

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 17, вып. 1359–1450 (2020)

УДК 517.956

DOI 10.33048/semi.2020.17.100

MSC 35D30, 35Q92, 74A10, 92B05

HOMOGENIZATION OF A SUBMERGED
TWO-LEVEL BRISTLE STRUCTURE
FOR MODELING IN BIOTECHNOLOGY

S.A. SAZHENKOV, E.V. SAZHENKOVA

ABSTRACT. The effective macroscopic model describing reciprocal motion of viscous weakly compressible fluid and two-level hierarchical fine bristle-like elastic structure is derived from microstructure via the Allaire-Briane homogenization method. This new model naturally generalizes the well-known system constructed by K.-H. Hoffmann, N. Botkin and V. Starovoitov for description of fine periodic elastic structures in fluids (2005). In applications, the established model can be used, for example, in description of airflow near surface of plant's leaf, in simulation of epithelium surfaces of blood vessels, and in design of biotechnological devices operating in liquids.

Keywords: homogenization, periodic structure, compressible fluid, elastic solid, Navier-Stokes equations, Lamé's equations, biotechnology, bionics

INTRODUCTION

In the present article, the linearized mathematical model of reciprocal motion of a viscous weakly compressible fluid and a linear elastic plate with attached bristles is considered. The precise basic formulation of the fluid-structure interaction is given further in Sec. 1 following the original outline in [12, Sec. 2.1]. It is called **Model A**. We bring this formulation to a dimensionless form in Sec. 2. In Sec.

SAZHENKOV, S.A., SAZHENKOVA, E.V., HOMOGENIZATION OF A SUBMERGED TWO-LEVEL BRISTLE STRUCTURE FOR MODELING IN BIOTECHNOLOGY.

© 2020 SAZHENKOV S.A., SAZHENKOVA E.V.

The work of the first co-author was carried out in accordance with the State Assignment of the Russian Ministry of Science and Higher Education entitled 'Modern methods of hydrodynamics for environmental management, industrial systems and polar mechanics' (Govt. contract code: FZMW-2020-0008, 24 January 2020).

Received April, 29, 2020, published September, 14, 2020.

3, we define a fine bristle-like elastic structure attached to the plate in the full detail. Without loss of generality, we assume that, in dimensionless variables, the viscous fluid and the elastic plate with attached bristles occupy the 3D unit cube $\Omega = \{\mathbf{x} \in \mathbb{R}^3 : 0 < x_1, x_2, x_3 < 1\}$. The plate lays on the bottom of the cube and fills in the layer $\Omega_{\text{pl}} = \{\mathbf{x} \in \Omega : 0 < x_3 < \Delta\}$ ($\Delta = \text{const} < 1$). The bristles are modeled as elastic cylinders that are very frequently periodically located on the flat upper surface of the plate, perpendicularly to this surface. Presence of cylinders of the two distinct sizes is assumed. The shorter and thinner cylinders are located one or several orders more frequently than the taller and thicker ones. The heights of the cylinders are fixed and equal to δ_* and δ^* , where $\delta_* < \delta^*$ and $\Delta + \delta^* < 1$. The distance between the symmetry axes of two neighboring tall cylinders is ε . At the same time, the distance between the symmetry axes of two neighboring short cylinders is ε^2 . Here $\varepsilon \ll 1$ is a small positive dimensionless parameter.

The fluid motion is described by the classical non-stationary Stokes equations and the motion of elastic body obeys the classical non-stationary Lamé equations of linear elasticity. On the fluid-solid interface, continuity of the velocity field and the normal tension field is prescribed. The system consisting of the Stokes and Lamé's equations and the fluid-solid interface conditions is endowed with the set of initial (in time t) conditions and boundary conditions on $\partial\Omega$. The distributions of velocity $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(\mathbf{x}, t)$ and pressure $p_\varepsilon = p_\varepsilon(\mathbf{x}, t)$ in the fluid phase and the distribution of displacements $\mathbf{v}_\varepsilon = \mathbf{v}_\varepsilon(\mathbf{x}, t)$ of the elastic particles are the sought functions in the system.

In this article, the above described dimensionless model incorporating the small parameter ε is called **Model A $_\varepsilon$** . Existence and uniqueness of weak solutions for any fixed $\varepsilon > 0$ follow from the well-known results established in [20, Chap. 1, Sec. 9.1], see in Sec. 2. In this formulation, each bristle is distinguished. Therefore Model A $_\varepsilon$ describes the *microscopic behavior* of the mechanical system under study.

The aim of the present work is to fulfill and justify the homogenization procedure, i.e., to pass to the limit in Model A $_\varepsilon$ as $\varepsilon \searrow 0$, and to derive a closed system of effective relations that describe the behavior of the mechanical system on the *macroscopic level*. To this end, firstly, in Sec. 2, the unified formulation is set up in terms of the sought velocity field \mathbf{u}_ε in the whole continuum, like in [20, Ch. 1, Sec. 9.1]. Secondly, in Sec. 4, we recall and, in Sec. 5, apply the Allaire-Briane homogenization method and derive the three-scale homogenized system of equations and boundary conditions for the triple consisting of the macroscopic velocity \mathbf{u} , mesoscopic velocity $\mathbf{u}^{(1)}$, and the microscopic velocity $\mathbf{u}^{(2)}$. In Sec. 5, this system is called **Model H-3sc**. Thirdly, in Secs. 6-12, we fulfill the full asymptotic decomposition, which consists in the gradual scale separation. As the result, we construct the desired effective limiting model for the macroscopic velocity \mathbf{u} solely. This model describes evolution of the mechanical system on the macroscopic level. It is a natural generalization of the Hoffmann-Botkin-Starovoitov system [12, Sec. 5.1], and we call it **Model HBS**.

Model HBS consists of Lamé's equations of the pure elastic component that occupies layer Ω_{pl} , the two systems of non-classical non-local in time integro-differential equations of viscoelastic layers $\Omega_{\delta_*} := \{\mathbf{x} \in \Omega : \Delta < x_3 < \Delta + \delta_*\}$ and $\Omega_{\delta^*} := \{\mathbf{x} \in \Omega : \Delta + \delta_* < x_3 < \Delta + \delta^*\}$, the classical Stokes equations of the pure fluid component $\Omega_{\text{fl}} := \{\mathbf{x} \in \Omega : \Delta + \delta^* < x_3 < 1\}$, and the set of the natural conditions on the discontinuity surfaces $\{\mathbf{x} \in \Omega : x_3 = \Delta\}$, $\{\mathbf{x} \in \Omega : x_3 = \Delta + \delta_*\}$,

and $\{\mathbf{x} \in \Omega: x_3 = \Delta + \delta^*\}$. Its full formulation is given in Sec. 12 and is written in the form of the system (12.2a)-(12.2s). Further, in Sec. 13, we formulate and prove the theorem on existence and uniqueness of weak solutions of Model HBS. In Sec. 14, we make some concluding remarks about the computational algorithm based on Model HBS and possible applications of Model HBS in biotechnology and bionics.

In the end of this introduction, we would like to notice that the content of the article involves very many notations and technical calculations, which is a usual thing in studies of homogenization problems. Therefore, for convenience of reading we add two appendices in the end of the article. In Appendix A, we briefly recall the notions and properties of the convolution integrals and the Laplace transform. In Appendix B, we aggregate the list of almost all notation used in the article.

1. BASIC FORMULATION OF FLUID-STRUCTURE INTERACTIONS

In this section, we state the basic formulation of the initial-boundary value problem in the **dimensional** independent and sought variables.

MODEL A. (THE DIMENSIONAL FORM.) Let $\Omega = (0, L)^3$ be divided into two disjoint subdomains Ω_F and Ω_S and the Lipschitz boundary Γ between them, so that the fluid occupies Ω_F and the plate with attached bristles occupies Ω_S . The fluid motion is governed by the non-stationary Stokes equations

$$(1.1a) \quad \rho_F \partial_t \mathbf{u} = -\nabla_x p + \operatorname{div}_x (\mathcal{P}_* : \nabla_x \mathbf{u}) + \rho_F \mathbf{f}, \quad \mathbf{x} \in \Omega_F, t > 0,$$

$$(1.1b) \quad \gamma \partial_t p = -\operatorname{div}_x \mathbf{u}, \quad \mathbf{x} \in \Omega_F, t > 0.$$

The motion of the elastic body is governed by the non-stationary Lamé equations

$$(1.1c) \quad \rho_S \partial_t^2 \mathbf{v} = \operatorname{div}_x (\mathcal{G} : \nabla_x \mathbf{v}) + \rho_S \mathbf{f}, \quad \mathbf{x} \in \Omega_S, t > 0.$$

On Γ , the fluid-solid system obeys the continuity conditions for velocities and normal tensions:

$$(1.1d) \quad \partial_t \mathbf{v} = \mathbf{u}, \quad \mathbf{x} \in \Gamma, t > 0,$$

$$(1.1e) \quad (\mathcal{G} : \nabla_x \mathbf{v}) \mathbf{n} = (-p \mathbb{I} + \mathcal{P}_* : \nabla_x \mathbf{u}) \mathbf{n}, \quad \mathbf{x} \in \Gamma, t > 0.$$

This system is endowed by imposing initial data for $t = 0$ and conditions on the fixed boundary $\partial\Omega$:

$$(1.1f) \quad \mathbf{u}|_{t=0} = \mathbf{u}^0(\mathbf{x}), \quad p|_{t=0} = p^0(\mathbf{x}), \quad \mathbf{x} \in \Omega_F,$$

$$(1.1g) \quad \mathbf{v}|_{t=0} = \mathbf{v}^0(\mathbf{x}), \quad \partial_t \mathbf{v}|_{t=0} = \mathbf{w}^0(\mathbf{x}), \quad \mathbf{x} \in \Omega_S,$$

$$(1.1h) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{u}^F(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega \cap \partial\Omega_F, t > 0,$$

$$(1.1i) \quad \mathbf{v}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega \cap \partial\Omega_S, t > 0.$$

In (1.1a)-(1.1i) the velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ in the fluid, the displacement field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ in the elastic body, and the pressure distribution $p = p(\mathbf{x}, t)$ in the fluid are the unknown functions; \mathbf{n} is the unit normal vector to surface Γ pointing into Ω_F ; ρ_F and ρ_S are the given densities of the fluid and elastic components, respectively; $\rho_F \mathbf{f}$ and $\rho_S \mathbf{f}$ are external forces; the coefficient γ characterizes the compressibility of the fluid; the fourth-rank tensor $\mathcal{P}_* = (\mathcal{P}_*^{ijkl})$ is the viscous stress tensor in the fluid defined by the relation

$$(1.2) \quad \mathcal{P}_* : \nabla_x \mathbf{u} = (\lambda \operatorname{div}_x \mathbf{u}) \mathbb{I} + 2\mu \mathbb{D}_x(\mathbf{u}),$$

i.e.,

$$\mathcal{P}_* = \lambda \mathbb{I} \otimes \mathbb{I} + 2\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn},$$

or, in the component-wise form,

$$\mathcal{P}_*^{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \sum_{m,n=1}^3 \frac{\delta_{im} \delta_{jn} + \delta_{jm} \delta_{in}}{2} \frac{\delta_{km} \delta_{ln} + \delta_{lm} \delta_{kn}}{2}.$$

Here, the unit second-rank tensor \mathbb{I} has the components $I_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, and the second-rank tensor \mathbb{J}^{mn} is defined by the formula

$$\mathbb{J}^{mn} = \frac{\mathbf{e}_m \otimes \mathbf{e}_n + \mathbf{e}_n \otimes \mathbf{e}_m}{2},$$

or, in the component-wise form,

$$(\mathbb{J}^{mn})_{ij} = \frac{\delta_{im} \delta_{jn} + \delta_{jm} \delta_{in}}{2}, \quad i, j = 1, 2, 3.$$

By \mathbf{e}_m ($m = 1, 2, 3$) we denote the standard Cartesian basis vectors in \mathbb{R}^3 . Thus $(\mathbf{e}_m)_j = \delta_{mj}$. The strain velocity tensor $\mathbb{D}_x(\mathbf{u})$ has the components

$$D_{xij}(\mathbf{u}) = (1/2) (\partial_{x_j} u_i + \partial_{x_i} u_j), \quad i, j = 1, 2, 3.$$

The quantities λ and μ are the positive bulk and dynamic viscosity coefficients of the fluid, respectively. The components \mathcal{G}^{ijkl} of the elastic stiffness tensor \mathcal{G} can be arbitrary up to base restrictions so that any anisotropic solid can be considered. We have

$$(\mathcal{G} : \nabla_x \mathbf{v})_{ij} = \sum_{k,l=1}^3 \mathcal{G}^{ijkl} \partial_{x_l} v_k,$$

the components \mathcal{G}^{ijkl} are constant; and the symmetry condition

$$(1.3) \quad \mathcal{G}^{ijkl} = \mathcal{G}^{ijlk} = \mathcal{G}^{klij} = \mathcal{G}^{jikl}, \quad i, j, k, l = 1, 2, 3,$$

and the positive-definiteness condition

$$(1.4a) \quad (\mathcal{G} : \mathbb{X}) : \mathbb{X} \equiv \sum_{i,j,k,l=1}^3 \mathcal{G}^{ijkl} X_{kl} X_{ij} \geq 0, \quad \forall \mathbb{X} = (X_{ij})_{i,j=1,2,3} \in \mathbb{R}^{3 \times 3},$$

$$(1.4b) \quad \sum_{i,j,k,l=1}^3 \mathcal{G}^{ijkl} X_{kl} X_{ij} = 0 \text{ if and only if } X_{kl} + X_{lk} = 0, \quad \forall k, l = 1, 2, 3,$$

are fulfilled. Demands (1.3), (1.4a), and (1.4b) perfectly meet the fundamental principles of Newtonian mechanics.

Notation 1. *In this section and further in the article, we use the conventional notation for the inner products of fourth-rank tensors and 3×3 -matrices and for the dyads of 3×3 -matrices and vectors. More precisely, $\mathcal{A} : \mathbb{B}$ is the inner product (convolution) of a fourth-rank tensor \mathcal{A} and a 3×3 -matrix \mathbb{B} . It is the 3×3 -matrix defined by the formula*

$$\mathcal{A} : \mathbb{B} = \left(\sum_{k,l=1,2,3} \mathcal{A}^{ijkl} B_{kl} \right)_{i,j=1,2,3}.$$

The inner product (convolution) of two 3×3 -matrices \mathbb{B} and \mathbb{C} is the scalar defined by the formula

$$\mathbb{B} : \mathbb{C} = \sum_{i,j=1}^3 B_{ij}C_{ij}.$$

In particular, we have

$$(\mathcal{A} : \mathbb{B}) : \mathbb{C} = \sum_{i,j,k,l=1}^3 \mathcal{A}^{ijkl} B_{kl}C_{ij}, \quad |\mathbb{B}|^2 = \mathbb{B} : \mathbb{B} = \sum_{i,j=1}^3 B_{ij}B_{ij}, \quad \text{tr } \mathbb{B} = \mathbb{B} : \mathbb{I}$$

for all fourth-rank tensors \mathcal{A} and 3×3 -matrices \mathbb{B} and \mathbb{C} .

The dyad $\mathbb{B} \otimes \mathbb{C}$ of two 3×3 -matrices \mathbb{B} and \mathbb{C} is the fourth-rank tensor defined by the formula

$$\mathbb{B} \otimes \mathbb{C} = (B_{ij}C_{kl})_{i,j,k,l=1,2,3},$$

and the dyad $\mathbf{a} \otimes \mathbf{b}$ of two vectors \mathbf{a} and \mathbf{b} is the 3×3 -matrix defined by the formula

$$\mathbf{a} \otimes \mathbf{b} = (a_i b_j)_{i,j=1,2,3}.$$

In particular, we have

$$(\mathbb{B} \otimes \mathbb{C}) : \mathbb{A} = (\mathbb{C} : \mathbb{A})\mathbb{B}, \quad (\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

for all 3×3 -matrices \mathbb{A} , \mathbb{B} , and \mathbb{C} , and for all vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

Notation 2. Throughout the whole article we deal with symmetric 3×3 -matrices, say, $\mathbb{X} = (X_{ij})_{i,j=1,2,3}$ such that $X_{ij} = X_{ji}$. We denote the class of these matrices by $\mathbb{R}_{\text{symm}}^{3 \times 3}$.

The following technical lemma will be quite useful in some further considerations.

Lemma 1. Let a constant fourth-rank tensor $\mathfrak{E} = (\mathfrak{E}^{ijkl})_{i,j,k,l=1,2,3}$ satisfy the positive definiteness conditions

$$(\mathfrak{E} : \mathbb{X}) : \mathbb{X} \equiv \sum_{i,j,k,l=1}^3 \mathfrak{E}^{ijkl} X_{kl}X_{ij} \geq 0, \quad \forall \mathbb{X} = (X_{ij})_{i,j=1,2,3} \in \mathbb{R}^{3 \times 3},$$

and

$$(1.5) \quad \sum_{i,j,k,l=1}^3 \mathfrak{E}^{ijkl} X_{kl}X_{ij} = 0 \text{ if and only if } X_{kl} + X_{lk} = 0, \quad \forall k, l = 1, 2, 3.$$

Then there exists a constant $c_{\mathfrak{E}} > 0$ such that

$$(\mathfrak{E} : \mathbb{X}) : \mathbb{X} \geq c_{\mathfrak{E}} |\mathbb{X}|^2, \quad \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}.$$

Proof. Notice that for symmetric matrices \mathbb{X} due to (1.5) we have that

$$(1.6) \quad (\mathfrak{E} : \mathbb{X}) : \mathbb{X} \equiv \sum_{i,j,k,l=1}^3 \mathfrak{E}^{ijkl} X_{kl}X_{ij} = 0 \text{ if and only if } \mathbb{X} = 0.$$

Now the proof of the lemma can be constructed by contradiction.

Let us suppose that such a constant $c_{\mathfrak{E}} > 0$ does not exist. Then for each $\xi > 0$ there exists $\mathbb{X}_{\xi} \neq 0$ such that $0 < (\mathfrak{E} : \mathbb{X}_{\xi}) : \mathbb{X}_{\xi} < \xi |\mathbb{X}_{\xi}|^2$ or, equivalently, $0 < (\mathfrak{E} : \mathbb{W}_{\xi}) : \mathbb{W}_{\xi} < \xi$ for $\mathbb{W}_{\xi} = \mathbb{X}_{\xi}/|\mathbb{X}_{\xi}|$. (Notice that $|\mathbb{W}_{\xi}| = 1$.) Passing to the limit as $\xi \rightarrow 0+$ and using local compactness of the unit sphere in $\mathbb{R}^{3 \times 3}$, we conclude that there exists $\mathbb{W}_* \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ such that $\mathbb{W}_{\xi} \xrightarrow{\xi \rightarrow 0} \mathbb{W}_*$ in $\mathbb{R}^{3 \times 3}$, $|\mathbb{W}_*| = 1$ and $(\mathfrak{E} : \mathbb{W}_*) : \mathbb{W}_* = 0$, which contradicts (1.6). The lemma is proved. \square

The following property of the elastic stiffness tensor \mathcal{G} is an immediate consequence of this lemma.

Corollary 1. *There exists a constant $c_G > 0$ such that*

$$(1.7) \quad (\mathcal{G} : \mathbb{X}) : \mathbb{X} \geq c_G |\mathbb{X}|^2, \quad \forall \mathbb{X} = (X_{ij})_{i,j=1,2,3} \in \mathbb{R}_{\text{symm}}^{3 \times 3}.$$

2. DIMENSIONLESS FORM OF MODEL A. NOTION OF WEAK SOLUTION

Our aim in this article is to apply the homogenization procedure to Model A in the case when the shape of Ω_S includes the fine bristle-like structure, which depends on the small dimensionless parameter $\varepsilon > 0$, i.e., to pass to the limit, as $\varepsilon \searrow 0$. In line with this, primarily, we bring Model A to a dimensionless form and absorb the interface conditions on Γ in the equations by introducing a uniform description of the both phases, as in [20, Ch. 1, Sec. 9.1].

Namely, choose the edge L (measured, for example, in meters: m) of the cube Ω , a characteristic duration of physical processes $\tau_{sc}(s)$, the atmosphere pressure $p_{sc}(kg \cdot m^{-1} \cdot s^{-2})$, the acceleration of free fall $g(m \cdot s^{-2})$, and the mean density of air $\rho_{sc}(kg \cdot m^{-3})$ at the temperature $273 K$ under atmosphere pressure as the characteristic scales of length, time, pressure, density of mass distributed forces, and density of matter, respectively. Introduce the dimensionless independent and dependent variables (with primes) by the formulas

$$(2.1) \quad \mathbf{x} = L\mathbf{x}', \quad t = \tau_{sc}t', \quad \mathbf{u} = \frac{L}{\tau_{sc}}\mathbf{u}', \quad \mathbf{v} = L\mathbf{v}', \quad p = p_{sc}p',$$

the given dimensionless vector of distributed mass forces, the elastic stiffness tensor, the mean densities of the solid and the fluid at rest, and the initial and boundary data (all with primes), respectively, by the formulas

$$(2.2) \quad \begin{aligned} \mathbf{f} &= g\mathbf{f}', & \mathcal{G} &= p_{sc}\mathcal{G}', & \rho_S &= \rho_{sc}\rho'_S, & \rho_F &= \rho_{sc}\rho'_F, \\ \mathbf{u}^0 &= \frac{L}{\tau_{sc}}\mathbf{u}^{0'}, & \mathbf{v}^0 &= L\mathbf{v}^{0'}, & p^0 &= p_{sc}p^{0'}, & \mathbf{w}^0 &= \frac{L}{\tau_{sc}}\mathbf{w}^{0'}, & \mathbf{u}^F &= \frac{L}{\tau_{sc}}\mathbf{u}^{F'}, \end{aligned}$$

the dimensionless ratios by the formulas

$$(2.3) \quad \alpha_\tau = \frac{\rho_{sc}L^2}{p_{sc}\tau_{sc}^2}, \quad \alpha_F = \frac{\rho_{sc}gL}{p_{sc}}, \quad \alpha_\lambda = \frac{\lambda}{\tau_{sc}p_{sc}}, \quad \alpha_\mu = \frac{\mu}{\tau_{sc}p_{sc}}, \quad \alpha_\gamma = \gamma p_{sc},$$

and the dimensionless viscous stress tensor by the formula

$$(2.4) \quad \mathcal{P}' = \alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}.$$

Now multiply (1.1a) and (1.1c) by L/p_{sc} , (1.1b) by τ_{sc} , and (1.1d), (1.1f)₁, (1.1g)₂, and (1.1h) by τ_{sc}/L , and divide (1.1e) by $p_{sc}L$, (1.1f)₂ and (1.2) by p_{sc} , and (1.1g)₁ and (1.1i) by L . After this, substitute (2.1)-(2.4) into the resulting equations and then **omit primes**. Thus Model A is brought to the **dimensionless** form.

MODEL A. (THE DIMENSIONLESS FORM.) Let $\Omega = (0, 1)^3$ be divided into two disjoint subdomains Ω_F and Ω_S and the Lipschitz boundary Γ between them, so that the fluid occupies the subdomain Ω_F , and the plate with attached bristles occupies the subdomain Ω_S . Let $T = \text{const} > 0$ be a given moment of time. It is necessary to find a velocity field $\mathbf{u}: \Omega_F \times (0, T) \mapsto \mathbb{R}^3$, a pressure distribution

$p: \Omega_F \times (0, T) \mapsto \mathbb{R}$, and a displacement field $\mathbf{v}: \Omega_S \times (0, T) \mapsto \mathbb{R}^3$ satisfying the equations

$$(2.5a) \quad \alpha_\tau \rho_F \partial_t \mathbf{u} = -\nabla_x p + \operatorname{div}_x(\mathcal{P} : \nabla_x \mathbf{u}) + \alpha_F \rho_F \mathbf{f}, \quad \mathbf{x} \in \Omega_F, t \in (0, T),$$

$$(2.5b) \quad \mathcal{P} : \nabla_x \mathbf{u} = (\alpha_\lambda \operatorname{div}_x \mathbf{u}) \mathbb{I} + 2\alpha_\mu \mathbb{D}_x(\mathbf{u}), \quad \mathbf{x} \in \Omega_F, t \in (0, T),$$

$$(2.5c) \quad \alpha_\gamma \partial_t p = -\operatorname{div}_x \mathbf{u}, \quad \mathbf{x} \in \Omega_F, t \in (0, T),$$

$$(2.5d) \quad \alpha_\tau \rho_S \partial_t^2 \mathbf{v} = \operatorname{div}_x(\mathcal{G} : \nabla_x \mathbf{v}) + \alpha_F \rho_S \mathbf{f}, \quad \mathbf{x} \in \Omega_S, t \in (0, T),$$

the interface conditions

$$(2.5e) \quad \partial_t \mathbf{v} = \mathbf{u}, \quad \mathbf{x} \in \Gamma, t \in (0, T),$$

$$(2.5f) \quad (\mathcal{G} : \nabla_x \mathbf{v}) \mathbf{n} = (-p \mathbb{I} + \mathcal{P} : \nabla_x \mathbf{u}) \mathbf{n}, \quad \mathbf{x} \in \Gamma, t \in (0, T),$$

the initial conditions

$$(2.5g) \quad \mathbf{u}|_{t=0} = \mathbf{u}^0(\mathbf{x}), \quad p|_{t=0} = p^0(\mathbf{x}), \quad \mathbf{x} \in \Omega_F,$$

$$(2.5h) \quad \mathbf{v}|_{t=0} = \mathbf{v}^0(\mathbf{x}), \quad \partial_t \mathbf{v}|_{t=0} = \mathbf{w}^0(\mathbf{x}), \quad \mathbf{x} \in \Omega_S,$$

and the conditions on the fixed boundary $\partial\Omega$

$$(2.5i) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{u}^F(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega \cap \partial\Omega_F, t \in (0, T),$$

$$(2.5j) \quad \mathbf{v}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega \cap \partial\Omega_S, t \in (0, T).$$

Definition 1. We assume that Ω_S and Ω_F depend on a small positive dimensionless parameter ε : $\Omega_F = \Omega_F^\varepsilon$ and $\Omega_S = \Omega_S^\varepsilon$. In line with this, the dimensionless form of Model A incorporating ε is named **Model A_ε** .

It is necessary to introduce the proper notion of weak solution to Model A_ε , uniform in the whole unit cube Ω . To this end, let us follow [12, Sec. 2.2] and [20, Sec. 9.1]. Introduce the Volterra operator J_t :

$$(2.6) \quad (J_t \mathbf{w})(t) = \int_0^t \mathbf{w}(s) ds.$$

Then equation (2.5d) and condition (2.5h)₁ can be equivalently rewritten as

$$\alpha_\tau \rho_S \partial_t \mathbf{u} = \operatorname{div}_x(\mathcal{G} : J_t \nabla_x \mathbf{u}) + \operatorname{div}_x(\mathcal{G} : \nabla_x \mathbf{v}^0) + \alpha_F \rho_S \mathbf{f}.$$

Here $\mathbf{u} = \partial_t \mathbf{v}$ is the velocity vector. Analogously, with the help of (2.6), equation (2.5c) and condition (2.5g)₂ can be rewritten in the equivalent form

$$p = -\alpha_\gamma^{-1} \operatorname{div}_x J_t \mathbf{u} + p^0.$$

We accept the following two assumptions on the initial data in Model A_ε .

Assumption 1. We suppose that the initial data for velocity

$$\mathbf{u}^0(\mathbf{x}) := \begin{cases} \mathbf{u}^0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega_F^\varepsilon, \\ \mathbf{w}^0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega_S^\varepsilon, \end{cases}$$

given in the whole cube Ω , does not depend on ε . In other words, we impose an independent of ε uniform initial velocity field \mathbf{u}^0 on Ω .

Assumption 1 is consistent with the requirement that the initial velocity field is continuous in the whole cube Ω .

Assumption 2. Also we suppose that the initial distributions of pressure p^0 and the initial displacement \mathbf{v}^0 are defined in the whole cube Ω , do not depend on ε , and along with \mathbf{u}^0 satisfy the compatibility condition

$$(2.7) \quad -p^0\mathbb{I} + \mathcal{P} : \nabla_x \mathbf{u}^0 = \mathcal{G} : \nabla_x \mathbf{v}^0 \quad \forall \mathbf{x} \in \Omega.$$

Relation (2.7) expresses continuity of the initial stress in the whole cube Ω . Now introduce into considerations the characteristic function of domain Ω_F^ε :

$$(2.8) \quad \chi^\varepsilon(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \in \Omega_F^\varepsilon, \\ 0 & \text{for } \mathbf{x} \in \Omega \setminus \Omega_F^\varepsilon. \end{cases}$$

Taking into account this notation, we rewrite the system (2.5a)-(2.5j) as the uniform momentum equation with the discontinuous coefficients in the whole Ω :

$$\alpha_\tau \rho_*^\varepsilon \partial_t \mathbf{u} = \operatorname{div}_x (\mathcal{M}_\varepsilon^t \nabla_x \mathbf{u} + \mathbb{M}_\varepsilon^0) + \alpha_F \rho_*^\varepsilon \mathbf{f}, \quad \mathbf{x} \in \Omega, \quad t > 0,$$

where

$$(2.9) \quad \mathcal{M}_\varepsilon^t \nabla_x \mathbf{u} = \chi^\varepsilon \mathcal{P} : \nabla_x \mathbf{u} + (\chi^\varepsilon \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \chi^\varepsilon) \mathcal{G}) : J_t \nabla_x \mathbf{u},$$

$$(2.10) \quad \rho_*^\varepsilon = \chi^\varepsilon \rho_F + (1 - \chi^\varepsilon) \rho_S,$$

$$(2.11) \quad \mathbb{M}_\varepsilon^0 = -\chi^\varepsilon p^0 \mathbb{I} + (1 - \chi^\varepsilon) \mathcal{G} : \nabla_x \mathbf{v}^0,$$

supplemented with the set of initial and boundary data

$$(2.12) \quad \mathbf{u}|_{t=0} = \mathbf{u}^0, \quad \mathbf{x} \in \Omega,$$

$$(2.13) \quad \mathbf{u}(\mathbf{x}, t) = \mathbf{u}^*(\mathbf{x}, t) \quad \text{for } (\mathbf{x}, t) \in \partial\Omega \times (0, T),$$

where $\mathbf{u}^* = \mathbf{u}^*(\mathbf{x}, t)$ is a given vector-function, which is defined in the whole closed space-time domain $\bar{\Omega} \times [0, T]$ and satisfies the boundary condition

$$(2.14) \quad \mathbf{u}^*|_{\partial\Omega} = \begin{cases} \mathbf{u}^F(\mathbf{x}, t) & \text{for } (x_1, x_2) \in (0, 1)^2, \quad x_3 = 1, \quad t \in [0, T], \\ & \text{and for } (x_1, x_2) \in \partial(0, 1)^2, \quad x_3 \in (\Delta, 1), \quad t \in [0, T]; \\ 0 & \text{for } (x_1, x_2) \in \partial(0, 1)^2, \quad x_3 \in (0, \Delta], \quad t \in [0, T], \\ & \text{and for } (x_1, x_2) \in (0, 1)^2, \quad x_3 = 0, \quad t \in [0, T]. \end{cases}$$

Now we are in a position to introduce a notion of weak solution to Model A_ε .

Definition 2. Function $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(\mathbf{x}, t)$ is a **weak solution of Model A_ε** , if it satisfies the regularity demands

$$\mathbf{u}_\varepsilon \in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega_F^\varepsilon)^3), \quad J_t \mathbf{u}_\varepsilon \in L^\infty(0, T; H^1(\Omega)^3),$$

conditions (2.12) and (2.13) in the trace sense, and the integral equality

$$(2.15) \quad \int_0^T \int_\Omega \left(-\alpha_\tau \rho_*^\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + [\mathcal{M}_\varepsilon^t \nabla_x \mathbf{u}_\varepsilon + \mathbb{M}_\varepsilon^0] : \nabla_x \boldsymbol{\varphi} - \alpha_F \rho_*^\varepsilon \mathbf{f} \cdot \boldsymbol{\varphi} \right) dx dt = \int_\Omega \alpha_\tau \rho_*^\varepsilon(\mathbf{x}) \mathbf{u}^0(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}, 0) dx$$

for all smooth test vector-functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x}, t)$ vanishing in the neighborhood of the plane $\{t = T\}$ and the boundary $\partial\Omega$.

Existence and uniqueness of weak solutions to Model A_ε is justified standardly for linear evolutionary problems. Quite similarly to [20, Ch. 1, Sec. 9.1], we establish that the following result is valid.

Proposition 1. *Whenever $\mathbf{u}^0 \in L^2(\Omega)^3$, $\mathbf{v}^0 \in H_0^1(\Omega)^3$, $p^0 \in L^2(\Omega)$, $\mathbf{f} \in L^2(\Omega \times (0, T))^3$, $\mathbf{u}^* \in C^2(\bar{\Omega} \times [0, T])^3$, for any fixed $\varepsilon > 0$ there exists a unique weak solution \mathbf{u}_ε to Model A_ε .*

Moreover, the energy estimate

$$(2.16) \quad \begin{aligned} & \operatorname{ess\,sup}_{t \in (0, T)} \left(\|\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^3}^2 + \|\mathbb{D}_x((J_t \mathbf{u}_\varepsilon)(t))\|_{L^2(\Omega_S^\varepsilon)^{3 \times 3}}^2 \right) \\ & + \int_0^T \|\mathbb{D}_x(\mathbf{u}_\varepsilon(t))\|_{L^2(\Omega_F^\varepsilon)^{3 \times 3}}^2 dt \leq C_0 \end{aligned}$$

and the additional estimate

$$\operatorname{ess\,sup}_{t \in [0, T]} \|(J_t \mathbf{u}_\varepsilon)(t)\|_{H^1(\Omega)^3} \leq C_1,$$

hold true, where the constants C_0 and C_1 depend only on $\|\mathbf{u}^0\|_{L^2(\Omega)^3}$, $\|\mathbf{v}^0\|_{H_0^1(\Omega)^3}$, $\|p^0\|_{L^2(\Omega)}$, $\|\mathbf{f}\|_{L^2(\Omega \times (0, T))^3}$, and $\|\mathbf{u}^*\|_{C^2(\bar{\Omega} \times [0, T])^3}$, and do not depend on ε .

In [12, Th. 2.6], the stronger estimate for the weak solution \mathbf{u}_ε , than in Proposition 1, was established. Namely, the following assertion holds true.

Proposition 2. *Suppose, in addition to the hypothesis of Proposition 1, that*

$$\mathbf{u}^0 \in H^1(\Omega)^3, \quad \operatorname{div}_x(\chi^\varepsilon \mathcal{P} : \nabla_x \mathbf{u}^0 + \mathbb{M}_\varepsilon^0) \in L^2(\Omega)^3, \quad \mathbf{f}, \partial_t \mathbf{f} \in L^2(\Omega \times (0, T))^3.$$

Then the weak solution of Problem A_ε satisfies the bound

$$(2.17) \quad \operatorname{ess\,sup}_{t \in (0, T)} \left(\|\partial_t \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^3} + \|\nabla_x \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)^{3 \times 3}} \right) \leq C_2,$$

where the constant C_2 has the same properties, as C_0 and C_1 . In particular, C_2 does not depend on ε .

Remark 1. *We emphasize that one of the reasons why C_2 does not depend on ε is that, due to Assumption 2, the expression $\chi^\varepsilon \mathcal{P} : \nabla_x \mathbf{u}^0 + \mathbb{M}_\varepsilon^0$, in fact, does not depend on ε .*

3. FINE GEOMETRY OF THE MICROSTRUCTURE

Let us define the geometrical forms of Ω_S^ε and Ω_F^ε precisely. By this we introduce a two-level bristle structure. Assume that taller bristles are located ε -periodically and shorter bristles are located ε^2 -periodically in x_1 and x_2 . Parameter ε is small and positive: $\varepsilon > 0$, $\varepsilon \ll 1$. In order to describe exact locations of the bristles, we introduce the pattern mesoscopic cell $\Sigma = (0, 1) \times (0, 1)$ and the pattern microscopic cell $\Theta = (0, 1) \times (0, 1)$, each consisting of the two nonempty subdomains and the interface between these subdomains:

$$\Sigma = \Sigma_F \cup \Sigma_S \cup \Gamma_\Sigma, \quad \Theta = \Theta_F \cup \Theta_S \cup \Gamma_\Theta.$$

Here Σ_S is the orthogonal projection of a taller bristle onto the flat surface $\{x_3 = \Delta\}$ of the plate Ω_{pl} taken in $\varepsilon^{-1} : 1$ scale, i.e., ε^{-1} -times stretched. Analogously, Θ_S is the orthogonal projection of a shorter bristle onto the flat surface $\{x_3 = \Delta\}$ of the plate Ω_{pl} taken in $\varepsilon^{-2} : 1$ scale.

We assume that both Σ_S and Θ_S are simply connected sets with smooth boundaries, each of them being locally situated on one side of the boundary. For simplicity, suppose that Σ_S and Θ_S do not have common points with $\partial\Sigma$ and $\partial\Theta$, respectively (see Figs. 1 and 2).

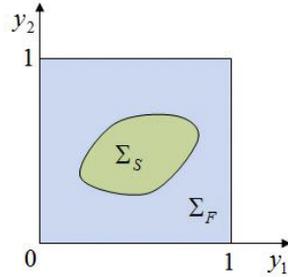


FIGURE 1. Mesoscopic pattern cell

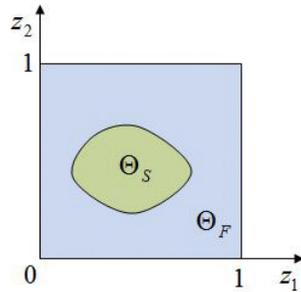


FIGURE 2. Microscopic pattern cell

Further, let us denote $\hat{\mathbf{x}} = (x_1, x_2)$, $\hat{\mathbf{y}} = (y_1, y_2)$, $\hat{\mathbf{z}} = (z_1, z_2)$ and introduce into consideration the characteristic functions $\hat{\zeta} = \hat{\zeta}(\hat{\mathbf{y}})$ and $\hat{\psi} = \hat{\psi}(\hat{\mathbf{z}})$ of the sets Σ_F and Θ_F , respectively:

$$(3.1) \quad \hat{\zeta}(\hat{\mathbf{y}}) = \begin{cases} 1 & \text{for } \hat{\mathbf{y}} \in \Sigma_F, \\ 0 & \text{for } \hat{\mathbf{y}} \in \bar{\Sigma} \setminus \Sigma_F, \end{cases} \quad \hat{\psi}(\hat{\mathbf{z}}) = \begin{cases} 1 & \text{for } \hat{\mathbf{z}} \in \Theta_F, \\ 0 & \text{for } \hat{\mathbf{z}} \in \bar{\Theta} \setminus \Theta_F. \end{cases}$$

Extend functions $\hat{\zeta}$ and $\hat{\psi}$ onto the whole spaces $\mathbb{R}_{\hat{\mathbf{y}}}^2$ and $\mathbb{R}_{\hat{\mathbf{z}}}^2$ 1-periodically.

Using the above introduced constructions, we set up the refined geometrical structure of domains $\Omega_S = \Omega_S^\varepsilon$ and $\Omega_F = \Omega_F^\varepsilon$ as follows.

Let $\Delta = \text{const} > 0$ be the sickness of the plate Ω_{p1} without taking bristles in account. Let $\delta^* = \text{const} > 0$ and $\delta_* = \text{const} > 0$ be the heights of taller and shorter bristles, respectively. We assume that $\delta^* + \Delta < 1$. Define the characteristic function $\chi(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})$ of domain $\Omega_F = \Omega_F^\varepsilon$ (in line with (2.8)) as follows:

$$(3.2) \quad \chi^\varepsilon(\mathbf{x}) = \zeta\left(\frac{\hat{\mathbf{x}}}{\varepsilon}, x_3\right) \psi\left(\frac{\hat{\mathbf{x}}}{\varepsilon^2}, x_3\right),$$

where

$$(3.3) \quad \zeta\left(\frac{\hat{\mathbf{x}}}{\varepsilon}, x_3\right) = \begin{cases} 1 & \text{for } \delta^* + \Delta < x_3 < 1, \\ \hat{\zeta}\left(\frac{\hat{\mathbf{x}}}{\varepsilon}\right) & \text{for } \Delta \leq x_3 \leq \delta^* + \Delta, \\ 0 & \text{for } 0 < x_3 < \Delta, \end{cases}$$

$$(3.4) \quad \psi\left(\frac{\hat{\mathbf{x}}}{\varepsilon^2}, x_3\right) = \begin{cases} 1 & \text{for } \delta_* + \Delta < x_3 < 1, \\ \hat{\psi}\left(\frac{\hat{\mathbf{x}}}{\varepsilon^2}\right) & \text{for } \Delta \leq x_3 \leq \delta_* + \Delta, \\ 0 & \text{for } 0 < x_3 < \Delta. \end{cases}$$

Thus, the structure of Ω_ε^ζ and Ω_ε^ψ is introduced. It is loosely shown on Fig. 3.

