \\ S_4(x,y) &= \alpha x^4 + \lambda x^3y + \delta x^2y^2 + \lambda xy^3 + \eta y^4, \\ R_8(x,y) &= -\frac{1}{3}\left(x^2+y^2\right)^2((3\alpha+2a-b)\,x^4 + (3\lambda-3c+9d)\,x^3y + (3\lambda-3c+9d)\,xy^3 \\ &\quad + (a+b+3\delta)\,x^2y^2 + (2b-a+3\eta)\,y^4), \end{aligned}$$

in which a, b, c, d, α, λ, δ, λ, η are real constants.

Moreover, we determine sufficient conditions for a polynomial differential system to possess two limit cycles, explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

We define the trigonometric functions

$$\begin{aligned} F(\theta) &= \frac{1}{8}(3\alpha+\delta+3\eta) + \frac{1}{2}\lambda\sin 2\theta + \frac{1}{2}(\alpha-\eta)\cos 2\theta + \frac{1}{8}(\alpha-\delta+\eta)\cos 4\theta, \\ G(\theta) &= \frac{1}{6}(a+b) + \frac{1}{2}(a-b)\cos 2\theta + \frac{1}{2}(3d-c)\sin 2\theta, \\ K(\theta) &= -\frac{1}{6}a - \frac{1}{6}b - \frac{3}{8}\alpha - \frac{1}{8}\delta - \frac{3}{8}\eta + \frac{1}{2}(c-3d-\lambda)\sin 2\theta + \frac{1}{8}(\delta-\alpha-\eta)\cos 4\theta \\ &\quad + \frac{1}{2}(b-a-\alpha+\eta)\cos 2\theta, \\ M(\theta) &= \int_0^\theta \left(\frac{2K(t)}{2d-c}\exp\left(\int_0^t \left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt, \end{aligned}$$

and $N(\theta) = \exp\left(\int_0^\theta \left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)$.

2. Main result

Our main result is contained in the following Theorem.

Theorem 1. *Consider a multi-parameter planar polynomial differential system* (3)*, then the following statements hold.*

1) If $2d-c\neq 0$, then the origin of coordinates $O(0,0)$ is the unique critical point at finite distance.

2) The curve $U(x,y) = x^2+y^2-1$, is an invariant algebraic curve of system (3) *with cofactor*

$$K(x,y) =$$

$$-\frac{2}{3}\left(x^2+y^2\right)\left((2a-b+3\alpha)x^6+3\alpha x^4+3\eta y^4+(9d-3c+3\lambda)xy\left(x^2+y^2\right)^2\right.$$
$$\left.+3x^2y^2\left((a+\alpha+\delta)x^2+(b+\delta+\eta)y^2+\delta\right)+3\lambda xy\left(x^2+y^2\right)+(2b-a+3\eta)y^6\right).$$

3) The system (3) *has the first integral*

$$H(x,y) = \frac{N\left(\arctan\frac{y}{x}\right)+\left(1-x^2-y^2\right)M\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1}.$$

4) The system (3) *has an explicit limit cycle, given in Cartesian coordinates by* (Γ_1)*:* $x^2+y^2-1=0$.

5) If

$$\tfrac{2}{3}a+\tfrac{2}{3}b+3\alpha+\delta+3\eta > 2\left|c-3d-2\lambda\right|+2\left|b-a-2\alpha+2\eta\right|+\left|\delta-\alpha-\eta\right|,$$

$$-\tfrac{1}{3}a-\tfrac{1}{3}b-\tfrac{3}{4}\alpha-\tfrac{1}{4}\delta-\tfrac{3}{4}\eta > \left|c-3d-\lambda\right|+\tfrac{1}{4}\left|\delta-\alpha-\eta\right|+\left|b-a-\alpha+\eta\right|, \tag{4}$$

$$\delta \neq \alpha+\eta \text{ and } c<2d,$$

then the system (3) *has another limit cycle* (Γ_2)*, explicitly given in polar coordinates* (r,θ) *by*

$$r(\theta,r_*) = \sqrt{\frac{(N(2\pi)-1)(N(\theta)+M(\theta))+M(2\pi)}{(N(2\pi)-1)M(\theta)+M(2\pi)}}.$$

Moreover, the limit cycle (Γ_1) *lies inside the limit cycle* (Γ_2).