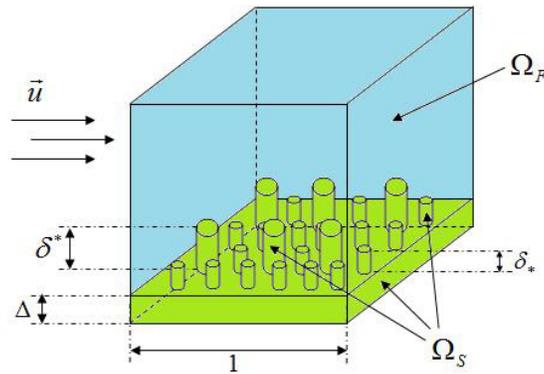


FIGURE 3. Plate with the bristles and the fluid flow

4. THE ALLAIRE-BRIANE THREE-SCALE CONVERGENCE METHOD

Our aim now is to pass to the limit in the integral equality (2.15) as $\varepsilon \searrow 0$. This limiting passage is based on the Allaire-Briane three-scale convergence method. We formulate the fundamentals of this method in the form adapted to the problem under consideration.

Proposition 3. (G. Allaire, M. Briane, [1, Th. 2.4].) *Let $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ be a bounded sequence in $L^2(\Omega \times (0, T))^3$. Then there exist a subsequence from $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ (still denoted by $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$) and a vector-function*

$$\mathbf{w} = \mathbf{w}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t), \quad \mathbf{w} \in L^2(\Omega \times \Sigma \times \Theta \times (0, T))^3,$$

such that the limiting relation

$$(4.1) \quad \begin{aligned} & \lim_{\varepsilon \searrow 0} \int_0^T \int_\Omega \mathbf{w}_\varepsilon(\mathbf{x}, t) \cdot \boldsymbol{\varphi}\left(\mathbf{x}, \frac{\hat{\mathbf{x}}}{\varepsilon}, \frac{\hat{\mathbf{x}}}{\varepsilon^2}, t\right) d\mathbf{x} dt \\ & = \int_0^T \int_\Omega \int_\Sigma \int_\Theta \mathbf{w}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) \cdot \boldsymbol{\varphi}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) d\mathbf{x} d\hat{\mathbf{y}} d\hat{\mathbf{z}} dt \end{aligned}$$

holds true for all smooth and 1-periodic in $\widehat{\mathbf{y}} = (y_1, y_2)$ and $\widehat{\mathbf{z}} = (z_1, z_2)$ test vector-functions $\varphi = \varphi(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}, t)$.

Definition 3. ([1, Defn. 2.3].) *If (4.1) holds then we say that the sequence $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ converges in the three-scale sense to \mathbf{w} :*

$$\mathbf{w}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \mathbf{w} \quad \text{in the 3-sc. sense.}$$

Proposition 4. (i) ([21, Sec. 3].) *The three-scale limit is unique, i.e., if sequence $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0} \subset L^2(\Omega \times (0, T))^3$ converges in the three-scale sense to vector-functions \mathbf{w}_* , $\mathbf{w}_{**} \in L^2(\Omega \times \Sigma \times \Theta \times (0, T))^3$ then $\mathbf{w}_* = \mathbf{w}_{**}$ a.e. in $\Omega \times \Sigma \times \Theta \times (0, T)$.*

(ii) ([1, Th. 2.6].) *Let sequences $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ and $\{\nabla_x \mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ be bounded in $L^2(\Omega \times (0, T))^3$. Then there exist a subsequence from $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ (still denoted by $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$) and a triple of vector-functions $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$, $\mathbf{w}_1 = \mathbf{w}_1(\mathbf{x}, \widehat{\mathbf{y}}, t)$ and $\mathbf{w}_2 = \mathbf{w}_2(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}, t)$ such that*

$$\begin{aligned} \mathbf{w} &\in L^2(0, T; H^1(\Omega)^3), & \mathbf{w}_1 &= (w_{11}, w_{12}, 0), & \mathbf{w}_2 &= (w_{21}, w_{22}, 0), \\ \mathbf{w}_1 &\in L^2(\Omega \times (0, T); (H^1_\#(\Sigma)/\mathbb{R})^3), & \mathbf{w}_2 &\in L^2(\Omega \times \Sigma \times (0, T); (H^1_\#(\Theta)/\mathbb{R})^3), \end{aligned}$$

and $\mathbf{w}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \mathbf{w}$, $\nabla_x \mathbf{w}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \nabla_x \mathbf{w} + \nabla_{\widehat{\mathbf{y}}} \mathbf{w}_1 + \nabla_{\widehat{\mathbf{z}}} \mathbf{w}_2$ in the 3-sc. sense.

(iii) ([1, Th. 4.6].) *Let sequences $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ and $\{\nabla_x \mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ be bounded in $L^2(\Omega \times (0, T))^3$. Let χ^ε be defined by formula (3.2). Then there exist a subsequence from $\{\mathbf{w}_\varepsilon\}_{\varepsilon \searrow 0}$ and functions $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$, $\mathbf{w}_1 = \mathbf{w}_1(\mathbf{x}, \widehat{\mathbf{y}}, t)$ and $\mathbf{w}_2 = \mathbf{w}_2(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}, t)$, with the same regularity properties as in assertion (ii), such that*

$$\begin{aligned} \chi^\varepsilon \mathbf{w}_\varepsilon &\xrightarrow{\varepsilon \searrow 0} \chi(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) \mathbf{w}(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}, t) \quad \text{in the 3-sc. sense,} \\ \chi^\varepsilon \nabla \mathbf{w}_\varepsilon &\xrightarrow{\varepsilon \searrow 0} \chi(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) (\nabla_x \mathbf{w} + \nabla_{\widehat{\mathbf{y}}} \mathbf{w}_1 + \nabla_{\widehat{\mathbf{z}}} \mathbf{w}_2) \quad \text{in the 3-sc. sense,} \end{aligned}$$

where χ is the characteristic function of the set $\Omega_F \times \Sigma_F \times \Theta_F$, i.e.,

$$(4.2) \quad \chi(\mathbf{x}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = \chi(x_3, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}) = \zeta(\widehat{\mathbf{y}}, x_3) \psi(\widehat{\mathbf{z}}, x_3),$$

where

$$(4.3) \quad \zeta(\widehat{\mathbf{y}}, x_3) = \begin{cases} 1 & \text{for } \delta^* + \Delta < x_3 < 1, \\ \widehat{\zeta}(\widehat{\mathbf{y}}) & \text{for } \Delta \leq x_3 \leq \delta^* + \Delta, \\ 0 & \text{for } 0 < x_3 < \Delta, \end{cases}$$

and

$$(4.4) \quad \psi(\widehat{\mathbf{z}}, x_3) = \begin{cases} 1 & \text{for } \delta_* + \Delta < x_3 < 1, \\ \widehat{\psi}(\widehat{\mathbf{z}}) & \text{for } \Delta \leq x_3 \leq \delta_* + \Delta, \\ 0 & \text{for } 0 < x_3 < \Delta. \end{cases}$$

In the formulation of Proposition 4 and further in the article, the standard notation for the spaces of periodic functions, which have gradients, is in use.

Notation 3. *By $H^1_\#(\Sigma)/\mathbb{R}$ and $H^1_\#(\Theta)/\mathbb{R}$ we denote the spaces of functions belonging to $H^1(\Sigma)$ and $H^1(\Theta)$, being 1-periodic in $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{z}}$, respectively, and satisfying the following calibration conditions:*

$$\int_\Sigma \varphi(\widehat{\mathbf{y}}) d\widehat{\mathbf{y}} = 0, \quad \int_\Theta \varphi(\widehat{\mathbf{z}}) d\widehat{\mathbf{z}} = 0.$$

By $\nabla_{\hat{\mathbf{y}}}$ and $\nabla_{\hat{\mathbf{z}}}$ we denote the gradient operators

$$(4.5) \quad \nabla_{\hat{\mathbf{y}}} = \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \\ 0 \end{pmatrix}, \quad \nabla_{\hat{\mathbf{z}}} = \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \\ 0 \end{pmatrix}.$$

5. THE LIMITING PASSAGE IN MODEL A_ε AS $\varepsilon \searrow 0$.
THREE-SCALE HOMOGENIZED EQUATIONS

On the strength of Propositions 1-4, we conclude that the family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of weak solutions of Model A_ε has the following properties.

Proposition 5. *There exist a subsequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon \searrow 0}$ from the family of weak solutions of Model A_ε and vector-functions*

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{u}^{(1)} = \mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t), \quad \mathbf{u}^{(2)} = \mathbf{u}^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t)$$

such that

$$(5.1) \quad \mathbf{u} \in L^\infty(0, T; H^1(\Omega)^3), \quad \partial_t \mathbf{u} \in L^\infty(0, T; L^2(\Omega)^3),$$

$$(5.2) \quad \mathbf{u}^{(1)} \in L^2(\Omega \times (0, T); (H^1_\#(\Sigma)/\mathbb{R})^3),$$

$$(5.3) \quad \mathbf{u}^{(2)} \in L^2(\Omega \times \Sigma \times (0, T); (H^1_\#(\Theta)/\mathbb{R})^3),$$

$$\mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \mathbf{u} \quad \text{in the 3-sc. sense and weakly in } L^2(\Omega \times (0, T))^3,$$

$$J_t \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} J_t \mathbf{u} \quad \text{in the 3-sc. sense and strongly in } L^2(\Omega \times (0, T))^3,$$

$$\nabla_x J_t \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \nabla_x J_t \mathbf{u} + \nabla_{\hat{\mathbf{y}}} J_t \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} J_t \mathbf{u}^{(2)} \quad \text{in the 3-sc. sense,}$$

$$\nabla_x \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)} \quad \text{in the 3-sc. sense,}$$

$$\chi^\varepsilon \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \mathbf{u}(\mathbf{x}, t) \quad \text{in the 3-sc. sense,}$$

$$\chi^\varepsilon J_t \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) (J_t \mathbf{u})(\mathbf{x}, t) \quad \text{in the 3-sc. sense,}$$

$$\chi^\varepsilon \nabla_x J_t \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) (\nabla_x J_t \mathbf{u} + \nabla_{\hat{\mathbf{y}}} J_t \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} J_t \mathbf{u}^{(2)}) \quad \text{in the 3-sc. sense,}$$

$$\chi^\varepsilon \nabla_x \mathbf{u}_\varepsilon \xrightarrow{\varepsilon \searrow 0} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) (\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) \quad \text{in the 3-sc. sense.}$$

Remark 2. *Due to the Rellich theorem [18, Ch. 1, Sec. 6] and bounds (2.16) and (2.17), the subsequence \mathbf{u}_ε converges to \mathbf{u} strongly in $L^2(\Omega \times (0, T))^3$, in fact.*

Now insert the test vector-function $\varphi = \varphi^\varepsilon(\mathbf{x}, t)$ of the form

$$(5.4) \quad \varphi^\varepsilon(\mathbf{x}, t) = \phi(\mathbf{x}, t) + \varepsilon \phi_1 \left(\mathbf{x}, \frac{\hat{\mathbf{x}}}{\varepsilon}, t \right) + \varepsilon^2 \phi_2 \left(\mathbf{x}, \frac{\hat{\mathbf{x}}}{\varepsilon}, \frac{\hat{\mathbf{x}}}{\varepsilon^2}, t \right)$$

into (2.15). In (5.4), $\phi = \phi(\mathbf{x}, t)$, $\phi_1 = \phi_1(\mathbf{x}, \hat{\mathbf{y}}, t)$ and $\phi_2 = \phi_2(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t)$ are arbitrary smooth test vector-functions such that ϕ , ϕ_1 , and ϕ_2 vanish in a neighborhood of $\partial\Omega$ and the section $\{t = T\}$, ϕ_1 is 1-periodic in $\hat{\mathbf{y}}$, and ϕ_2 is 1-periodic in $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. Pass to the limit as $\varepsilon \searrow 0$. Using Proposition 5, we derive the integral equality

$$(5.5) \quad \int_0^T \int_\Omega \int_\Sigma \int_\Theta \left\{ -\alpha_\tau \rho \mathbf{u} \cdot \partial_t \phi \right.$$

$$\begin{aligned}
& + \left[\mathcal{M}^t(\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) + \mathbb{M}^0 \right] : \nabla_x \phi - \alpha_F \rho \mathbf{f} \cdot \phi \Big\} d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\
& + \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \left[\mathcal{M}^t(\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) + \mathbb{M}^0 \right] : \nabla_{\hat{\mathbf{y}}} \phi_1 d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\
& + \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \left[\mathcal{M}^t(\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) + \mathbb{M}^0 \right] : \nabla_{\hat{\mathbf{z}}} \phi_2 d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\
& = \int_{\Omega} \int_{\Sigma} \int_{\Theta} \alpha_{\tau} \rho \mathbf{u}^0 \cdot \phi|_{t=0} d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x}.
\end{aligned}$$

Here the linear operator \mathcal{M}^t is defined by (2.9), with $\chi = \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ on the place of χ^ε . The viscous stresses have the forms

$$\begin{aligned}
\mathbf{P} : \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} &= (\alpha_{\lambda} \operatorname{div}_{\hat{\mathbf{y}}} \mathbf{u}^{(1)}) \mathbb{I} + 2\alpha_{\mu} \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}), \\
\mathbf{P} : \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)} &= (\alpha_{\lambda} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) \mathbb{I} + 2\alpha_{\mu} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{u}^{(2)}),
\end{aligned}$$

where

$$(5.6) \quad \mathbb{D}_{\hat{\mathbf{y}}}(\varphi) = \begin{pmatrix} D_{\hat{\mathbf{y}}11}(\varphi) & D_{\hat{\mathbf{y}}12}(\varphi) & 0 \\ D_{\hat{\mathbf{y}}21}(\varphi) & D_{\hat{\mathbf{y}}22}(\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{D}_{\hat{\mathbf{z}}}(\psi) = \begin{pmatrix} D_{\hat{\mathbf{z}}11}(\psi) & D_{\hat{\mathbf{z}}12}(\psi) & 0 \\ D_{\hat{\mathbf{z}}21}(\psi) & D_{\hat{\mathbf{z}}22}(\psi) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(5.7) \quad \operatorname{div}_{\hat{\mathbf{y}}} \varphi = \frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial y_2}, \quad \operatorname{div}_{\hat{\mathbf{z}}} \psi = \frac{\partial \psi_1}{\partial z_1} + \frac{\partial \psi_2}{\partial z_2}$$

for arbitrary admissible vector-functions $\varphi = \varphi(\hat{\mathbf{y}})$ and $\psi = \psi(\hat{\mathbf{z}})$. By ρ we denote the homogenized three-scale density:

$$(5.8) \quad \rho(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) = \rho(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) := \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \rho_F + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \rho_S.$$

The matrix-valued function \mathbb{M}^0 is defined by (2.11), with $\chi = \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ on the place of χ^ε .

Remark 3. Since ϕ , ϕ_1 , and ϕ_2 are arbitrary, in the sense of distributions, (5.5) is equivalent to the system consisting of the macroscopic integro-differential equation

$$(5.9) \quad \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta} \partial_t \mathbf{u} - \operatorname{div}_x \left\{ \int_{\Sigma} \int_{\Theta} \left[\mathcal{M}^t(\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) + \mathbb{M}^0 \right] d\hat{\mathbf{z}} d\hat{\mathbf{y}} \right\} \\
= \alpha_F \langle \rho \rangle_{\Sigma \times \Theta} \mathbf{f}, \quad (\mathbf{x}, t) \in \Omega \times (0, T),$$

the mesoscopic integro-differential equation

$$(5.10) \quad \operatorname{div}_{\hat{\mathbf{y}}} \left\{ \int_{\Theta} \left[\mathcal{M}^t(\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) + \mathbb{M}^0 \right] d\hat{\mathbf{z}} \right\} = 0, \quad (\mathbf{x}, t, \hat{\mathbf{y}}) \in \Omega \times (0, T) \times \Sigma,$$

the microscopic differential equation

$$(5.11) \quad \operatorname{div}_{\hat{\mathbf{z}}} \left\{ \mathcal{M}^t(\nabla_x \mathbf{u} + \nabla_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{u}^{(2)}) + \mathbb{M}^0 \right\} = 0, \quad (\mathbf{x}, t, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \in \Omega \times (0, T) \times \Sigma \times \Theta,$$

and the initial condition

$$(5.12) \quad \mathbf{u}|_{t=0} = \mathbf{u}^0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

This system is supplemented with the boundary condition

$$(5.13) \quad \mathbf{u} = \mathbf{u}^* \quad \text{for } \mathbf{x} \in \partial\Omega, t \in (0, T),$$

and the periodicity conditions for the mesoscopic velocity $\mathbf{u}^{(1)}$ and the microscopic velocity $\mathbf{u}^{(2)}$:

$$(5.14) \quad \mathbf{u}^{(1)} = \mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t) \text{ is 1-periodic in } \hat{\mathbf{y}},$$

$$(5.15) \quad \mathbf{u}^{(2)} = \mathbf{u}^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) \text{ is 1-periodic in } \hat{\mathbf{y}} \text{ and } \hat{\mathbf{z}}.$$

In (5.9) and further in the article, by $\langle \rho \rangle_{\Sigma \times \Theta}$ the mean value on $\Sigma \times \Theta$ of the three-scale density ρ is denoted, i.e.,

$$(5.16) \quad \langle \rho \rangle_{\Sigma \times \Theta}(x_3) = \int_{\Sigma} \int_{\Theta} \rho(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) d\hat{\mathbf{z}} d\hat{\mathbf{y}},$$

or, in the detailed form,

$$\langle \rho \rangle_{\Sigma \times \Theta}(x_3) = \begin{cases} \rho_F & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \rho_{\sigma} & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \rho_{\theta} & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \rho_S & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(5.17) \quad \sigma = \langle \hat{\zeta} \rangle_{\Sigma}, \quad \theta = \langle \hat{\psi} \rangle_{\Theta}$$

are the mean values of $\hat{\zeta}$ and $\hat{\psi}$, respectively, and

$$(5.18) \quad \rho_{\sigma} = \sigma \rho_F + (1 - \sigma) \rho_S, \quad \rho_{\theta} = \sigma \theta \rho_F + (1 - \sigma \theta) \rho_S$$

are the constant mean densities of the averaged viscoelastic medium in Ω_{δ^*} and Ω_{δ_*} , respectively.

Definition 4. System (5.9)-(5.15) is the closed three-scale homogenized model of the fluid-structure interactions. Let us call it **Model H-3sc**.

Theorem 1. Assume $\mathbf{u}^0 \in H^1(\Omega)^3$, $\mathbf{v}^0 \in H_0^1(\Omega)^3$, $p^0 \in H^1(\Omega)$, $\mathbf{f}, \partial_t \mathbf{f} \in L^2(\Omega \times (0, T))^3$, $\mathbf{u}^* \in C^2(\bar{\Omega} \times [0, T])^3$, and Assumptions 1 and 2 hold; then there exists a unique weak solution $\{\mathbf{u}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$ to Model H-3sc.

Definition 5. The triple of vector-functions $\{\mathbf{u}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$ is a **weak solution of Model H-3sc**, if it satisfies inclusions (5.1)-(5.3), the boundary condition (5.12) in the trace sense, and the integral equality (5.5) for arbitrary smooth test vector-functions $\phi = \phi(\mathbf{x}, t)$, $\phi_1 = \phi_1(\mathbf{x}, \hat{\mathbf{y}}, t)$, and $\phi_2 = \phi_2(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t)$ such that ϕ , ϕ_1 , and ϕ_2 vanish in a neighborhood of $\partial\Omega$ and the section $\{t = T\}$, ϕ_1 is 1-periodic in $\hat{\mathbf{y}}$, and ϕ_2 is 1-periodic in $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$.

Proof of Theorem 1. Existence of weak solutions $\{\mathbf{u}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$ has already been proved by means of the above fulfilled limiting passage as $\varepsilon \searrow 0$. Let us turn to justification of the uniqueness assertion.

Let $\{\mathbf{u}_*, \mathbf{u}_*^{(1)}, \mathbf{u}_*^{(2)}\}$ and $\{\mathbf{u}_{**}, \mathbf{u}_{**}^{(1)}, \mathbf{u}_{**}^{(2)}\}$ be two weak solutions of Model H-3sc corresponding to the same set of initial and boundary data $\mathbf{u}^0, \mathbb{M}^0$, and \mathbf{u}^* . Denote $\mathbf{U} := \mathbf{u}_* - \mathbf{u}_{**}$, $\mathbf{U}^{(1)} := \mathbf{u}_*^{(1)} - \mathbf{u}_{**}^{(1)}$ and $\mathbf{U}^{(2)} := \mathbf{u}_*^{(2)} - \mathbf{u}_{**}^{(2)}$. Subtract (5.5) with $\{\mathbf{u}_{**}, \mathbf{u}_{**}^{(1)}, \mathbf{u}_{**}^{(2)}\}$ from (5.5) with $\{\mathbf{u}_*, \mathbf{u}_*^{(1)}, \mathbf{u}_*^{(2)}\}$ to get

$$(5.19) \quad \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \left\{ -\alpha_{\tau} \rho \mathbf{U} \cdot \partial_t \phi + \mathcal{M}^t (\nabla_x \mathbf{U} + \nabla_{\hat{\mathbf{y}}} \mathbf{U}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{U}^{(2)}) : (\nabla_x \phi + \nabla_{\hat{\mathbf{y}}} \phi_1 + \nabla_{\hat{\mathbf{z}}} \phi_2) \right\} d\hat{\mathbf{z}} d\hat{\mathbf{y}} dx dt = 0.$$

Remark that $\phi_1 \in L^2(\Omega \times (0, T); H_{\sharp}^1(\Sigma)^3)$, $\phi_2 \in L^2(\Omega \times \Sigma \times (0, T); H_{\sharp}^1(\Theta)^3)$, and $\phi \in L^\infty(0, T; H^1(\Omega)^3)$, such that $\phi|_{\partial\Omega} = \phi|_{t=T} = 0$, are admissible test vector-functions for (5.5) and (5.19). Now, let us follow the track of considerations in [20, Ch. 2, Sec. 5.2]. Fix $\varkappa \in (0, T]$. Fix arbitrary $\varkappa_*, \varkappa_{**} \in (0, \varkappa)$, $\varkappa_* < \varkappa_{**}$. Take a continuous piece-wise linear function θ_m on $[0, \varkappa]$ such that $\theta_m(t) = 1$ if $\varkappa_*(2/m) < t < \varkappa_{**}(2/m)$, and $\theta_m(t) = 0$ if $t > \varkappa_{**}(1/m)$ and $t < \varkappa_*(1/m)$. Take a regularizing sequence $\omega_n \in C_0^\infty(\mathbb{R})$ such that

$$\omega_n(t) = \omega_n(-t), \quad \omega_n(t) \geq 0, \quad \int_{-\infty}^{\infty} \omega_n(t) dt = 1, \quad \text{supp } \omega_n \subset \left[-\frac{1}{n}, \frac{1}{n}\right].$$

For $n > 2m$, set $\phi_{mn} = ((\theta_m \mathbf{U}) * \omega_n * \omega_n) \theta_m$ ($m, n \in \mathbb{N}$), where the asterisk $*$ means the integral convolution in t on the whole space \mathbb{R} , i.e.,

$$(\Phi * \Psi)(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \Phi(t - \tau) \Psi(\tau) d\tau \quad \text{for integrable } \Phi \text{ and } \Psi.$$

(Remark that this is not the same convolution integral as in Definition A.1 in Appendix A.) Substitute ϕ_{mn} for ϕ into (5.19). Clearly, this choice of test vector-function ϕ is valid. Furthermore, in (5.19) take $\phi_1 = \mathbf{U}^{(1)}$ and $\phi_2 = \mathbf{U}^{(2)}$, which is also a valid choice of the test vector-functions due to the remark stated after (5.19).

Applying the arguments of [20, Ch. 2, Sec. 5.2], after some technical transformations and passage to the limit as $n \nearrow \infty$, $m \nearrow \infty$, $\varkappa_* \searrow 0$ and $\varkappa_{**} \nearrow \varkappa$ (successively), we arrive at the integral identity

$$\begin{aligned} (5.20) \quad & \frac{1}{2} \int_{\Omega} \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) |\mathbf{U}(\mathbf{x}, \varkappa)|^2 d\mathbf{x} \\ & + \int_0^\varkappa \int_{\Omega} \int_{\Sigma} \int_{\Theta} \mathcal{M}^t (\nabla_x \mathbf{U} + \nabla_{\hat{\mathbf{y}}} \mathbf{U}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{U}^{(2)}) : \\ & \quad : (\nabla_x \mathbf{U} + \nabla_{\hat{\mathbf{y}}} \mathbf{U}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{U}^{(2)}) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\ & = \frac{1}{2} \int_{\Omega} \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) |\mathbf{U}(\mathbf{x}, 0)|^2 d\mathbf{x} \equiv 0. \end{aligned}$$

Representing $\mathcal{M}^t (\nabla_x \mathbf{U} + \nabla_{\hat{\mathbf{y}}} \mathbf{U}^{(1)} + \nabla_{\hat{\mathbf{z}}} \mathbf{U}^{(2)})$ in the detailed form, using the symmetry conditions (1.3) and the initial conditions $\mathbf{U}|_{t=0} = \mathbf{U}^{(1)}|_{t=0} = \mathbf{U}^{(2)}|_{t=0} \equiv 0$, we rewrite (5.20) as follows:

$$\begin{aligned} (5.21) \quad & \frac{1}{2} \int_{\Omega} \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) |\mathbf{U}(\mathbf{x}, \varkappa)|^2 d\mathbf{x} \\ & + \frac{1}{2} \int_{\Omega} \int_{\Sigma} \int_{\Theta} \chi \alpha_\gamma^{-1} |J_\varkappa (\text{div}_x \mathbf{U} + \text{div}_{\hat{\mathbf{y}}} \mathbf{U}^{(1)} + \text{div}_{\hat{\mathbf{z}}} \mathbf{U}^{(2)})|^2 d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} \\ & + \frac{1}{2} \int_{\Omega} \int_{\Sigma} \int_{\Theta} (1 - \chi) \left[\mathcal{G} : J_\varkappa (\mathbb{D}_x(\mathbf{U}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{U}^{(1)}) + \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{U}^{(2)})) \right] : \\ & \quad : J_\varkappa (\mathbb{D}_x(\mathbf{U}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{U}^{(1)}) + \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{U}^{(2)})) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} \\ & + \alpha_\lambda \int_0^\varkappa \int_{\Omega} \int_{\Sigma} \int_{\Theta} \chi |\text{div}_x \mathbf{U} + \text{div}_{\hat{\mathbf{y}}} \mathbf{U}^{(1)} + \text{div}_{\hat{\mathbf{z}}} \mathbf{U}^{(2)}|^2 d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\ & + 2\alpha_\mu \int_0^\varkappa \int_{\Omega} \int_{\Sigma} \int_{\Theta} \chi |\mathbb{D}_x(\mathbf{U}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{U}^{(1)}) + \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{U}^{(2)})|^2 d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt = 0, \\ & \forall \varkappa \in (0, T]. \end{aligned}$$

Every integral in (5.21) is nonnegative. For the third integral this is true due to nonnegativity of the tensor \mathcal{G} (see (1.4a)), and for the remaining integrals this is quite evident. Hence all integrals in (5.21) vanish. This immediately yields that $\mathbf{U} \equiv 0$, i.e., $\mathbf{u}_* = \mathbf{u}_{**}$. Next, introducing $\mathbf{U} \equiv 0$ into (5.21) we derive that

$$\mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{U}^{(1)}) + \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{U}^{(2)}) = 0 \quad \text{for a.e. } (\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) \in \Omega \times \Sigma \times \Theta \times (0, T).$$

Represent this identity in the form

$$\frac{1}{2} \left(\frac{\partial}{\partial z_j} \left[U_i^{(2)} + \sum_{k=1}^2 \frac{\partial U_i^{(1)}}{\partial y_k} z_k \right] + \frac{\partial}{\partial z_i} \left[U_j^{(2)} + \sum_{k=1}^2 \frac{\partial U_j^{(1)}}{\partial y_k} z_k \right] \right) = 0, \quad i, j = 1, 2,$$

or, briefly, $\mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{U}^{(2)} + (\nabla_{\hat{\mathbf{y}}} \mathbf{U}^{(1)}) \hat{\mathbf{z}}) = 0$. Resolving this expression as equation $\mathbb{D}_{\hat{\mathbf{z}}}(\hat{\mathbf{U}}) = 0$ for $\hat{\mathbf{U}} = \mathbf{U}^{(2)} + (\nabla_{\hat{\mathbf{y}}} \mathbf{U}^{(1)}) \hat{\mathbf{z}}$, we arrive at the system of two scalar equalities:

$$\begin{aligned} \hat{U}_1 &\equiv U_1^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) + \sum_{k=1}^2 \frac{\partial U_1^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t)}{\partial y_k} z_k = C_3(\mathbf{x}, \hat{\mathbf{y}}, t) z_2 + C_4(\mathbf{x}, \hat{\mathbf{y}}, t), \\ \hat{U}_2 &\equiv U_2^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) + \sum_{k=1}^2 \frac{\partial U_2^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t)}{\partial y_k} z_k = -C_3(\mathbf{x}, \hat{\mathbf{y}}, t) z_1 + C_4(\mathbf{x}, \hat{\mathbf{y}}, t), \end{aligned}$$

with some functions C_3 and C_4 .

Because of 1-periodicity of $\mathbf{U}^{(2)}$ in $\hat{\mathbf{z}}$ we deduce

$$(5.22) \quad \frac{\partial U_1^{(1)}}{\partial y_1} = 0, \quad \frac{\partial U_2^{(1)}}{\partial y_2} = 0, \quad C_3 = \frac{\partial U_1^{(1)}}{\partial y_2} = -\frac{\partial U_2^{(1)}}{\partial y_1}$$

for a.e. $(\mathbf{x}, \hat{\mathbf{y}}, t) \in \Omega \times \Sigma \times (0, T)$, and therefore $U_1^{(2)} = U_2^{(2)} = C_4$. As $\int_{\Theta} \mathbf{U}^{(2)} d\hat{\mathbf{z}} = 0$ the latter implies $\mathbf{U}^{(2)} \equiv 0$, i.e., $\mathbf{u}_*^{(2)} = \mathbf{u}_{**}^{(2)}$. Finally, from (5.22) it follows that $\mathbf{U}^{(1)}$ does not depend on $\hat{\mathbf{y}}$, which along with $\int_{\Sigma} \mathbf{U}^{(1)} d\hat{\mathbf{y}} = 0$ yields that $\mathbf{U}^{(1)} \equiv 0$, i.e., $\mathbf{u}_*^{(1)} = \mathbf{u}_{**}^{(1)}$. Thus the uniqueness assertion in Theorem 1 is proved. \square

The further aim of our research is to separate macroscopic, mesoscopic, and microscopic scales from each other. In the modern homogenization theory, this procedure is called the *asymptotic decomposition*. In this article, we fulfill and rigorously justify the complete asymptotic decomposition.

6. ASYMPTOTIC DECOMPOSITION I: THE MICROSCOPIC CELL PROBLEMS

We will seek for a representation of the vector-function $\mathbf{u}^{(2)}$ in the form

$$(6.1) \quad \begin{aligned} \mathbf{u}^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) &:= \sum_{i,j=1}^3 \left[\left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) + \frac{\partial u_i^{(1)}}{\partial y_j}(\mathbf{x}, \hat{\mathbf{y}}, t) \right) \mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3) \right. \\ &\quad \left. + \int_0^t \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, \tau) + \frac{\partial u_i^{(1)}}{\partial y_j}(\mathbf{x}, \hat{\mathbf{y}}, \tau) \right) \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t - \tau, x_3) d\tau \right] \\ &\quad + p^0(\mathbf{x}) \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3), \end{aligned}$$

where

$$(6.2) \quad \mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3) := \zeta(\hat{\mathbf{y}}, x_3) \mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3) + (1 - \zeta(\hat{\mathbf{y}}, x_3)) \mathbf{Z}_{01}^{ij}(\hat{\mathbf{z}}, x_3),$$

$$(6.3) \quad \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) := \zeta(\hat{\mathbf{y}}, x_3) \mathbf{Z}_{10}^{ij}(\hat{\mathbf{z}}, t, x_3) + (1 - \zeta(\hat{\mathbf{y}}, x_3)) \mathbf{Z}_{11}^{ij}(\hat{\mathbf{z}}, t, x_3),$$

$$(6.4) \quad \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) := \zeta(\hat{\mathbf{y}}, x_3) \mathbf{Z}_{20}(\hat{\mathbf{z}}, t, x_3) + (1 - \zeta(\hat{\mathbf{y}}, x_3)) \mathbf{Z}_{21}(\hat{\mathbf{z}}, t, x_3),$$

$$(6.5) \quad \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) := \zeta(\hat{\mathbf{y}}, x_3) \mathbf{Z}_{30}^{ij}(\hat{\mathbf{z}}, t, x_3) + (1 - \zeta(\hat{\mathbf{y}}, x_3)) \mathbf{Z}_{31}^{ij}(\hat{\mathbf{z}}, t, x_3).$$

Here \mathbf{u} and $\mathbf{u}^{(1)}$, along with the vector-function $\mathbf{u}^{(2)}$, constitute the weak solution of Model H-3sc. Vector-functions \mathbf{Z}_{00}^{ij} , \mathbf{Z}_{01}^{ij} , \mathbf{Z}_{10}^{ij} , \mathbf{Z}_{11}^{ij} , \mathbf{Z}_{30}^{ij} , \mathbf{Z}_{31}^{ij} ($i, j = 1, 2, 3$), \mathbf{Z}_{20} , and \mathbf{Z}_{21} are the sought functions.

In (5.5), take the test vector-functions ϕ , ϕ_1 , and ϕ_2 of the following form:

$$\phi(\mathbf{x}, t) := 0, \quad \phi_1(\mathbf{x}, \hat{\mathbf{y}}, t) := 0, \quad \phi_2(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) := \frac{\partial \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t)}{\partial t} \phi_{21}(\hat{\mathbf{z}}),$$

where ϕ_{20} is an arbitrary smooth 1-periodic in $\hat{\mathbf{y}}$ scalar function, vanishing in a neighborhood of $\partial\Omega$ and the section $\{t = T\}$, and ϕ_{21} is an arbitrary 1-periodic in $\hat{\mathbf{z}}$ smooth vector-function. After quite routine but rather lengthy calculations involving the use of Proposition A.1 (see in Appendix A), we arrive at the integral equality

$$(6.6) \quad \begin{aligned} & - \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \left(\frac{\partial^2 u_i}{\partial x_j \partial t} + \frac{\partial^2 u_i^{(1)}}{\partial y_j \partial t} \right) \times \\ & \quad \times \sum_{k,l=1}^3 \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left(\alpha_{\lambda} \delta_{ij} \delta_{kl} + \alpha_{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right) \\ & \quad + \alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_0^{ij} + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl}(\mathbf{Z}_0^{ij}) \left. \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \right. \\ & - \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) + \frac{\partial u_i^{(1)}}{\partial y_j}(\mathbf{x}, \hat{\mathbf{y}}, 0) \right) \times \\ & \quad \times \sum_{k,l=1}^3 \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left(\alpha_{\lambda} \delta_{ij} \delta_{kl} + \alpha_{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right) \\ & \quad + \alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_0^{ij} + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl}(\mathbf{Z}_0^{ij}) \left. \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, 0) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} \right. \\ & - \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i^{(1)}}{\partial y_j} \right) \times \\ & \quad \times \sum_{k,l=1}^3 \left[\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_{\gamma}^{-1} \sum_{p,q=1}^3 \left(\delta_{kl} \delta_{pq} \delta_{ip} \delta_{jq} + \delta_{kl} \delta_{pq} \frac{\partial \mathbf{Z}_{0p}^{ij}}{\partial z_q} \right) \right. \right. \\ & \quad \left. \left. + \alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3) + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl}(\mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3)) \right\} \times \right. \\ & \quad \left. \left. \times \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t) \right] d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \end{aligned}$$

$$\begin{aligned}
 & - \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i^{(1)}}{\partial y_j} \right) \sum_{k,l=1}^3 (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \times \\
 & \quad \times \sum_{p,q=1}^3 \mathcal{G}^{klpq} \left(\delta_{ip} \delta_{jq} + \frac{\partial Z_{0p}^{ij}}{\partial z_q} \right) \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\
 & - \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \left(\int_0^t \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, \tau) + \frac{\partial u_i^{(1)}}{\partial y_j}(\mathbf{x}, \hat{\mathbf{y}}, \tau) \right) \times \right. \\
 & \quad \times \sum_{k,l=1}^3 \left\{ \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left[\alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \frac{\partial \mathbf{Z}_1^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t - \tau, x_3) \right. \right. \\
 & \quad \left. \left. + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl} \left(\frac{\partial \mathbf{Z}_1^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t - \tau, x_3) \right) + \alpha_{\gamma}^{-1} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t - \tau, x_3) \right] \right. \\
 & \quad \left. \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \sum_{p,q=1}^3 \mathcal{G}^{klpq} \frac{\partial Z_{1p}^{ij}}{\partial z_q}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t - \tau, x_3) \right\} d\tau \right) \times \\
 & \quad \times \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\
 & - \int_{\Omega} \int_{\Sigma} \int_{\Theta} p^0(\mathbf{x}) \sum_{k,l=1}^3 \left\{ \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left[-\delta_{kl} + \alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3) \right. \right. \\
 & \quad \left. \left. + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl}(\mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3)) \right] \right\} \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, 0) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} \\
 & - \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} p^0(\mathbf{x}) \sum_{k,l=1}^3 \left\{ \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left[2\alpha_{\mu} D_{\hat{\mathbf{z}}kl} \left(\frac{\partial \mathbf{Z}_2}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right) \right. \right. \\
 & \quad \left. \left. + \alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \frac{\partial \mathbf{Z}_2}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) + \alpha_{\gamma}^{-1} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right] \right. \\
 & \quad \left. \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \sum_{p,q=1}^3 \mathcal{G}^{klpq} \frac{\partial Z_{2p}}{\partial z_q}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} dt \\
 & - \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \times \\
 & \quad \times \sum_{k,l=1}^3 \left(\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left[\alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3) + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl}(\mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3)) \right] \right. \\
 & \quad \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \mathcal{G}^{ijkl} \right) \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, 0) d\hat{\mathbf{z}} d\hat{\mathbf{y}} d\mathbf{x} \\
 & - \int_0^T \int_{\Omega} \int_{\Sigma} \int_{\Theta} \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \sum_{k,l=1}^3 \left\{ \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left[\alpha_{\lambda} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \frac{\partial \mathbf{Z}_3^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right. \right. \\
 & \quad \left. \left. + 2\alpha_{\mu} D_{\hat{\mathbf{z}}kl} \left(\frac{\partial \mathbf{Z}_3^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right) + \alpha_{\gamma}^{-1} \delta_{kl} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right] \right\}
 \end{aligned}$$

$$+(1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \sum_{p,q=1}^3 \mathcal{G}^{klpq} \frac{\partial Z_{3p}^{ij}}{\partial z_q}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \left\} \frac{\partial \phi_{21k}}{\partial z_l}(\hat{\mathbf{z}}) \phi_{20}(\mathbf{x}, \hat{\mathbf{y}}, t) d\hat{\mathbf{z}} d\hat{\mathbf{y}} dx dt = 0.$$

From the structure of this integral equality it is clear that it holds true independently of any possible values of \mathbf{u} , $\mathbf{u}^{(1)}$, \mathbf{v}^0 , and p^0 if \mathbf{Z}_0^{ij} , \mathbf{Z}_1^{ij} , \mathbf{Z}_2 , and \mathbf{Z}_3^{ij} satisfy the following integral equalities for all admissible test vector-functions ϕ_{21} :

$$(6.7) \quad \int_{\Theta} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_\lambda \delta_{ij} \mathbb{I} + 2\alpha_\mu \mathbb{J}^{ij} + \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3) \right. \\ \left. + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3)) \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}} = 0, \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$(6.8) \quad \int_{\Theta} (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \left\{ \mathcal{G} : (\mathbb{J}^{ij} + \nabla_{\hat{\mathbf{z}}} \mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3)) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} d\hat{\mathbf{z}} = 0, \\ \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$(6.9) \quad \int_{\Theta} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_\gamma^{-1} (\delta_{ij} + \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3)) \mathbb{I} + \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3) \right. \\ \left. + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3)) \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}} = 0, \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$(6.10) \quad \int_{\Theta} \left[\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \frac{\partial \mathbf{Z}_1^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}\left(\frac{\partial \mathbf{Z}_1^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3)\right) \right. \right. \\ \left. \left. + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right. \\ \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \left\{ \mathcal{G} : \nabla_{\hat{\mathbf{z}}} \mathbf{Z}_1^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right] d\hat{\mathbf{z}} = 0, \\ \forall t \in (0, T], \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$(6.11) \quad \int_{\Theta} \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ -\mathbb{I} + \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3) + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3)) \right\} : \\ : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}} = 0, \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$(6.12) \quad \int_{\Theta} \left[\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \frac{\partial \mathbf{Z}_2}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}\left(\frac{\partial \mathbf{Z}_2}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3)\right) \right. \right. \\ \left. \left. + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right. \\ \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \left\{ \mathcal{G} : \nabla_{\hat{\mathbf{z}}} \mathbf{Z}_2(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right] d\hat{\mathbf{z}} = 0, \\ \forall t \in (0, T], \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$(6.13) \quad \int_{\Theta} \left[\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3) + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, 0, x_3)) \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) \right. \\ \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \left\{ \mathcal{G} : \mathbb{J}^{ij} \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right] d\hat{\mathbf{z}} = 0, \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1];$$

$$\begin{aligned}
 (6.14) \quad & \int_{\Theta} \left[\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \left\{ \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \frac{\partial \mathbf{Z}_3^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}} \left(\frac{\partial \mathbf{Z}_3^{ij}}{\partial t}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right) \right. \right. \\
 & \quad \left. \left. + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right. \\
 & \quad \left. + (1 - \chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}})) \left\{ \mathcal{G} : \nabla_{\hat{\mathbf{z}}} \mathbf{Z}_3^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} \right] d\hat{\mathbf{z}} = 0, \\
 & \quad \forall t \in (0, T], \quad \forall \hat{\mathbf{y}} \in \Sigma, \quad \forall x_3 \in [0, 1].
 \end{aligned}$$

In (6.8) and (6.13) we have remarked that $\mathcal{G} : (\mathbf{e}_i \otimes \mathbf{e}_j) = \mathcal{G} : \mathbb{J}^{ij}$, due to (1.3). Notice that (6.12) and (6.14) formally coincide.

Remark 4. Since the vector-function $\hat{\mathbf{z}} \mapsto \mathbf{u}^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t)$ belongs to $(H_{\#}^1(\Theta)/\mathbb{R})^3$ for a.e. $(\mathbf{x}, \hat{\mathbf{y}}, t)$, the vector-functions \mathbf{Z}_0^{ij} , \mathbf{Z}_1^{ij} , \mathbf{Z}_2 , and \mathbf{Z}_3^{ij} should belong to $(H_{\#}^1(\Theta)/\mathbb{R})^3$ as vector-functions of $\hat{\mathbf{z}}$.

Taking into account this remark, on the basis of (6.7)-(6.14) we formulate the problems on the pattern cell Θ for finding the independent of the time variable t vector-functions \mathbf{Z}_{00}^{ij} , $\mathbf{Z}_{10}^{ij}|_{t=0}$, $\mathbf{Z}_{20}|_{t=0}$, and $\mathbf{Z}_{30}^{ij}|_{t=0}$, and the time-dependent vector-functions \mathbf{Z}_{10}^{ij} , \mathbf{Z}_{20} , and \mathbf{Z}_{30}^{ij} for $t > 0$. For the functions \mathbf{Z}_{01}^{ij} , \mathbf{Z}_{11}^{ij} , \mathbf{Z}_{21} , and \mathbf{Z}_{31}^{ij} we show that all of them should be taken identically equal to zero.

At first, we formulate the problem for the vector-function $\hat{\mathbf{z}} \mapsto \mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3)$.

Notice that, for $x_3 \in (0, \Delta)$, the integral equality (6.7) turns into the trivial identity $0 = 0$, since $\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \equiv 0$. Therefore it holds true for any \mathbf{Z}_0^{ij} , in particular, for $\mathbf{Z}_0^{ij} \equiv 0$. The integral equality (6.8) also holds true for

$$\mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3) \equiv 0 \text{ on } \Sigma \times \Theta \times (0, \Delta),$$

since $\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) = \zeta(\hat{\mathbf{y}}, x_3)\psi(\hat{\mathbf{z}}, x_3) \equiv 0$ for $x_3 \in (0, \Delta)$ and ϕ_{21} is a 1-periodic vector-function. Analogously, (6.7) and (6.8) are valid if

$$\mathbf{Z}_0^{ij}(\hat{\mathbf{y}}, \hat{\mathbf{z}}, x_3) \equiv 0 \text{ on } \Sigma \times \Theta \times (\Delta + \delta_*, 1),$$

since $\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) = \zeta(\hat{\mathbf{y}}, x_3)\psi(\hat{\mathbf{z}}, x_3) \equiv 1$ for $x_3 \in (\Delta + \delta^*, 1)$, $\chi(x_3, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \equiv \zeta(\hat{\mathbf{y}}, x_3)$ for $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$, and ϕ_{21} is a 1-periodic function.

Inserting (6.2) into (6.7) we conclude that the vector-function $\hat{\mathbf{z}} \mapsto \mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3)$ should resolve the following Stokes system in Θ_F for $x_3 \in [\Delta, \Delta + \delta_*]$:

$$\begin{aligned}
 (6.15a) \quad & \int_{\Theta_F} \left\{ \alpha_\lambda \delta_{ij} \mathbb{I} + 2\alpha_\mu \mathbb{J}^{ij} + \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3) \right. \\
 & \quad \left. + 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3)) \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}} = 0, \quad \forall \phi_{21} \in H_{\#}^1(\Theta)^3,
 \end{aligned}$$

$$(6.15b) \quad \mathbf{Z}_{00}^{ij}(\cdot, x_3) \in (H_{\#}^1(\Theta)/\mathbb{R})^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].$$

Inserting (6.2) into (6.8), we deduce the integral equality for \mathbf{Z}_{00}^{ij} in Θ_S and for \mathbf{Z}_{01}^{ij} in Θ :

$$\begin{aligned}
 (6.16) \quad & \zeta(\hat{\mathbf{y}}, x_3) \int_{\Theta_S} \left\{ \mathcal{G} : (\mathbb{J}^{ij} + \nabla_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3)) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} d\hat{\mathbf{z}} \\
 & \quad + (1 - \zeta(\hat{\mathbf{y}}, x_3)) \int_{\Theta} \left\{ \mathcal{G} : (\mathbb{J}^{ij} + \nabla_{\hat{\mathbf{z}}} \mathbf{Z}_{01}^{ij}(\hat{\mathbf{z}}, x_3)) \right\} : \nabla_{\hat{\mathbf{z}}} \phi_{21} d\hat{\mathbf{z}} = 0,
 \end{aligned}$$

$$\forall \widehat{\mathbf{y}} \in \Sigma, \forall x_3 \in [\Delta, \Delta + \delta_*].$$

Set

$$\mathbf{Z}_{01}^{ij}(\widehat{\mathbf{z}}, x_3) \equiv 0 \text{ on } \Theta \times [0, 1].$$

With this choice of \mathbf{Z}_{01}^{ij} , on the strength of 1-periodicity of $\phi_{21} = \phi_{21}(\widehat{\mathbf{z}})$, by Green's formula we deduce that the second integral in (6.16) vanishes. Consequently, the vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)$ must satisfy the integral equality

$$(6.17) \quad \int_{\Theta_S} \left\{ \mathcal{G} : (\mathbb{J}^{ij} + \nabla_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)) \right\} : \nabla_{\widehat{\mathbf{z}}} \phi_{21} d\widehat{\mathbf{z}} = 0, \\ \forall \phi_{21} \in H_{\#}^1(\Theta)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].$$

Thus, the vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)$ should be a weak solution of the Dirichlet problem for the linear elasticity equation:

$$(6.18) \quad \begin{cases} \operatorname{div}_{\widehat{\mathbf{z}}} \left\{ \mathcal{G} : (\nabla_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3) + \mathbb{J}^{ij}) \right\} = 0 \text{ on } \Theta_S, \\ \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)|_{\partial\Theta_S} = \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)|_{\partial\Theta_F \setminus \partial\Theta}, \\ \mathbf{Z}_{00}^{ij}(\cdot, x_3) \in H_{\#}^1(\Theta_S)^3, \quad \Delta \leq x_3 \leq \Delta + \delta_*. \end{cases}$$

In this formulation, $\mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)|_{\partial\Theta_F \setminus \partial\Theta}$ is the trace of the solution of the system (6.15a)-(6.15b) on the interface between Θ_F and Θ_S . Notice that the vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)$, defined by the solutions of the problems (6.15a)-(6.15b) and (6.18) on the whole pattern cell Θ , is 1-periodic and belongs to $H^1(\Theta)^3$ due to the boundary condition (6.18)₂.

Now we are in a position to formulate the first of the problems on the microscopic pattern cell Θ .

Problem Z1. It is necessary to find a vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3)$ ($i, j = 1, 2, 3$), which vanishes for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$(6.19) \quad \mathbf{Z}_{00}^{ij}(\cdot, x_3) \in (H_{\#}^1(\Theta)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

and resolves the Stokes system (6.15a)-(6.15b) and the linear elasticity problem (6.18) in the weak sense, i.e., in the sense of the integral equality (6.17).

Remark 5. From the above outline it is clear that if \mathbf{Z}_{00}^{ij} is the solution of Problem Z1 and \mathbf{Z}_{01}^{ij} is identically equal to zero then the vector-function \mathbf{Z}_0^{ij} defined by equality (6.2) satisfies (6.7) and (6.8) ($i, j = 1, 2, 3$).

Remark 6. On the strength of the well-known theory of generalized solutions of boundary value problems in mathematical physics [18, 19], Problem Z1 has the unique weak solution for every $x_3 \in (0, 1)$.

The formulations of the rest of the problems on the pattern cell Θ are constructed by means of the similar arguments as for Problem Z1. Therefore we omit the details and just state these problems and the results on their well-posedness.

Problem Z2. It is necessary to find a vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ ($i, j = 1, 2, 3$), defined in Θ_F for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$\mathbf{Z}_{10}^{ij}(\cdot, 0, x_3) \in (H_{\#}^1(\Theta_F)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

and the integral equality

$$(6.20) \quad \int_{\Theta_F} \left\{ \alpha_\gamma^{-1} \left(\delta_{ij} + \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{ij}(\widehat{\mathbf{z}}, x_3) \right) \mathbb{I} + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, 0, x_3) \right. \\ \left. + 2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, 0, x_3)) \right\} : \mathbb{D}_{\widehat{\mathbf{z}}}(\boldsymbol{\phi}_{21}) d\widehat{\mathbf{z}} = 0, \\ \forall \boldsymbol{\phi}_{21} \in H_{\#}^1(\Theta)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].$$

In (6.20), \mathbf{Z}_{00}^{ij} is the solution of Problem Z1.

Remark 7. If $\mathbf{Z}_{11}^{ij}(\widehat{\mathbf{z}}, 0, x_3) \equiv 0$ and the vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ is the solution of Problem Z2 then the vector-function $\mathbf{Z}_1^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ defined by (6.3) satisfies (6.9) ($i, j = 1, 2, 3$).

Problem Z3. It is necessary to find a vector-function $(\widehat{\mathbf{z}}, t) \mapsto \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3)$ ($i, j = 1, 2, 3$), defined in $\Theta \times (0, T)$ for all $x_3 \in [0, 1]$, which vanishes for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$(6.21a) \quad \mathbf{Z}_{10}^{ij}(\cdot, \cdot, x_3) \in L^\infty(0, T; (H_{\#}^1(\Theta)/\mathbb{R})^3) \cap H^1(0, T; H_{\#}^1(\Theta_F)^3),$$

the integral equality

$$(6.21b) \quad \int_{\Theta_F} \left\{ \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}} \frac{\partial \mathbf{Z}_{10}^{ij}}{\partial t}(\widehat{\mathbf{z}}, t, x_3) + 2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}\left(\frac{\partial \mathbf{Z}_{10}^{ij}}{\partial t}(\widehat{\mathbf{z}}, t, x_3)\right) \right. \\ \left. + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3) \right\} : \mathbb{D}_{\widehat{\mathbf{z}}}(\boldsymbol{\phi}_{21}) d\widehat{\mathbf{z}} \\ + \int_{\Theta_S} \left\{ \boldsymbol{\mathcal{G}} : \nabla_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3) \right\} : \nabla_{\widehat{\mathbf{z}}} \boldsymbol{\phi}_{21} d\widehat{\mathbf{z}} = 0, \\ \forall \boldsymbol{\phi}_{21} \in H_{\#}^1(\Theta)^3 \quad \text{for a.e. } t \in (0, T], \quad x_3 \in [\Delta, \Delta + \delta_*],$$

and the initial condition

$$(6.21c) \quad \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3)|_{t=0} = \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, 0, x_3) \quad \text{for } \widehat{\mathbf{z}} \in \Theta_F, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

where $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ is the solution of Problem Z2.

Remark 8. In the sense of distributions, the integral equality (6.21b) is equivalent to the system consisting of the Stokes equations

$$\operatorname{div}_{\widehat{\mathbf{z}}}\left[2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}\left(\frac{\partial \mathbf{Z}_{10}^{ij}}{\partial t}(\widehat{\mathbf{z}}, t, x_3)\right) + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}}\frac{\partial \mathbf{Z}_{10}^{ij}}{\partial t}(\widehat{\mathbf{z}}, t, x_3) + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}}\mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3)\right] = 0, \\ \widehat{\mathbf{z}} \in \Theta_F, \quad t > 0,$$

(written in terms of the ‘displacements’ \mathbf{Z}_{10}^{ij}), the linear elasticity equations

$$\operatorname{div}_{\widehat{\mathbf{z}}}\left[\boldsymbol{\mathcal{G}} : \nabla_{\widehat{\mathbf{z}}}\mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3)\right] = 0, \quad \widehat{\mathbf{z}} \in \Theta_S, \quad t > 0,$$

and the interface condition

$$\left[2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}\left(\frac{\partial \mathbf{Z}_{10}^{ij}}{\partial t}(\widehat{\mathbf{z}}, t, x_3)\right) + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}}\frac{\partial \mathbf{Z}_{10}^{ij}}{\partial t}(\widehat{\mathbf{z}}, t, x_3) \right. \\ \left. + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}}\mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3)\right] \mathbf{n}_\Theta \Big|_{\partial\Theta_F \setminus \partial\Theta} = \left[\boldsymbol{\mathcal{G}} : \nabla_{\widehat{\mathbf{z}}}\mathbf{Z}_{10}^{ij}(\widehat{\mathbf{z}}, t, x_3)\right] \mathbf{n}_\Theta \Big|_{\partial\Theta_S}, \quad t > 0.$$

Here and further in Sec. 6, by \mathbf{n}_Θ we denote the unit outward normal to $\partial\Theta_S$. The initial condition (6.21c) is understood in the trace sense.

Remark 9. If $Z_{11}^{ij}(\widehat{z}, t, x_3) \equiv 0$ and Z_{10}^{ij} is the solution of Problem Z3 then the vector-function Z_1^{ij} defined by (6.3) satisfies (6.10) ($i, j = 1, 2, 3$).

Problem Z4. It is necessary to find a vector-function $\widehat{z} \mapsto Z_{20}(\widehat{z}, 0, x_3)$, defined in Θ_F for all $x_3 \in [0, 1]$, which is identically equal to zero for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$Z_{20}(\cdot, 0, x_3) \in (H_{\#}^1(\Theta_F)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

and the integral equality

$$\int_{\Theta_F} \left\{ -\mathbb{I} + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{z}} Z_{20}(\widehat{z}, 0, x_3) + 2\alpha_\mu \mathbb{D}_{\widehat{z}}(Z_{20}(\widehat{z}, 0, x_3)) \right\} : \mathbb{D}_{\widehat{z}}(\phi_{21}) d\widehat{z} = 0, \\ \forall \phi_{21} \in H_{\#}^1(\Theta)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].$$

Remark 10. If $Z_{21}^{ij}(\widehat{z}, 0, x_3) \equiv 0$ and the vector-function $\widehat{z} \mapsto Z_{20}^{ij}(\widehat{z}, 0, x_3)$ is the solution of Problem Z4 then the vector-function $Z_2^{ij}(\widehat{z}, 0, x_3)$ defined by (6.4) satisfies (6.11) ($i, j = 1, 2, 3$).

Problem Z5. It is necessary to find a vector-function $(\widehat{z}, t) \mapsto Z_{20}(\widehat{z}, t, x_3)$, defined in $\Theta \times (0, T)$ for all $x_3 \in [0, 1]$, which is identically equal to zero for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$(6.23a) \quad Z_{20}(\cdot, \cdot, x_3) \in L^\infty(0, T; (H_{\#}^1(\Theta)/\mathbb{R})^3) \cap H^1(0, T; H_{\#}^1(\Theta_F)^3),$$

the integral equality

$$(6.23b) \quad \int_{\Theta_F} \left\{ 2\alpha_\mu \mathbb{D}_{\widehat{z}} \left(\frac{\partial Z_{20}}{\partial t}(\widehat{z}, t, x_3) \right) + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{z}} \frac{\partial Z_{20}}{\partial t}(\widehat{z}, t, x_3) + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\widehat{z}} Z_{20}(\widehat{z}, t, x_3) \right\} : \\ : \mathbb{D}_{\widehat{z}}(\phi_{21}) d\widehat{z} \\ + \int_{\Theta_S} \left\{ \mathcal{G} : \nabla_{\widehat{z}} Z_{20}(\widehat{z}, t, x_3) \right\} : \nabla_{\widehat{z}} \phi_{21} d\widehat{z} = 0,$$

$$\forall \phi_{21} \in H_{\#}^1(\Theta)^3 \quad \text{for a.e. } t \in (0, T], \quad \forall x_3 \in [\Delta, \Delta + \delta_*],$$

and the initial condition

$$Z_{20}(\widehat{z}, t, x_3)|_{t=0} = Z_{20}(\widehat{z}, 0, x_3) \quad \text{for } \widehat{z} \in \Theta_F, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

where $\widehat{z} \mapsto Z_{20}(\widehat{z}, 0, x_3)$ is the solution of Problem Z4.

Remark 11. If Z_{20}^{ij} is the solution of Problem Z5 and Z_{21}^{ij} is identically equal to zero then the vector-function Z_2^{ij} , defined by (6.4), satisfies (6.12) ($i, j = 1, 2, 3$).

Problem Z6. It is necessary to find a vector-function $\widehat{z} \mapsto Z_{30}^{ij}(\widehat{z}, 0, x_3)$ ($i, j = 1, 2, 3$), defined in Θ_F for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$Z_{30}^{ij}(\cdot, 0, x_3) \in (H_{\#}^1(\Theta_F)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

and the integral equality

$$(6.24) \quad \int_{\Theta_F} \left\{ 2\alpha_\mu \mathbb{D}_{\widehat{z}}(Z_{30}^{ij}(\widehat{z}, 0, x_3)) + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{z}} Z_{30}^{ij}(\widehat{z}, 0, x_3) \right\} : \mathbb{D}_{\widehat{z}}(\phi_{21}) d\widehat{z} \\ + \int_{\Theta_S} \left\{ \mathcal{G} : \mathbb{J}^{ij} \right\} : \nabla_{\widehat{z}} \phi_{21} d\widehat{z} = 0, \quad \forall \phi_{21} \in H_{\#}^1(\Theta)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].$$

Remark 12. *In the sense of distributions, (6.24) is equivalent to the mixed boundary value problem for the linear elliptic equation:*

$$\begin{cases} \operatorname{div}_{\widehat{\mathbf{z}}} \left[2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, 0, x_3)) + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, 0, x_3) \right] = 0, & \widehat{\mathbf{z}} \in \Theta_F, \\ \left[2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{30}(\widehat{\mathbf{z}}, 0, x_3)) + \alpha_\lambda \mathbb{I} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, 0, x_3) \right] \mathbf{n}_\Theta \Big|_{\partial\Theta_F \setminus \partial\Theta} = (\mathcal{G} : \mathbb{J}^{ij}) \mathbf{n}_\Theta \Big|_{\partial\Theta_S}, \\ \mathbf{Z}_{30}^{ij}(\cdot, 0, x_3) \in (H_{\#}^1(\Theta)/\mathbb{R})^3. \end{cases}$$

Remark 13. *If the vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ is the solution of Problem Z6 and $\mathbf{Z}_{31}^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ is identically equal to zero then the vector-function $\mathbf{Z}_3^{ij}(\widehat{\mathbf{z}}, 0, x_3)$, defined by (6.5), satisfies (6.13) ($i, j = 1, 2, 3$).*

Problem Z7. It is necessary to find a vector-function $(\widehat{\mathbf{z}}, t) \mapsto \mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, t, x_3)$ ($i, j = 1, 2, 3$), defined in the space-time parallelepiped $\Theta \times (0, T)$ for all $x_3 \in [0, 1]$, which is identically equal to zero for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$, satisfies the regularity condition

$$(6.25) \quad \mathbf{Z}_{30}^{ij}(\cdot, \cdot, x_3) \in L^\infty(0, T; (H_{\#}^1(\Theta)/\mathbb{R})^3) \cap H^1(0, T; H_{\#}^1(\Theta_F)^3),$$

the integral equality (6.23b) (with \mathbf{Z}_{30}^{ij} on the place of \mathbf{Z}_{20}) and the initial condition

$$\mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, t, x_3)|_{t=0} = \mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, 0, x_3) \quad \text{for } \widehat{\mathbf{z}} \in \Theta_F, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

where the vector-function $\widehat{\mathbf{z}} \mapsto \mathbf{Z}_{30}^{ij}(\widehat{\mathbf{z}}, 0, x_3)$ is the solution of Problem Z6.

Remark 14. *If \mathbf{Z}_{30}^{ij} is the solution of Problem Z7 and \mathbf{Z}_{31}^{ij} is identically equal to zero then the vector-function \mathbf{Z}_3^{ij} , defined by (6.5), satisfies (6.14) ($i, j = 1, 2, 3$).*

Remark 15. *Quite similarly as for Problem Z1, we have that each of Problems Z2, Z4, and Z6 has a unique weak solution due to the well-known theory of generalized solutions of equations of mathematical physics [18, 19]. Existence and uniqueness of weak solutions of Problems Z3, Z5 and Z7 is justified quite similarly to the well-posedness result for the cell problems in [10, Lem. 10].*

Remark 16. *Since $\mathbb{J}^{ij} = \mathbb{J}^{ji}$, $\delta_{ij} = \delta_{ji}$, and the elastic stiffness tensor \mathcal{G} is symmetric, the solutions of the cell problems \mathbf{Z}_{k0}^{ij} possess the symmetry property $\mathbf{Z}_{k0}^{ij} = \mathbf{Z}_{k0}^{ji}$ ($k = 0, 1, 3$, $i, j = 1, 2, 3$).*

Remark 17. *Notice that the vector-functions \mathbf{Z}_{00}^{ij} , \mathbf{Z}_{10}^{ij} , \mathbf{Z}_{20} , and \mathbf{Z}_{30}^{ij} do not vary with change of x_3 on the segment $\{\Delta \leq x_3 \leq \Delta + \delta_*\}$, i.e.,*

$$\frac{\partial \mathbf{Z}_{00}^{ij}}{\partial x_3} = \frac{\partial \mathbf{Z}_{10}^{ij}}{\partial x_3} = \frac{\partial \mathbf{Z}_{20}}{\partial x_3} = \frac{\partial \mathbf{Z}_{30}^{ij}}{\partial x_3} = 0 \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*],$$

since the given data in Problems Z1-Z7 do not vary with change of x_3 on this segment.

Let us complete outline in Sec. 6 by the following observation.

Remark 18. *We have established that if \mathbf{Z}_{00}^{ij} , $\mathbf{Z}_{10}^{ij}|_{t=0}$, \mathbf{Z}_{10}^{ij} , $\mathbf{Z}_{20}|_{t=0}$, \mathbf{Z}_{20} , $\mathbf{Z}_{30}^{ij}|_{t=0}$, and \mathbf{Z}_{30}^{ij} are the solutions of Problems Z1-Z7, respectively, then (6.6) holds true independently of the possible values of the vector-functions \mathbf{u} and $\mathbf{u}^{(1)}$ and the macroscopic boundary conditions (5.12)-(5.13). In particular, this means that the solutions of Problems Z1-Z7 fully define dynamics of the mechanical system on the microscale, i.e., on the characteristic scale of the shorter bristles.*

7. TWO-SCALE HOMOGENIZED COEFFICIENTS AND THEIR PROPERTIES

In this and the next sections, using the solutions of Problems Z1-Z7 we derive a two-scale system of equations for the two sought vector-functions $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\mathbf{u}^{(1)} = \mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t)$. This system does not incorporate variable $\hat{\mathbf{z}}$ explicitly, which means separation of the microscopic scale from the meso- and macroscopic ones in the homogenized description of the mechanical system. The microscopic velocity $\mathbf{u}^{(2)}$ and the variable $\hat{\mathbf{z}}$ get sublimed in the coefficients of the two-scale model.

With account of the identities $\mathbf{Z}_{01}^{ij} \equiv 0$, $\mathbf{Z}_{11}^{ij} \equiv 0$, $\mathbf{Z}_{21} \equiv 0$, and $\mathbf{Z}_{31}^{ij} \equiv 0$, the representation (6.1) takes the form

$$(7.1) \quad \mathbf{u}^{(2)}(\mathbf{x}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, t) = \zeta(\hat{\mathbf{y}}, x_3) \sum_{i,j=1}^3 \left[\left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) + \frac{\partial u_i^{(1)}}{\partial y_j}(\mathbf{x}, \hat{\mathbf{y}}, t) \right) \mathbf{Z}_{00}^{ij}(\hat{\mathbf{z}}, x_3) \right. \\ \left. + \int_0^t \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, \tau) + \frac{\partial u_i^{(1)}}{\partial y_j}(\mathbf{x}, \hat{\mathbf{y}}, \tau) \right) \mathbf{Z}_{10}^{ij}(\hat{\mathbf{z}}, t - \tau, x_3) d\tau \right] \\ + \zeta(\hat{\mathbf{y}}, x_3) p^0(\mathbf{x}) \mathbf{Z}_{20}(\hat{\mathbf{z}}, t, x_3) + \zeta(\hat{\mathbf{y}}, x_3) \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \mathbf{Z}_{30}^{ij}(\hat{\mathbf{z}}, t, x_3).$$

In (5.5), we take $\phi_2 \equiv 0$ and insert (7.1). Taking into account (1.3), (2.9), (2.11), (4.2)-(4.4), (5.8), Remark 16, and assertion (iii) of Proposition A.1, and properly collecting terms, after simple but rather lengthy calculations, we establish that, for arbitrary test vector-functions $\phi = \phi(\mathbf{x}, t)$ and $\phi_1 = \phi_1(\mathbf{x}, \hat{\mathbf{y}}, t)$ such that ϕ and ϕ_1 vanish in the neighborhood of $\partial\Omega$ and the section $\{t = T\}$, and ϕ_1 is 1-periodic in $\hat{\mathbf{y}}$, the integral equality

$$(7.2) \quad \int_0^T \int_{\Omega} \left\{ -\alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{u}(\mathbf{x}, t) \cdot \partial_t \phi(\mathbf{x}, t) \right. \\ \left. + \int_{\Sigma} \left[\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : \left(\mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t)) \right) \right. \right. \\ \left. \left. + \mathcal{B}_0(\hat{\mathbf{y}}, x_3) : \left(\mathbb{D}_x((J_t \mathbf{u})(\mathbf{x}, t)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_t \mathbf{u}^{(1)})(\mathbf{x}, \hat{\mathbf{y}}, t)) \right) \right. \right. \\ \left. \left. + \int_0^t \mathcal{A}_1(\hat{\mathbf{y}}, x_3, t - \tau) : \left(\mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, \tau)) \right) \right. \right. \\ \left. \left. + \int_0^t \mathcal{B}_1(\hat{\mathbf{y}}, x_3, t - \tau) : \left[\mathbb{D}_x((J_{\tau} \mathbf{u})(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_{\tau} \mathbf{u}^{(1)})(\mathbf{x}, \hat{\mathbf{y}}, \tau)) \right] \right. \right. \\ \left. \left. + \mathbb{F}^0(\hat{\mathbf{y}}, \mathbf{x}, t) \right] : \left(\mathbb{D}_x(\phi(\mathbf{x}, t)) + \mathbb{D}_{\hat{\mathbf{y}}}(\phi_1(\mathbf{x}, \hat{\mathbf{y}}, t)) \right) d\hat{\mathbf{y}} \right\} d\mathbf{x} dt \\ = \int_0^T \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{f} \cdot \phi d\mathbf{x} dt + \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{u}^0(\mathbf{x}) \cdot \phi(\mathbf{x}, 0) d\mathbf{x}$$

holds true. Here the fourth-rank tensors \mathcal{A}_0 , \mathcal{B}_0 , \mathcal{A}_1 , and \mathcal{B}_1 , and the matrix \mathbb{F}^0 are defined via the solutions of the microscopic cell problems by the formulas

$$(7.3) \quad \mathcal{A}_0(\hat{\mathbf{y}}, x_3) = \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_{\lambda} \mathbb{I} \otimes \left(\langle \psi \rangle_{\Theta}(x_3) \mathbb{I} + \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_{\Theta}(x_3) \mathbb{J}^{mn} \right) \right. \\ \left. + 2\alpha_{\mu} \sum_{m,n=1}^3 \left(\langle \psi \rangle_{\Theta}(x_3) \mathbb{J}^{mn} + \langle \hat{\psi} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}) \rangle_{\Theta}(x_3) \right) \otimes \mathbb{J}^{mn} \right],$$

(7.4)

$$\begin{aligned} \mathcal{B}_0(\hat{\mathbf{y}}, x_3) &= \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\gamma^{-1} \mathbb{I} \otimes \left(\langle \psi \rangle_\Theta(x_3) \mathbb{I} + \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_\Theta(x_3) \mathbb{J}^{mn} \right) \right. \\ &\quad \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \hat{\psi}) \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}) \rangle_\Theta(x_3) \right) \otimes \mathbb{J}^{mn} \right] + \left(1 - \langle \psi \rangle_\Theta(x_3) \zeta(\hat{\mathbf{y}}, x_3) \right) \mathcal{G}, \end{aligned}$$

(7.5)

$$\begin{aligned} \mathcal{A}_1(\hat{\mathbf{y}}, x_3, t) &= \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{10}^{mn} \rangle_\Theta(t, x_3) \mathbb{J}^{mn} \right. \\ &\quad \left. + 2\alpha_\mu \sum_{m,n=1}^3 \langle \hat{\psi} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}) \rangle_\Theta(t, x_3) \otimes \mathbb{J}^{mn} \right], \end{aligned}$$

(7.6)

$$\begin{aligned} \mathcal{B}_1(\hat{\mathbf{y}}, x_3, t) &= \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{10}^{mn} \rangle_\Theta(t, x_3) \mathbb{J}^{mn} \right. \\ &\quad \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \hat{\psi}) \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}) \rangle_\Theta(t, x_3) \right) \otimes \mathbb{J}^{mn} \right], \end{aligned}$$

(7.7)

$$\begin{aligned} \mathbb{F}^0(\hat{\mathbf{y}}, \mathbf{x}, t) &= \zeta(\hat{\mathbf{y}}, x_3) \left[p^0(\mathbf{x}) \left(\alpha_\lambda \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{20} \rangle_\Theta(t, x_3) \mathbb{I} + 2\alpha_\mu \langle \hat{\psi} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{20}) \rangle_\Theta(t, x_3) \right) \right. \\ &\quad \left. + \alpha_\gamma^{-1} \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} J_t \mathbf{Z}_{20} \rangle_\Theta(t, x_3) \mathbb{I} + \mathcal{G} : \langle (1 - \hat{\psi}) \mathbb{D}_{\hat{\mathbf{z}}}(J_t \mathbf{Z}_{20}) \rangle_\Theta(t, x_3) - \langle \psi \rangle_\Theta(x_3) \mathbb{I} \right) \\ &\quad + \sum_{m,n=1}^3 \frac{\partial v_m^0}{\partial x_n}(\mathbf{x}) \left(\alpha_\lambda \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{30}^{mn} \rangle_\Theta(t, x_3) \mathbb{I} + 2\alpha_\mu \langle \hat{\psi} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{30}^{mn}) \rangle_\Theta(t, x_3) \right. \\ &\quad \left. + \alpha_\gamma^{-1} \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} J_t \mathbf{Z}_{30}^{mn} \rangle_\Theta(t, x_3) \mathbb{I} + \mathcal{G} : \langle (1 - \hat{\psi}) \mathbb{D}_{\hat{\mathbf{z}}}(J_t \mathbf{Z}_{30}^{mn}) \rangle_\Theta(t, x_3) \right) \Big] \\ &\quad + \left(1 - \langle \psi \rangle_\Theta(x_3) \zeta(\hat{\mathbf{y}}, x_3) \right) \mathcal{G} : \nabla_{\mathbf{x}} \mathbf{v}^0. \end{aligned}$$

In the component-wise form, (7.3)-(7.7) are as follows ($i, j, k, l = 1, 2, 3$):

(7.8)

$$\begin{aligned} \mathcal{A}_0^{ijkl}(\hat{\mathbf{y}}, x_3) &= \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\lambda \delta_{ij} \left(\langle \psi \rangle_\Theta(x_3) \delta_{kl} + \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_\Theta(x_3) (\mathbb{J}^{mn})_{kl} \right) \right. \\ &\quad \left. + 2\alpha_\mu \sum_{m,n=1}^3 \left(\langle \psi \rangle_\Theta(x_3) (\mathbb{J}^{mn})_{ij} + \langle \hat{\psi} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}) \rangle_\Theta(x_3) (\mathbb{J}^{mn})_{kl} \right) \right], \end{aligned}$$

(7.9)

$$\begin{aligned} \mathcal{B}_0^{ijkl}(\hat{\mathbf{y}}, x_3) &= \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\gamma^{-1} \delta_{ij} \left(\langle \psi \rangle_\Theta(x_3) \delta_{kl} + \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_\Theta(x_3) (\mathbb{J}^{mn})_{kl} \right) \right. \\ &\quad \left. + \sum_{m,n=1}^3 \left(\sum_{q,r=1}^3 \mathcal{G}^{ijqr} \langle (1 - \hat{\psi}) \mathbb{D}_{\hat{\mathbf{z}}_{qr}}(\mathbf{Z}_{00}^{mn}) \rangle_\Theta(x_3) (\mathbb{J}^{mn})_{kl} \right) \right] + \left(1 - \langle \psi \rangle_\Theta(x_3) \zeta(\hat{\mathbf{y}}, x_3) \right) \mathcal{G}^{ijkl}, \end{aligned}$$

$$(7.10) \quad \mathcal{A}_1^{ijkl}(\hat{\mathbf{y}}, x_3, t) = \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\lambda \delta_{ij} \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{10}^{mn} \rangle_{\Theta}(t, x_3) (\mathbb{J}^{mn})_{kl} \right. \\ \left. + 2\alpha_\mu \sum_{m,n=1}^3 \langle \hat{\psi} D_{\hat{\mathbf{z}}_{ij}}(\mathbf{Z}_{10}^{mn}) \rangle_{\Theta}(t, x_3) (\mathbb{J}^{mn})_{kl} \right],$$

$$(7.11) \quad \mathcal{B}_1^{ijkl}(\hat{\mathbf{y}}, x_3, t) = \zeta(\hat{\mathbf{y}}, x_3) \left[\alpha_\gamma^{-1} \delta_{ij} \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{10}^{mn} \rangle_{\Theta}(t, x_3) (\mathbb{J}^{mn})_{kl} \right. \\ \left. + \sum_{m,n=1}^3 \left(\sum_{q,r=1}^3 \mathcal{G}^{ijqr} \langle (1 - \hat{\psi}) D_{\hat{\mathbf{z}}_{qr}}(\mathbf{Z}_{10}^{mn}) \rangle_{\Theta}(t, x_3) \right) (\mathbb{J}^{mn})_{kl} \right],$$

(7.12)

$$F_{ij}^0(\hat{\mathbf{y}}, \mathbf{x}, t) = \zeta(\hat{\mathbf{y}}, x_3) \left[p^0(\mathbf{x}) \left(\alpha_\lambda \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{20} \rangle_{\Theta}(t, x_3) \delta_{ij} \right. \right. \\ \left. \left. + 2\alpha_\mu \langle \hat{\psi} D_{\hat{\mathbf{z}}_{ij}}(\mathbf{Z}_{20}) \rangle_{\Theta}(t, x_3) + \alpha_\gamma^{-1} \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} J_t \mathbf{Z}_{20} \rangle_{\Theta}(t, x_3) \delta_{ij} \right. \right. \\ \left. \left. + \sum_{q,r=1}^3 \mathcal{G}^{ijqr} \langle (1 - \hat{\psi}) D_{\hat{\mathbf{z}}_{qr}}(J_t \mathbf{Z}_{20}) \rangle_{\Theta}(t, x_3) - \langle \psi \rangle_{\Theta}(x_3) \delta_{ij} \right) \right. \\ \left. + \sum_{m,n=1}^3 \frac{\partial v_m^0}{\partial x_n}(\mathbf{x}) \left(\alpha_\lambda \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{30}^{mn} \rangle_{\Theta}(t, x_3) \delta_{ij} + 2\alpha_\mu \langle \hat{\psi} D_{\hat{\mathbf{z}}_{ij}}(\mathbf{Z}_{30}^{mn}) \rangle_{\Theta}(t, x_3) \right. \right. \\ \left. \left. + \alpha_\gamma^{-1} \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} J_t \mathbf{Z}_{30}^{mn} \rangle_{\Theta}(t, x_3) \delta_{ij} + \sum_{q,r=1}^3 \mathcal{G}^{ijqr} \langle (1 - \hat{\psi}) D_{\hat{\mathbf{z}}_{qr}}(J_t \mathbf{Z}_{30}^{mn}) \rangle_{\Theta}(t, x_3) \right) \right] \\ \left. + \left(1 - \langle \psi \rangle_{\Theta}(x_3) \zeta(\hat{\mathbf{y}}, x_3) \right) \sum_{q,r=1}^3 \mathcal{G}^{ijqr} \frac{\partial v_q^0}{\partial x_r}(\mathbf{x}). \right.$$

Furthermore, let us write out (7.3)-(7.7) in more detail, inserting (4.3) and (4.4) and using the property that \mathbf{Z}_{00}^{mn} , \mathbf{Z}_{10}^{mn} , \mathbf{Z}_{20} , and \mathbf{Z}_{30}^{mn} vanish for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$. We get

(7.13)

$$\mathcal{A}_0(\hat{\mathbf{y}}, x_3) = \begin{cases} \alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \hat{\zeta}(\hat{\mathbf{y}}) \left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \hat{\zeta}(\hat{\mathbf{y}}) \mathcal{A}_0^f & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(7.14) \quad \mathcal{A}_0^f \stackrel{def}{=} \alpha_\lambda \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \hat{\psi} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_{\Theta} \mathbb{J}^{mn} \right) \\ + 2\alpha_\mu \sum_{m,n=1}^3 \left(\theta \mathbb{J}^{mn} + \langle \hat{\psi} \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn};$$

$$(7.15) \quad \mathcal{B}_0(\widehat{\mathbf{y}}, x_3) = \begin{cases} \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \widehat{\zeta}(\widehat{\mathbf{y}}) \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{B}_0^f + (1 - \theta \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \mathcal{G} & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(7.16) \quad \mathcal{B}_0^f \stackrel{def}{=} \alpha_\gamma^{-1} \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_{\Theta} \mathbb{J}^{mn} \right) + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn};$$

$$(7.17) \quad \mathcal{A}_1(\widehat{\mathbf{y}}, x_3, t) = \begin{cases} \widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{A}_1^f(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1), \end{cases}$$

where

$$(7.18) \quad \mathcal{A}_1^f(t) \stackrel{def}{=} \alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn} \rangle_{\Theta}(t) \mathbb{J}^{mn} + 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi} \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}) \rangle_{\Theta}(t) \otimes \mathbb{J}^{mn};$$

$$(7.19) \quad \mathcal{B}_1(\widehat{\mathbf{y}}, x_3, t) = \begin{cases} \widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{B}_1^f(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1), \end{cases}$$

where

$$(7.20) \quad \mathcal{B}_1^f(t) \stackrel{def}{=} \alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn} \rangle_{\Theta}(t) \mathbb{J}^{mn} + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}) \rangle_{\Theta}(t) \right) \otimes \mathbb{J}^{mn};$$

(7.21)

$$\mathbb{F}^0(\widehat{\mathbf{y}}, \mathbf{x}, t) = \begin{cases} -p^0(\mathbf{x}) \mathbb{I} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ -\widehat{\zeta}(\widehat{\mathbf{y}}) p^0(\mathbf{x}) \mathbb{I} + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \nabla_x \mathbf{v}^0 & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \widehat{\zeta}(\widehat{\mathbf{y}}) \left(p^0(\mathbf{x}) \mathbb{F}_f^0(t) + \sum_{m,n=1}^3 \frac{\partial v_m^0}{\partial x_n}(\mathbf{x}) \mathbb{F}_{sol}^{0mn}(t) \right) + (1 - \theta \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \nabla_x \mathbf{v}^0 & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \mathcal{G} : \nabla_x \mathbf{v}^0 & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(7.22) \quad \mathbb{F}_f^0(t) \stackrel{def}{=} \alpha_\lambda \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{20} \rangle_{\Theta}(t) \mathbb{I} + 2\alpha_\mu \langle \widehat{\psi} \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{20}) \rangle_{\Theta}(t) + \alpha_\gamma^{-1} \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} J_t \mathbf{Z}_{20} \rangle_{\Theta}(t) \mathbb{I} + \mathcal{G} : \langle (1 - \widehat{\psi}) \mathbb{D}_{\widehat{\mathbf{z}}}(J_t \mathbf{Z}_{20}) \rangle_{\Theta}(t) - \theta \mathbb{I}$$

and

$$(7.23) \quad \mathbb{F}_{sol}^{0mn}(t) \stackrel{def}{=} \alpha_\lambda \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{30}^{mn} \rangle_{\Theta}(t) \mathbb{I} + 2\alpha_\mu \langle \widehat{\psi} \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{30}^{mn}) \rangle_{\Theta}(t)$$

$$+ \alpha_\gamma^{-1} \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} J_t \mathbf{Z}_{30}^{mn} \rangle_{\Theta}(t) \mathbb{I} + \mathcal{G} : \langle (1 - \widehat{\psi}) \mathbb{D}_{\widehat{\mathbf{z}}} (J_t \mathbf{Z}_{30}^{mn}) \rangle_{\Theta}(t),$$

$$m, n = 1, 2, 3.$$

Remark 19. Due to Remark 17, the tensors $\mathcal{A}_0^f, \mathcal{B}_0^f, \mathcal{A}_1^f(t),$ and $\mathcal{B}_1^f(t),$ and the matrices $\mathbb{F}_f^0(t)$ and $\mathbb{F}_{sol}^{0mn}(t),$ defined in (7.13)-(7.22), do not depend on $x_3.$

We establish the following main properties of the two-scale coefficients.