Proof.

Proof of statement (1).

By definition, $A(x_*,y_*)\in\mathbb{R}^2$ is a critical point of system (3) if

$$\begin{cases} x_*S_4(x_*,y_*)+P_7(x_*,y_*)+x_*R_8(x_*,y_*)=0,\\ \\ y_*S_4(x_*,y_*)+Q_7(x_*,y_*)+y_*R_8(x_*,y_*)=0,\end{cases}$$

we have $y_*P_7(x_*,y_*)-x_*Q_7(x_*,y_*)=(2d-c)\left(x_*^2+y_*^2\right)^4=0$. According to the condition $2d-c\neq 0$, then $x_*=0, y_*=0$ is the unique of this equation. Thus the origin is the unique critical point at finite distance.

This completes the proof of statement (1) of Theorem 1.

Proof of statement (2).

A computation shows that $U(x,y) = x^2+y^2-1$ satisfies the linear partial differential equation (2), the associated cofactor being

$$K(x,y) = -\frac{2}{3}\left(x^2+y^2\right)\left((2a-b+3\alpha)x^6+3\alpha x^4+3\eta y^4\right.$$
$$+(9d-3c+3\lambda)xy\left(x^2+y^2\right)^2$$
$$\left.+3x^2y^2\left((a+\alpha+\delta)x^2+(b+\delta+\eta)y^2+\delta\right)+3\lambda xy\left(x^2+y^2\right)+(2b-a+3\eta)y^6\right),$$

then the curve $U(x,y)=0$ is an invariant algebraic curve of system (3) with cofactor $K(x,y)$.

This completes the proof of statement (2) of Theorem 1.

Proof of statements (3), (4) and (5) of Theorem 1.

In order to prove our results (3), (4) and (5) we write the polynomial differential system (3) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the system becomes

$$(5) \qquad \begin{cases} r' = \dfrac{dr}{dt} = F(\theta)\, r^5 + G(\theta)\, r^7 + K(\theta)\, r^9, \\ \\ \theta' = \dfrac{d\theta}{dt} = (c - 2d)\, r^6. \end{cases}$$

where the trigonometric functions $F(\theta)$, $G(\theta)$ and $K(\theta)$ are given in introduction.

According to $c < 2d$, we get θ' is negative for all $t \in \mathbb{R}$, the orbits $(r(t), \theta(t))$ of system (5) have the opposite orientation with respect to those $(x(t), y(t))$ of system (3).

Taking θ as an independent variable, we obtain the equation

$$(6) \qquad \frac{dr}{d\theta} = \frac{F(\theta)}{c - 2d}\frac{1}{r} + \frac{G(\theta)}{c - 2d} r + \frac{K(\theta)}{c - 2d} r^3.$$

Via the change of variables $\rho = r^2$, this equation (6) is transformed into the Riccati equation

$$(7) \qquad \frac{d\rho}{d\theta} = \frac{2F(\theta)}{c - 2d} + \frac{2G(\theta)}{c - 2d}\rho + \frac{2K(\theta)}{c - 2d}\rho^2.$$

This equation is integrable, since it possesses the particular solution $\rho = 1$.

By introducing the standard change of variables $\rho = z+1$ we obtain the Bernoulli equation

$$(8) \qquad \frac{dz}{d\theta} = \left(\frac{2G(\theta) + 4K(\theta)}{c - 2d}\right) z + \frac{2K(\theta)}{c - 2d} z^2.$$

We note that $z = 0$ is solutions for (8), assume now that $z \neq 0$ by introducing the standard change of variables $y = \dfrac{1}{z}$ we obtain the linear equation

$$(9) \qquad \frac{dy}{d\theta} = \left(\frac{2G(\theta) + 4K(\theta)}{2d - c}\right) y + \frac{2K(\theta)}{2d - c}.$$

The general solution of linear equation (9) is

$$y(\theta) = \frac{\mu + M(\theta)}{N(\theta)},$$

where $\mu \in \mathbb{R}$.

Consequently, the general solution of equation (8) is

$$z(\theta) = 0, \quad z(\theta) = \frac{N(\theta)}{\mu + M(\theta)},$$

where $\mu \in \mathbb{R}$.