Proposition 6. (i) *The tensor-valued functions*

$$\mathcal{A}_0 = \mathcal{A}_0(\widehat{\mathbf{y}}, x_3), \mathcal{B}_0 = \mathcal{B}_0(\widehat{\mathbf{y}}, x_3), \mathcal{A}_1 = \mathcal{A}_1(\widehat{\mathbf{y}}, x_3, t), \mathcal{B}_1 = \mathcal{B}_1(\widehat{\mathbf{y}}, x_3, t),$$

and the matrix-valued function $\mathbb{F}^0 = \mathbb{F}^0(\widehat{\mathbf{y}}, \mathbf{x}, t)$ satisfy the regularity conditions

$$(7.24) \quad \mathcal{A}_0, \mathcal{B}_0, \mathcal{A}_1, \mathcal{B}_1, \text{ and } \mathbb{F}^0 \text{ are 1-periodic in } \widehat{\mathbf{y}},$$

$$(7.25) \quad \mathcal{A}_0, \mathcal{B}_0 \in L^\infty(\Sigma \times (0, 1))^{3 \times 3 \times 3 \times 3},$$

$$(7.26) \quad \mathcal{A}_1, \mathcal{B}_1, \partial_t \mathcal{A}_1, \partial_t \mathcal{B}_1 \in L^\infty(\Sigma \times (0, 1) \times (0, T))^{3 \times 3 \times 3 \times 3},$$

$$(7.27) \quad \mathbb{F}^0, \partial_t \mathbb{F}^0 \in L^\infty(\Sigma \times (0, T); L^2(\Omega))^{3 \times 3}.$$

(ii) *The tensor-valued function \mathcal{A}_0 satisfies the finiteness property*

$$(7.28) \quad \mathcal{A}_0 = 0 \text{ for } (\widehat{\mathbf{y}}, x_3) \in (\Sigma \times [0, \Delta]) \cup (\overline{\Sigma}_S \times [\Delta, \Delta + \delta^*]),$$

the symmetry property

$$(7.29) \quad \mathcal{A}_0^{ijkl} = \mathcal{A}_0^{jikl} = \mathcal{A}_0^{jilk} = \mathcal{A}_0^{klij}, \quad \forall (\widehat{\mathbf{y}}, x_3) \in \Sigma \times [0, 1], \quad \forall i, j, k, l = 1, 2, 3,$$

and the uniform positive definiteness property:

$$(7.30) \quad \text{there exists a constant } C_5 > 0 \text{ such that } (\mathcal{A}_0(\widehat{\mathbf{y}}, x_3) : \mathbb{X}) : \mathbb{X} \geq C_5 |\mathbb{X}|^2,$$

$$\forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall (\widehat{\mathbf{y}}, x_3) \in (\Sigma_F \times [\Delta, \Delta + \delta^*]) \cup (\Sigma \times [\Delta + \delta^*, 1]).$$

(iii) *The tensor-valued function*

$$(7.31) \quad \mathcal{H}^s(\widehat{\mathbf{y}}, x_3) \stackrel{\text{def}}{=} s \mathcal{A}_0(\widehat{\mathbf{y}}, x_3) + \mathcal{B}_0(\widehat{\mathbf{y}}, x_3) + s \overline{\mathcal{A}}_1(\widehat{\mathbf{y}}, x_3, s) + \overline{\mathcal{B}}_1(\widehat{\mathbf{y}}, x_3, s)$$

satisfies the symmetry properties

$$(7.32) \quad \mathcal{H}^{sijkl} = \mathcal{H}^{sjikl} = \mathcal{H}^{sjilk}, \quad \forall (\widehat{\mathbf{y}}, x_3) \in \Sigma \times [0, 1], \quad \forall s > 0,$$

$\forall i, j, k, l = 1, 2, 3,$ and

$$(7.33) \quad (\mathcal{H}^s : \mathbb{X}) : \mathbb{W} = (\mathcal{H}^s : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

and the uniform positive definiteness property:

$$(7.34) \quad \text{there exists a constant } C_6^s > 0 \text{ such that } (\mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : \mathbb{X}) : \mathbb{X} \geq C_6^s |\mathbb{X}|^2,$$

$$\forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall (\widehat{\mathbf{y}}, x_3) \in \Sigma \times [0, 1], \quad \forall s > 0.$$

(iv) *For a.e. $(\widehat{\mathbf{y}}, \mathbf{x}, t) \in \Sigma \times \Omega \times [0, T]$ matrix $\mathbb{F}^0(\widehat{\mathbf{y}}, \mathbf{x}, t)$ is symmetric.*

Notation 4. In (7.31), by $\overline{\mathcal{A}}_1$ and $\overline{\mathcal{B}}_1$ the respective Laplace transforms in t of \mathcal{A}_1 and \mathcal{B}_1 are denoted. It is assumed that \mathcal{A}_1 and \mathcal{B}_1 vanish for $t > T.$

Recall that the Laplace transform of a function $\varphi(t)$ is defined by the formula

$$(7.35) \quad \overline{\varphi}(s) = \mathcal{L}[\varphi](s) = \int_0^\infty \varphi(t) e^{-st} dt, \quad s > 0.$$

Some basic facts about the Laplace transform and the inverse Laplace transform are given in Appendix A in the end of the article.

Proof of Proposition 6. (i) The periodicity conditions (7.24) hold since the characteristic function ζ is 1-periodic in $\widehat{\mathbf{y}}$ in representations (7.3)-(7.7) (equivalently, in (7.8)-(7.12)). The regularity properties (7.25)-(7.27) hold true due to the regularity properties (6.19), (6.21a), (6.23a), and (6.25) of the functions \mathbf{Z}_{00}^{ij} , \mathbf{Z}_{10}^{ij} , \mathbf{Z}_{20} , and \mathbf{Z}_{30} , respectively, and due to the conditions $p^0 \in L^2(\Omega)$ and $\mathbf{v}^0 \in H^1(\Omega)^3$, imposed in the assumptions of Propositions 1 and 2.

(ii) The finiteness property (7.28), the symmetry property (7.29) for $(\widehat{\mathbf{y}}, x_3) \in \Sigma \times ((0, \Delta) \cup (\Delta + \delta_*, 1))$, and the uniform positive definiteness property (7.30) for $(\widehat{\mathbf{y}}, x_3) \in (\Sigma_F \times (\Delta + \delta_*, \Delta + \delta^*)) \cup (\Sigma \times [\Delta + \delta^*, 1))$ directly follow from (7.13).

In order to establish (7.29) and (7.30) for $x_3 \in [\Delta, \Delta + \delta_*]$, we recall the formulation of Problem Z1. Substituting $\phi_{21} = \mathbf{Z}_{00}^{kl}(\widehat{\mathbf{z}}, x_3)$ for the test vector-function into (6.15a), which is legal, we deduce

$$(7.36) \quad \int_{\Theta_F} \left(\alpha_\lambda \delta_{ij} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{kl} + 2\alpha_\mu \mathbb{J}^{ij} : \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{kl}) + \alpha_\lambda \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{ij} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{kl} + 2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{ij}) : \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{kl}) \right) d\widehat{\mathbf{z}} = 0.$$

Here, using the symmetry property $\mathbf{Z}_{00}^{mn} = \mathbf{Z}_{00}^{nm}$ (see Remark 16), we can represent

$$(7.37) \quad \int_{\Theta_F} \alpha_\lambda \delta_{ij} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{kl} d\widehat{\mathbf{z}} = \sum_{m,n=1}^3 \alpha_\lambda \delta_{ij} \delta_{mk} \delta_{nl} \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_\Theta = \alpha_\lambda \delta_{ij} \sum_{m,n=1}^3 \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_\Theta (\mathbb{J}^{mn})_{kl},$$

$$(7.38) \quad \int_{\Theta_F} 2\alpha_\mu \mathbb{J}^{ij} : \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{kl}) d\widehat{\mathbf{z}} = \int_{\Theta_F} 2\alpha_\mu D_{\widehat{\mathbf{z}}ij}(\mathbf{Z}_{00}^{kl}) d\widehat{\mathbf{z}} = 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi} D_{\widehat{\mathbf{z}}ij}(\mathbf{Z}_{00}^{mn}) \rangle_\Theta \delta_{mk} \delta_{nl} = 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi} D_{\widehat{\mathbf{z}}ij}(\mathbf{Z}_{00}^{mn}) \rangle_\Theta (\mathbb{J}^{mn})_{kl}.$$

Inserting (7.37) and (7.38) into (7.36), we arrive at the identity

$$\alpha_\lambda \delta_{ij} \sum_{m,n=1}^3 \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn} \rangle_\Theta (\mathbb{J}^{mn})_{kl} + 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi} D_{\widehat{\mathbf{z}}ij}(\mathbf{Z}_{00}^{mn}) \rangle_\Theta (\mathbb{J}^{mn})_{kl} = -\alpha_\lambda \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{ij} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{kl} \rangle_\Theta - 2\alpha_\mu \langle \widehat{\psi} \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{ij}) : \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{kl}) \rangle_\Theta, \quad i, j, k, l = 1, 2, 3.$$

Now, inserting this into (7.8) we establish the following representation for \mathcal{A}_0 on $\Sigma \times [\Delta, \Delta + \delta_*]$:

$$(7.39) \quad \mathcal{A}_0^{ijkl}(\widehat{\mathbf{y}}, x_3) = \widehat{\zeta}(\widehat{\mathbf{y}}) \left(\alpha_\lambda \theta \delta_{ij} \delta_{kl} + 2\alpha_\mu \theta (\mathbb{J}^{ij})_{kl} - \alpha_\lambda \langle \widehat{\psi} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{ij} \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{kl} \rangle_\Theta - 2\alpha_\mu \langle \widehat{\psi} \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{ij}) : \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{kl}) \rangle_\Theta \right), \quad i, j, k, l = 1, 2, 3.$$

Due to the symmetry property $\mathbf{Z}_{00}^{ij} = \mathbf{Z}_{00}^{ji}$, from (7.39) we immediately deduce (7.29) on $\Sigma \times [\Delta, \Delta + \delta_*]$.

Further, multiplying (7.36) by $\widehat{\zeta} X_{ij}X_{kl}$, where $\mathbb{X} = (X_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ is arbitrary, and taking the sum in i, j, k, l from 1 to 3, on $\Sigma \times [\Delta, \Delta + \delta_*]$ we get the identity

$$(7.40) \quad \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_F} \left[\alpha_\lambda \operatorname{tr} \mathbb{X} \operatorname{div}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) + 2\alpha_\mu \mathbb{X} : \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) \right. \\ \left. + \alpha_\lambda \operatorname{div}_{\widehat{\mathbf{z}}} \left(\sum_{i,j=1}^3 X_{ij} \mathbf{Z}_{00}^{ij} \right) \operatorname{div}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) \right. \\ \left. + 2\alpha_\mu \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{i,j=1}^3 X_{ij} \mathbf{Z}_{00}^{ij} \right) : \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) \right] d\widehat{\mathbf{z}} = 0.$$

Multiplying (7.8) by $X_{ij}X_{kl}$ and taking the sum in i, j, k, l from 1 to 3, we get (7.41)

$$(\mathcal{A}_0 : \mathbb{X}) : \mathbb{X} = \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_F} \left[\alpha_\lambda \operatorname{tr} \mathbb{X} \operatorname{tr} \mathbb{X} + 2\alpha_\mu \mathbb{X} : \mathbb{X} + \alpha_\lambda \operatorname{tr} \mathbb{X} \operatorname{div}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) \right. \\ \left. + 2\alpha_\mu \mathbb{X} : \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) \right] d\widehat{\mathbf{z}} \quad \text{on } \Sigma \times [\Delta, \Delta + \delta_*].$$

Now, by swapping the left and the right hand sides in (7.40) followed by adding (7.41), we establish the identity

$$(7.42) \quad (\mathcal{A}_0 : \mathbb{X}) : \mathbb{X} = \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_F} \left[\alpha_\lambda \left| \operatorname{div}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) + \operatorname{tr} \mathbb{X} \right|^2 \right. \\ \left. + 2\alpha_\mu \left| \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl} \right) + \mathbb{X} \right|^2 \right] d\widehat{\mathbf{z}} \quad \text{on } \Sigma \times [\Delta, \Delta + \delta_*].$$

From this representation it is clear that $(\mathcal{A}_0 : \mathbb{X}) : \mathbb{X} \geq 0$ for all symmetric $\mathbb{X} \in \mathbb{R}^{3 \times 3}$ for $x_3 \in [\Delta, \Delta + \delta_*]$. Further, notice that, due to Lemma 1, the property (7.30) is equivalent to the following one:

$$(7.43) \quad (\mathcal{A}_0 : \mathbb{X}) : \mathbb{X} \geq 0, \quad \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}; \\ (\mathcal{A}_0 : \mathbb{X}) : \mathbb{X} = 0 \text{ and } \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3} \text{ if and only if } \mathbb{X} = 0.$$

Now, the strict positive definiteness property (7.30) for \mathcal{A}_0 (with $x_3 \in [\Delta, \Delta + \delta_*]$) is justified by the contradiction method, following the lines of [10, Lem. 8] (see also [24, proof of Th. 2]). Suppose that for some $\mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ such that $\mathbb{X} \neq 0$, the equality $(\mathcal{A}_0 : \mathbb{X}) : \mathbb{X} = 0$ takes place. Hence due to (7.42) one has

$$(7.44) \quad \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl}(\widehat{\mathbf{z}}, x_3) \right) = -\mathbb{X}, \quad \widehat{\mathbf{z}} \in \Theta_F, \quad x_3 \in [\Delta, \Delta + \delta_*].$$

This equality immediately implies that the sum $\sum_{k,l=1}^3 X_{kl} \mathbf{Z}_{00}^{kl}(\widehat{\mathbf{z}}, x_3)$ is linear in

$\widehat{\mathbf{z}}$, that is, has the form $\mathbf{c}_0 + \sum_{k=1}^2 \mathbf{c}_k z_k$, where \mathbf{c}_k ($k = 0, 1, 2$) are some constant vectors. However, on the strength of 1-periodicity of \mathbf{Z}_{00}^{kl} in $\widehat{\mathbf{z}}$ and the condition

that the set $\bigcup_{\mathbf{k} \in \mathbb{Z}^2} (\Theta_F + \mathbf{k}) \subset \mathbb{R}_z^2$ is connected (see Sec. 3), this is possible only if $\mathbf{c}_k = 0$ for $k = 1, 2$. From this and equality (7.44) it follows that $\mathbb{X} = 0$, which contradicts the initial assumption $\mathbb{X} \neq 0$. Therefore (7.43) holds true. Hence by Lemma 1 the uniform positive definiteness property (7.30) is valid for \mathcal{A}_0 for $(\hat{\mathbf{y}}, x_3) \in \Sigma_F \times [\Delta, \Delta + \delta_*]$. Thus the proof of assertion (ii) is completed.

(iii) Substituting (7.13), (7.15), (7.17), and (7.19) into (7.31) and using linearity of the Laplace transform and formula $\mathcal{L}[1](s) = 1/s$ (see assertions (i) and (iv) of Proposition A.3), we arrive at the following representation for \mathcal{H}^s ($s > 0$):

$$(7.45) \quad \mathcal{H}^s(\hat{\mathbf{y}}, x_3) = \begin{cases} s\left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}\right) + \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ s\hat{\zeta}(\hat{\mathbf{y}})\left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}\right) \\ + \hat{\zeta}(\hat{\mathbf{y}})\alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \hat{\zeta}(\hat{\mathbf{y}}))\mathcal{G} & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ s\hat{\zeta}(\hat{\mathbf{y}})\mathcal{A}_0^f + \hat{\zeta}(\hat{\mathbf{y}})\mathcal{B}_0^f + (1 - \theta\hat{\zeta}(\hat{\mathbf{y}}))\mathcal{G} + s\hat{\zeta}(\hat{\mathbf{y}})\overline{\mathcal{A}}_1^f(s) + \hat{\zeta}(\hat{\mathbf{y}})\overline{\mathcal{B}}_1^f(s) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \mathcal{G} & \text{for } x_3 \in (0, \Delta). \end{cases}$$

By the straightforward calculation, we easily deduce that (7.32)-(7.34) follow for $(\hat{\mathbf{y}}, x_3) \in \Sigma \times ((0, \Delta) \cup (\Delta + \delta_*, 1))$ directly from (7.45).

Now let us establish (7.32)-(7.34) for $(\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*]$, which requires further careful and rather lengthy calculations. Applying the Laplace transform in t to the integral equality (6.21b) we get

$$(7.46) \quad \int_{\Theta_F} \left\{ s\alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \overline{\mathbf{Z}}_{10}^{ij}(\hat{\mathbf{z}}, s, x_3) - \alpha_\lambda \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \mathbf{Z}_{10}^{ij}(\hat{\mathbf{z}}, 0, x_3) + 2s\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\overline{\mathbf{Z}}_{10}^{ij}(\hat{\mathbf{z}}, s, x_3)) \right. \\ \left. - 2\alpha_\mu \mathbb{D}_{\hat{\mathbf{z}}}(\mathbf{Z}_{10}^{ij}(\hat{\mathbf{z}}, 0, x_3)) + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_{\hat{\mathbf{z}}} \overline{\mathbf{Z}}_{10}^{ij}(\hat{\mathbf{z}}, s, x_3) \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}} \\ + \int_{\Theta_S} \left\{ \mathcal{G} : \mathbb{D}_{\hat{\mathbf{z}}}(\overline{\mathbf{Z}}_{10}^{ij}(\hat{\mathbf{z}}, s, x_3)) \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}} = 0, \\ \forall \phi_{21} \in H_{\sharp}^1(\Theta)^3, \quad \forall s > 0, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].$$

Next, we multiply (6.15a) by sX_{ij} and (6.17), (6.20), and (7.46) by X_{ij} , where $\mathbb{X} = (X_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ is arbitrary. We take the sum of the resulting equalities considering the test vector-functions ϕ_{21} are the same, and then take the sum in i and j from 1 to 3. Suitably aggregating the terms and remarking that all terms with the initial function $\mathbf{Z}_{10}^{ij}(\hat{\mathbf{z}}, 0, x_3)$ cancel, we write out the resulting equality as follows:

$$(7.47) \quad \int_{\Theta_F} \left\{ (s\alpha_\lambda + \alpha_\gamma^{-1}) \left[\operatorname{div}_{\hat{\mathbf{z}}} \left(\sum_{i,j=1}^3 X_{ij} (\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \operatorname{tr} \mathbb{X} \right] \mathbb{I} \right. \\ \left. + 2s\alpha_\mu \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{i,j=1}^3 X_{ij} (\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \mathbb{X} \right] \right\} : \mathbb{D}_{\hat{\mathbf{z}}}(\phi_{21}) d\hat{\mathbf{z}}$$

$$+ \int_{\Theta_S} \left\{ \mathcal{G} : \left[\mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{i,j=1}^3 X_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \mathbb{X} \right] \right\} : \mathbb{D}_{\widehat{\mathbf{z}}}(\phi_{21}) d\widehat{\mathbf{z}} = 0.$$

In view of the further technical manipulations, let us substitute m and n for i and j in (7.47). After this, let us take $\phi_{21} = \sum_{i,j=1}^3 W_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij})$, which is a legal choice of a test-function. Here $\mathbb{W} = (W_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ is taken arbitrarily. Thus, from (7.47) we derive the integral identity

$$(7.48) \quad \int_{\Theta_F} \left\{ (s\alpha_\lambda + \alpha_\gamma^{-1}) \left[\text{div}_{\widehat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \text{tr } \mathbb{X} \right] \mathbb{I} \right. \\ \left. + 2s\alpha_\mu \left[\mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] \right\} : \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{i,j=1}^3 W_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) d\widehat{\mathbf{z}} \\ + \int_{\Theta_S} \left\{ \mathcal{G} : \left[\mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] \right\} : \mathbb{D}_{\widehat{\mathbf{z}}} \left(\sum_{i,j=1}^3 W_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) d\widehat{\mathbf{z}} = 0$$

for $s > 0$ and $x_3 \in [\Delta, \Delta + \delta_*]$. Now applying the Laplace transform in t to (7.10) and (7.11) and combining the results with (7.8) and (7.9), we write out the expression for the components of \mathcal{H}^s in $\Sigma \times [\Delta, \Delta + \delta_*]$ as follows ($i, j, m, n = 1, 2, 3$):

$$(7.49) \quad \mathcal{H}^{sijmn}(\widehat{\mathbf{y}}, x_3) = \\ = s\mathcal{A}_0^{ijmn}(\widehat{\mathbf{y}}, x_3) + \mathcal{B}_0^{ijmn}(\widehat{\mathbf{y}}, x_3) + s\overline{\mathcal{A}}_1^{ijmn}(\widehat{\mathbf{y}}, x_3, s) + \overline{\mathcal{B}}_1^{ijmn}(\widehat{\mathbf{y}}, x_3, s) \\ = \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_F} \left\{ \delta_{ij} (s\alpha_\lambda + \alpha_\gamma^{-1}) (\delta_{mn} + \text{div}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn})) \right. \\ \left. + 2s\alpha_\mu \left((\mathbb{J}^{mn})_{ij} + D_{\widehat{\mathbf{z}}ij}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) \right\} d\widehat{\mathbf{z}} \\ + \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_S} \left\{ \sum_{q,r=1}^3 \mathcal{G}^{ijqr} D_{\widehat{\mathbf{z}}qr}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) + \mathcal{G}^{ijmn} \right\} d\widehat{\mathbf{z}} + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G}^{ijmn},$$

In order to establish this representation, we have noticed in (7.9) that

$$(1 - \theta \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G}^{ijmn} = \widehat{\zeta}(\widehat{\mathbf{y}}) (1 - \theta) \mathcal{G}^{ijmn} + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G}^{ijmn} \\ = \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_S} \mathcal{G}^{ijmn} d\widehat{\mathbf{z}} + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G}^{ijmn}.$$

Due to the symmetry properties of \mathbf{Z}_{00}^{mn} , $\overline{\mathbf{Z}}_{10}^{mn}$, and \mathcal{G} , see Remark 16 and identity (1.3), from (7.49) it is clear that (7.32) holds true for $(\widehat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*]$.

Let $\mathbb{X} = (X_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ and $\mathbb{W} = (W_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ be arbitrary. Multiplying (7.49) by $X_{mn}W_{ij}$ and taking the sum in i, j, m , and n from 1 to 3, we get

$$(7.50) \quad (\mathcal{H}^s : \mathbb{X}) : \mathbb{W} = \\ = \widehat{\zeta}(\widehat{\mathbf{y}}) \int_{\Theta_F} \left\{ (s\alpha_\lambda + \alpha_\gamma^{-1}) \left[\text{div}_{\widehat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \text{tr } \mathbb{X} \right] \mathbb{I} \right.$$

$$\begin{aligned}
 &+ 2s\alpha_\mu \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] : \mathbb{W} d\hat{\mathbf{z}} \\
 &+ \widehat{\zeta}(\hat{\mathbf{y}}) \int_{\Theta_S} \left\{ \mathcal{G} : \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] \right\} : \mathbb{W} d\hat{\mathbf{z}} \\
 &+ (1 - \widehat{\zeta}(\hat{\mathbf{y}}))(\mathcal{G} : \mathbb{X}) : \mathbb{W}, \quad \forall (\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*].
 \end{aligned}$$

Further, we multiply (7.48) by $\widehat{\zeta}$ and swap the left and the right hand sides. After this, adding the resulting equality to (7.50) we establish the representation

(7.51)

$$\begin{aligned}
 (\mathcal{H}^s : \mathbb{X}) : \mathbb{W} = & \\
 = \widehat{\zeta}(\hat{\mathbf{y}}) \int_{\Theta_F} & \left\{ (s\alpha_\lambda + \alpha_\gamma^{-1}) \left[\operatorname{div}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \operatorname{tr} \mathbb{X} \right] \times \right. \\
 & \left. \times \left[\operatorname{div}_{\hat{\mathbf{z}}} \left(\sum_{i,j=1}^3 W_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \operatorname{tr} \mathbb{W} \right] \right\} \\
 + 2s\alpha_\mu & \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] : \\
 & : \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{i,j=1}^3 W_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \mathbb{W} \right] d\hat{\mathbf{z}} \\
 + \widehat{\zeta}(\hat{\mathbf{y}}) \int_{\Theta_S} & \left\{ \mathcal{G} : \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] \right\} : \\
 & : \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{i,j=1}^3 W_{ij}(\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \mathbb{W} \right] d\hat{\mathbf{z}} + (1 - \widehat{\zeta}(\hat{\mathbf{y}}))(\mathcal{G} : \mathbb{X}) : \mathbb{W}, \\
 & \forall (\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*].
 \end{aligned}$$

From this representation and identity (1.3) it follows that (7.33) holds true on $\Sigma \times [\Delta, \Delta + \delta_*]$ for $s > 0$.

Finally, inserting $\mathbb{W} := \mathbb{X}$ into (7.51) we obtain the identity

(7.52)

$$\begin{aligned}
 (\mathcal{H}^s : \mathbb{X}) : \mathbb{X} = & \widehat{\zeta}(\hat{\mathbf{y}}) \int_{\Theta_F} \left\{ (s\alpha_\lambda + \alpha_\gamma^{-1}) \left| \operatorname{div}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \operatorname{tr} \mathbb{X} \right|^2 \right. \\
 & \left. + 2s\alpha_\mu \left| \mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right|^2 \right\} d\hat{\mathbf{z}} \\
 + \widehat{\zeta}(\hat{\mathbf{y}}) \int_{\Theta_S} & \left\{ \mathcal{G} : \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{m,n=1}^3 X_{mn}(\mathbf{Z}_{00}^{mn} + \overline{\mathbf{Z}}_{10}^{mn}) \right) + \mathbb{X} \right] \right\} :
 \end{aligned}$$

$$: \left[\mathbb{D}_{\hat{\mathbf{z}}} \left(\sum_{i,j=1}^3 X_{ij} (\mathbf{Z}_{00}^{ij} + \overline{\mathbf{Z}}_{10}^{ij}) \right) + \mathbb{X} \right] d\hat{\mathbf{z}}$$

$$+ (1 - \widehat{\zeta}(\hat{\mathbf{y}})) (\mathcal{G} : \mathbb{X}) : \mathbb{X}, \quad \forall (s, \hat{\mathbf{y}}, x_3) \in (0, +\infty) \times \Sigma \times [\Delta, \Delta + \delta_*].$$

From (7.52) due to non-negativeness of \mathcal{G} it follows that the quadratic form $\mathbb{X} \mapsto (\mathcal{H}^s : \mathbb{X}) : \mathbb{X}$ is nonnegative for $(\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*]$. Keeping track of justification of (7.30), in the case when $(\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*]$, we conclude that $(\mathcal{H}^s : \mathbb{X}) : \mathbb{X} = 0$ for $\mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ if and only if $\mathbb{X} = 0$. On the strength of Lemma 1, from this it follows that (7.34) holds true on $\Sigma \times [\Delta, \Delta + \delta_*]$. Thus assertion (iii) is completely proved.

(iv) The symmetry property of matrix \mathbb{F}^0 follows from (7.12) due to the symmetry property (1.3) of the tensor \mathcal{G} . Proposition 6 is proved. \square

Corollary 2. *The following assertions hold true.*

(i) *The constant tensor \mathcal{A}_0^f defined in (7.14) satisfies the symmetry property*

$$\mathcal{A}_0^{fijkl} = \mathcal{A}_0^{fjikl} = \mathcal{A}_0^{fjilk} = \mathcal{A}_0^{fklij} \quad \forall i, j, k, l = 1, 2, 3,$$

and the positive definiteness property

$$(\mathcal{A}_0^f : \mathbb{X}) : \mathbb{X} \geq C_5 |\mathbb{X}|^2 \quad \forall \mathbb{X} \in \mathbb{R}^{3 \times 3} \text{ (}\mathbb{X} \text{ is symmetric)}.$$

(ii) *The tensor-valued function*

$$\mathcal{H}_f^s(\hat{\mathbf{y}}) := \widehat{\zeta}(\hat{\mathbf{y}}) (s\mathcal{A}_0^f + \mathcal{B}_0^f + s\overline{\mathcal{A}}_1^f(s) + \overline{\mathcal{B}}_1^f(s)) + (1 - \theta\widehat{\zeta}(\hat{\mathbf{y}}))\mathcal{G},$$

which is the restriction of \mathcal{H}^s on $[\Delta, \Delta + \delta_]$, admits the symmetry properties*

$$\mathcal{H}_f^{sijkl} = \mathcal{H}_f^{sjikl} = \mathcal{H}_f^{sjilk}, \quad \forall (\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*], \quad \forall s > 0,$$

and

$$(\mathcal{H}_f^s : \mathbb{X}) : \mathbb{W} = (\mathcal{H}_f^s : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

and the uniform positive definiteness property:

$$\text{there exists a constant } C_6^s > 0 \text{ such that } (\mathcal{H}_f^s(\hat{\mathbf{y}}, x_3) : \mathbb{X}) : \mathbb{X} \geq C_6^s |\mathbb{X}|^2,$$

$$\forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall (\hat{\mathbf{y}}, x_3) \in \Sigma \times [\Delta, \Delta + \delta_*], \quad \forall s > 0.$$

(The constant C_6^s is the same as in assertion (iii) of Proposition 6.)

(iii) *Matrices $\mathbb{F}_f^0(t)$ and $\mathbb{F}_{sol}^{0mn}(t)$ ($m, n = 1, 2, 3$) are symmetric and satisfy the regularity conditions*

$$(7.53) \quad \mathbb{F}_f^0, \mathbb{F}_{sol}^{0mn} \in W_\infty^1(0, T) \quad (m, n = 1, 2, 3).$$

Proof. All assertions directly follow from Proposition 6. \square

8. FORMULATION OF THE TWO-SCALE HOMOGENIZED MODEL – MODEL H-2SC

Due to arbitrariness of the test vector-functions ϕ and ϕ_1 , using Green’s theorem we conclude that, in the sense of distributions, the integral equality (7.2) is equivalent to the system consisting of the *macroscopic* integro-differential equation

(8.1a)

$$\alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta} \partial_t \mathbf{u} - \text{div}_x \left\{ \int_\Sigma \left[\mathcal{A}_0 : (\mathbb{D}_x(\mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)})) + \mathcal{B}_0 : (\mathbb{D}_x(J_t \mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)})) \right] \right\}$$

$$\begin{aligned}
 & + \int_0^t \mathcal{A}_1(t - \tau) : (\mathbb{D}_x(\mathbf{u}(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\tau))) d\tau \\
 & + \int_0^t \mathcal{B}_1(t - \tau) : \{ \mathbb{D}_x((J_\tau \mathbf{u})(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_\tau \mathbf{u}^{(1)})(\tau)) \} d\tau + \mathbb{F}^0 \Big] d\hat{\mathbf{y}} \Big\} = \alpha_F \langle \rho \rangle_{\Sigma \times \Theta} \mathbf{f}, \\
 & (\mathbf{x}, t) \in \left[\Omega \setminus (\{x_3 = \Delta\} \cup \{x_3 = \Delta + \delta_*\} \cup \{x_3 = \Delta + \delta^*\}) \right] \times (0, T),
 \end{aligned}$$

the *mesoscopic* integro-differential equation

$$\begin{aligned}
 (8.1b) \quad \operatorname{div}_{\hat{\mathbf{y}}} & \left[\mathcal{A}_0 : (\mathbb{D}_x(\mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)})) + \mathcal{B}_0 : (\mathbb{D}_x(J_t \mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)})) \right. \\
 & + \int_0^t \mathcal{A}_1(t - \tau) : (\mathbb{D}_x(\mathbf{u}(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\tau))) d\tau \\
 & \left. + \int_0^t \mathcal{B}_1(t - \tau) : \{ \mathbb{D}_x((J_\tau \mathbf{u})(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_\tau \mathbf{u}^{(1)})(\tau)) \} d\tau + \mathbb{F}^0 \right] = 0, \\
 & (\mathbf{x}, t, \hat{\mathbf{y}}) \in \Omega \times (0, T) \times \Sigma,
 \end{aligned}$$

the following conditions on the strong discontinuity sections $\{x_3 = \Delta\}$, $\{x_3 = \Delta + \delta_*\}$, and $\{x_3 = \Delta + \delta^*\}$:

$$\begin{aligned}
 (8.1c) \quad & \left\{ \int_{\Sigma} \left[\mathcal{A}_0 : (\mathbb{D}_x(\mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)})) + \mathcal{B}_0 : (\mathbb{D}_x(J_t \mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)})) \right. \right. \\
 & + \int_0^t \mathcal{A}_1(t - \tau) : (\mathbb{D}_x(\mathbf{u}(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\tau))) d\tau \\
 & \left. \left. + \int_0^t \mathcal{B}_1(t - \tau) : \{ \mathbb{D}_x((J_\tau \mathbf{u})(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_\tau \mathbf{u}^{(1)})(\tau)) \} d\tau + \mathbb{F}^0 \right] d\hat{\mathbf{y}} \right\} \mathbf{e}_3 \Big|_{x_3=x_3^{(k)}+0} \\
 & = \left\{ \int_{\Sigma} \left[\mathcal{A}_0 : (\mathbb{D}_x(\mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)})) + \mathcal{B}_0 : (\mathbb{D}_x(J_t \mathbf{u}) + \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)})) \right. \right. \\
 & + \int_0^t \mathcal{A}_1(t - \tau) : (\mathbb{D}_x(\mathbf{u}(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\tau))) d\tau \\
 & \left. \left. + \int_0^t \mathcal{B}_1(t - \tau) : \{ \mathbb{D}_x((J_\tau \mathbf{u})(\tau)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_\tau \mathbf{u}^{(1)})(\tau)) \} d\tau + \mathbb{F}^0 \right] d\hat{\mathbf{y}} \right\} \mathbf{e}_3 \Big|_{x_3=x_3^{(k)}-0}, \\
 & \hat{\mathbf{x}} \in (0, 1)^2, \quad t \in [0, T], \quad k = 1, 2, 3,
 \end{aligned}$$

where

$$x_3^{(1)} = \Delta, \quad x_3^{(2)} = \Delta + \delta_*, \quad x_3^{(3)} = \Delta + \delta^*,$$

and the initial condition (5.12).

We supplement the system (8.1a)-(8.1c) & (5.12) with the boundary condition (5.13) for \mathbf{u} and by the periodicity condition (5.14) for $\mathbf{u}^{(1)}$. Considering the tensors \mathcal{A}_0 , \mathcal{B}_0 , \mathcal{A}_1 , and \mathcal{B}_1 and the matrix \mathbb{F}^0 given, we obtain that the system (8.1a)-(8.1c) & (5.12)-(5.14) constitutes the closed two-scale homogenized model of the fluid-structure interactions.

Definition 6. We call the system (8.1a)-(8.1c) & (5.12)-(5.14) **Model H-2sc**.

We naturally formulate the notion of weak solution of Model H-2sc as follows.

Definition 7. *The pair of vector-functions $\{\mathbf{u}, \mathbf{u}^{(1)}\}$ is a weak solution of Model H-2sc, if it satisfies the inclusions (5.1) and (5.2), the boundary condition (5.13) in the trace sense, and the integral equality (7.2) for arbitrary sufficiently smooth test vector-functions $\phi = \phi(\mathbf{x}, t)$ and $\phi_1 = \phi_1(\mathbf{x}, \hat{\mathbf{y}}, t)$ such that ϕ and ϕ_1 vanish in a neighborhood of $\partial\Omega$ and the section $\{t = T\}$, and ϕ_1 is 1-periodic in $\hat{\mathbf{y}}$.*

Remark 20. *Substituting (5.8), (7.13), (7.15), (7.17), (7.19), and (7.21) into equation (8.1a), we unfold this equation as the compressible Stokes system*

$$\alpha_\tau \rho_F \partial_t \mathbf{u} - \operatorname{div}_x (2\alpha_\mu \mathbb{D}_x(\mathbf{u}) + \alpha_\lambda \mathbb{I} \operatorname{div}_x \mathbf{u} + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_x J_t \mathbf{u} - \mathbb{I} p^0) = \alpha_F \rho_F \mathbf{f}$$

in $\{0 < x_1, x_2 < 1\} \times \{\Delta + \delta^* < x_3 < 1\} \times (0, T)$,

the nonclassical integro-differential momentum equation of linear viscoelasticity

$$\begin{aligned} &\alpha_\tau \rho_\sigma \partial_t \mathbf{u} - \sigma \operatorname{div}_x \left[2\alpha_\mu \mathbb{D}_x(\mathbf{u}) + \alpha_\lambda \mathbb{I} \operatorname{div}_x \mathbf{u} + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_x J_t \mathbf{u} - \mathbb{I} p^0 \right] \\ &\quad - (1 - \sigma) \operatorname{div}_x \left[\mathcal{G} : \mathbb{D}_x(J_t \mathbf{u}) + \mathcal{G} : \nabla_x \mathbf{v}^0 \right] \\ &\quad - \operatorname{div}_x \left[2\alpha_\mu \langle \hat{\zeta} \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}) \rangle_\Sigma + \alpha_\lambda \mathbb{I} \langle \hat{\zeta} \operatorname{div}_{\hat{\mathbf{y}}} \mathbf{u}^{(1)} \rangle_\Sigma + \alpha_\gamma^{-1} \mathbb{I} \langle \hat{\zeta} \operatorname{div}_{\hat{\mathbf{y}}} J_t \mathbf{u}^{(1)} \rangle_\Sigma \right. \\ &\quad \left. + \mathcal{G} : \langle (1 - \hat{\zeta}) \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)}) \rangle_\Sigma \right] = \alpha_F \rho_\sigma \mathbf{f} \end{aligned}$$

in $\{0 < x_1, x_2 < 1\} \times \{\Delta + \delta_* < x_3 < \Delta + \delta^*\} \times (0, T)$,

the nonclassical integro-differential momentum equation of linear viscoelasticity with the memory of shape

$$\begin{aligned} &\alpha_\tau \rho_\theta \partial_t \mathbf{u} - \sigma \operatorname{div}_x \left[\mathcal{A}_0^f : \mathbb{D}_x(\mathbf{u}) + \mathcal{B}_0^f : \mathbb{D}_x(J_t \mathbf{u}) + \int_0^t \mathcal{A}_1^f(t - \tau) : \mathbb{D}_x(\mathbf{u}(\tau)) d\tau \right. \\ &\quad \left. + \int_0^t \mathcal{B}_1^f(t - \tau) : \mathbb{D}_x((J_\tau \mathbf{u})(\tau)) d\tau + p^0 \mathbb{F}_f^0 + \sum_{m,n=1}^3 (\partial_{x_n} v_m^0) \mathbb{F}_{sol}^{0mn} \right] \\ &\quad - (1 - \sigma\theta) \operatorname{div}_x \left[\mathcal{G} : \mathbb{D}_x(J_t \mathbf{u}) + \mathcal{G} : \nabla_x \mathbf{v}^0 \right] \\ &\quad - \operatorname{div}_x \left[\mathcal{A}_0^f : \langle \hat{\zeta} \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}) \rangle_\Sigma + \mathcal{B}_0^f : \langle \hat{\zeta} \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)}) \rangle_\Sigma \right. \\ &\quad \left. + \int_0^t \mathcal{A}_1^f(t - \tau) : \langle \hat{\zeta} \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\tau)) \rangle_\Sigma d\tau + \int_0^t \mathcal{B}_1^f(t - \tau) : \langle \hat{\zeta} \mathbb{D}_{\hat{\mathbf{y}}}((J_\tau \mathbf{u}^{(1)})(\tau)) \rangle_\Sigma d\tau \right. \\ &\quad \left. + \mathcal{G} : \langle (1 - \theta \hat{\zeta}) \mathbb{D}_{\hat{\mathbf{y}}}(J_t \mathbf{u}^{(1)}) \rangle_\Sigma \right] = \alpha_F \rho_\theta \mathbf{f} \end{aligned}$$

in $\{0 < x_1, x_2 < 1\} \times \{\Delta < x_3 < \Delta + \delta_*\} \times (0, T)$,

and Lamé's system of linear elasticity

$$\begin{aligned} &\alpha_\tau \rho_S \partial_t \mathbf{u} - \operatorname{div}_x (\mathcal{G} : \mathbb{D}_x(J_t \mathbf{u}) + \mathcal{G} : \nabla_x \mathbf{v}^0) = \alpha_F \rho_S \mathbf{f} \\ &\quad \text{in } \{0 < x_1, x_2 < 1\} \times \{0 < x_3 < \Delta\} \times (0, T). \end{aligned}$$

9. WELL-POSEDNESS OF MODEL H-2SC

By the considerations fulfilled in Secs. 5-8 we have proved the following existence result.

Proposition 7. *Assume that the tensor-valued functions \mathcal{A}_0 , \mathcal{B}_0 , \mathcal{A}_1 , and \mathcal{B}_1 and the matrix-valued function \mathbb{F}^0 are given by the formulas (7.3)-(7.7), where the*

vector-functions \mathbf{Z}_{00}^{mn} , \mathbf{Z}_{10}^{mn} , \mathbf{Z}_{20} , and \mathbf{Z}_{30}^{mn} are defined as the solutions of Problems Z1-Z7, and $\mathbf{v}^0 \in H^1(\Omega)^3$ and $p^0 \in H^1(\Omega)$ in (7.7) are given.

Then, for any given $\mathbf{u}^0 \in H^1(\Omega)^3$, $\mathbf{u}^* \in C^2(\bar{\Omega} \times [0, T])^3$, $\mathbf{f} \in L^2(\Omega \times (0, T))^3$ such that $\mathbf{u}^0(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}, 0)$ for $\mathbf{x} \in \partial\Omega$ and $\partial_t \mathbf{f} \in L^2(\Omega \times (0, T))^3$, there exists at least one weak solution of Model H-2sc in the sense of Definition 7.

Assumptions on the tensors and the matrix \mathbb{F}^0 in Proposition 7 mean that the coefficients in (8.1a) and (8.1b) are defined by the specific microstructure and therefore are specific. In a sense, the existence result in Proposition 7 strongly depends on the microstructure given in the layer $\{\Delta \leq x_3 \leq \Delta + \delta_*\}$, since the coefficients in (8.1a) and (8.1b) explicitly depend on \mathbf{Z}_{00}^{mn} , \mathbf{Z}_{10}^{mn} , \mathbf{Z}_{20} , and \mathbf{Z}_{30}^{mn} . Furthermore, the limiting passage in Model A_ε as $\varepsilon \searrow 0$ followed by the asymptotic decomposition leads merely to the existence result, while the question of uniqueness remains open.

The following theorem asserts the existence and uniqueness of weak solutions to Model H-2sc. We emphasize that these existence and uniqueness results hold true independently of whether Model H-2sc is connected with the certain microstructure via representations (7.3)-(7.7), or not.

Theorem 2. *Assume that the tensor-valued functions \mathcal{A}_0 , \mathcal{B}_0 , \mathcal{A}_1 , and \mathcal{B}_1 and the matrix-valued function \mathbb{F}^0 in equations (8.1a) and (8.1b) are given and have the properties, stated in assertions (i)-(iv) of Proposition 6. Let all of them be, in principle, irrelevant to the data given for Model A_ε .*

Then, for any given $\mathbf{u}^0 \in H^1(\Omega)^3$, $\mathbf{u}^ \in C^2(\bar{\Omega} \times [0, T])^3$, $\mathbf{f} \in L^2(\Omega \times (0, T))^3$ such that $\mathbf{u}^0(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}, 0)$ for $\mathbf{x} \in \partial\Omega$ and $\partial_t \mathbf{f} \in L^2(\Omega \times (0, T))^3$, there exists a unique weak solution of Model H-2sc in the sense of Definition 7.*

Proof. We divide the proof into five steps.

Step 1. Let us formulate the definition of weak solution of Model H-2sc in an alternative way, which is equivalent to Definition 7 and is more preferable for justification of the theorem for technical reasons.

We integrate by parts in the first integral in the left-hand side of (7.2), which makes sense due to (5.1)₂, then in (7.2) we take the test vector-functions of the form $\phi(\mathbf{x}, t) = \xi(t)\Phi(\mathbf{x})$ and $\phi_1(\mathbf{x}, \hat{\mathbf{y}}, t) = \xi(t)\Phi_1(\mathbf{x}, \hat{\mathbf{y}})$. Due to arbitrariness of ξ , from (7.2) we derive the integral equality

$$\begin{aligned}
 (9.1) \quad & \int_{\Omega} \left\{ \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{u}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) \right. \\
 & + \int_{\Sigma} \left[\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t))) \right. \\
 & \quad + \mathcal{B}_0(\hat{\mathbf{y}}, x_3) : \{ \mathbb{D}_x((J_t \mathbf{u})(\mathbf{x}, t)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_t \mathbf{u}^{(1)})(\mathbf{x}, \hat{\mathbf{y}}, t)) \} \\
 & \quad + \int_0^t \mathcal{A}_1(\hat{\mathbf{y}}, x_3, t - \tau) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, \tau))) d\tau \\
 & \quad + \int_0^t \mathcal{B}_1(\hat{\mathbf{y}}, x_3, t - \tau) : \{ \mathbb{D}_x((J_\tau \mathbf{u})(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{\mathbf{y}}}((J_\tau \mathbf{u}^{(1)})(\mathbf{x}, \hat{\mathbf{y}}, \tau)) \} d\tau \\
 & \quad \left. \left. + \mathbb{F}^0(\hat{\mathbf{y}}, \mathbf{x}, t) \right] : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\Phi_1(\mathbf{x}, \hat{\mathbf{y}}))) d\hat{\mathbf{y}} \right\} d\mathbf{x} \\
 & = \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{f}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) d\mathbf{x} \quad \text{for a.e. } t \in [0, T],
 \end{aligned}$$

where Φ and Φ_1 are sufficiently smooth test vector-functions such that Φ and Φ_1 vanish in a neighborhood of $\partial\Omega$, and Φ_1 is 1-periodic in \hat{y} .

Also, let us notice that (5.12) holds in the strong trace sense due to the boundedness of $\|\partial_t \mathbf{u}\|_{L^\infty(0,T;L^2(\Omega)^3)}$ (see inclusion (5.1)₂). Thus we make the following conclusion.

Remark 21. *The pair of vector-functions $\{\mathbf{u}, \mathbf{u}^{(1)}\}$ is a weak solution of Model H-2sc in the sense of Definition 7 if and only if it satisfies inclusions (5.1) and (5.2), the initial condition (5.12) and the boundary condition (5.13) in the strong trace sense, and the integral equality (9.1) for a.e. $t \in [0, T]$ for all admissible test vector-functions Φ and Φ_1 .*

Step 2. On the strength of the initial condition (5.12) and the property of absolute continuity of the Lebesgue integral with respect to the limits of integration [27, Exercise 1.6.49(vi)], from (9.1) it follows that

$$\begin{aligned}
 (9.2) \quad & \int_{\Omega} \left\{ \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{u}(\mathbf{x}, t)|_{t=0} \cdot \Phi(\mathbf{x}) \right. \\
 & + \int_{\Sigma} \left[\mathcal{A}_0(\hat{y}, x_3) : (\mathbb{D}_x(\mathbf{u}^0(\mathbf{x})) + \mathbb{D}_{\hat{y}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{y}, 0))) \right. \\
 & \left. \left. + \mathbb{F}^0(\hat{y}, \mathbf{x}, 0) \right] : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{y}}(\Phi_1(\mathbf{x}, \hat{y}))) d\hat{y} \right\} dx \\
 & - \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{f}(\mathbf{x}, 0) \cdot \Phi(\mathbf{x}) dx \\
 & = \lim_{t \searrow 0} \left\{ - \int_{\Omega} \int_{\Sigma} \left[\mathcal{B}_0(\hat{y}, x_3) : \{ \mathbb{D}_x((J_t \mathbf{u})(\mathbf{x}, t)) + \mathbb{D}_{\hat{y}}((J_t \mathbf{u}^{(1)})(\mathbf{x}, \hat{y}, t)) \} \right. \right. \\
 & + \int_0^t \mathcal{A}_1(\hat{y}, x_3, t - \tau) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{y}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{y}, \tau))) d\tau \\
 & \left. \left. + \int_0^t \mathcal{B}_1(\hat{y}, x_3, t - \tau) : \{ \mathbb{D}_x((J_{\tau} \mathbf{u})(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{y}}((J_{\tau} \mathbf{u}^{(1)})(\mathbf{x}, \hat{y}, \tau)) \} d\tau \right] : \right. \\
 & \left. : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{y}}(\Phi_1(\mathbf{x}, \hat{y}))) d\hat{y} dx \right\} = 0.
 \end{aligned}$$

Further, differentiating (9.1) with respect to t and using assertion (ii) of Proposition A.1, we establish the following integral equality:

$$\begin{aligned}
 (9.3) \quad & \frac{d}{dt} \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{u}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) dx \\
 & + \frac{d}{dt} \int_{\Omega} \int_{\Sigma} \left[\mathcal{A}_0(\hat{y}, x_3) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathbb{D}_{\hat{y}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{y}, t))) \right] : \\
 & \quad : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{y}}(\Phi_1(\mathbf{x}, \hat{y}))) d\hat{y} dx \\
 & + \int_{\Omega} \int_{\Sigma} \left[\mathcal{B}_0(\hat{y}, x_3) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathbb{D}_{\hat{y}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{y}, t))) \right. \\
 & + \int_0^t \frac{\partial \mathcal{A}_1(\hat{y}, x_3, t - \tau)}{\partial t} : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{y}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{y}, \tau))) d\tau \\
 & \left. + \mathcal{A}_1(\hat{y}, x_3, 0) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathbb{D}_{\hat{y}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{y}, t))) \right] :
 \end{aligned}$$

$$\begin{aligned} & + \int_0^t \mathcal{B}_1(\hat{\mathbf{y}}, x_3, t - \tau) : (\mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, \tau))) d\tau \\ & + \partial_t \mathbb{F}^0(\hat{\mathbf{y}}, \mathbf{x}, t) : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\Phi_1(\mathbf{x}, \hat{\mathbf{y}}))) d\hat{\mathbf{y}} d\mathbf{x} \\ & = \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{f}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

This integral equality is understood in the sense of distributions on $[0, T]$.

Step 3. We extend all functions in (9.3) for $t > T$ by zero. On the strength of the well-known properties of the Laplace transform (see in Appendix A), applying the Laplace transform in t to (9.3), taking into account the given initial data (5.12), using representation (7.31), and remarking that the integrals containing the term $\mathcal{A}_1(\hat{\mathbf{y}}, x_3, 0)$ cancel, we arrive at the integral equality

$$\begin{aligned} (9.4) \quad & \int_{\Omega} \left\{ \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (s^2 \bar{\mathbf{u}}(\mathbf{x}, s) - s \mathbf{u}^0(\mathbf{x}) - \partial_t \mathbf{u}(\mathbf{x}, t)|_{t=0}) \cdot \Phi(\mathbf{x}) \right. \\ & + \int_{\Sigma} [\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : (\mathbb{D}_x(\bar{\mathbf{u}}(\mathbf{x}, s)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{u}}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, s))) \\ & + s \bar{\mathbb{F}}^0(\hat{\mathbf{y}}, \mathbf{x}, s) - \mathcal{A}_0(\hat{\mathbf{y}}, x_3) : (\mathbb{D}_x(\mathbf{u}^0(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, 0))) \\ & \left. - \mathbb{F}^0(\hat{\mathbf{y}}, \mathbf{x}, 0) : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\Phi_1(\mathbf{x}, \hat{\mathbf{y}}))) d\hat{\mathbf{y}} \right\} d\mathbf{x} \\ & = \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (s \bar{\mathbf{f}}(\mathbf{x}, s) - \mathbf{f}(\mathbf{x}, 0)) \cdot \Phi(\mathbf{x}) d\mathbf{x}, \quad \forall s > 0. \end{aligned}$$

Combining (9.4) and (9.2) and slightly re-arranging terms, we get the integral equality

$$\begin{aligned} (9.5) \quad & \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) s^2 \bar{\mathbf{u}}(\mathbf{x}, s) \cdot \Phi(\mathbf{x}) d\mathbf{x} \\ & + \int_{\Omega} \int_{\Sigma} [\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : (\mathbb{D}_x(\bar{\mathbf{u}}(\mathbf{x}, s)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{u}}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, s)))] : \\ & \quad : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\Phi_1(\mathbf{x}, \hat{\mathbf{y}}))) d\hat{\mathbf{y}} d\mathbf{x} \\ & = s \int_{\Omega} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (\alpha_{\tau} \mathbf{u}^0(\mathbf{x}) + \alpha_F \bar{\mathbf{f}}(\mathbf{x}, s)) \cdot \Phi(\mathbf{x}) d\mathbf{x} \\ & \quad - s \int_{\Omega} \int_{\Sigma} \bar{\mathbb{F}}^0(\hat{\mathbf{y}}, \mathbf{x}, s) : (\mathbb{D}_x(\Phi(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\Phi_1(\mathbf{x}, \hat{\mathbf{y}}))) d\hat{\mathbf{y}} d\mathbf{x}, \quad \forall s > 0, \end{aligned}$$

where the test vector-functions Φ and Φ_1 meet the regularity requirements from (9.1). Moreover, we can omit the demand that $\Phi_1 \equiv 0$ in a neighborhood of $\partial\Omega$, using the standard density arguments.

Step 4. On this step we aim to prove that for an arbitrarily fixed $s > 0$ there exists a unique pair $\{\bar{\mathbf{u}}, \bar{\mathbf{u}}^{(1)}\}$ satisfying the regularity requirements $\bar{\mathbf{u}}(\cdot, s) \in H^1(\Omega)^3$ and $\bar{\mathbf{u}}^{(1)}(\cdot, \cdot, s) \in L^2(\Omega; (H_{\#}^1(\Sigma)/\mathbb{R})^3)$, the boundary condition $\bar{\mathbf{u}} = \bar{\mathbf{u}}^*$ on $\partial\Omega$ in the strong trace sense, and the integral equality (9.5) for all admissible test vector-functions Φ and Φ_1 . To this end, we implement the Lax-Milgram theorem [4, Th. 4.6]. The variable s plays the role of a parameter in this formulation.

We introduce the Hilbert space

$$\mathcal{H} := H_0^1(\Omega)^3 \times L^2(\Omega; (H_{\#}^1(\Sigma)/\mathbb{R})^3)$$

equipped with the norm

$$\|\mathbf{B}\|_{\mathcal{H}} := \left(\|\mathbf{b}_1\|_{H_0^1(\Omega)^3}^2 + \|\mathbf{b}_2\|_{L^2(\Omega; (H_{\sharp}^1(\Sigma)/\mathbb{R})^3)}^2 \right)^{1/2}, \quad \forall \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2) \in \mathcal{H}.$$

We clarify that $\mathbf{b}_1 \in H_0^1(\Omega)^3$ and $\mathbf{b}_2 \in L^2(\Omega; (H_{\sharp}^1(\Sigma)/\mathbb{R})^3)$ here.

Next, we introduce the bilinear form $a^s: \mathcal{H} \times \mathcal{H} \mapsto \mathbb{R}$ and the linear functional $\Gamma^s: \mathcal{H} \mapsto \mathbb{R}$ by the formulas

$$\begin{aligned} a^s(\mathbf{B}, \mathbf{W}) &:= s^2 \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{b}_1(\mathbf{x}) \cdot \mathbf{w}_1(\mathbf{x}) d\mathbf{x} \\ &+ \int_{\Omega} \int_{\Sigma} \left[\mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : (\mathbb{D}_x(\mathbf{b}_1(\mathbf{x})) + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \widehat{\mathbf{y}}))) \right] : \\ &\quad : (\mathbb{D}_x(\mathbf{w}_1(\mathbf{x})) + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{w}_2(\mathbf{x}, \widehat{\mathbf{y}}))) d\widehat{\mathbf{y}} d\mathbf{x}, \\ \forall \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2) \in \mathcal{H}, \quad \forall \mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2) \in \mathcal{H} \end{aligned}$$

and

$$\begin{aligned} \Gamma^s(\mathbf{W}) &:= s \int_{\Omega} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (\alpha_{\tau} \mathbf{u}^0(\mathbf{x}) + \alpha_F \bar{\mathbf{f}}(\mathbf{x}, s) - \alpha_{\tau} s \bar{\mathbf{u}}^*(\mathbf{x}, s)) \cdot \mathbf{w}_1(\mathbf{x}) d\mathbf{x} \\ &- \int_{\Omega} \int_{\Sigma} [s \bar{\mathbb{F}}^0(\widehat{\mathbf{y}}, \mathbf{x}, s) + \mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : \mathbb{D}_x(\bar{\mathbf{u}}^*)] : (\mathbb{D}_x(\mathbf{w}_1(\mathbf{x})) + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{w}_2(\mathbf{x}, \widehat{\mathbf{y}}))) d\widehat{\mathbf{y}} d\mathbf{x}, \\ \forall \mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2) \in \mathcal{H}. \end{aligned}$$

In terms of a^s and Γ^s the question of finding the pair $\{\bar{\mathbf{u}}, \bar{\mathbf{u}}^{(1)}\}$ satisfying (9.5) and the above stated set of requirements is formulated as the following variational problem:

$$(9.6) \quad \text{Find } \mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2) \in \mathcal{H} \text{ satisfying the variational equation} \\ a^s(\mathbf{U}, \mathbf{W}) = \Gamma^s(\mathbf{W}), \quad \forall \mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2) \in \mathcal{H}.$$

Upon finding such a \mathbf{U} , it simply suffices to set $\bar{\mathbf{u}} := \mathbf{U}_1 + \bar{\mathbf{u}}^*$ and $\bar{\mathbf{u}}^{(1)} := \mathbf{U}_2$.

Let us find out the properties of a^s and Γ^s sufficient for the well-posedness of problem (9.6). Due to (7.24)-(7.26) we have that $\mathcal{H}^s \in L^\infty(\Sigma \times (0, 1))^{3 \times 3 \times 3 \times 3} \forall s > 0$. Using this, (7.32), the Cauchy-Bunyakovsky inequality, and the inequalities

$$(9.7) \quad 0 \leq \max_{x_3 \in [0, 1]} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \leq \max\{\rho_F, \rho_S\}, \quad \sum_{k=1}^4 a_k \leq 2 \sqrt{\sum_{k=1}^4 a_k^2}, \quad \forall a_k \in \mathbb{R},$$

we construct the estimate

$$\begin{aligned} &|a^s(\mathbf{B}, \mathbf{W})| \stackrel{(7.32)}{=} \left| s^2 \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{b}_1(\mathbf{x}) \cdot \mathbf{w}_1(\mathbf{x}) d\mathbf{x} \right. \\ &+ \left. \int_{\Omega} \int_{\Sigma} \left[\mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : (\nabla_x \mathbf{b}_1(\mathbf{x}) + \nabla_{\widehat{\mathbf{y}}} \mathbf{b}_2(\mathbf{x}, \widehat{\mathbf{y}})) \right] : (\nabla_x \mathbf{w}_1(\mathbf{x}) + \nabla_{\widehat{\mathbf{y}}} \mathbf{w}_2(\mathbf{x}, \widehat{\mathbf{y}})) d\widehat{\mathbf{y}} d\mathbf{x} \right| \\ &\leq s^2 \alpha_{\tau} \max\{\rho_F, \rho_S\} \|\mathbf{b}_1\|_{L^2(\Omega)^3} \|\mathbf{w}_1\|_{L^2(\Omega)^3} \\ &+ \|\mathcal{H}^s\|_{L^\infty(\Sigma \times (0, 1))^{3 \times 3 \times 3 \times 3}} \|\nabla_x \mathbf{b}_1 + \nabla_{\widehat{\mathbf{y}}} \mathbf{b}_2\|_{L^2(\Omega \times \Sigma)^{3 \times 3}} \|\nabla_x \mathbf{w}_1 + \nabla_{\widehat{\mathbf{y}}} \mathbf{w}_2\|_{L^2(\Omega \times \Sigma)^{3 \times 3}} \\ &\leq \max\{s^2 \alpha_{\tau} \rho_F, s^2 \alpha_{\tau} \rho_S, \|\mathcal{H}^s\|_{L^\infty(\Sigma \times (0, 1))^{3 \times 3 \times 3 \times 3}}\} \times \\ &\quad \times (\|\mathbf{b}_1\|_{L^2(\Omega)^3} + \|\nabla_x \mathbf{b}_1\|_{L^2(\Omega)^{3 \times 3}} + \|\nabla_{\widehat{\mathbf{y}}} \mathbf{b}_2\|_{L^2(\Omega \times \Sigma)^{3 \times 3}} + \|\mathbf{b}_2\|_{L^2(\Omega \times \Sigma)^3}) \times \\ &\quad \times (\|\mathbf{w}_1\|_{L^2(\Omega)^3} + \|\nabla_x \mathbf{w}_1\|_{L^2(\Omega)^{3 \times 3}} + \|\nabla_{\widehat{\mathbf{y}}} \mathbf{w}_2\|_{L^2(\Omega \times \Sigma)^{3 \times 3}} + \|\mathbf{w}_2\|_{L^2(\Omega \times \Sigma)^3}) \end{aligned}$$

$$\leq 4 \max \left\{ s^2 \alpha_\tau \rho_F, s^2 \alpha_\tau \rho_S, \|\mathcal{H}^s\|_{L^\infty(\Sigma \times (0,1))^{3 \times 3 \times 3 \times 3}} \right\} \|\mathbf{B}\|_{\mathcal{H}} \|\mathbf{W}\|_{\mathcal{H}},$$

$$\forall \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2) \in \mathcal{H}, \quad \forall \mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2) \in \mathcal{H}, \quad \forall s > 0.$$

This estimate yields that

(9.8) the bilinear form a^s is continuous on $\mathcal{H} \times \mathcal{H}$ for all $s > 0$.

In order to proceed further, let us notice that the identity

(9.9)
$$\int_{\Omega} \int_{\Sigma} |\mathbb{D}_x(\mathbf{b}_1(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \hat{\mathbf{y}}))|^2 d\hat{\mathbf{y}} d\mathbf{x}$$

$$= \int_{\Omega} |\mathbb{D}_x(\mathbf{b}_1(\mathbf{x}))|^2 d\mathbf{x} + \int_{\Omega} \int_{\Sigma} |\mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \hat{\mathbf{y}}))|^2 d\hat{\mathbf{y}} d\mathbf{x}$$

holds for any $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2) \in \mathcal{H}$, due to the identities $\int_{\Sigma} \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0$ and $\int_{\Sigma} d\hat{\mathbf{y}} = 1$. The former holds due to Green’s formula and 1-periodicity of \mathbf{b}_2 in $\hat{\mathbf{y}}$.

Also, let us recall the well-known Korn and Poincaré inequalities.

Lemma 2. (THE FIRST KORN INEQUALITY.) Assume \mathcal{O} is a connected bounded open set in \mathbb{R}_x^N with Lipschitz continuous boundary $\partial\mathcal{O}$; then

(9.10)
$$\|\nabla_x \phi\|_{L^2(\mathcal{O})^{N \times N}} \leq \sqrt{2} \|\mathbb{D}_x(\phi)\|_{L^2(\mathcal{O})^{N \times N}}, \quad \forall \phi \in H_0^1(\mathcal{O})^N.$$

Proof of this lemma can be found in [22, Ch. I, Sec. 2.1].

Lemma 3. (THE KORN INEQUALITY FOR THE PERIODIC CASE.) Assume $K = (0, 1)^N$; then there exists the constant $c_K = c_K(K)$ such that

(9.11)
$$\|\phi_1\|_{H_{\sharp}^1(K)^N} \leq c_K \|\mathbb{D}_{\hat{\mathbf{y}}}(\phi_1)\|_{L^2(K)^{N \times N}}, \quad \forall \phi_1 \in (H_{\sharp}^1(K)/\mathbb{R})^N.$$

Proof of this lemma can be found in [22, Ch. I, Sec. 2.3].

Lemma 4. (POINCARÉ INEQUALITY.) Let \mathcal{O} be a connected bounded open set in \mathbb{R}_x^N such that $\partial\mathcal{O}$ is Lipschitz continuous. Then there exists a constant $c_{\mathcal{O}}$ such that

(9.12)
$$\|\phi\|_{L^2(\mathcal{O})^N} \leq c_{\mathcal{O}} \|\nabla_x \phi\|_{L^2(\mathcal{O})^{N \times N}}, \quad \forall \phi \in H_0^1(\mathcal{O})^N.$$

Proof of this lemma can be found in [4, Ch. 3, Sec. 3.3].

Using (7.34), (9.9), and Lemmas 2-4, for an arbitrary $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2) \in \mathcal{H}$ we establish the following estimate:

$$a^s(\mathbf{B}, \mathbf{B}) \equiv s^2 \int_{\Omega} \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) |\mathbf{b}_1(\mathbf{x})|^2 d\mathbf{x}$$

$$+ \int_{\Omega} \int_{\Sigma} [\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : (\mathbb{D}_x(\mathbf{b}_1(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \hat{\mathbf{y}})))] :$$

$$: (\mathbb{D}_x(\mathbf{b}_1(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \hat{\mathbf{y}}))) d\hat{\mathbf{y}} d\mathbf{x}$$

$$\stackrel{(7.34)}{\geq} C_6^s \int_{\Omega} \int_{\Sigma} |\mathbb{D}_x(\mathbf{b}_1(\mathbf{x})) + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \hat{\mathbf{y}}))|^2 d\hat{\mathbf{y}} d\mathbf{x}$$

$$\stackrel{(9.9)}{=} C_6^s \left(\int_{\Omega} |\mathbb{D}_x(\mathbf{b}_1(\mathbf{x}))|^2 d\mathbf{x} + \int_{\Omega} \int_{\Sigma} |\mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{b}_2(\mathbf{x}, \hat{\mathbf{y}}))|^2 d\hat{\mathbf{y}} d\mathbf{x} \right)$$

$$\begin{aligned}
& \stackrel{(9.10)}{\geq} \stackrel{(9.11)}{C_6^s} \left(\frac{1}{2} \|\nabla_x \mathbf{b}_1\|_{L^2(\Omega)^{3 \times 3}}^2 + \frac{1}{c_K^2} \|\mathbf{b}_2\|_{L^2(\Omega; (H_{\sharp}^1(\Sigma)/\mathbb{R})^3)}^2 \right) \\
& \stackrel{(9.12)}{\geq} C_6^s \left(\frac{1}{2(1+c_{\Omega}^2)} \|\mathbf{b}_1\|_{H_0^1(\Omega)}^2 + \frac{1}{c_K^2} \|\mathbf{b}_2\|_{L^2(\Omega; (H_{\sharp}^1(\Sigma)/\mathbb{R})^3)}^2 \right) \geq \\
& \geq \min \left\{ \frac{C_6^s}{2(1+c_{\Omega}^2)}, \frac{C_6^s}{c_K^2} \right\} \|\mathbf{B}\|_{\mathcal{H}}^2.
\end{aligned}$$

This estimate yields that

$$\begin{aligned}
& \text{for } s > 0 \text{ the bilinear form } a^s \text{ is } \mathcal{H}\text{-elliptic} \\
(9.13) \quad & \text{with the constant } C_7^s := \min \left\{ \frac{C_6^s}{2(1+c_{\Omega}^2)}, \frac{C_6^s}{c_K^2} \right\}.
\end{aligned}$$

Finally, on the strength of the assumptions in the theorem imposed on the given data, using the standard technics we get the estimate

$$(9.14) \quad |\Gamma^s(\mathbf{W})| \leq C_8^s \|\mathbf{W}\|_{\mathcal{H}}, \quad \forall \mathbf{W} \in \mathcal{H}, \quad \forall s > 0,$$

where the positive constant C_8^s depends on s , α_{τ} , α_F , ρ_F , ρ_S , $\|\mathbf{u}^0\|_{L^2(\Omega)^3}$, $\|\mathbf{f}\|_{L^2(\Omega \times (0,T))^3}$, $\|\mathbf{u}^*\|_{C^1(\bar{\Omega} \times [0,T])^3}$, and $\|\mathbb{F}^0\|_{L^\infty(\Sigma \times (0,T); L^2(\Omega))}$. This estimate yields that

$$(9.15) \quad \text{the linear functional } \Gamma^s \text{ is continuous on } \mathcal{H} \text{ for } s > 0, \text{ and } \|\Gamma^s\|_{\mathcal{H}^*} \leq C_8^s.$$

On the strength of (9.8), (9.13), (9.14), and (9.15), by the Lax-Milgram theorem we conclude that there exists the unique solution $\mathbf{U} \in \mathcal{H}$ to the variational problem (9.6), and this solution admits the estimate

$$(9.16) \quad \|\mathbf{U}\|_{\mathcal{H}} \leq (1/C_7^s) \|\Gamma^s\|_{\mathcal{H}^*} \leq C_8^s/C_7^s, \quad s > 0.$$

Hence there exists the unique pair of functions $\bar{\mathbf{u}} = \mathbf{U}_1(\mathbf{x}; s) + \bar{\mathbf{u}}^*(\mathbf{x}, s)$ and $\bar{\mathbf{u}}^{(1)} = \mathbf{U}_2(\mathbf{x}, \hat{\mathbf{y}}; s)$ satisfying the integral equality (9.5), the boundary condition $\bar{\mathbf{u}}(\mathbf{x}, s) = \bar{\mathbf{u}}^*(\mathbf{x}, s)$ on $\partial\Omega$ for $s > 0$, and the set of the regularity requirements formulated at the beginning of step 4.

Step 5. In completion of the proof of the theorem, we rely on the theory of inverse Laplace transform. On the strength of the bound (9.16), there exist the inverse Laplace transforms in s of \mathbf{U}_1 and \mathbf{U}_2 , which are given by the Bromwich integrals (for details, see Appendix A):

$$\begin{aligned}
\mathcal{L}^{-1}[\mathbf{U}_1](\mathbf{x}, t) &= \frac{1}{2\pi i} \int_{s_1-i\infty}^{s_1+i\infty} \mathbf{U}_1(\mathbf{x}; s) e^{st} ds, \\
\mathcal{L}^{-1}[\mathbf{U}_2](\mathbf{x}, \hat{\mathbf{y}}, t) &= \frac{1}{2\pi i} \int_{s_1-i\infty}^{s_1+i\infty} \mathbf{U}_2(\mathbf{x}, \hat{\mathbf{y}}; s) e^{st} ds,
\end{aligned}$$

where $s_1 > 0$ is an arbitrarily fixed real value. Applying the inverse Laplace transform in s to (9.6) and fulfilling the steps 1-3 of the present proof in the reverse order, we conclude that the pair of functions $\mathbf{u} = \mathcal{L}^{-1}[\mathbf{U}_1](\mathbf{x}, t) + \mathbf{u}^*(\mathbf{x}, t)$ and $\mathbf{u}^{(1)} = \mathcal{L}^{-1}[\mathbf{U}_2](\mathbf{x}, \hat{\mathbf{y}}, t)$ is a weak solution to Model H-2sc in the sense of Definition 7. At the same time, we notice that the formal application of the Laplace transform in t to (9.3) is rigorous, in fact.

Finally, the solution $\{\mathbf{u}, \mathbf{u}^{(1)}\}$ is unique due to one-to-oneness of \mathcal{L} and \mathcal{L}^{-1} .

Theorem 2 is proved. \square

10. ASYMPTOTIC DECOMPOSITION II: THE MESOSCOPIC CELL PROBLEMS

We will seek for the representation of the vector-function $\mathbf{u}^{(1)}$ in the form

$$\begin{aligned}
 (10.1) \quad \mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t) &= \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) \mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3) \\
 &+ \sum_{i,j=1}^3 \int_0^t \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t - \tau) \mathbf{Y}_1^{ij}(\hat{\mathbf{y}}, \tau, x_3) d\tau \\
 &+ p^0(\mathbf{x}) \mathbf{Y}_2(\hat{\mathbf{y}}, t, x_3) + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, t, x_3).
 \end{aligned}$$

Here \mathbf{u} and $\mathbf{u}^{(1)}$ constitute the weak solution of Model H-2sc in the sense of Definition 7. Vector-functions \mathbf{Y}_0^{ij} , \mathbf{Y}_1^{ij} , \mathbf{Y}_3^{ij} ($i, j = 1, 2, 3$), and \mathbf{Y}_2 are the sought functions.

Remark 22. *Since the vector-function $\hat{\mathbf{y}} \mapsto \mathbf{u}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, t)$ belongs to $(H_{\#}^1(\Sigma)/\mathbb{R})^3$ for a.e. $(\mathbf{x}, t) \in \Omega \times (0, T)$, the vector-functions \mathbf{Y}_0^{ij} , \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} should belong to $(H_{\#}^1(\Sigma)/\mathbb{R})^3$ as vector-functions of $\hat{\mathbf{y}}$ for a.e. $(x_3, t) \in (0, 1) \times (0, T)$.*

Assuming that \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} admit the Laplace transform in t , from (10.1) we derive the equivalent representation for the image with respect to the Laplace transform:

$$\begin{aligned}
 \bar{\mathbf{u}}^{(1)}(\mathbf{x}, \hat{\mathbf{y}}, s) &= \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) [\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)] + p^0(\mathbf{x}) \bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3) \\
 &+ \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3), \quad (\mathbf{x}, \hat{\mathbf{y}}) \in \Omega \times \Sigma, \quad s > 0.
 \end{aligned}$$

In order to find \mathbf{Y}_0^{ij} , \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} , we use the integral equalities (9.2) and (9.5). Let us recall that (9.2) and (9.5) directly follow from the integral equality (7.2) in Definition 7. Derivation of (9.2) and (9.5) from (7.2) was fulfilled in full detail within steps 1-3 of the proof of Theorem 2.

At first, we substitute (10.1), (7.13), (7.21), $\Phi \equiv 0$, and $\Phi_1(\mathbf{x}, \hat{\mathbf{y}}) := \phi_{11}(\mathbf{x})\phi_{12}(\hat{\mathbf{y}})$ into (9.2), where ϕ_{11} is an arbitrary smooth scalar test function and ϕ_{12} is an arbitrary test vector-function belonging to $H_{\#}^1(\Sigma)^3$. Thus we get the integral equality

$$\begin{aligned}
 (10.2) \quad &\int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \left\{ \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) \int_{\Sigma} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \right. \right. \\
 &\quad \left. \left. : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3))) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 &+ p^0(\mathbf{x}) \int_{\Sigma} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)) \right. \\
 &\quad \left. \left. - \mathbb{I} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right.
 \end{aligned}$$

$$\begin{aligned}
 (10.6) \quad & \int_{\Sigma_S} [\mathcal{G} : \mathbb{J}^{ij}] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \stackrel{(1.3)}{=} \int_{\Sigma_S} [\mathcal{G} : \mathbb{J}^{ij}] : \nabla_{\hat{\mathbf{y}}} \phi_{12}(\hat{\mathbf{y}}) d\hat{\mathbf{y}} \\
 & = \int_{\partial\Sigma_S} [\mathcal{G} : \mathbb{J}^{ij}] \phi_{12}(\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) \cdot \mathbf{n}_{\Sigma}(\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) d\boldsymbol{\sigma}_{\hat{\mathbf{y}}}, \quad i, j = 1, 2, 3.
 \end{aligned}$$

Here $\mathbf{n}_{\Sigma} = (n_{\Sigma 1}, n_{\Sigma 2}, 0)$ is the unit outward normal to the boundary of domain $\Sigma_S \subset \Sigma$, and $d\boldsymbol{\sigma}_{\hat{\mathbf{y}}}$ stands for an infinitesimal element of $\partial\Sigma_F$.