Then the general solution of equation (7) is

$$\rho(\theta) = 1, \quad \rho(\theta) = \frac{\mu + N(\theta) + M(\theta)}{\mu + M(\theta)},$$

where $\mu \in \mathbb{R}$.

Consequently, the general solution of (6) is

$$r(\theta, \mu) = 1, \quad r(\theta, \mu) = \left(\frac{\mu + N(\theta) + M(\theta)}{\mu + M(\theta)}\right)^{\frac{1}{2}},$$

where $\mu \in \mathbb{R}$.

From this solution we obtain a first integral in the variables (x, y) of the form

$$H(x,y) = \frac{N\left(\arctan\frac{y}{x}\right) + \left(1 - x^2 - y^2\right) M\left(\arctan\frac{y}{x}\right)}{x^2 + y^2 - 1}.$$

Hence, statement (3) of Theorem 1 is proved.

The curves $H = \mu$ with $\mu \in \mathbb{R}$, which are formed by trajectories of the differential system (3), in Cartesian coordinates are written as

$$\begin{aligned} x^2 + y^2 &= 1, \\ x^2 + y^2 &= \frac{\mu + N\left(\arctan\frac{y}{x}\right) + M\left(\arctan\frac{y}{x}\right)}{\mu + M\left(\arctan\frac{y}{x}\right)}, \end{aligned}$$

where $\mu \in \mathbb{R}$.

Notice that system (3) has a periodic orbit if and only if equation (6) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (6) that fulfills $r(0, r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$.

The solution $r(\theta, r_0)$ of the differential equation (6) such that $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = \sqrt{\frac{N(\theta) + M(\theta) + \frac{1}{-1+r_0^2}}{M(\theta) + \frac{1}{-1+r_0^2}}}$$

where $r_0 = r(0)$.

We have the particular solution $\rho(\theta) = 1$ of the differential equation (6), from this solution we obtain $r^2(\theta, 1) = 1 > 0$, for all $\theta \in [0, \pi]$ is a particular solution of the differential equation (6). This is the limit cycle for the differential systems (3), corresponding of course to an invariant algebraic curve $U(x, y) = x^2 + y^2 - 1 = 0$.

More precisely, in Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, the curve (Γ_1) defined by this limit cycle is (Γ_1): $x^2 + y^2 - 1 = 0$.

Hence, statement (4) of Theorem 1 is proved.

A periodic solution of system (3) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$, which leads to unique value $r_0 = r_*$, given by

$$r_* = \sqrt{\frac{N(2\pi) + M(2\pi) - 1}{M(2\pi)}},$$

r_* is the intersection of the periodic orbit with the OX_+ axis.

After the substitution of this value of r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \sqrt{\frac{(N(2\pi) - 1)(N(\theta) + M(\theta)) + M(2\pi)}{(N(2\pi) - 1) M(\theta) + M(2\pi)}}.$$

In what follows it is proved that $r(\theta, r_*) > 0$. Indeed

$$\begin{aligned} M(2\pi) - M(\theta) &= \int_0^{2\pi} \left(\frac{2K(t)}{2d-c} \exp\left(\int_0^t \left(\frac{2G(w) + 4K(w)}{c - 2d}\right) dw\right)\right) dt \\ &\quad + \int_\theta^0 \left(\frac{2K(t)}{2d-c} \exp\left(\int_0^t \left(\frac{2G(w) + 4K(w)}{c - 2d}\right) dw\right)\right) dt \\ &= \int_\theta^{2\pi} \left(\frac{2K(t)}{2d-c} \exp\left(\int_0^t \left(\frac{2G(w) + 4K(w)}{c - 2d}\right) dw\right)\right) dt \end{aligned}$$

According to the conditions (4), hence $\dfrac{G(\theta)+2K(\theta)}{2d-c}<0$ and $\dfrac{K(\theta)}{2d-c}>0$ for all $\theta\in(0,\pi)$, then we have $M(2\pi)-M(\theta)>0$ and $N(2\pi)>1$, this ensures that r_* and $r(\theta,r_*)$ are well defined for all $\theta\in(0,\pi)$, therefore we have $r_*>0$ and $r(\theta,r_*)>0$ for all $\theta\in[0,\pi]$ and the limit cycle do not pass through the equilibrium point $O(0,0)$ of system (3). This is the second limit cycle for the differential system (3), we note it by (Γ_2).