With account of (10.3)-(10.6), equality (10.2) takes the following form:

$$\begin{aligned}
 (10.7) \quad & \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \left\{ \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) \int_{\Sigma} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \right. \right. \\
 & \qquad \qquad \qquad \left. \left. : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + p^0(\mathbf{x}) \int_{\Sigma} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
 & + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \right. \\
 & \qquad \qquad \qquad \left. : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, 0, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} \\
 & + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta_*}^{\Delta + \delta^*} \left\{ \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) \int_{\Sigma_F} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \right. \right. \\
 & \qquad \qquad \qquad \left. \left. : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3))) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + p^0(\mathbf{x}) \int_{\Sigma_F} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)) \right. \\
 & \qquad \qquad \qquad \left. \left. - \mathbb{I} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \left(\int_{\Sigma_F} \left[(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \right. \right. \\
 & \qquad \qquad \qquad \left. \left. : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, 0, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + \int_{\partial\Sigma_S} [\mathcal{G} : \mathbb{J}^{ij}] \phi_{12}(\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) \cdot \mathbf{n}_{\Sigma}(\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) d\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} \\
 & + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta}^{\Delta + \delta_*} \left\{ \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j}(\mathbf{x}, 0) \int_{\Sigma_F} [\mathcal{A}_0^f : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)))] : \right. \\
 & \qquad \qquad \qquad \left. : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + p^0(\mathbf{x}) \int_{\Sigma_F} [\mathcal{A}_0^f : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)) + \mathbb{F}_f^0(0)] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma_F} [\mathcal{A}_0^f : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, 0, x_3)) + \mathbb{F}_{sol}^{0ij}(0) \\
& \quad - \theta \mathcal{G} : \mathbb{J}^{ij}] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} = 0.
\end{aligned}$$

From the structure of this integral equality it is clear that it holds true independently of any possible values of the vector-functions \mathbf{u} and \mathbf{v}^0 and the scalar function p^0 if the vector-functions \mathbf{Y}_0^{ij} , $\mathbf{Y}_2|_{t=0}$, and $\mathbf{Y}_3^{ij}|_{t=0}$ satisfy the identities

$$(10.8) \quad \mathbf{Y}_0^{ij} = \mathbf{Y}_2|_{t=0} = \mathbf{Y}_3^{ij}|_{t=0} \equiv 0 \text{ for } (\hat{\mathbf{y}}, x_3) \in \Sigma \times ((0, 1) \setminus [\Delta, \Delta + \delta^*]),$$

and the following integral equalities:

$$\begin{aligned}
(10.9) \quad & \int_{\Sigma_F} \left[(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3))) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0, \\
& \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*];
\end{aligned}$$

$$\begin{aligned}
(10.10) \quad & \int_{\Sigma_F} \left[\mathcal{A}_0^f : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3))) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0, \\
& \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*];
\end{aligned}$$

$$\begin{aligned}
(10.11) \quad & \int_{\Sigma_F} \left[(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)) - \mathbb{I} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0, \\
& \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*];
\end{aligned}$$

$$\begin{aligned}
(10.12) \quad & \int_{\Sigma_F} \left[\mathcal{A}_0^f : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)) + \mathbb{F}_f^0(0) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0, \\
& \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*];
\end{aligned}$$

$$\begin{aligned}
(10.13) \quad & \int_{\Sigma_F} \left[(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, 0, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
& + \int_{\partial \Sigma_S} \left[(\mathcal{G} : \mathbb{J}^{ij}) \phi_{12}(\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) \right] \cdot \mathbf{n}_{\Sigma}(\boldsymbol{\sigma}_{\hat{\mathbf{y}}}) d\boldsymbol{\sigma}_{\hat{\mathbf{y}}} = 0, \\
& \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*];
\end{aligned}$$

$$\begin{aligned}
(10.14) \quad & \int_{\Sigma_F} \left[\mathcal{A}_0^f : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, 0, x_3)) + \mathbb{F}_{sol}^{0ij}(0) - \theta \mathcal{G} : \mathbb{J}^{ij} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0, \\
& \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].
\end{aligned}$$

At second, we substitute representation (7.45) (for $x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1)$), the image of (7.21) with respect to the Laplace transform in t , $\Phi \equiv 0$, and $\Phi_1(\mathbf{x}, \hat{\mathbf{y}}) = \phi_{11}(\mathbf{x})\phi_{12}(\hat{\mathbf{y}})$ into (9.5). Here, as before, ϕ_{11} is an arbitrary smooth scalar test

function and ϕ_{12} is an arbitrary test vector-function belonging to $H_{\sharp}^1(\Sigma)^3$. Thus we get the integral equality

$$\begin{aligned}
 (10.15) \quad & \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \right. \\
 & \quad \times \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \right. \\
 & \quad \left. \left. \left(\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + p^0(\mathbf{x}) \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) \right. \\
 & \quad \left. \left. - \mathbb{I} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) \right. \right. \\
 & \quad \left. \left. + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} \\
 & + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta_*}^{\Delta + \delta^*} \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \right. \\
 & \quad \times \int_{\Sigma} \left[\left\{ \hat{\zeta}(\hat{\mathbf{y}}) \left(s \left[\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right] + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) + (1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} \right\} : \right. \\
 & \quad \left. \left. \left(\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right. \\
 & + p^0(\mathbf{x}) \int_{\Sigma} \left[\left\{ \hat{\zeta}(\hat{\mathbf{y}}) \left[s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right] \right. \right. \\
 & \quad \left. \left. + (1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) - \hat{\zeta}(\hat{\mathbf{y}}) \mathbb{I} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
 & + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\left\{ \hat{\zeta}(\hat{\mathbf{y}}) \left[s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right] \right. \right. \\
 & \quad \left. \left. + (1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) \right. \\
 & \quad \left. \left. + (1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} : \mathbb{J}^{ij} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} \\
 & + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta}^{\Delta + \delta_*} \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \right. \\
 & \quad \times \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \left[\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right] \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
 & \quad \left. + p^0(\mathbf{x}) \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) + s \hat{\zeta}(\hat{\mathbf{y}}) \bar{\mathbb{F}}_f^0(s) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \right.
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) \right. \\
& \quad \left. + s \left(\hat{\zeta}(\hat{\mathbf{y}}) \left(\bar{\mathbb{F}}_{sol}^{0ij}(s) - \frac{1}{s} \theta \mathcal{G} : \mathbb{J}^{ij} \right) + \frac{1}{s} \mathcal{G} : \mathbb{J}^{ij} \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} \\
& + \int_{\{0 < x_1, x_2 < 1\}} \int_0^{\Delta} \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \int_{\Sigma} \left[\mathcal{G} : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3))) \right. \right. \\
& \quad \left. \left. + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
& + p^0(\mathbf{x}) \int_{\Sigma} \left[\mathcal{G} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
& + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\mathcal{G} : \mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) \right] : \\
& \quad : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} = 0, \quad \forall s > 0.
\end{aligned}$$

Let us consider the first integral in (10.15). In this integral, using Green's theorem and 1-periodicity of ϕ_{12} , we obtain that

(10.16)

$$\begin{aligned}
& \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \mathbb{J}^{ij} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0, \\
& \quad i, j = 1, 2, 3, \quad s > 0.
\end{aligned}$$

With account of (10.4), (10.5), and (10.16), equality (10.15) takes the following shorter form:

(10.17)

$$\begin{aligned}
& \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) \right. \right. \right. \\
& \quad \left. \left. + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \left(\mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
& + p^0(\mathbf{x}) \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) \right. \right. \\
& \quad \left. \left. + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \\
& + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\left\{ s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) \right. \right. \\
& \quad \left. \left. + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\hat{\mathbf{x}} \\
& + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta_*}^{\Delta + \delta^*} \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \int_{\Sigma} \left[\left\{ \widehat{\zeta}(\widehat{\mathbf{y}}) \left(s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} \right\} : \right. \\
 & \quad \left. : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) + \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \\
 & + p^0(\mathbf{x}) \int_{\Sigma} \left[\left\{ \widehat{\zeta}(\widehat{\mathbf{y}}) \left(s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) \right. \right. \\
 & \quad \left. \left. + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_2(\widehat{\mathbf{y}}, s, x_3)) - \widehat{\zeta}(\widehat{\mathbf{y}}) \mathbb{I} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \\
 & + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\left\{ \widehat{\zeta}(\widehat{\mathbf{y}}) \left(s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) \right. \right. \\
 & \quad \left. \left. + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_3^{ij}(\widehat{\mathbf{y}}, s, x_3)) - \widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{G} : \mathbb{J}^{ij} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\widehat{\mathbf{x}} \\
 & + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta}^{\Delta + \delta_*} \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \right. \\
 & \quad \left. \times \int_{\Sigma} \left[\mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) + \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \right. \\
 & \quad \left. + p^0(\mathbf{x}) \int_{\Sigma} \left[\mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_2(\widehat{\mathbf{y}}, s, x_3)) + s \widehat{\zeta}(\widehat{\mathbf{y}}) \overline{\mathbb{F}}_f^0(s) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \right. \\
 & \quad \left. + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_3^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right. \right. \\
 & \quad \quad \left. \left. + \widehat{\zeta}(\widehat{\mathbf{y}}) (s \overline{\mathbb{F}}_{sol}^{0ij}(s) - \theta \mathcal{G} : \mathbb{J}^{ij}) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\widehat{\mathbf{x}} \\
 & + \int_{\{0 < x_1, x_2 < 1\}} \int_0^{\Delta} \left\{ \sum_{i,j=1}^3 \frac{\partial \bar{u}_i}{\partial x_j}(\mathbf{x}, s) \right. \\
 & \quad \left. \times \int_{\Sigma} \left[\mathcal{G} : \left(\mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) + \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \right. \\
 & \quad \left. + p^0(\mathbf{x}) \int_{\Sigma} \left[\mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_2(\widehat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \right. \\
 & \quad \left. + \sum_{i,j=1}^3 \frac{\partial v_i^0}{\partial x_j}(\mathbf{x}) \int_{\Sigma} \left[\mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_3^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \Big\} \phi_{11}(\mathbf{x}) dx_3 d\widehat{\mathbf{x}} = 0.
 \end{aligned}$$

From the structure of this integral equality it is clear that it holds true independently of any possible values of \mathbf{u} , \mathbf{v}^0 , and p^0 if the vector-functions \mathbf{Y}_0^{ij} , $\overline{\mathbf{Y}}_1^{ij}$, $\overline{\mathbf{Y}}_2$, and $\overline{\mathbf{Y}}_3^{ij}$ satisfy the identities (10.8)₁ and

$$(10.18) \quad \overline{\mathbf{Y}}_1^{ij} = \overline{\mathbf{Y}}_2 = \overline{\mathbf{Y}}_3^{ij} \equiv 0 \quad \text{for } (\widehat{\mathbf{y}}, s, x_3) \in \Sigma \times (0, +\infty) \times ((0, 1) \setminus [\Delta, \Delta + \delta^*]),$$

and the following integral equalities:

$$(10.19)$$

$$\int_{\Sigma} \left[\left\{ \widehat{\zeta}(\widehat{\mathbf{y}}) \left(s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} \right\} : \right.$$

$$\begin{aligned} & : \left(\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta_*^*], \end{aligned}$$

(10.20)

$$\begin{aligned} & \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) + \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in [\Delta, \Delta + \delta_*^*]; \end{aligned}$$

(10.21)

$$\begin{aligned} & \int_{\Sigma} \left[\left\{ \hat{\zeta}(\hat{\mathbf{y}}) \left(s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) \right. \right. \\ & \quad \left. \left. + (1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) - \hat{\zeta}(\hat{\mathbf{y}}) \mathbb{I} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta_*^*]; \end{aligned}$$

$$\begin{aligned} (10.22) \quad & \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)) + s \hat{\zeta}(\hat{\mathbf{y}}) \bar{\mathbb{F}}_f^0(s) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in [\Delta, \Delta + \delta_*^*]; \end{aligned}$$

(10.23)

$$\begin{aligned} & \int_{\Sigma} \left[\left\{ \hat{\zeta}(\hat{\mathbf{y}}) \left(s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) \right. \right. \\ & \quad \left. \left. + (1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) - \hat{\zeta}(\hat{\mathbf{y}}) \mathcal{G} : \mathbb{J}^{ij} \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta_*^*]; \end{aligned}$$

(10.24)

$$\begin{aligned} & \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)) + \hat{\zeta}(\hat{\mathbf{y}}) (s \bar{\mathbb{F}}_{sol}^{0ij}(s) - \theta \mathcal{G} : \mathbb{J}^{ij}) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in [\Delta, \Delta + \delta_*^*]. \end{aligned}$$

Now we aim to formulate the problems for determination of \mathbf{Y}_0^{ij} , \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} . We start with the notice that the integral equality (10.19) holds true if \mathbf{Y}_0^{ij} and $\bar{\mathbf{Y}}_1^{ij}$ satisfy the following three integral equalities:

(10.25)

$$\begin{aligned} & \int_{\Sigma} \left[\hat{\zeta}(\hat{\mathbf{y}}) s \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) \right) \right] : \\ & \quad \quad \quad : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta_*^*]; \end{aligned}$$

$$\begin{aligned} (10.26) \quad & \int_{\Sigma} \left[(1 - \hat{\zeta}(\hat{\mathbf{y}})) \mathcal{G} : \left(\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}))d\hat{\mathbf{y}} = 0, \\ & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta_*^*]; \end{aligned}$$

and

$$\begin{aligned}
 (10.27) \quad & \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \alpha_{\gamma}^{-1} (\mathbb{I} \otimes \mathbb{I}) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) \right) \right. \\
 & \quad + \left\{ \widehat{\zeta}(\widehat{\mathbf{y}}) \left(s \left[\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right] + \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} \right) \right. \\
 & \quad \left. \left. + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} \right\} : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
 & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*].
 \end{aligned}$$

Analogously, we substitute the representation of $\mathcal{H}^s(\widehat{\mathbf{y}}, x_3)$ (see (7.45)) into (10.20) and then notice that (10.20) holds true if \mathbf{Y}_0^{ij} and $\overline{\mathbf{Y}}_1^{ij}$ satisfy the following three integral equalities:

$$\begin{aligned}
 (10.28) \quad & s \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{A}_0^f : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) \right) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
 & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in [\Delta, \Delta + \delta_*];
 \end{aligned}$$

$$\begin{aligned}
 (10.29) \quad & \int_{\Sigma} \left[(1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) \right) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
 & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*];
 \end{aligned}$$

and

$$\begin{aligned}
 (10.30) \quad & \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) (\mathcal{B}_0^f + (1 - \theta) \mathcal{G}) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) \right) \right. \\
 & \quad \left. + \mathcal{H}^s(\widehat{\mathbf{y}}, x_3) : \mathbb{D}_{\widehat{\mathbf{y}}}(\overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
 & \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall s > 0, \quad \forall x_3 \in [\Delta, \Delta + \delta_*].
 \end{aligned}$$

Indeed, summing up (10.25), (10.26), and (10.27), we get (10.19), and summing up (10.28), (10.29), and (10.30), we get (10.20).

Remark 23. *The integral equality (10.25) is equivalent to (10.9), and the integral equality (10.28) is equivalent to (10.10). The integral equalities (10.26) and (10.29) coincide except for the matter that (10.26) is set on the segment $\{\Delta + \delta_* < x_3 \leq \Delta + \delta^*\}$ and (10.29) is set on the segment $\{\Delta \leq x_3 \leq \Delta + \delta_*\}$.*

On the basis of identity (10.8)₁, the integral equalities (10.9), (10.10), (10.26), and (10.29), and Remarks 22 and 23, we formulate *the cell problem on Σ* for determination of $\mathbf{Y}_0^{ij} = \mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)$ as follows.

Problem Y1. It is necessary to find a vector-function $\widehat{\mathbf{y}} \mapsto \mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)$ ($i, j = 1, 2, 3$), defined in the pattern cell Σ for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.31) \quad \mathbf{Y}_0^{ij}(\cdot, x_3) \in (H_{\#}^1(\Sigma)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta^*],$$

and the integral equalities (10.9), (10.10), (10.26), and (10.29).

Proposition 8. *Assume that the tensor \mathcal{A}_0^f in the formulation of Problem Y1 is given and has the properties stated in assertion (i) of Corollary 2; then Problem Y1 has the unique solution.*

Proof. In view of (10.31), the vector-function \mathbf{Y}_0^{ij} on Σ_F should meet the demand

$$(10.32) \quad \mathbf{Y}_0^{ij}(\cdot, x_3) \in H_{\#}^1(\Sigma_F)^3, \quad x_3 \in [\Delta, \Delta + \delta^*].$$

Due to the properties of \mathcal{A}_0^f (in assertion (i) of Corollary 2), the system (10.10) & (10.32) for $x_3 \in [\Delta, \Delta + \delta_*]$ is the standard variational elliptic problem. Therefore, applying the Lax-Milgram theorem, by means of the standard considerations we conclude that this problem has a solution $\mathbf{Y}_0^{ij}(\cdot, x_3)$ on Σ_F for $x_3 \in [\Delta, \Delta + \delta_*]$ and this solution is unique up to an arbitrary fixed constant.

Further, the integral equality (10.26) supplemented with the Dirichlet condition

$$(10.33) \quad \mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)|_{\partial\Sigma_S} = \mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3)|_{\partial\Sigma_F \setminus \partial\Sigma}, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

is the standard variational elliptic problem on Σ_S for $x_3 \in [\Delta, \Delta + \delta_*]$. Here $\mathbf{Y}_0^{ij}|_{\partial\Sigma_F \setminus \partial\Sigma}$ is the trace of the solution of the system (10.10) & (10.32) on $\partial\Sigma_S$. On the strength of the theory of generalized solutions of boundary value problems in mathematical physics [18,19], there exists the unique solution of (10.26) & (10.33).

Thus we construct the solution \mathbf{Y}_0^{ij} on the whole cell Σ . Due to the boundary condition (10.33) we have that $\mathbf{Y}_0^{ij} \in H_{\#}^1(\Sigma)^3$. Calibrating so that $\langle \mathbf{Y}_0^{ij}(\cdot, x_3) \rangle_{\Sigma} = 0$ we obtain the *unique* solution of Problem Y1 for $x_3 \in [\Delta, \Delta + \delta_*]$. Here we additionally notice that if \mathbf{Y}_0^{ij} resolves (10.26) & (10.33) then $\mathbf{Y}_0^{ij} + c$ resolves (10.26) supplemented with the boundary data $\mathbf{Y}_0^{ij}|_{\partial\Sigma_F \setminus \partial\Sigma} + c$, where c is an arbitrary constant, which allows calibration on the whole Σ .

Analogously, since the fourth-rank tensor $\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}$ is symmetric and positive definite, the system (10.9) & (10.32) for $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$ is the standard variational elliptic problems, as well. Keeping track of the above consideration, this time with $(\Delta + \delta_*, \Delta + \delta^*)$ on the place of $[\Delta, \Delta + \delta_*]$ and (10.29) on the place of (10.26), we conclude that there exists a unique solution of Problem Y1 for $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$. By this we complete the proof of Proposition 8. \square

Next, we formulate *the cell problems on Σ* for determination of the independent of t vector-functions $\mathbf{Y}_2|_{t=0} = \mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)$ and $\mathbf{Y}_3|_{t=0} = \mathbf{Y}_3(\hat{\mathbf{y}}, 0, x_3)$ on the basis of identities (10.8)_{2,3}, the integral equalities (10.11)-(10.14) and Remark 22 in the quite natural way as follows.

Problem Y2. It is necessary to find a vector-function $\hat{\mathbf{y}} \mapsto \mathbf{Y}_2(\hat{\mathbf{y}}, 0, x_3)$, defined in Σ_F for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.34) \quad \mathbf{Y}_2(\cdot, 0, x_3) \in (H_{\#}^1(\Sigma_F)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta^*],$$

and the integral equalities (10.11) and (10.12).

Problem Y3. It is necessary to find a vector-function $\hat{\mathbf{y}} \mapsto \mathbf{Y}_3^{ij}(\hat{\mathbf{y}}, 0, x_3)$ ($i, j = 1, 2, 3$), defined in Σ_F for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.35) \quad \mathbf{Y}_3(\cdot, 0, x_3) \in (H_{\#}^1(\Sigma_F)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta + \delta^*],$$

and the integral equalities (10.13) and (10.14).

Proposition 9. *Assume that the tensor \mathcal{A}_0^f and the matrices \mathbb{F}_f^0 and \mathbb{F}_{sol}^{0ij} in the formulations of Problems Y2-Y3 are given and have the properties stated in*

assertions (i) and (ii) of Corollary 2; then each of Problems Y2-Y3 has a unique solution.

Proof. Justification of this proposition is analogous to the proof of Proposition 8. Namely, due to the properties of \mathcal{A}_0^f , $\mathbb{F}_f^0(0)$ and \mathbb{F}_{sol}^{0ij} , we have that the systems (10.12) & (10.34) and (10.14) & (10.35) for $x_3 \in [\Delta, \Delta + \delta_*]$ are the standard variational elliptic problems. Therefore, applying the Lax-Milgram theorem, by means of the standard considerations we conclude that these problems have unique solutions $\mathbf{Y}_2(\cdot, 0, x_3)$ and $\mathbf{Y}_3^{ij}(\cdot, 0, x_3)$, respectively, for $x_3 \in [\Delta, \Delta + \delta_*]$.

Analogously, since the fourth-rank tensor $\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}$ is symmetric and positive definite, we have that the systems (10.11) & (10.34) and (10.13) & (10.35) for $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$ are the standard variational elliptic problems, as well. Therefore, as before, we conclude that these problems have unique solutions $\mathbf{Y}_2(\cdot, 0, x_3)$ and $\mathbf{Y}_3^{ij}(\cdot, 0, x_3)$, respectively, for $x_3 \in [\Delta + \delta_*, \Delta + \delta^*]$.

This observation completes the proof of Proposition 9. □

Further, let us consider each of the integral equalities (10.27), (10.30), (10.21)-(10.24) as the variational equation, where $\hat{\mathbf{y}} \mapsto \bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)$, $\hat{\mathbf{y}} \mapsto \bar{\mathbf{Y}}_2(\hat{\mathbf{y}}, s, x_3)$, and $\hat{\mathbf{y}} \mapsto \bar{\mathbf{Y}}_3^{ij}(\hat{\mathbf{y}}, s, x_3)$ are the sought vector-functions, $s > 0$ and $x_3 \in [\Delta, \Delta + \delta^*]$ are real parameters, and \mathbf{Y}_0^{ij} ($i, j = 1, 2, 3$) is given as the solution of Problem Y1.

Proposition 10. *The following assertions hold true.*

- (i) For any fixed pair of parameters $(s, x_3) \in (0, +\infty) \times (\Delta + \delta_*, \Delta + \delta^*]$, the variational equation (10.27) has a unique solution $\bar{\mathbf{Y}}_1^{ij}(\cdot, s, x_3)$ belonging to $(H_{\sharp}^1(\Sigma)/\mathbb{R})^3$, $i, j = 1, 2, 3$.
- (ii) For any fixed pair of parameters $(s, x_3) \in (0, +\infty) \times [\Delta, \Delta + \delta_*]$, the variational equation (10.30) has a unique solution $\bar{\mathbf{Y}}_1^{ij}(\cdot, s, x_3)$ belonging to $(H_{\sharp}^1(\Sigma)/\mathbb{R})^3$, $i, j = 1, 2, 3$.
- (iii) For any fixed pair of parameters $(s, x_3) \in (0, +\infty) \times (\Delta + \delta_*, \Delta + \delta^*]$, the variational equation (10.21) has a unique solution $\bar{\mathbf{Y}}_2(\cdot, s, x_3)$ belonging to $(H_{\sharp}^1(\Sigma)/\mathbb{R})^3$.
- (iv) For any fixed pair of parameters $(s, x_3) \in (0, +\infty) \times [\Delta, \Delta + \delta_*]$, the variational equation (10.22) has a unique solution $\bar{\mathbf{Y}}_2(\cdot, s, x_3)$ belonging to $(H_{\sharp}^1(\Sigma)/\mathbb{R})^3$.
- (v) For any fixed pair of parameters $(s, x_3) \in (0, +\infty) \times (\Delta + \delta_*, \Delta + \delta^*]$, the variational equation (10.23) has a unique solution $\bar{\mathbf{Y}}_3^{ij}(\cdot, s, x_3)$ belonging to $(H_{\sharp}^1(\Sigma)/\mathbb{R})^3$, $i, j = 1, 2, 3$.
- (vi) For any fixed pair of parameters $(s, x_3) \in (0, +\infty) \times [\Delta, \Delta + \delta_*]$, the variational equation (10.24) has a unique solution $\bar{\mathbf{Y}}_3^{ij}(\cdot, s, x_3)$ belonging to $(H_{\sharp}^1(\Sigma)/\mathbb{R})^3$, $i, j = 1, 2, 3$.

Proof. Due to assertion (iii) of Proposition 6, representation (7.45), the regularity condition (10.31), and assertion (ii) of Corollary 2, we have that each of the variational equations (10.27), (10.30), (10.21)-(10.24) is a standard variational elliptic problem. Therefore, applying the Lax-Milgram theorem, by means of the standard considerations we conclude that each of these equations has a unique solution. Thus all assertions of Proposition 10 hold true. □

As the result of the above considerations, we have formulated the cell problems on Σ in terms of the images with respect to the Laplace transform in t of the sought vector-functions \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} . More certainly, the system (10.18)₁ & (10.27) & (10.30) is the cell problem for determination of $\overline{\mathbf{Y}}_1^{ij} = \mathcal{L}[\mathbf{Y}_1^{ij}]$ ($i, j = 1, 2, 3$), the system (10.18)₂ & (10.21) & (10.22) is the cell problem for determination of $\overline{\mathbf{Y}}_2 = \mathcal{L}[\mathbf{Y}_2]$, and the system (10.18)₃ & (10.23) & (10.24) is the cell problem for determination of $\overline{\mathbf{Y}}_3^{ij} = \mathcal{L}[\mathbf{Y}_3^{ij}]$ ($i, j = 1, 2, 3$). In this line, Proposition 10 is the assertion on existence and uniqueness of these problems.

Now let us transform these formulations into the boundary value problems for the sought vector-functions \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} themselves. For $s > 0$, we divide (10.27) by s and apply the inverse Laplace transform \mathcal{L}^{-1} in s . Using the basic properties of \mathcal{L}^{-1} , we arrive at the variational equation

$$(10.36) \quad \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \left\{ \alpha_{\gamma}^{-1}(\mathbb{I} \otimes \mathbb{I}) : (\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3))) \right. \right. \\ \left. \left. + \left(\alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, t, x_3)) \right. \right. \\ \left. \left. + \alpha_{\gamma}^{-1}(\mathbb{I} \otimes \mathbb{I}) : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_1^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right\} \right. \\ \left. + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_1^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\ \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall t \in [0, T], \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*), \quad \forall i, j = 1, 2, 3.$$

Further, dividing by s (for $s > 0$), substituting (7.45) for \mathcal{H}^s , and applying the inverse Laplace transform \mathcal{L}^{-1} in s , from (10.30) we deduce the variational equation

$$(10.37) \quad \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) (\mathcal{B}_0^f + (1 - \theta) \mathcal{G}) : (\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3))) \right. \\ \left. + \widehat{\zeta}(\widehat{\mathbf{y}}) \left\{ \mathcal{A}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, t, x_3)) + \mathcal{B}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_1^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right. \right. \\ \left. \left. + \int_0^t [\mathcal{A}_1^f(t - \tau) + (J_t \mathcal{B}_1^f)(t - \tau)] : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, \tau, x_3)) d\tau \right\} \right. \\ \left. + (1 - \theta) \widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_1^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\ \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall t \in [0, T], \quad \forall x_3 \in [\Delta, \Delta + \delta_*], \quad \forall i, j = 1, 2, 3.$$

Having equations (10.36) and (10.37) derived, we are in a position to formulate the problems of finding $\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, 0, x_3)$ and $\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, t, x_3)$ for $t > 0$ ($i, j = 1, 2, 3$).

We substitute $t = 0$ into (10.36) and (10.37). Thus we set up the following.

Problem Y4. It is necessary to find a vector-function $\widehat{\mathbf{y}} \mapsto \mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, 0, x_3)$ ($i, j = 1, 2, 3$), defined in the set Σ_F for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.38) \quad \mathbf{Y}_1^{ij}(\cdot, 0, x_3) \in (H_{\#}^1(\Sigma_F)/\mathbb{R})^3, \quad \forall x_3 \in [\Delta, \Delta + \delta^*],$$

and resolves the variational equations

(10.39)

$$\int_{\Sigma_F} \left\{ \alpha_\gamma^{-1}(\mathbb{I} \otimes \mathbb{I}) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) \right) \right. \\ \left. + \left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, 0, x_3)) \right\} : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\ \forall \phi_{12} \in H_\#^1(\Sigma)^3, \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \quad \forall i, j = 1, 2, 3,$$

and

$$(10.40) \quad \int_{\Sigma_F} \left[(\mathcal{B}_0^f + (1 - \theta)\mathcal{G}) : \left(\mathbb{J}^{ij} + \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)) \right) \right. \\ \left. + \mathcal{A}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, 0, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\ \forall \phi_{12} \in H_\#^1(\Sigma)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta_*], \quad \forall i, j = 1, 2, 3.$$

Further, for $t > 0$ we set up the following.

Problem Y5. It is necessary to find a vector-function $(\widehat{\mathbf{y}}, t) \mapsto \mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, t, x_3)$ ($i, j = 1, 2, 3$), defined in the set $\Sigma \times (0, T]$ for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.41) \quad \mathbf{Y}_1^{ij}(\cdot, \cdot, x_3) \in L^\infty(0, T; (H_\#^1(\Sigma)/\mathbb{R})^3) \cap H^1(0, T; H_\#^1(\Sigma_F)^3)$$

for $x_3 \in [\Delta, \Delta + \delta^*]$ and resolves the variational equations (10.36) and (10.37).

In (10.39), (10.40), (10.36), and (10.37), the function \mathbf{Y}_0^{ij} ($i, j = 1, 2, 3$) is the solution of Problem Y1.

Now we divide each of the variational equations (10.21)-(10.24) by s (for $s > 0$) and, after this, apply the inverse Laplace transform \mathcal{L}^{-1} in s . Also, in these transformations of (10.22) and (10.24) we use the explicit representation (7.45) of the tensor \mathcal{H}^s . Thus we derive the four variational equations (see equations (10.43), (10.44), (10.46), and (10.47) below), in which terms we formulate the problems of determining \mathbf{Y}_2 and \mathbf{Y}_3^{ij} ($i, j = 1, 2, 3$) for $t > 0$ as follows.

Problem Y6. It is necessary to find a vector-function $(\widehat{\mathbf{y}}, t) \mapsto \mathbf{Y}_2(\widehat{\mathbf{y}}, t, x_3)$, defined in the set $\Sigma \times (0, T]$ for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.42) \quad \mathbf{Y}_2(\cdot, \cdot, x_3) \in L^\infty(0, T; (H_\#^1(\Sigma)/\mathbb{R})^3) \cap H^1(0, T; H_\#^1(\Sigma_F)^3)$$

for all $x_3 \in [\Delta, \Delta + \delta^*]$ and resolves the variational equations

$$(10.43) \quad \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \left\{ \left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_2(\widehat{\mathbf{y}}, t, x_3)) \right. \right. \\ \left. \left. + \alpha_\gamma^{-1}(\mathbb{I} \otimes \mathbb{I}) : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\widehat{\mathbf{y}}, t, x_3)) \right\} \right. \\ \left. + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\widehat{\mathbf{y}}, t, x_3)) - \widehat{\zeta}(\widehat{\mathbf{y}}) \mathbb{I} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\ \forall \phi_{12} \in H_\#^1(\Sigma)^3, \quad \forall t \in (0, T], \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*],$$

and

$$\begin{aligned}
(10.44) \quad & \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \left\{ \mathcal{A}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_2(\widehat{\mathbf{y}}, t, x_3)) + \mathcal{B}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\widehat{\mathbf{y}}, t, x_3)) \right. \right. \\
& \quad \left. \left. + \int_0^t \left(\mathcal{A}_1^f(t - \tau) + (J_t \mathcal{B}_1^f)(t - \tau) \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_2(\widehat{\mathbf{y}}, \tau, x_3)) d\tau \right\} \right. \\
& \quad \left. + (1 - \theta \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\widehat{\mathbf{y}}, t, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \\
& \quad \quad \quad + \int_{\Sigma} \widehat{\zeta}(\widehat{\mathbf{y}}) \mathbb{F}_f^0(t) : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
& \quad \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall t \in (0, T], \quad \forall x_3 \in [\Delta, \Delta + \delta_*].
\end{aligned}$$

Problem Y7. It is necessary to find a vector-function $(\widehat{\mathbf{y}}, t) \mapsto \mathbf{Y}_3^{ij}(\widehat{\mathbf{y}}, t, x_3)$ ($i, j = 1, 2, 3$), defined in the set $\Sigma \times (0, T]$ for all $x_3 \in [0, 1]$, which vanishes for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, satisfies the regularity condition

$$(10.45) \quad \mathbf{Y}_3^{ij}(\cdot, \cdot, x_3) \in L^\infty(0, T; (H_{\#}^1(\Sigma)/\mathbb{R})^3) \cap H^1(0, T; H_{\#}^1(\Sigma_F)^3), \quad \forall x_3 \in [\Delta, \Delta + \delta^*],$$

and resolves the variational equation

$$\begin{aligned}
(10.46) \quad & \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \left\{ \left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\widehat{\mathbf{y}}, t, x_3)) \right. \right. \\
& \quad \left. \left. + \alpha_\gamma^{-1} (\mathbb{I} \otimes \mathbb{I}) : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_3^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right\} \right. \\
& \quad \left. + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_3^{ij})(\widehat{\mathbf{y}}, t, x_3)) - \widehat{\zeta}(\widehat{\mathbf{y}}) \mathcal{G} : \mathbb{J}^{ij} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
& \quad \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall t \in (0, T], \quad \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*],
\end{aligned}$$

and the variational equation

$$\begin{aligned}
(10.47) \quad & \int_{\Sigma} \left[\widehat{\zeta}(\widehat{\mathbf{y}}) \left\{ \mathcal{A}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\widehat{\mathbf{y}}, t, x_3)) + \mathcal{B}_0^f : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_3^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right. \right. \\
& \quad \left. \left. + \int_0^t \left(\mathcal{A}_1^f(t - \tau) + (J_t \mathcal{B}_1^f)(t - \tau) \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{ij}(\widehat{\mathbf{y}}, \tau, x_3)) d\tau \right\} \right. \\
& \quad \left. + (1 - \theta \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_3^{ij})(\widehat{\mathbf{y}}, t, x_3)) \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} \\
& \quad \quad \quad + \int_{\Sigma} \widehat{\zeta}(\widehat{\mathbf{y}}) \left(\mathbb{F}_{sol}^{0ij}(t) - \theta \mathcal{G} : \mathbb{J}^{ij} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\phi_{12}(\widehat{\mathbf{y}})) d\widehat{\mathbf{y}} = 0, \\
& \quad \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall t \in (0, T], \quad \forall x_3 \in [\Delta, \Delta + \delta_*].
\end{aligned}$$

Proposition 11. Let \mathbf{Y}_0^{ij} ($i, j = 1, 2, 3$) be the solution of Problem Y1. Assume that the tensors \mathcal{A}_0^f , \mathcal{B}_0^f , $\mathcal{A}_1^f(t)$, and $\mathcal{B}_1^f(t)$ and the matrices \mathbb{F}_f^0 and \mathbb{F}_{sol}^{0ij} in the formulations of Problems Y4–Y7 are given and have the properties in items (i)–(iii) of Corollary 2; then each of Problems Y4–Y7 has a unique solution.

Proof. On the strength of Proposition 10 and one-to-oneness of \mathcal{L} and \mathcal{L}^{-1} , each of the variational equations (10.36), (10.37), (10.39), (10.40), (10.43), (10.44), (10.46), and (10.47) has a unique solution. Furthermore, on the strength of linearity and one-on-oneness of \mathcal{L} and \mathcal{L}^{-1} , identities (10.18) yield that \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} vanish for all $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$, which completes the proof. \square

Remark 24. Since $\mathbb{J}^{ij} = \mathbb{J}^{ji}$ and $\mathbb{F}_s^{0ij} = \mathbb{F}_s^{0ji}$, the solutions $\mathbf{Y}_0^{ij} = \mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3)$, $\mathbf{Y}_1^{ij} = \mathbf{Y}_1^{ij}(\widehat{\mathbf{y}}, t, x_3)$, and $\mathbf{Y}_3^{ij} = \mathbf{Y}_3^{ij}(\widehat{\mathbf{y}}, t, x_3)$ ($t \in [0, T]$) possess the symmetry property $\mathbf{Y}_k^{ij} = \mathbf{Y}_k^{ji}$ ($k = 0, 1, 3, i, j = 1, 2, 3$).

Remark 25. Notice that the vector-functions \mathbf{Y}_0^{ij} , \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} do not vary with change of x_3 on the intervals $(\Delta, \Delta + \delta_*)$ and $(\Delta + \delta_*, \Delta + \delta^*)$, i.e.,

$$\frac{\partial \mathbf{Y}_0^{ij}}{\partial x_3} = \frac{\partial \mathbf{Y}_1^{ij}}{\partial x_3} = \frac{\partial \mathbf{Y}_2}{\partial x_3} = \frac{\partial \mathbf{Y}_3^{ij}}{\partial x_3} = 0 \quad \text{for } x_3 \in (\Delta, \Delta + \delta_*) \cup (\Delta + \delta_*, \Delta + \delta^*),$$

since the given data in Problems Y1-Y7 do not vary with change of x_3 on $(\Delta, \Delta + \delta_*)$ and on $(\Delta + \delta_*, \Delta + \delta^*)$.

Remark 26. We have established that if \mathbf{Y}_0^{ij} , \mathbf{Y}_1^{ij} , \mathbf{Y}_2 , and \mathbf{Y}_3^{ij} are the solutions of Problems Y1-Y7 then equalities (10.7) and (10.17) hold true independently of the values of the vector-function \mathbf{u} and the macroscopic boundary conditions (5.12)-(5.13). In particular, this means that the solutions of Problems Y1-Y7 fully define dynamics of the mechanical system on the mesoscale, i.e., on the characteristic scale of the taller bristles.

11. MACROSCOPIC EFFECTIVE COEFFICIENTS AND THEIR PROPERTIES

In this and the next section, with the help of solutions of Problems Y1-Y7, from the equations of Model H-2sc we derive the system of effective homogenized macroscopic equations for the macroscopic velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. This system does not incorporate the variable $\widehat{\mathbf{y}}$ explicitly, which means separation of the mesoscopic scale from the macroscopic one in the homogenized description of the mechanical system. The mesoscopic velocity $\mathbf{u}^{(1)}$ and the variable $\widehat{\mathbf{y}}$ get sublimed in the coefficients of the macroscopic equations.

In (7.2), we take $\phi_1 \equiv 0$ and substitute representation (10.1) for $\mathbf{u}^{(1)}$. Taking into account (7.21), Remarks 24-26, and assertions (i) and (iii) of Proposition A.1 and properly collecting terms, after considerably simple but rather lengthy calculations, we establish that the integral equality

$$\begin{aligned} (11.1) \quad & \int_0^T \int_{\Omega} \left\{ -\alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{u}(\mathbf{x}, t) \cdot \partial_t \phi(\mathbf{x}, t) \right. \\ & + \left(\mathcal{V}(x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathcal{E}(x_3) : \mathbb{D}_x((J_t \mathbf{u})(\mathbf{x}, t)) \right. \\ & + \int_0^t \mathcal{K}(t - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \\ & + \int_0^t \int_0^{\tau'} \mathcal{Q}(t - \tau', \tau' - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' \\ & + \int_0^t \int_0^{\tau'} \mathcal{W}(t - \tau', \tau' - \tau, x_3) : \mathbb{D}_x((J_{\tau} \mathbf{u})(\mathbf{x}, \tau)) d\tau d\tau' \\ & \left. \left. + \mathbb{F}_p(t, x_3) p^0(\mathbf{x}) + \mathcal{F}_v(t, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \right) : \mathbb{D}_x(\phi(\mathbf{x}, t)) \right\} d\mathbf{x} dt \\ & = \int_0^T \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{f}(\mathbf{x}, t) \cdot \phi(\mathbf{x}, t) d\mathbf{x} dt \\ & \quad + \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{u}^0(\mathbf{x}) \cdot \phi(\mathbf{x}, 0) d\mathbf{x} \end{aligned}$$

holds true for an arbitrary smooth test vector-function $\phi = \phi(\mathbf{x}, t)$ vanishing in the neighborhood of boundary $\partial\Omega$ and section $\{t = T\}$. Here the fourth-rank tensor-valued functions \mathcal{V} , \mathcal{E} , \mathcal{K} , \mathcal{Q} , \mathcal{W} , and \mathcal{F}_v , and the matrix-valued function \mathbb{F}_p are defined via the solutions of Problems Y1-Y7 by the formulas

$$(11.2) \quad \mathcal{V}(x_3) = \langle \mathcal{A}_0(\cdot, x_3) \rangle_{\Sigma} + \sum_{p,q=1}^3 \langle \mathcal{A}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq},$$

$$(11.3) \quad \mathcal{E}(x_3) = \langle \mathcal{B}_0(\cdot, x_3) \rangle_{\Sigma} + \sum_{p,q=1}^3 \langle \mathcal{B}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq},$$

$$(11.4) \quad \mathcal{K}(t, x_3) = \sum_{p,q=1}^3 \langle \mathcal{A}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} \\ + \langle \mathcal{A}_1(\cdot, t, x_3) \rangle_{\Sigma} + \sum_{p,q=1}^3 \langle \mathcal{A}_1(\cdot, t, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq},$$

$$(11.5) \quad \mathcal{Q}(t, \tau, x_3) = \sum_{p,q=1}^3 \langle \mathcal{B}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} \\ + \sum_{p,q=1}^3 \langle \mathcal{A}_1(\cdot, t, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} + \langle \mathcal{B}_1(\cdot, \tau, x_3) \rangle_{\Sigma} \\ + \sum_{p,q=1}^3 \langle \mathcal{B}_1(\cdot, \tau, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq},$$

$$(11.6) \quad \mathcal{W}(t, \tau, x_3) = \sum_{p,q=1}^3 \langle \mathcal{B}_1(\cdot, t, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq},$$

$$(11.7) \quad \mathcal{F}_v(t, x_3) = \sum_{p,q=1}^3 \langle \mathcal{A}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} \\ + \sum_{p,q=1}^3 \langle \mathcal{B}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}((J_t \mathbf{Y}_3^{pq})(\cdot, t, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} \\ + \sum_{p,q=1}^3 \int_0^t \langle \mathcal{A}_1(\cdot, t - \tau, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_3^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} d\tau \\ + \sum_{p,q=1}^3 \int_0^t \langle \mathcal{B}_1(\cdot, t - \tau, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}((J_{\tau} \mathbf{Y}_3^{pq})(\cdot, \tau, x_3)) \rangle_{\Sigma} \otimes \mathbb{J}^{pq} d\tau \\ + \langle \mathcal{F}_v^0(\cdot, t, x_3) \rangle_{\Sigma},$$

where

$$(11.8a) \quad \mathcal{F}_v^0(\hat{\mathbf{y}}, t, x_3) = 0 \text{ for } x_3 \in (\Delta + \delta^*, 1),$$

$$(11.8b) \quad \mathcal{F}_v^0(\hat{\mathbf{y}}, t, x_3) = (1 - \hat{\zeta}(\hat{\mathbf{y}}))\mathcal{G} \text{ for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*],$$

$$(11.8c) \quad \mathcal{F}_v^0(\hat{\mathbf{y}}, t, x_3) = \widehat{\zeta}(\hat{\mathbf{y}}) \sum_{p,q=1}^3 \mathbb{F}_{sol}^{0pq}(t) \otimes \mathbb{J}^{pq} + (1 - \theta \widehat{\zeta}(\hat{\mathbf{y}})) \mathcal{G}$$

for $x_3 \in [\Delta, \Delta + \delta_*]$,

$$(11.8d) \quad \mathcal{F}_v^0(\hat{\mathbf{y}}, t, x_3) = \mathcal{G} \text{ for } x_3 \in (0, \Delta),$$

and

$$(11.9) \quad \begin{aligned} \mathbb{F}_p(t, x_3) &= \langle \mathcal{A}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\cdot, t, x_3)) \rangle_{\Sigma} + \langle \mathcal{B}_0(\cdot, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\cdot, t, x_3)) \rangle_{\Sigma} \\ &+ \int_0^t \langle \mathcal{A}_1(\cdot, t - \tau, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_2(\cdot, \tau, x_3)) \rangle_{\Sigma} d\tau \\ &+ \int_0^t \langle \mathcal{B}_1(\cdot, t - \tau, x_3) : \mathbb{D}_{\hat{\mathbf{y}}}((J_{\tau} \mathbf{Y}_2)(\cdot, \tau, x_3)) \rangle_{\Sigma} d\tau + \langle \mathbb{F}_p^0(\cdot, t, x_3) \rangle_{\Sigma}, \end{aligned}$$

where

$$(11.10) \quad \mathbb{F}_p^0(\hat{\mathbf{y}}, t, x_3) = \begin{cases} -\mathbb{I} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ -\widehat{\zeta}(\hat{\mathbf{y}})\mathbb{I} & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*), \\ \widehat{\zeta}(\hat{\mathbf{y}})\mathbb{F}_f^0(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta). \end{cases}$$

In the component-wise form, (11.2)-(11.10) are as follows ($k, l, m, n = 1, 2, 3$):

$$(11.11) \quad \begin{aligned} \mathcal{V}^{klmn}(x_3) &= \langle \mathcal{A}_0^{klmn}(\cdot, x_3) \rangle_{\Sigma} + \sum_{i,j,p,q=1}^3 \langle \mathcal{A}_0^{kl ij}(\cdot, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn}, \\ \mathcal{E}^{klmn}(x_3) &= \langle \mathcal{B}_0^{klmn}(\cdot, x_3) \rangle_{\Sigma} + \sum_{i,j,p,q=1}^3 \langle \mathcal{B}_0^{kl ij}(\cdot, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn}, \\ \mathcal{K}^{klmn}(t, x_3) &= \sum_{i,j,p,q=1}^3 \langle \mathcal{A}_0^{kl ij}(\cdot, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_1^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} \\ &+ \langle \mathcal{A}_1^{klmn}(\cdot, t, x_3) \rangle_{\Sigma} + \sum_{i,j,p,q=1}^3 \langle \mathcal{A}_1^{kl ij}(\cdot, t, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn}, \\ \mathcal{Q}^{klmn}(t, \tau, x_3) &= \sum_{i,j,p,q=1}^3 \langle \mathcal{B}_0^{kl ij}(\cdot, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} \\ &+ \sum_{i,j,p,q=1}^3 \langle \mathcal{A}_1^{kl ij}(\cdot, t, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} + \langle \mathcal{B}_1^{klmn}(\cdot, \tau, x_3) \rangle_{\Sigma} \\ &+ \sum_{i,j,p,q=1}^3 \langle \mathcal{B}_1^{kl ij}(\cdot, \tau, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn}, \\ \mathcal{W}^{klmn}(t, \tau, x_3) &= \sum_{i,j,p,q=1}^3 \langle \mathcal{B}_1^{kl ij}(\cdot, t, x_3) D_{\hat{\mathbf{y}}_{ij}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn}, \end{aligned}$$

$$\begin{aligned}
(11.12) \quad \mathcal{F}_v^{klmn}(t, x_3) &= \sum_{i,j,p,q=1}^3 \langle \mathcal{A}_0^{kl ij}(\cdot, x_3) D_{\widehat{\mathbf{y}}_{ij}}(\mathbf{Y}_3^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} \\
&+ \sum_{i,j,p,q=1}^3 \langle \mathcal{B}_0^{kl ij}(\cdot, x_3) D_{\widehat{\mathbf{y}}_{ij}}((J_t \mathbf{Y}_3^{pq})(\cdot, t, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} \\
&+ \sum_{i,j,p,q=1}^3 \int_0^t \langle \mathcal{A}_1^{kl ij}(\cdot, t - \tau, x_3) D_{\widehat{\mathbf{y}}_{ij}}(\mathbf{Y}_3^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} d\tau \\
&+ \sum_{i,j,p,q=1}^3 \int_0^t \langle \mathcal{B}_1^{kl ij}(\cdot, t - \tau, x_3) D_{\widehat{\mathbf{y}}_{ij}}((J_{\tau} \mathbf{Y}_3^{pq})(\cdot, \tau, x_3)) \rangle_{\Sigma} (\mathbb{J}^{pq})_{mn} d\tau \\
&+ \langle \mathcal{F}_v^0{}^{klmn}(\cdot, t, x_3) \rangle_{\Sigma},
\end{aligned}$$

where

$$\mathcal{F}_v^0{}^{klmn}(\widehat{\mathbf{y}}, t, x_3) = \begin{cases} 0 & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G}^{klmn} & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \widehat{\zeta}(\widehat{\mathbf{y}}) (\mathbb{F}_{sol}^0{}^{mn})_{kl}(t) + (1 - \langle \widehat{\psi} \rangle_{\Theta} \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G}^{klmn} & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \mathcal{G}^{klmn} & \text{for } x_3 \in (0, \Delta), \end{cases}$$

and

$$\begin{aligned}
(11.13) \quad F_{pkl}(t, x_3) &= \sum_{i,j=1}^3 \langle \mathcal{A}_0^{kl ij}(\cdot, x_3) D_{\widehat{\mathbf{y}}_{ij}}(\mathbf{Y}_2(\cdot, t, x_3)) \rangle_{\Sigma} \\
&+ \sum_{i,j=1}^3 \langle \mathcal{B}_0^{kl ij}(\cdot, x_3) D_{\widehat{\mathbf{y}}_{ij}}((J_t \mathbf{Y}_2)(\cdot, t, x_3)) \rangle_{\Sigma} \\
&+ \sum_{i,j=1}^3 \int_0^t \langle \mathcal{A}_1^{kl ij}(\cdot, t - \tau, x_3) D_{\widehat{\mathbf{y}}_{ij}}(\mathbf{Y}_2(\cdot, \tau, x_3)) \rangle_{\Sigma} d\tau \\
&+ \sum_{i,j=1}^3 \int_0^t \langle \mathcal{B}_1^{kl ij}(\cdot, t - \tau, x_3) D_{\widehat{\mathbf{y}}_{ij}}((J_{\tau} \mathbf{Y}_2)(\cdot, \tau, x_3)) \rangle_{\Sigma} d\tau \\
&+ \langle F_{pkl}^0(\cdot, t, x_3) \rangle_{\Sigma},
\end{aligned}$$

where

$$(11.14) \quad F_{pkl}^0(\widehat{\mathbf{y}}, t, x_3) = \begin{cases} -\delta_{kl} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ -\widehat{\zeta}(\widehat{\mathbf{y}}) \delta_{kl} & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \widehat{\zeta}(\widehat{\mathbf{y}}) F_{fkl}^0(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta). \end{cases}$$

Furthermore, let us write out (11.2)-(11.7) and (11.9) in more detail, using the property that \mathbf{Y}_0^{pq} , \mathbf{Y}_1^{pq} , \mathbf{Y}_2 , and \mathbf{Y}_3^{pq} ($p, q = 1, 2, 3$) vanish for $x_3 \in (0, \Delta) \cup (\Delta +$

$\delta^*, 1)$ and inserting (7.13)–(7.23), (11.8) and (11.10). We get

$$(11.15) \quad \mathbf{v}(x_3) = \begin{cases} \alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \sigma\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\sigma\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} + \mathbf{v}_{\text{corr}}^\sigma & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \sigma\theta\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\sigma\theta\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} + \mathbf{v}_{\text{corr}}^\theta & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

(11.16)

$$\begin{aligned} \mathbf{v}_{\text{corr}}^\sigma &= \sum_{p,q=1}^3 \alpha_\lambda \mathbb{I} \langle \widehat{\zeta}(\cdot) \operatorname{div}_{\widehat{\mathbf{y}}} \mathbf{Y}_0^{pq}(\cdot, x_3) \rangle_\Sigma \otimes \mathbb{J}^{pq} \\ &+ \sum_{p,q=1}^3 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\zeta}(\cdot) D_{\widehat{\mathbf{y}}^{mn}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_\Sigma \mathbb{J}^{mn} \otimes \mathbb{J}^{pq}, \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \end{aligned}$$

and

$$(11.17) \quad \begin{aligned} \mathbf{v}_{\text{corr}}^\theta &= \sigma\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_\Theta \mathbb{J}^{mn} \\ &+ 2\sigma\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_\Theta \otimes \mathbb{J}^{mn} \\ &+ \sum_{p,q=1}^3 \left[\left(\theta\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + \alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_\Theta \right) \mathbb{J}^{pq} \right. \\ &+ \left. 2\alpha_\mu \sum_{m,n=1}^3 \left(\theta \mathbb{J}^{mn} + \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_\Theta \right) \otimes \mathbb{J}^{mn} \right] : \\ &\quad : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_\Sigma \otimes \mathbb{J}^{pq}, \quad x_3 \in [\Delta, \Delta + \delta_*]; \end{aligned}$$

$$(11.18) \quad \mathcal{E}(x_3) = \begin{cases} \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \sigma\alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \sigma)\mathcal{G} + \mathcal{E}_{\text{corr}}^\sigma & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \sigma\theta\alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \sigma\theta)\mathcal{G} + \mathcal{E}_{\text{corr}}^\theta & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \mathcal{G} & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(11.19) \quad \begin{aligned} \mathcal{E}_{\text{corr}}^\sigma &= \sum_{p,q=1}^3 \langle [\widehat{\zeta}(\cdot)\alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\cdot))\mathcal{G}] : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_\Sigma \otimes \mathbb{J}^{pq}, \\ &\quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \end{aligned}$$

$$\begin{aligned}
(11.20) \quad \mathcal{E}_{\text{corr}}^\theta &= \sigma \alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \\
&+ \sigma \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \\
&+ \sum_{p,q=1}^3 \left\langle \left[\widehat{\zeta}(\cdot) \left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \right. \right. \\
&+ \left. \left. \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right\} \right. \\
&\left. \left. + (1 - \theta \widehat{\zeta}(\cdot)) \mathcal{G} : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \right]_{\Sigma} \right\rangle_{\Sigma} \otimes \mathbb{J}^{pq}, \quad x_3 \in [\Delta, \Delta + \delta_*];
\end{aligned}$$

$$(11.21) \quad \mathcal{K}(t, x_3) = \begin{cases} 0 & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \mathcal{K}^\sigma(t) & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathcal{K}^\theta(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$\begin{aligned}
(11.22) \quad \mathcal{K}^\sigma(t) &= \sum_{p,q=1}^3 \left[\left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \right. \\
&\left. \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq}, \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*),
\end{aligned}$$

$$\begin{aligned}
(11.23) \quad \mathcal{K}^\theta(t) &= \sum_{p,q=1}^3 \left[\left\{ \alpha_\lambda \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \right. \\
&+ \left. \left. 2\alpha_\mu \sum_{m,n=1}^3 \left(\theta \mathbb{J}^{mn} + \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right\} : \right. \\
&\left. \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq} \\
&+ \sigma \left(\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right. \\
&+ \left. 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t, x_3)) \rangle_{\Theta} \otimes \mathbb{J}^{mn} \right) \\
&+ \sum_{p,q=1}^3 \left[\left(\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right. \right. \\
&+ \left. \left. 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t, x_3)) \rangle_{\Theta} \otimes \mathbb{J}^{mn} \right) : \right. \\
&\left. \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq}, \quad x_3 \in [\Delta, \Delta + \delta_*];
\end{aligned}$$

$$(11.24) \quad \mathcal{Q}(t, \tau, x_3) = \begin{cases} 0 & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ \mathcal{Q}^\sigma(\tau) & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*), \\ \mathcal{Q}_0^\theta(\tau) + \sum_{p,q=1}^3 (\mathcal{Q}_1^\theta(t) : \mathbb{Q}^{pq}(\tau)) \otimes \mathbb{J}^{pq} & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(11.25) \quad \mathcal{Q}^\sigma(\tau) = \sum_{p,q=1}^3 \left\langle \left(\widehat{\zeta}(\cdot) \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\cdot)) \mathcal{G} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \right\rangle_\Sigma \otimes \mathbb{J}^{pq},$$

$$x_3 \in (\Delta + \delta_*, \Delta + \delta^*),$$

$$(11.26) \quad \begin{aligned} \mathcal{Q}_0^\theta(\tau) &= \sum_{p,q=1}^3 \left\langle \left[\widehat{\zeta}(\cdot) \left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_\Theta \mathbb{J}^{mn} \right) \right. \right. \\ &\quad \left. \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_\Theta \right) \otimes \mathbb{J}^{mn} \right\} \right. \\ &\quad \left. + (1 - \theta \widehat{\zeta}(\cdot)) \mathcal{G} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \right\rangle_\Sigma \otimes \mathbb{J}^{pq} \\ &\quad + \sigma \left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, \tau, x_3) \rangle_\Theta \mathbb{J}^{mn} \right. \\ &\quad \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, \tau, x_3)) \rangle_\Theta \right) \otimes \mathbb{J}^{mn} \right\} \\ &\quad + \sum_{p,q=1}^3 \left[\left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, \tau, x_3) \rangle_\Theta \mathbb{J}^{mn} \right. \right. \\ &\quad \left. \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, \tau, x_3)) \rangle_\Theta \right) \otimes \mathbb{J}^{mn} \right\} : \right. \\ &\quad \left. : \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\cdot, x_3)) \right\rangle_\Sigma \otimes \mathbb{J}^{pq}, \quad x_3 \in [\Delta, \Delta + \delta_*]; \end{aligned}$$

$$(11.27) \quad \begin{aligned} \mathcal{Q}_1^\theta(t) &= \alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t, x_3) \rangle_\Theta \mathbb{J}^{mn} \\ &\quad + 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t, x_3)) \rangle_\Theta \otimes \mathbb{J}^{mn}, \quad x_3 \in [\Delta, \Delta + \delta_*]; \end{aligned}$$

$$(11.28) \quad \mathbb{Q}^{pq}(\tau) = \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_1^{pq}(\cdot, \tau, x_3)) \rangle_\Sigma, \quad x_3 \in [\Delta, \Delta + \delta_*];$$

$$(11.29) \quad \mathcal{W}(t, \tau, x_3) = \begin{cases} 0 & \text{for } x_3 \in (0, 1) \setminus [\Delta, \Delta + \delta_*], \\ \sum_{p,q=1}^3 (\mathcal{W}_1^\theta(t) : \mathbb{Q}^{pq}(\tau)) \otimes \mathbb{J}^{pq} & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \end{cases}$$

where

$$(11.30) \quad \mathbf{W}_1^\theta(t) = \alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \\ + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn}, \quad x_3 \in [\Delta, \Delta + \delta_*],$$

and $\mathbb{Q}^{pq}(\tau)$ is given by formula (11.28);

$$(11.31) \quad \mathcal{F}_v(t, x_3) = \begin{cases} 0 & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ (1 - \sigma) \mathcal{G} + \mathcal{F}_{v\text{-corr}}^\sigma(t) & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*), \\ (1 - \sigma\theta) \mathcal{G} + \mathcal{F}_{v\text{-corr}}^\theta(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ \mathcal{G} & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$(11.32) \quad \mathcal{F}_{v\text{-corr}}^\sigma(t) = \\ = \sum_{p,q=1}^3 \left[\left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq} \\ + \sum_{p,q=1}^3 \left\langle \left(\widehat{\zeta}(\cdot) \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\cdot)) \mathcal{G} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_3^{pq})(\cdot, t, x_3)) \right\rangle_{\Sigma} \otimes \mathbb{J}^{pq}, \\ x_3 \in (\Delta + \delta_*, \Delta + \delta^*),$$

and

$$(11.33) \quad \mathcal{F}_{v\text{-corr}}^\theta(t) = \sum_{p,q=1}^3 \left[\left\{ \alpha_\lambda \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \right. \\ \left. \left. + 2\alpha_\mu \sum_{m,n=1}^3 \left(\theta \mathbb{J}^{mn} + \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right\} : \right. \\ \left. \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq} \\ + \sum_{p,q=1}^3 \left\langle \left[\widehat{\zeta}(\cdot) \left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \right. \right. \\ \left. \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right\} \right. \\ \left. \left. + (1 - \theta \widehat{\zeta}(\cdot)) \mathcal{G} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_3^{pq})(\cdot, t, x_3)) \right\rangle_{\Sigma} \otimes \mathbb{J}^{pq} \\ + \sum_{p,q=1}^3 \int_0^t \left[\left(\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \\ \left. \left. + \sum_{m,n=1}^3 \left(\theta \mathbb{J}^{mn} + \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right] : \right. \\ \left. \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{pq}(\cdot, t, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq} \tau \, d\tau$$

$$\begin{aligned}
 & + 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3)) \rangle_{\Theta} \otimes \mathbb{J}^{mn} \Big) : \\
 & \qquad \qquad \qquad : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_3^{pq}(\cdot, \tau, x_3)) \rangle_{\Sigma} \Big] \otimes \mathbb{J}^{pq} d\tau \\
 & + \sum_{p,q=1}^3 \int_0^t \left[\left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right. \right. \\
 & \quad \left. \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right\} : \right. \\
 & \qquad \qquad \qquad \left. : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}((J_\tau \mathbf{Y}_3^{pq})(\cdot, \tau, x_3)) \rangle_{\Sigma} \right] \otimes \mathbb{J}^{pq} d\tau \\
 & + \sigma \sum_{p,q=1}^3 \left(\alpha_\lambda \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{30}^{pq}(\cdot, t, x_3) \rangle_{\Theta} \mathbb{I} + 2\alpha_\mu \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{30}^{pq}(\cdot, t, x_3)) \rangle_{\Theta} \right. \\
 & \left. + \alpha_\gamma^{-1} \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}}(J_t \mathbf{Z}_{30}^{pq})(\cdot, t, x_3) \rangle_{\Theta} \mathbb{I} + \mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}((J_t \mathbf{Z}_{30}^{pq})(\cdot, t, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{pq}, \\
 & \qquad \qquad \qquad x_3 \in (\Delta, \Delta + \delta_*];
 \end{aligned}$$

$$(11.34) \quad \mathbb{F}_p(t, x_3) = \begin{cases} -\mathbb{I} & \text{for } x_3 \in (\Delta + \delta^*, 1), \\ -\sigma \mathbb{I} + \mathbb{F}_{p\text{-corr}}^\sigma(t) & \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*), \\ -\sigma \theta \mathbb{I} + \mathbb{F}_{p\text{-corr}}^\theta(t) & \text{for } x_3 \in [\Delta, \Delta + \delta_*], \\ 0 & \text{for } x_3 \in (0, \Delta), \end{cases}$$

where

$$\begin{aligned}
 (11.35) \quad \mathbb{F}_{p\text{-corr}}^\sigma(t) & = \left(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right) : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_2(\cdot, t, x_3)) \rangle_{\Sigma} \\
 & \quad + \left\langle \left[\widehat{\zeta}(\cdot) \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\cdot)) \mathcal{G} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\cdot, t, x_3)) \right\rangle_{\Sigma}, \\
 & \qquad \qquad \qquad x_3 \in (\Delta + \delta_*, \Delta + \delta^*],
 \end{aligned}$$

and

$$\begin{aligned}
 (11.36) \quad \mathbb{F}_{p\text{-corr}}^\theta(t) & = \left[\alpha_\lambda \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \\
 & \quad \left. + 2\alpha_\mu \sum_{m,n=1}^3 \left(\theta \mathbb{J}^{mn} + \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right] : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_2(\cdot, t, x_3)) \rangle_{\Sigma} \\
 & \quad + \left\langle \left[\widehat{\zeta}(\cdot) \left\{ \alpha_\gamma^{-1} \mathbb{I} \otimes \left(\theta \mathbb{I} + \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{00}^{mn}(\cdot, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right) \right. \right. \right. \\
 & \quad \left. \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{00}^{mn}(\cdot, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right\} \right. \\
 & \quad \left. \left. + (1 - \theta \widehat{\zeta}(\cdot)) \mathcal{G} \right] : \mathbb{D}_{\widehat{\mathbf{y}}}((J_t \mathbf{Y}_2)(\cdot, t, x_3)) \right\rangle_{\Sigma}
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \left[\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right. \\
 & \quad \left. + 2\alpha_\mu \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3)) \rangle_{\Theta} \otimes \mathbb{J}^{mn} \right] : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_2(\cdot, \tau, x_3)) \rangle_{\Sigma} d\tau \\
 & + \int_0^t \left[\alpha_\gamma^{-1} \mathbb{I} \otimes \sum_{m,n=1}^3 \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3) \rangle_{\Theta} \mathbb{J}^{mn} \right. \\
 & \quad \left. + \sum_{m,n=1}^3 \left(\mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{10}^{mn}(\cdot, t - \tau, x_3)) \rangle_{\Theta} \right) \otimes \mathbb{J}^{mn} \right] : \\
 & \qquad \qquad \qquad : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{\mathbf{y}}}((J_\tau \mathbf{Y}_2)(\cdot, \tau, x_3)) \rangle_{\Sigma} d\tau \\
 & + \sigma \left(\alpha_\lambda \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}} \mathbf{Z}_{20}(\cdot, t, x_3) \rangle_{\Theta} \mathbb{I} + 2\alpha_\mu \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{\mathbf{z}}}(\mathbf{Z}_{20}(\cdot, t, x_3)) \rangle_{\Theta} \right. \\
 & \quad \left. + \alpha_\gamma^{-1} \langle \widehat{\psi}(\cdot) \operatorname{div}_{\widehat{\mathbf{z}}}(J_t \mathbf{Z}_{20})(\cdot, t, x_3) \rangle_{\Theta} \mathbb{I} + \mathcal{G} : \langle (1 - \widehat{\psi}(\cdot)) \mathbb{D}_{\widehat{\mathbf{z}}}((J_t \mathbf{Z}_{20})(\cdot, t, x_3)) \rangle_{\Theta} \right), \\
 & \qquad \qquad \qquad x_3 \in (\Delta, \Delta + \delta_*].
 \end{aligned}$$

Remark 27. Due to Remarks 17 and 25, the tensors $\mathbf{V}_{\text{corr}}^\sigma, \mathbf{V}_{\text{corr}}^\theta, \mathcal{E}_{\text{corr}}^\sigma, \mathcal{E}_{\text{corr}}^\theta$ are constant and the tensors $\mathcal{K}^\sigma(t), \mathcal{K}^\theta(t), \mathcal{Q}^\sigma(\tau), \mathcal{Q}^\theta(\tau), \mathcal{Q}_1^\theta(t), \mathcal{W}_1^\theta(t), \mathcal{F}_{v\text{-corr}}^\sigma(t), \mathcal{F}_{v\text{-corr}}^\theta(t)$, and the matrices $\mathbb{Q}^{pq}(\tau), \mathbb{F}_{p\text{-corr}}^\sigma(t), \mathbb{F}_{p\text{-corr}}^\theta(t)$ depend only on t (or τ).

Proposition 12. (i) The tensor-valued functions $\mathbf{V} = \mathbf{V}(x_3), \mathcal{E} = \mathcal{E}(x_3), \mathcal{K} = \mathcal{K}(t, x_3), \mathcal{Q} = \mathcal{Q}(t, \tau, x_3), \mathcal{W} = \mathcal{W}(t, \tau, x_3)$, and $\mathcal{F}_v = \mathcal{F}_v(t, x_3)$ and the matrix-valued function $\mathbb{F}_p = \mathbb{F}_p(t, x_3)$ satisfy the following regularity conditions:

$$\begin{aligned}
 (11.37) \quad & \mathbf{V}, \mathcal{E} \in L^\infty(0, 1)^{3 \times 3 \times 3 \times 3}, \\
 (11.38) \quad & \mathcal{K}, \mathcal{F}_v, \partial_t \mathcal{K}, \partial_t \mathcal{F}_v \in L^\infty((0, T) \times (0, 1))^{3 \times 3 \times 3 \times 3}, \\
 (11.39) \quad & \mathcal{Q}, \mathcal{W}, \partial_t \mathcal{Q} \in L^\infty((0, T) \times (0, T) \times (0, 1))^{3 \times 3 \times 3 \times 3}, \\
 (11.40) \quad & \mathbb{F}_p, \partial_t \mathbb{F}_p \in L^\infty((0, T) \times (0, 1))^{3 \times 3}.
 \end{aligned}$$

(ii) The tensor-valued function \mathbf{V} satisfies the finiteness property

$$(11.41) \quad \mathbf{V} = 0 \quad \text{for } x_3 \in (0, \Delta),$$

the symmetry property

$$(11.42) \quad \mathcal{V}^{ijkl} = \mathcal{V}^{jikl} = \mathcal{V}^{jilk} = \mathcal{V}^{klij}, \quad \forall x_3 \in [0, 1], \quad \forall i, j, k, l = 1, 2, 3,$$

and the uniform positive definiteness property:

$$\begin{aligned}
 (11.43) \quad & \text{there exists a constant } C_9 > 0 \text{ such that } (\mathbf{V}(x_3) : \mathbb{X}) : \mathbb{X} \geq C_9 |\mathbb{X}|^2, \\
 & \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall x_3 \in \Sigma \times [\Delta, 1].
 \end{aligned}$$

(iii) The tensor-valued function

$$\begin{aligned}
 (11.44) \quad & \mathfrak{H}^s(x_3) := s\mathbf{V}(x_3) + \mathcal{E}(x_3) + s\overline{\mathcal{K}}(s, x_3) + 1_{x_3 \in (\Delta + \delta_*, \Delta + \delta_*^*]}(\mathbf{x}) \overline{\mathcal{Q}}^\sigma(s) \\
 & \quad + 1_{x_3 \in (\Delta, \Delta + \delta_*^*]}(\mathbf{x}) \left\{ \overline{\mathcal{Q}}_0^\theta(s) + \sum_{p,q=1}^3 \left[(s\overline{\mathcal{Q}}_1^\theta(s) + \overline{\mathcal{W}}_1^\theta(s)) : \overline{\mathbb{Q}}^{pq}(s) \right] \otimes \mathbb{J}^{pq} \right\}
 \end{aligned}$$

satisfies the symmetry properties

$$(11.45) \quad \mathfrak{H}^s ijkl = \mathfrak{H}^s jikl = \mathfrak{H}^s jilk, \quad \forall x_3 \in [0, 1], \quad \forall s > 0, \quad \forall i, j, k, l = 1, 2, 3,$$

$$(11.46) \quad (\mathfrak{H}^s : \mathbb{X}) : \mathbb{W} = (\mathfrak{H}^s : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

and the uniform positive definiteness property:

$$(11.47) \quad \text{there exists a constant } C_{10}^s > 0 \text{ such that } (\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{X} \geq C_{10}^s |\mathbb{X}|^2, \\ \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall x_3 \in [0, 1], \quad \forall s > 0.$$

(iv) For a.e. $(t, x_3) \in (0, T) \times (0, 1)$ the tensor $\mathcal{F}_v(t, x_3)$ satisfies the symmetry properties

$$(11.48) \quad \mathcal{F}_v^{ijkl} = \mathcal{F}_v^{jikl} = \mathcal{F}_v^{jilk}, \quad \forall i, j, k, l = 1, 2, 3,$$

$$(11.49) \quad (\mathcal{F}_v : \mathbb{X}) : \mathbb{W} = (\mathcal{F}_v : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}.$$

(v) For a.e. $(t, x_3) \in (0, T) \times (0, 1)$ the matrix $\mathbb{F}_p(t, x_3)$ is symmetric.

Notation 5. In (11.44) and further, by $1_{x_3 \in \mathcal{I}}$ we denote the indicator function of the set \mathcal{I} :

$$1_{x_3 \in \mathcal{I}}(\mathbf{x}) = \begin{cases} 1 & \text{if } x_3 \in \mathcal{I}, \\ 0 & \text{if } x_3 \notin \mathcal{I}. \end{cases}$$

Proof of Proposition 12 is analogous to the proof of Proposition 6. Let us divide it into five steps.

(i) Inclusions (11.37)-(11.40) directly follow from (11.2)-(11.10), (10.31), (10.34), (10.35), (10.38), (10.41), (10.42), (10.45), (7.53), and Remark 25.

(ii) The finiteness property (11.41) immediately follows from (11.15). Also, the properties (11.42) and (11.43) follow for $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$ directly from (11.15). In order to establish (11.42) and (11.43) for $x_3 \in [\Delta, \Delta + \delta^*]$, we recall the integral equalities (10.9) and (10.10) from the formulation of Problem Y1. Using (7.13), we give the uniform formulation of (10.9) and (10.10) in terms of \mathcal{A}_0 as follows:

$$\int_{\Sigma} \left[\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : (\mathbb{J}^{ij} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3))) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}} = 0, \\ \forall \phi_{12} \in H_{\#}^1(\Sigma)^3, \quad \forall x_3 \in [\Delta, \Delta + \delta^*].$$

Here we re-assign indices (i, j) to (k, l) for technical reasons, and take the test vector-function $\phi = \mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)$, which is legal. Thus we get

$$(11.50) \quad \int_{\Sigma} \left[\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : (\mathbb{J}^{kl} + \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3))) \right] : \mathbb{D}_{\hat{\mathbf{y}}}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}} = 0, \\ \forall k, l, m, n = 1, 2, 3, \quad \forall x_3 \in [\Delta, \Delta + \delta^*],$$

or, equivalently in the component-wise form,

$$\sum_{i,j,p,q=1}^3 \int_{\Sigma} \mathcal{A}_0^{ijpq}(\hat{\mathbf{y}}, x_3) (\mathbb{J}^{kl})_{pq} D_{\hat{\mathbf{y}}ij}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}} \\ + \sum_{i,j,p,q=1}^3 \int_{\Sigma} \mathcal{A}_0^{ijpq}(\hat{\mathbf{y}}, x_3) D_{\hat{\mathbf{y}}pq}(\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3)) D_{\hat{\mathbf{y}}ij}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}} = 0, \\ \forall k, l, m, n = 1, 2, 3, \quad \forall x_3 \in [\Delta, \Delta + \delta^*].$$

Calculating the sum in p and q in the first integral and using the symmetry of \mathcal{A}_0 , see (7.29), we arrive at the equality

$$(11.51) \quad \begin{aligned} & \sum_{i,j=1}^3 \int_{\Sigma} \mathcal{A}_0^{ijkl}(\hat{\mathbf{y}}, x_3) D_{\hat{\mathbf{y}}ij}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}} \\ &= - \sum_{i,j,p,q=1}^3 \int_{\Sigma} \mathcal{A}_0^{ijpq}(\hat{\mathbf{y}}, x_3) D_{\hat{\mathbf{y}}pq}(\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3)) D_{\hat{\mathbf{y}}ij}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}}, \\ & \quad \forall k, l, m, n = 1, 2, 3, \quad \forall x_3 \in [\Delta, \Delta + \delta^*]. \end{aligned}$$

Calculating the sum in p and q in (11.11) and using the symmetry properties $\mathbf{Y}_0^{kl} = \mathbf{Y}_0^{lk}$ and (7.29) and Remark 24, we represent (11.52)

$$\mathcal{V}^{klmn}(x_3) = \int_{\Sigma} \mathcal{A}_0^{klmn}(\hat{\mathbf{y}}, x_3) d\hat{\mathbf{y}} + \sum_{i,j=1}^3 \int_{\Sigma} \mathcal{A}_0^{ijkl}(\hat{\mathbf{y}}, x_3) D_{\hat{\mathbf{y}}ij}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}}.$$

Combining (11.51) and (11.52) we establish the following representation for the components of \mathcal{V} on $[\Delta, \Delta + \delta^*]$:

$$(11.53) \quad \begin{aligned} \mathcal{V}^{klmn}(x_3) &= \int_{\Sigma} \mathcal{A}_0^{klmn}(\hat{\mathbf{y}}, x_3) d\hat{\mathbf{y}} \\ & \quad - \sum_{i,j,p,q=1}^3 \int_{\Sigma} \mathcal{A}_0^{ijpq}(\hat{\mathbf{y}}, x_3) D_{\hat{\mathbf{y}}pq}(\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3)) D_{\hat{\mathbf{y}}ij}(\mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3)) d\hat{\mathbf{y}}, \\ & \quad \forall k, l, m, n = 1, 2, 3, \quad \forall x_3 \in [\Delta, \Delta + \delta^*]. \end{aligned}$$

Using again (7.29) and Remark 24, from (11.53) we immediately deduce (11.42) on $[\Delta, \Delta + \delta^*]$.

Further, multiplying (11.50) by $X_{kl}X_{mn}$, where $\mathbb{X} = (X_{kl}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ is arbitrary, and taking the sum in k, l, m , and n from 1 to 3, on $[\Delta, \Delta + \delta^*]$, we get the identity

$$(11.54) \quad \begin{aligned} & \int_{\Sigma} \left[\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : \left(\mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) \right) \right) \right] : \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{m,n=1}^3 X_{mn} \mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3) \right) d\hat{\mathbf{y}} \\ & \quad = 0. \end{aligned}$$

Multiplying (11.11) by $X_{kl}X_{mn}$ and taking the sum in k, l, m , and n from 1 to 3, we arrive at the identity

$$(11.55) \quad \begin{aligned} (\mathcal{V} : \mathbb{X}) : \mathbb{X} &= \int_{\Sigma} [\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : \mathbb{X}] : \mathbb{X} d\hat{\mathbf{y}} + \int_{\Sigma} \left[\mathcal{A}_0 : \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) \right) \right] : \mathbb{X} d\hat{\mathbf{y}}, \\ & \quad \forall x_3 \in [\Delta, \Delta + \delta^*]. \end{aligned}$$

Now, by swapping the left and the right hand sides in (11.54) followed by adding (11.55), we establish that

$$(\mathcal{V}(x_3) : \mathbb{X}) : \mathbb{X} = \int_{\Sigma} \left[\mathcal{A}_0(\hat{\mathbf{y}}, x_3) : \left(\mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) \right) \right) \right] :$$

$$\begin{aligned}
 & : \left[\mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{m,n=1}^3 X_{mn} \mathbf{Y}_0^{mn}(\hat{\mathbf{y}}, x_3) \right) \right] d\hat{\mathbf{y}} \\
 & \stackrel{(7.30)}{\geq} C_5 \int_{\Sigma} \left| \mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{k,l=1}^3 X_{kl} \mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) \right) \right|^2 d\hat{\mathbf{y}}, \quad \forall x_3 \in [\Delta, \Delta + \delta^*].
 \end{aligned}$$

Thus $(\mathcal{V}(x_3) : \mathbb{X}) : \mathbb{X} \geq 0$ for all $\mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ for $x_3 \in [\Delta, \Delta + \delta^*]$. At last, the strict positive definiteness property (11.43) on $[\Delta, \Delta + \delta^*]$ is justified quite analogously to the proof of the positive definiteness of the tensor \mathcal{A}_0 in Proposition 6. By this, the proof of assertion (ii) is complete.

(iii) By the straightforward calculation, we easily deduce that the properties (11.45), (11.46), and (11.47) follow for $x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1)$ directly from (1.3), (1.7), (11.15), (11.18), (11.21), and (11.44).

Now let us establish (11.45)-(11.47) for $x_3 \in [\Delta, \Delta + \delta^*]$.

At first, we consider the case when $x_3 \in (\Delta + \delta_*, \Delta + \delta^*)$. Recall that (10.19) is the sum of (10.25) (or, equivalently, (10.9)), (10.26) and (10.27). Therefore it holds true due to Proposition 8 and item (i) of Proposition 10. We multiply (10.19) by X_{ij} , where $\mathbb{X} = (X_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ is arbitrary, and take the sum in i and j from 1 to 3. Using (7.45), we write out the resulting equality as follows:

$$\begin{aligned}
 (11.56) \quad & \int_{\Sigma} \left\{ \mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \left(\mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{i,j=1}^3 X_{ij} (\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right) \right\} : \\
 & \mathbb{D}_{\hat{\mathbf{y}}}(\phi_{12}(\hat{\mathbf{y}})) d\hat{\mathbf{y}} = 0
 \end{aligned}$$

for all $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$ and $s > 0$.

Let $\mathbb{W} = (W_{kl}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ be arbitrary. Inserting

$$\phi_{12} := \sum_{k,l=1}^3 W_{kl} (\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{kl}(\hat{\mathbf{y}}, s, x_3))$$

into (11.56), which is the legal choice of test vector-function, we get the identity

$$\begin{aligned}
 (11.57) \quad & \int_{\Sigma} \left\{ \mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \left(\mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{i,j=1}^3 X_{ij} (\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right) \right\} : \\
 & \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{k,l=1}^3 W_{kl} (\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{kl}(\hat{\mathbf{y}}, s, x_3)) \right) d\hat{\mathbf{y}} = 0, \\
 & \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \quad \forall s > 0.
 \end{aligned}$$

Next, substituting (11.15), (11.16), (11.18), (11.19), (11.21), (11.22), and (11.25) (restricted to $(\Delta + \delta_*, \Delta + \delta^*) \times \{0 < s < +\infty\}$) into (11.44), we establish the representation

$$\begin{aligned}
 (11.58) \quad & \mathfrak{H}^s(x_3) = s \left(\sigma \alpha_{\lambda} \mathbb{I} \otimes \mathbb{I} + 2\sigma \alpha_{\mu} \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\Sigma} \left[\alpha_{\lambda} \mathbb{I} \sum_{p,q=1}^3 \widehat{\zeta}(\widehat{\mathbf{y}}) \operatorname{div}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{pq}(\widehat{\mathbf{y}}, s, x_3)) \otimes \mathbb{J}^{pq} \right. \\
 & \quad \left. + \sum_{p,q=1}^3 2\alpha_{\mu} \sum_{m,n=1}^3 \widehat{\zeta}(\widehat{\mathbf{y}}) D_{\widehat{\mathbf{y}}mn}(\mathbf{Y}_0^{pq}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{pq}(\widehat{\mathbf{y}}, s, x_3)) \mathbb{J}^{mn} \otimes \mathbb{J}^{pq} \right] d\widehat{\mathbf{y}} \\
 & + \sigma \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \sigma) \mathcal{G} \\
 & + \int_{\Sigma} \sum_{p,q=1}^3 \left(\widehat{\zeta}(\widehat{\mathbf{y}}) \alpha_{\gamma}^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\widehat{\mathbf{y}})) \mathcal{G} \right) : \mathbb{D}_{\widehat{\mathbf{y}}}(\mathbf{Y}_0^{pq}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{pq}(\widehat{\mathbf{y}}, s, x_3)) \otimes \mathbb{J}^{pq} d\widehat{\mathbf{y}}.
 \end{aligned}$$

Using (7.45), representation $\sigma = \langle \widehat{\zeta} \rangle_{\Sigma} \equiv \int_{\Sigma} \widehat{\zeta}(\widehat{\mathbf{y}}) d\widehat{\mathbf{y}}$, and representation (11.58), we write out the quadratic expression $(\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{W}$ on the segment $(\Delta + \delta_*, \Delta + \delta^*]$ as follows:

(11.59)

$$\begin{aligned}
 (\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{W} & = s\sigma(\alpha_{\lambda} \operatorname{tr} \mathbb{X} \operatorname{tr} \mathbb{W} + 2\alpha_{\mu} \mathbb{X} : \mathbb{W}) \\
 & + \int_{\Sigma} \left[\mathfrak{H}^s(\widehat{\mathbf{y}}, x_3) : \left(\mathbb{X} + \mathbb{D}_{\widehat{\mathbf{y}}} \left(\sum_{i,j=1}^3 X_{ij}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right) \right) \right] : \mathbb{W} d\widehat{\mathbf{y}}, \\
 \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \quad \forall s > 0, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}.
 \end{aligned}$$

Now, by swapping the left and the right hand sides in (11.57) followed by adding (11.59), we establish the representation

(11.60)

$$\begin{aligned}
 (\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{W} & = s\sigma(\alpha_{\lambda} \operatorname{tr} \mathbb{X} \operatorname{tr} \mathbb{W} + 2\alpha_{\mu} \mathbb{X} : \mathbb{W}) \\
 & + \int_{\Sigma} \left\{ \mathfrak{H}^s(\widehat{\mathbf{y}}, x_3) : \left[\mathbb{X} + \mathbb{D}_{\widehat{\mathbf{y}}} \left(\sum_{i,j=1}^3 X_{ij}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] \right\} : \\
 & \quad : \left[\mathbb{W} + \mathbb{D}_{\widehat{\mathbf{y}}} \left(\sum_{k,l=1}^3 W_{kl}(\mathbf{Y}_0^{kl}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{kl}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] d\widehat{\mathbf{y}}, \\
 \forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \quad \forall s > 0, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}.
 \end{aligned}$$

The symmetry properties (11.45) and (11.46) on the segment $(\Delta + \delta_*, \Delta + \delta^*]$ for $s > 0$ follow immediately from the above representation due to (7.32) and (7.33).