According to the conditions (4), we get

$$\begin{aligned} M(\theta) &= \int_0^\theta \left(\frac{2K(t)}{2d-c}\exp\left(\int_0^t \left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)\right)dt>0 \\ \text{and } N(\theta) &= \exp\left(\int_0^\theta \left(\frac{2G(w)+4K(w)}{c-2d}\right)dw\right)>1, \end{aligned}$$

for all $\theta\in[0,\pi]$, then we have $r_*=\sqrt{1+\frac{N(2\pi)-1}{M(2\pi)}}>1$. Moreover, $r(\theta,r_*)=\sqrt{1+\dfrac{(N(2\pi)-1)N(\theta)}{(N(2\pi)-1)M(\theta)+M(2\pi)}}>1$, this justified that the limit cycle (Γ_1) lies inside the limit cycle (Γ_2).

We conclude that system (3) has two limit cycles (Γ_1) and (Γ_2).

This completes the proof of statement (5) of Theorem 1. □

3. Example

The following examples are given to illustrate our result.

Example 1 If we take $a=b=-50, c=-3, d=-1, \alpha=\eta=10, \lambda=1$ and $\delta=28$, then system (3) reads

$$(10)\quad \begin{cases} x'=x\left(10x^4+x^3y+28x^2y^2+xy^3+10y^4\right)+\frac{1}{3}(3y-50x)\left(x^2+y^2\right)^3 \\ -\frac{1}{3}x\left(x^2+y^2\right)^2\left(-20x^4+3x^3y-16x^2y^2+3xy^3-20y^4\right), \\ \\ y'=y\left(10x^4+x^3y+28x^2y^2+xy^3+10y^4\right)-\frac{1}{3}(3x+50y)\left(x^2+y^2\right)^3 \\ -\frac{1}{3}y\left(x^2+y^2\right)^2\left(-20x^4+3x^3y-16x^2y^2+3xy^3-20y^4\right), \end{cases}$$

The curve $x^2+y^2-1=0$ is an invariant algebraic curve of system (10) with cofactor

$$\begin{aligned} K(x,y) &= 2\left(x^2+y^2\right)(8\left(x^6+y^6\right)-12\left(x^4+y^4\right)+3xy\left(x^4+x^2+y^2+y^4\right) \\ &\quad +2x^2y^2\left(11x^2+3xy+11y^2-13\right)). \end{aligned}$$

The system (10) has the first integral

$$H(x,y)=\frac{N\left(\arctan\frac{y}{x}\right)+\left(1-x^2-y^2\right)M\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1},$$

where $N(\theta)=\exp\left(1+\frac{32}{3}\theta-\cos 2\theta-\sin 4\theta\right)$ and

$M(\theta)=-\frac{1}{2}\exp\left(\frac{32}{3}\theta-\cos 2\theta-\sin 4\theta\right)+\frac{50e}{3}\int_0^\theta \exp\left(\frac{32}{3}t-\cos 2t-\sin 4t\right)dt.$

The system (10) has the limit cycle (Γ_1) whose expression is (Γ_1): $x^2+y^2-1=0$.

This system (10) has another limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r\left(\theta,r_*\right)=\sqrt{\frac{\left(N\left(2\pi\right)-1\right)\left(N\left(\theta\right)+M\left(\theta\right)\right)+M\left(2\pi\right)}{\left(N\left(2\pi\right)-1\right)M\left(\theta\right)+M\left(2\pi\right)}},$$

where $\theta\in\mathbb{R}$. The intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_*=\sqrt{\frac{e^{\frac{64}{3}\pi}+2.4047\times10^{29}-1}{2.4047\times10^{29}}}=1.2376$$

We conclude that system (10) has two limit cycles (Γ_1) and (Γ_2). Since $r_*=1.2376>1$, the limit cycle (Γ_1) lies inside the limit cycle (Γ_2).

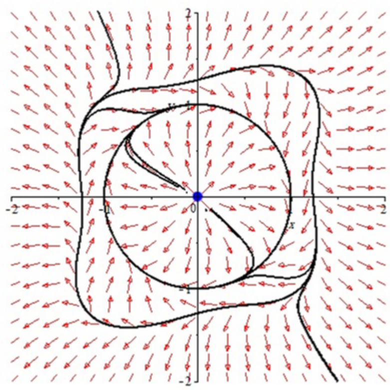

FIG. 1. Limit Cycles of System (10).

Example 2 If we take $a=b=-60, c=3, d=1, \alpha=\eta=12, \lambda=-3$ and $\delta=26$, then system (3) reads

(11)

$$\begin{cases} x'=\dfrac{dx}{dt}=x\left(12x^4-3x^3y+26x^2y^2-3xy^3+12y^4\right)-\left(x^2+y^2\right)^3\left(20x+y\right)+ \\ x\left(x^2+y^2\right)^2\left(8x^4+8y^4+3xy^3+3x^3y+14x^2y^2\right), \\ \\ y'=\dfrac{dy}{dt}=y\left(12x^4-3x^3y+26x^2y^2-3xy^3+12y^4\right)+\left(x^2+y^2\right)^3\left(x-20y\right)+ \\ y\left(x^2+y^2\right)^2\left(8x^4+8y^4+3xy^3+3x^3y+14x^2y^2\right), \end{cases}$$

The curve $x^2+y^2-1=0$ is an invariant algebraic curve of system (11) with cofactor

$$\begin{aligned} K(x,y) &= -\frac{2}{3}\left(x^2+y^2\right)(20\left(x^6+y^6\right)-30\left(x^4+y^4\right)- \\ &\qquad 3xy\left(x^2+y^2+1\right)\left(x^2+y^2\right)+36x^2y^2\left(x^2+y^2-\frac{7}{3}\right)). \end{aligned}$$

The system (11) has the first integral

$$H\left(x,y\right)=\frac{N\left(\arctan\frac{y}{x}\right)+\left(1-x^2-y^2\right)M\left(\arctan\frac{y}{x}\right)}{x^2+y^2-1},$$

where $N(\theta) = \exp\left(3 - 9\theta + \frac{1}{4}\sin 4\theta - 3\cos 2\theta\right)$ and

$$M(\theta) = -\frac{e^2}{2}\exp\left(-9\theta + \frac{1}{4}\sin 4\theta - 3\cos 2\theta\right)$$
$$-20e^3\int_0^\theta \exp\left(-9t + \frac{1}{4}\sin 4t - 3\cos 2t\right)dt.$$

The system (11) has the limit cycle (Γ_1) whose expression is (Γ_1): $x^2+y^2-1=0$.

This system (11) has another limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta, r_*) = \sqrt{\frac{(N(2\pi)-1)(N(\theta)+M(\theta))+M(2\pi)}{(N(2\pi)-1)M(\theta)+M(2\pi)}},$$

where $\theta \in \mathbb{R}$. The intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \sqrt{\frac{2.762\times 10^{-25} - 2.6042 - 1}{-2.6042}} = 1.1764$$

We conclude that system (11) has two limit cycles (Γ_1) and (Γ_2). Since $r_* = 1.1764 > 1$, the limit cycle (Γ_1) lies inside the limit cycle (Γ_2).

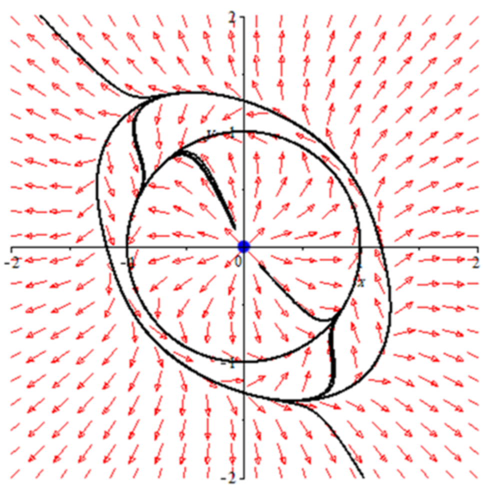

FIG. 2. Limit Cycles of System (11).

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