Finally, inserting $\mathbb{W} := \mathbb{X}$ into (11.60) we establish the inequality

$$\begin{aligned}
 (\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{X} & = s\sigma(\alpha_{\lambda} |\operatorname{tr} \mathbb{X}|^2 + 2\alpha_{\mu} |\mathbb{X}|^2) \\
 & + \int_{\Sigma} \left\{ \mathfrak{H}^s(\widehat{\mathbf{y}}, x_3) : \left[\mathbb{X} + \mathbb{D}_{\widehat{\mathbf{y}}} \left(\sum_{i,j=1}^3 X_{ij}(\mathbf{Y}_0^{ij}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{ij}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] \right\} : \\
 & \quad : \left[\mathbb{X} + \mathbb{D}_{\widehat{\mathbf{y}}} \left(\sum_{k,l=1}^3 X_{kl}(\mathbf{Y}_0^{kl}(\widehat{\mathbf{y}}, x_3) + \overline{\mathbf{Y}}_1^{kl}(\widehat{\mathbf{y}}, s, x_3)) \right) \right] d\widehat{\mathbf{y}} \\
 & \stackrel{(7.34)}{\geq} s\sigma(\alpha_{\lambda} |\operatorname{tr} \mathbb{X}|^2 + 2\alpha_{\mu} |\mathbb{X}|^2) \geq s\sigma(\alpha_{\lambda} + 2\alpha_{\mu}) |\mathbb{X}|^2,
 \end{aligned}$$

$$\forall x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \quad \forall s > 0, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

which immediately yields that (11.47) holds true for $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$. Thus the properties (11.45)-(11.47) hold true on $(\Delta + \delta_*, \Delta + \delta^*]$.

At second, we consider the case when $x_3 \in [\Delta, \Delta + \delta_*]$. Justification of (11.45)-(11.47) in this case is analogous to the case when $x_3 \in (\Delta + \delta_*, \Delta + \delta^*]$. We recall that (10.20) is the sum of (10.28) (or, equivalently, (10.10)), (10.29), and (10.30). Therefore it holds true due to Proposition 8 and item (ii) of Proposition 10. We multiply (10.20) by X_{ij} , where $\mathbb{X} = (X_{ij}) \in \mathbb{R}_{\text{symm}}^{3 \times 3}$ is arbitrary, and take the sum in i and j from 1 to 3. Thus we arrive at the identity (11.56) on $[\Delta, \Delta + \delta_*] \times \{0 < s < +\infty\}$. Further, inserting

$$\phi_{12} = \sum_{k,l=1}^3 W_{kl}(\mathbf{Y}_0^{kl}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{kl}(\hat{\mathbf{y}}, s, x_3))$$

into (11.56) we arrive at (11.57) on $[\Delta, \Delta + \delta_*] \times \{0 < s < +\infty\}$.

Next, suitably combining (7.14), (7.16), (7.18), (7.20), (7.45), (11.15), (11.18), (11.21), (11.26), (11.27), (11.28), (11.30), (11.44), and the representation $\sigma = \int_{\Sigma} \hat{\zeta}(\hat{\mathbf{y}}) d\hat{\mathbf{y}}$, we write out the quadratic expression $(\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{W}$ on the segment $[\Delta, \Delta + \delta_*]$ as follows:

(11.61)

$$\begin{aligned} (\mathfrak{H}^s(x_3) : \mathbb{X}) : \mathbb{W} &= s\sigma\theta(\alpha_\lambda \text{tr } \mathbb{X} \text{ tr } \mathbb{W} + 2\alpha_\mu \mathbb{X} : \mathbb{W}) \\ &+ \int_{\Sigma} \left[\mathcal{H}^s(\hat{\mathbf{y}}, x_3) : \left(\mathbb{X} + \mathbb{D}_{\hat{\mathbf{y}}} \left(\sum_{i,j=1}^3 X_{ij}(\mathbf{Y}_0^{ij}(\hat{\mathbf{y}}, x_3) + \bar{\mathbf{Y}}_1^{ij}(\hat{\mathbf{y}}, s, x_3)) \right) \right) \right] : \mathbb{W} d\hat{\mathbf{y}}, \\ &\forall x_3 \in (\Delta, \Delta + \delta_*], \quad \forall s > 0, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}. \end{aligned}$$

At last, we swap the left and right hand sides in (11.57) (restricted to $[\Delta, \Delta + \delta_*] \times \{0 < s < +\infty\}$), add the result to (11.61), and complete justification of (11.45)-(11.47) (on $[\Delta, \Delta + \delta_*] \times \{0 < s < +\infty\}$) quite analogously to the case $(x_3, s) \in (\Delta + \delta_*, \Delta + \delta^*) \times (0, +\infty)$.

Thus, all assertions of item (iii) are proved.

(iv) The symmetry properties (11.48) and (11.49) directly follow from (11.12) due to the symmetry of $\mathcal{A}_0, \mathcal{B}_0, \mathcal{A}_1, \mathcal{B}_1, \mathcal{G}, \mathbb{J}^{pq}$, and \mathbb{F}_{sol}^0 , see Proposition 6 and identity (1.3).

(v) The symmetry of \mathbb{F}_p directly follows from (11.13) and (11.14) due to the symmetry of $\mathcal{A}_0, \mathcal{B}_0, \mathcal{A}_1$, and \mathcal{B}_1 , and matrix \mathbb{F}_f^0 , see Proposition 6. \square

Let us formulate the additional result, which will be useful further.

Proposition 13. *Assume that the tensor-valued functions $\mathcal{V}, \mathcal{E}, \mathcal{K}, \mathcal{Q}, \mathcal{W}$, and \mathcal{F}_v , and the matrix-valued function \mathbb{F}_p admit the representations (11.15), (11.18), (11.21), (11.24), (11.29), (11.31), and (11.34), respectively; then assertions (i)-(v) of Proposition 12 are equivalent to Conditions C1-C5, stated below.*

Conditions C1-C5. (C1) The fourth-rank tensor-valued functions $\mathcal{K}^\sigma, \mathcal{K}^\theta, \mathcal{Q}^\sigma, \mathcal{Q}_0^\theta, \mathcal{Q}_1^\theta, \partial_t \mathcal{Q}_1^\theta, \mathcal{W}_1^\theta, \mathcal{F}_{v\text{-corr}}^\sigma, \mathcal{F}_{v\text{-corr}}^\theta, \partial_t \mathcal{F}_{v\text{-corr}}^\sigma$, and $\partial_t \mathcal{F}_{v\text{-corr}}^\theta$ belong to the space $L^\infty(0, T)^{3 \times 3 \times 3 \times 3}$. The matrix-valued functions $\mathbb{Q}^{pq}, \mathbb{F}_{p\text{-corr}}^\sigma, \mathbb{F}_{p\text{-corr}}^\theta, \partial_t \mathbb{F}_{p\text{-corr}}^\sigma$, and $\partial_t \mathbb{F}_{p\text{-corr}}^\theta$ belong to the space $L^\infty(0, T)^{3 \times 3}$.

(C2) The constant tensors $\mathcal{V}_{\text{corr}}^\sigma$ and $\mathcal{V}_{\text{corr}}^\theta$ satisfy the symmetry properties

$$\mathcal{V}_{\text{corr}}^{\sigma ijkl} = \mathcal{V}_{\text{corr}}^{\sigma jikl} = \mathcal{V}_{\text{corr}}^{\sigma jilk} = \mathcal{V}_{\text{corr}}^{\sigma klij}, \quad \mathcal{V}_{\text{corr}}^{\theta ijkl} = \mathcal{V}_{\text{corr}}^{\theta jikl} = \mathcal{V}_{\text{corr}}^{\theta jilk} = \mathcal{V}_{\text{corr}}^{\theta klij},$$

$i, j, k, l = 1, 2, 3$, and the uniform positive definiteness properties: there exists a constant $C_9 > 0$ such that

$$2\sigma\alpha_\mu|\mathbb{X}|^2 + \sigma\alpha_\lambda|\text{tr } \mathbb{X}|^2 + (\mathcal{V}_{\text{corr}}^\sigma : \mathbb{X}) : \mathbb{X} \geq C_9|\mathbb{X}|^2, \quad \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

$$2\sigma\theta\alpha_\mu|\mathbb{X}|^2 + \sigma\theta\alpha_\lambda|\text{tr } \mathbb{X}|^2 + (\mathcal{V}_{\text{corr}}^\theta : \mathbb{X}) : \mathbb{X} \geq C_9|\mathbb{X}|^2, \quad \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

Here the constant C_9 is the same as in (11.43).

(C3) The tensors

$$(11.62) \quad \mathfrak{H}_\sigma^s := s \left(\sigma\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\sigma\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} + \mathcal{V}_{\text{corr}}^\sigma \right) \\ + \sigma\alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \sigma)\mathcal{G} + \mathcal{E}_{\text{corr}}^\sigma + s\overline{\mathcal{K}}^\sigma(s) + \overline{\mathcal{Q}}^\sigma(s), \quad s > 0,$$

and

$$(11.63) \quad \mathfrak{H}_\theta^s := s \left(\sigma\theta\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\sigma\theta\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} + \mathcal{V}_{\text{corr}}^\theta \right) \\ + \sigma\theta\alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \sigma\theta)\mathcal{G} + \mathcal{E}_{\text{corr}}^\theta + s\overline{\mathcal{K}}^\theta(s) \\ + \overline{\mathcal{Q}}_0^\theta(s) + \sum_{p,q=1}^3 \left[(s\overline{\mathcal{Q}}_1^\theta(s) + \overline{\mathcal{W}}_1^\theta(s)) : \overline{\mathcal{Q}}^{pq}(s) \right] \otimes \mathbb{J}^{pq}, \quad s > 0,$$

satisfy the symmetry properties

$$\mathfrak{H}_\sigma^{s ijkl} = \mathfrak{H}_\sigma^{s jikl} = \mathfrak{H}_\sigma^{s jilk}, \quad \forall s > 0, \quad \forall i, j, k, l = 1, 2, 3,$$

$$\mathfrak{H}_\theta^{s ijkl} = \mathfrak{H}_\theta^{s jikl} = \mathfrak{H}_\theta^{s jilk}, \quad \forall s > 0, \quad \forall i, j, k, l = 1, 2, 3,$$

$$(\mathfrak{H}_\sigma^s : \mathbb{X}) : \mathbb{W} = (\mathfrak{H}_\sigma^s : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall s > 0,$$

$$(\mathfrak{H}_\theta^s : \mathbb{X}) : \mathbb{W} = (\mathfrak{H}_\theta^s : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad \forall s > 0,$$

and the positive definiteness properties: there exists a constant $C_{10}^s > 0$ such that

$$(\mathfrak{H}_\sigma^s : \mathbb{X}) : \mathbb{X} \geq C_{10}^s |\mathbb{X}|^2, \quad (\mathfrak{H}_\theta^s : \mathbb{X}) : \mathbb{X} \geq C_{10}^s |\mathbb{X}|^2, \quad \forall \mathbb{X} \in \mathbb{R}_{\text{symm}}^{3 \times 3}, \quad s > 0.$$

Here the constant C_{10}^s is the same as in (11.47).

(C4) For a.e. $t \in (0, T)$ the tensors $\mathcal{F}_{\text{v-corr}}^\sigma(t)$ and $\mathcal{F}_{\text{v-corr}}^\theta(t)$ satisfy the symmetry properties

$$\mathcal{F}_{\text{v-corr}}^{\sigma ijkl} = \mathcal{F}_{\text{v-corr}}^{\sigma jikl} = \mathcal{F}_{\text{v-corr}}^{\sigma jilk}, \quad \forall i, j, k, l = 1, 2, 3,$$

$$\mathcal{F}_{\text{v-corr}}^{\theta ijkl} = \mathcal{F}_{\text{v-corr}}^{\theta jikl} = \mathcal{F}_{\text{v-corr}}^{\theta jilk}, \quad \forall i, j, k, l = 1, 2, 3,$$

$$(\mathcal{F}_{\text{v-corr}}^\sigma : \mathbb{X}) : \mathbb{W} = (\mathcal{F}_{\text{v-corr}}^\sigma : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3},$$

$$(\mathcal{F}_{\text{v-corr}}^\theta : \mathbb{X}) : \mathbb{W} = (\mathcal{F}_{\text{v-corr}}^\theta : \mathbb{W}) : \mathbb{X}, \quad \forall \mathbb{X}, \mathbb{W} \in \mathbb{R}_{\text{symm}}^{3 \times 3}.$$

(C5) The matrices $\mathbb{Q}^{pq}(\tau)$, $\mathbb{F}_{\text{p-corr}}^\sigma(t)$, and $\mathbb{F}_{\text{p-corr}}^\theta(t)$ are symmetric.

Proof of Proposition 13. On the strength of representations (11.15), (11.18), (11.21), (11.24), (11.29), (11.31), and (11.34), Conditions C1-C5 are just the paraphrase of assertions (i)-(v) of Proposition 12. \square

12. FORMULATION OF THE EFFECTIVE MACROSCOPIC MODEL – MODEL HBS

Inserting (5.8), (5.16), (11.15), (11.18), (11.21), (11.24), (11.29), (11.31), and (11.34) into the integral equality (11.1), and applying formula (A.6) to the terms containing \mathcal{Q}^σ , \mathcal{Q}_0^θ , and \mathcal{W}_1^θ , we unfold this integral equality as follows:

$$\begin{aligned}
 (12.1) \quad & \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \left\{ -\alpha_\tau \rho_F \mathbf{u} \cdot \partial_t \phi \right. \\
 & \quad \left. + \left(\alpha_\lambda \mathbb{I} \operatorname{div}_x \mathbf{u} + 2\alpha_\mu \mathbb{D}_x(\mathbf{u}) + \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_x(J_t \mathbf{u}) - \mathbb{I} p^0 \right) : \mathbb{D}_x(\phi) \right\} dx_3 d\hat{\mathbf{x}} dt \\
 & + \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta_*}^{\Delta + \delta^*} \left\{ -\alpha_\tau \rho_\sigma \mathbf{u} \cdot \partial_t \phi \right. \\
 & \quad \left. + \left(\sigma \alpha_\lambda \mathbb{I} \operatorname{div}_x \mathbf{u} + 2\sigma \alpha_\mu \mathbb{D}_x(\mathbf{u}) + \mathcal{V}_{\text{corr}}^\sigma : \mathbb{D}_x(\mathbf{u}) + \sigma \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_x(J_t \mathbf{u}) \right. \right. \\
 & \quad \left. \left. + (1 - \sigma) \mathcal{G} : \mathbb{D}_x(J_t \mathbf{u}) + \mathcal{E}_{\text{corr}}^\sigma : \mathbb{D}_x(J_t \mathbf{u}) \right. \right. \\
 & \quad \left. \left. + \int_0^t \boldsymbol{\kappa}^\sigma(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau + \int_0^t (J_{t-\tau} \mathcal{Q}^\sigma)(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \right. \right. \\
 & \quad \left. \left. - \left(\sigma \mathbb{I} - \mathbb{F}_{\text{p-corr}}^\sigma(t) \right) p^0 + \left((1 - \sigma) \mathcal{G} + \mathcal{F}_{\text{v-corr}}^\sigma(t) : \mathbb{D}_x(\mathbf{v}^0) \right) : \mathbb{D}_x(\phi) \right\} dx_3 d\hat{\mathbf{x}} dt \\
 & + \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta}^{\Delta + \delta_*} \left\{ -\alpha_\tau \rho_\theta \mathbf{u} \cdot \partial_t \phi \right. \\
 & \quad \left. + \left(\sigma \theta \alpha_\lambda \mathbb{I} \operatorname{div}_x \mathbf{u} + 2\sigma \theta \alpha_\mu \mathbb{D}_x(\mathbf{u}) + \mathcal{V}_{\text{corr}}^\theta : \mathbb{D}_x(\mathbf{u}) + \sigma \theta \alpha_\gamma^{-1} \mathbb{I} \operatorname{div}_x(J_t \mathbf{u}) \right. \right. \\
 & \quad \left. \left. + (1 - \sigma \theta) \mathcal{G} : \mathbb{D}_x(J_t \mathbf{u}) + \mathcal{E}_{\text{corr}}^\theta : \mathbb{D}_x(J_t \mathbf{u}) \right. \right. \\
 & \quad \left. \left. + \int_0^t \boldsymbol{\kappa}^\theta(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau + \int_0^t (J_{t-\tau} \mathcal{Q}_0^\theta)(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \right. \right. \\
 & \quad \left. \left. + \int_0^t \int_0^{\tau'} \left[\sum_{p,q=1}^3 \left(\mathcal{Q}_1^\theta(t - \tau') : \mathbb{Q}^{pq}(\tau' - \tau) \right) \otimes \mathbb{J}^{pq} \right] : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' \right. \right. \\
 & \quad \left. \left. + \int_0^t \int_0^{\tau'} \left[\sum_{p,q=1}^3 \left(\mathcal{W}_1^\theta(t - \tau') : (J_{\tau'-\tau} \mathbb{Q}^{pq})(\tau' - \tau) \right) \otimes \mathbb{J}^{pq} \right] : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' \right. \right. \\
 & \quad \left. \left. - \left(\sigma \theta \mathbb{I} - \mathbb{F}_{\text{p-corr}}^\theta(t) \right) p^0 + \left((1 - \sigma \theta) \mathcal{G} + \mathcal{F}_{\text{v-corr}}^\theta(t) : \mathbb{D}_x(\mathbf{v}^0) \right) : \mathbb{D}_x(\phi) \right\} dx_3 d\hat{\mathbf{x}} dt \\
 & + \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_0^\Delta \left\{ -\alpha_\tau \rho_S \mathbf{u} \cdot \partial_t \phi + \mathcal{G} : \left(\mathbb{D}_x(J_t \mathbf{u}) + \mathbb{D}_x(\mathbf{v}^0) \right) : \right. \\
 & \quad \left. : \mathbb{D}_x(\phi) \right\} dx_3 d\hat{\mathbf{x}} dt \\
 & = \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \alpha_F \rho_F \mathbf{f} \cdot \phi dx_3 d\hat{\mathbf{x}} dt \\
 & + \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta_*}^{\Delta + \delta^*} \alpha_F \rho_\sigma \mathbf{f} \cdot \phi dx_3 d\hat{\mathbf{x}} dt \\
 & + \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta}^{\Delta + \delta_*} \alpha_F \rho_\theta \mathbf{f} \cdot \phi dx_3 d\hat{\mathbf{x}} dt
 \end{aligned}$$

$$\begin{aligned}
& + \int_0^T \int_{\{0 < x_1, x_2 < 1\}} \int_0^\Delta \alpha_F \rho_S \mathbf{f} \cdot \boldsymbol{\phi} \, dx_3 \, d\widehat{\mathbf{x}} \, dt \\
& + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta^*}^1 \alpha_\tau \rho_F \mathbf{u}^0(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}, 0) \, dx_3 \, d\widehat{\mathbf{x}} \\
& + \int_{\{0 < x_1, x_2 < 1\}} \int_{\Delta + \delta_*}^{\Delta + \delta^*} \alpha_\tau \rho_\sigma \mathbf{u}^0(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}, 0) \, dx_3 \, d\widehat{\mathbf{x}} \\
& + \int_{\{0 < x_1, x_2 < 1\}} \int_\Delta^{\Delta + \delta_*} \alpha_\tau \rho_\theta \mathbf{u}^0(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}, 0) \, dx_3 \, d\widehat{\mathbf{x}} \\
& + \int_{\{0 < x_1, x_2 < 1\}} \int_0^\Delta \alpha_\tau \rho_S \mathbf{u}^0(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}, 0) \, dx_3 \, d\widehat{\mathbf{x}}.
\end{aligned}$$

Due to sufficient arbitrariness of the test vector-function $\boldsymbol{\phi}$, the integral equality (12.1) in the sense of distributions can be equivalently written as the system consisting of the following equations and initial and interfacial conditions.

■ In $\Omega_\Pi = \{\mathbf{x} \in \Omega: \Delta + \delta^* < x_3 < 1\}$, the fluid motion is governed by the Stokes system

$$(12.2a) \quad \alpha_\tau \rho_F \partial_t \mathbf{u} - \operatorname{div}_x (\mathcal{P} : \mathbb{D}_x(\mathbf{u})) - \alpha_\gamma^{-1} \nabla_x \operatorname{div}_x J_t \mathbf{u} = -\nabla_x p^0 + \alpha_F \rho_F \mathbf{f}.$$

■ In $\Omega_{\delta^*} = \{\mathbf{x} \in \Omega: \Delta + \delta_* < x_3 < \Delta + \delta^*\}$, the fluid-structure interactions are governed by the equations of linear viscoelasticity with memory effects

$$(12.2b) \quad \begin{aligned} & \alpha_\tau \rho_\sigma \partial_t \mathbf{u} - \operatorname{div}_x (\mathcal{P}^\sigma : \mathbb{D}_x(\mathbf{u})) - \operatorname{div}_x (\mathcal{G}^\sigma : \mathbb{D}_x(J_t \mathbf{u})) \\ & - \operatorname{div}_x \int_0^t \mathcal{R}^\sigma(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau = \operatorname{div}_x \mathbb{S}_0^\sigma + \alpha_F \rho_\sigma \mathbf{f}, \end{aligned}$$

where

$$(12.2c) \quad \mathcal{P}^\sigma = \sigma \mathcal{P} + \mathcal{V}_{\text{corr}}^\sigma$$

is the constant effective instantaneous viscous stress tensor in Ω_{δ^*} ,

$$(12.2d) \quad \mathcal{G}^\sigma = \sigma \alpha_\gamma^{-1} \mathbb{I} \otimes \mathbb{I} + (1 - \sigma) \mathcal{G} + \mathcal{E}_{\text{corr}}^\sigma$$

is the constant effective instantaneous elastic stiffness tensor in Ω_{δ^*} ,

$$(12.2e) \quad \mathcal{R}^\sigma(t) = \mathcal{K}^\sigma(t) + (J_t \mathcal{Q}^\sigma)(t)$$

is the time-dependent viscoelastic relaxation tensor in Ω_{δ^*} , and

$$(12.2f) \quad \mathbb{S}_0^\sigma(\mathbf{x}, t) = -(\sigma \mathbb{I} - \mathbb{F}_{\text{p-corr}}^\sigma(t)) p^0(\mathbf{x}) + ((1 - \sigma) \mathcal{G} + \mathcal{F}_{\text{v-corr}}^\sigma(t)) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x}))$$

is the additional space- and time-dependent partial stress tensor arising from the initial fluid-structure interactions in Ω_{δ^*} .

■ In $\Omega_{\delta_*} = \{\mathbf{x} \in \Omega: \Delta < x_3 < \Delta + \delta_*\}$, the fluid-structure interactions are governed by the equations of linear viscoelasticity with memory effects

$$(12.2g) \quad \begin{aligned} & \alpha_\tau \rho_\theta \partial_t \mathbf{u} - \operatorname{div}_x (\mathcal{P}^\theta : \mathbb{D}_x(\mathbf{u})) - \operatorname{div}_x (\mathcal{G}^\theta : \mathbb{D}_x(J_t \mathbf{u})) \\ & - \operatorname{div}_x \int_0^t \mathcal{R}_1^\theta(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \\ & - \operatorname{div}_x \int_0^t \int_0^{\tau'} \mathcal{R}_2^\theta(t - \tau', \tau' - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' = \operatorname{div}_x \mathbb{S}_0^\theta + \alpha_F \rho_\theta \mathbf{f}, \end{aligned}$$

where

$$(12.2h) \quad \mathcal{P}^\theta = \sigma\theta\mathcal{P} + \mathcal{V}_{\text{corr}}^\theta$$

is the constant effective instantaneous viscous stress tensor in Ω_{δ_*} ,

$$(12.2i) \quad \mathcal{G}^\theta = \sigma\theta\alpha_\gamma^{-1}\mathbb{I} \otimes \mathbb{I} + (1 - \sigma\theta)\mathcal{G} + \mathcal{E}_{\text{corr}}^\theta$$

is the constant effective instantaneous elastic stiffness tensor in Ω_{δ_*} ,

$$(12.2j) \quad \mathcal{R}_1^\theta(t) = \mathcal{K}^\theta(t) + (J_t\mathcal{Q}_0^\theta)(t)$$

is the time-dependent first viscoelastic relaxation tensor in Ω_{δ_*} ,

$$(12.2k) \quad \mathcal{R}_2^\theta(t_1, t_2) = \sum_{p,q=1}^3 \left[\mathcal{Q}_1^\theta(t_1) : \mathbb{Q}^{pq}(t_2) + \mathcal{W}_1^\theta(t_1) : (J_{t_2}\mathbb{Q}^{pq})(t_2) \right] \otimes \mathbb{J}^{pq}$$

is the time-dependent second viscoelastic relaxation tensor in Ω_{δ_*} , and

$$(12.2l) \quad \mathbb{S}_0^\theta(\mathbf{x}, t) = -(\sigma\theta\mathbb{I} - \mathbb{F}_{\text{p-corr}}^\theta(t))p^0(\mathbf{x}) + ((1 - \sigma\theta)\mathcal{G} + \mathcal{F}_{\text{v-corr}}^\theta(t)) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x}))$$

is the additional space- and time-dependent partial stress tensor arising from the initial fluid-structure interactions in Ω_{δ_*} .

■ In $\Omega_{\text{pl}} = \{\mathbf{x} \in \Omega : 0 < x_3 < \Delta\}$, the motion of elastic plate is governed by the Lamé system of linear elasticity

$$(12.2m) \quad \alpha_\tau \rho_S \partial_t \mathbf{u} - \text{div}_x(\mathcal{G} : \mathbb{D}_x(J_t \mathbf{u})) = \text{div}_x(\mathcal{G} : \mathbb{D}_x(\mathbf{v}^0)) + \alpha_F \rho_S \mathbf{f}.$$

■ The macroscopic velocity satisfies the initial condition

$$(12.2n) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

i.e., the initial condition (5.12).

■ Using the standard arguments (see, for example [9, Sec. 11.3] or [28, Sec. 6.2]), from (12.1) we derive the interface conditions on the sections $\{x_3 = \Delta + \delta^*\}$, $\{x_3 = \Delta + \delta_*\}$, and $\{x_3 = \Delta\}$ for the normal stress:

$$(12.2o) \quad \left[\mathcal{P} : \mathbb{D}_x(\mathbf{u}) + \alpha_\gamma^{-1} \mathbb{I} \text{div}_x J_t \mathbf{u} - \mathbb{I} p^0 \right] \mathbf{e}_3 \Big|_{x_3=\Delta+\delta^*+0} \\ = \left[\mathcal{P}^\sigma : \mathbb{D}_x(\mathbf{u}) + \mathcal{G}^\sigma : \mathbb{D}_x(J_t \mathbf{u}) \right. \\ \left. + \int_0^t \mathcal{R}^\sigma(t-\tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau - \mathbb{S}_0^\sigma \right] \mathbf{e}_3 \Big|_{x_3=\Delta+\delta^*-0} \quad \text{for } \hat{\mathbf{x}} \in (0, 1)^2,$$

(12.2p)

$$\left[\mathcal{P}^\sigma : \mathbb{D}_x(\mathbf{u}) + \mathcal{G}^\sigma : \mathbb{D}_x(J_t \mathbf{u}) \right. \\ \left. + \int_0^t \mathcal{R}^\sigma(t-\tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau - \mathbb{S}_0^\sigma \right] \mathbf{e}_3 \Big|_{x_3=\Delta+\delta_*+0} \\ = \left[\mathcal{P}^\theta : \mathbb{D}_x(\mathbf{u}) + \mathcal{G}^\theta : \mathbb{D}_x(J_t \mathbf{u}) + \int_0^t \mathcal{R}_1^\theta(t-\tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \right. \\ \left. + \int_0^t \int_0^{\tau'} \mathcal{R}_2^\theta(t-\tau', \tau'-\tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau - \mathbb{S}_0^\theta \right] \mathbf{e}_3 \Big|_{x_3=\Delta+\delta_*-0} \quad \text{for } \hat{\mathbf{x}} \in (0, 1)^2,$$

and

$$\begin{aligned}
 (12.2q) \quad & \left[\mathcal{P}^\theta : \mathbb{D}_x(\mathbf{u}) + \mathcal{G}^\theta : \mathbb{D}_x(J_t \mathbf{u}) + \int_0^t \mathcal{R}_1^\theta(t - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \right. \\
 & \left. + \int_0^t \int_0^{\tau'} \mathcal{R}_2^\theta(t - \tau', \tau' - \tau) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau - \mathbb{S}_0^\theta \right] \mathbf{e}_3 \Big|_{x_3=\Delta+0} \\
 & = \left[\mathcal{G} : \mathbb{D}_x(J_t \mathbf{u}) - \mathcal{G} : \mathbb{D}_x(\mathbf{v}^0) \right] \mathbf{e}_3 \Big|_{x_3=\Delta-0} \quad \text{for } \widehat{\mathbf{x}} \in (0, 1)^2.
 \end{aligned}$$

The system (12.2a)-(12.2q) is endowed with the following continuity and boundary conditions on the macroscopic velocity.

■ From inclusion $\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^3)$ (see (5.1)), we derive the interfacial conditions

$$(12.2r) \quad \mathbf{u} \Big|_{x_3=x_3^{(k)}+0} = \mathbf{u} \Big|_{x_3=x_3^{(k)}-0} \quad \text{for } \widehat{\mathbf{x}} \in (0, 1)^2, \quad t \in [0, T], \quad k = 1, 2, 3,$$

where $x_3^{(1)} = \Delta$, $x_3^{(2)} = \Delta + \delta_*$, and $x_3^{(3)} = \Delta + \delta^*$.

■ On the outer boundary of Ω , the macroscopic velocity satisfies the condition

$$(12.2s) \quad \mathbf{u} = \mathbf{u}^* \quad \text{for } \mathbf{x} \in \partial\Omega, \quad t \in (0, T),$$

i.e., the boundary condition (5.13), which is inherited from Problems A $_\varepsilon$, H-2sc, and H-3sc, successively.

Remark 28. \mathcal{P}^σ and \mathcal{P}^θ are the restrictions of \mathcal{V} to the segments $\{\Delta + \delta_* < x_3 \leq \Delta + \delta^*\}$ and $\{\Delta \leq x_3 \leq \Delta + \delta_*\}$, respectively; and \mathcal{G}^σ and \mathcal{G}^θ are the restrictions of \mathcal{E} to the segments $\{\Delta + \delta_* < x_3 \leq \Delta + \delta^*\}$ and $\{\Delta \leq x_3 \leq \Delta + \delta_*\}$, respectively.

Considering the tensors $\mathcal{V}_{\text{corr}}^\sigma$, $\mathcal{E}_{\text{corr}}^\sigma$, \mathcal{K}^σ , \mathcal{Q}^σ , $\mathcal{F}_{\text{v-corr}}^\sigma$, $\mathcal{V}_{\text{corr}}^\theta$, $\mathcal{E}_{\text{corr}}^\theta$, \mathcal{K}^θ , \mathcal{Q}_0^θ , \mathcal{Q}_1^θ , \mathcal{W}_1^θ , and $\mathcal{F}_{\text{v-corr}}^\theta$, the matrices $\mathbb{F}_{\text{p-corr}}^\sigma$, $\mathbb{F}_{\text{p-corr}}^\theta$, and \mathbb{Q}^{pq} , the initial displacement distribution \mathbf{v}^0 and the initial pressure distribution p^0 given, we obtain that the system (12.2a)-(12.2s) constitutes the closed effective macroscopic homogenized model of ‘the compressible fluid – two-level fine elastic structure’ interactions, where \mathbf{u} is the sought vector-function.

Definition 8. We call the system (12.2a)-(12.2s) **Model HBS** after the well-known homogenized model of ‘the compressible fluid – one-level fine elastic structure’ interactions derived from the microstructure by K.-H. Hoffmann, N.D. Botkin and

V.N. Starovoitov in [12].

We naturally formulate the notion of weak solutions of Model HBS as follows.

Definition 9. Vector-function \mathbf{u} is a **weak solution of Model HBS** if it satisfies inclusion (5.1), the boundary condition (12.2s) in the trace sense, and the integral equality (12.1) for arbitrary smooth test vector-function $\phi = \phi(\mathbf{x}, t)$ vanishing in the neighborhood of $\partial\Omega$ and the section $\{t = T\}$.

13. WELL-POSEDNESS OF MODEL HBS

By the considerations fulfilled in Secs. 5-12, we have proved the following existence result for Model HBS.

Proposition 14. *Assume that $\mathcal{V}_{\text{corr}}^\sigma, \mathcal{E}_{\text{corr}}^\sigma, \mathcal{K}^\sigma, \mathcal{Q}^\sigma, \mathcal{F}_{v\text{-corr}}^\sigma, \mathcal{V}_{\text{corr}}^\theta, \mathcal{E}_{\text{corr}}^\theta, \mathcal{K}^\theta, \mathcal{Q}_0^\theta, \mathcal{Q}_1^\theta, \mathcal{W}_1^\theta, \mathcal{F}_{v\text{-corr}}^\theta, \mathbb{F}_{p\text{-corr}}^\sigma, \mathbb{F}_{p\text{-corr}}^\theta$, and \mathbb{Q}^{pq} are given by the formulas (11.15)-(11.36) (or, equivalently, by the formulas (11.2)-(11.10)), where the vector-functions $\mathbf{Z}_{00}^{mn}, \mathbf{Z}_{10}^{mn}, \mathbf{Z}_{20}, \mathbf{Z}_{30}^{mn}, \mathbf{Y}_0^{mn}, \mathbf{Y}_1^{mn}, \mathbf{Y}_2$, and \mathbf{Y}_3^{mn} are the solutions of Problems Z1-Z7 and Y1-Y7. Assume that $v^0 \in H^1(\Omega)^3$ and $p^0 \in H^1(\Omega)$ are given in the integral equality (12.1) and, correspondingly, in equations (12.2a), (12.2f), (12.2l), and (12.2m).*

Then, for any given $\mathbf{u}^0 \in H^1(\Omega)^3, \mathbf{u}^ \in C^2(\bar{\Omega} \times [0, T])^3, \mathbf{f} \in L^2(\Omega \times (0, T))^3$ such that $\mathbf{u}^0(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}, 0)$ for $\mathbf{x} \in \partial\Omega$ and $\partial_t \mathbf{f} \in L^2(\Omega \times (0, T))^3$, there exists at least one weak solution of Model HBS in the sense of Definition 9.*

The assumptions on the tensors and matrices imposed in Proposition 14 mean that the coefficients of equations (12.2b) and (12.2g) are defined by the specific mesoscopic and microscopic structures. In this sense, the existence result in Proposition 14 essentially depends on the mesoscopic and macroscopic structures given in the layer $\{\Delta \leq x_3 \leq \Delta + \delta^*\}$, since the coefficients of equation (12.2b) explicitly depend on $\mathbf{Y}_0^{mn}, \mathbf{Y}_1^{mn}, \mathbf{Y}_2$, and \mathbf{Y}_3^{mn} , and the coefficients of equation (12.2g) explicitly depend on $\mathbf{Z}_{00}^{mn}, \mathbf{Z}_{10}^{mn}, \mathbf{Z}_{20}, \mathbf{Z}_{30}^{mn}, \mathbf{Y}_0^{mn}, \mathbf{Y}_1^{mn}, \mathbf{Y}_2$, and \mathbf{Y}_3^{mn} . Furthermore, the limiting passage in Model A_ε as $\varepsilon \searrow 0$ followed by the two successive asymptotic decomposition procedures leads merely to the existence result, while the question of uniqueness remains open.

The following theorem asserts the existence and uniqueness of weak solutions to Model HBS. We emphasize that these existence and uniqueness results hold true independently of whether Model HBS is connected with the certain microscopic and mesoscopic structures via representations (11.15)-(11.36), or not.

Theorem 3. *Assume that the tensors $\mathcal{V}_{\text{corr}}^\sigma, \mathcal{E}_{\text{corr}}^\sigma, \mathcal{K}^\sigma, \mathcal{Q}^\sigma, \mathcal{F}_{v\text{-corr}}^\sigma, \mathcal{V}_{\text{corr}}^\theta, \mathcal{E}_{\text{corr}}^\theta, \mathcal{K}^\theta, \mathcal{Q}_0^\theta, \mathcal{Q}_1^\theta, \mathcal{W}_1^\theta$, and $\mathcal{F}_{v\text{-corr}}^\theta$, and the matrices $\mathbb{F}_{p\text{-corr}}^\sigma, \mathbb{F}_{p\text{-corr}}^\theta$, and \mathbb{Q}^{pq} are given and meet Conditions C1-C5. Let all of them be, in principle, irrelevant to the data given for Model A_ε . Assume that the initial displacement distribution $v^0 \in H^1(\Omega)^3$ and the initial pressure distribution $p^0 \in H^1(\Omega)$ are given in the integral equality (11.1) and, correspondingly, (12.2a), (12.2f), (12.2l), and (12.2m).*

Then, for any given $\mathbf{u}^0 \in H^1(\Omega)^3, \mathbf{u}^ \in C^2(\bar{\Omega} \times [0, T])^3, \mathbf{f} \in L^2(\Omega \times (0, T))^3$ such that $\mathbf{u}^0(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}, 0)$ for $\mathbf{x} \in \partial\Omega$ and $\partial_t \mathbf{f} \in L^2(\Omega \times (0, T))^3$, there exists a unique weak solution of Model HBS in the sense of Definition 9.*

Proof. We divide the proof into five steps. It is quite analogous to justification of Theorem 2 and therefore we outline it in a rather concise way.

Step 1. Let us formulate the definition of weak solution of Model HBS in an alternative way, which is equivalent to Definition 9 and is more preferable for justification of the theorem for technical reasons.

For the sole purpose of brevity, we consider the integral (11.1) instead of its unfolded version (12.1). We integrate by parts in the first integral in the left-hand side of (11.1), which makes sense due to (5.1)₂, then in (11.1) we take the test vector-function of the form $\phi(\mathbf{x}, t) = \xi(t)\Phi(\mathbf{x})$. Due to arbitrariness of ξ , from (11.1) we derive the integral equality

$$(13.1) \quad \int_{\Omega} \left\{ \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{u}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) + \left[\mathcal{V}(x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \mathcal{E}(x_3) : \mathbb{D}_x((J_t \mathbf{u})(\mathbf{x}, t)) \right] \right.$$

$$\begin{aligned}
 & + \int_0^t \mathcal{K}(t - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \\
 & + \int_0^t \int_0^{\tau'} \mathcal{Q}(t - \tau', \tau' - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' \\
 & + \int_0^t \int_0^{\tau'} \mathcal{W}(t - \tau', \tau' - \tau, x_3) : \mathbb{D}_x((J_\tau \mathbf{u})(\mathbf{x}, \tau)) d\tau d\tau' \\
 & + \mathbb{F}_p(t, x_3) p^0(\mathbf{x}) + \mathcal{F}_v(t, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \Big] : \mathbb{D}_x(\Phi(\mathbf{x})) \Big\} d\mathbf{x} \\
 & = \int_\Omega \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{f}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) d\mathbf{x} \quad \text{for a.e. } t \in [0, T],
 \end{aligned}$$

where Φ is a sufficiently smooth test vector-function vanishing in a neighborhood of boundary $\partial\Omega$.

Also, let us notice that the initial condition (12.2n) holds in the strong trace sense due to the boundedness of $\|\partial_t \mathbf{u}\|_{L^\infty(0, T; L^2(\Omega)^3)}$ (see inclusion (5.1)₂). Thus we make the following conclusion.

Remark 29. *Vector-function \mathbf{u} is a weak solution of Model HBS in the sense of Definition 9 if and only if it satisfies inclusions (5.1), the initial condition (12.2n) and the boundary condition (12.2s) in the strong trace sense, and the integral equality (13.1) for a.e. $t \in [0, T]$ for all admissible test vector-functions Φ .*

Step 2. On the strength of the initial condition (12.2n) and the property of absolute continuity of the Lebesgue integral with respect to the limits of integration, from (13.1) it follows that

$$\begin{aligned}
 (13.2) \quad & \int_\Omega \left\{ \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{u}(\mathbf{x}, t) \Big|_{t=0} \cdot \Phi(\mathbf{x}) \right. \\
 & + \left[\mathcal{V}(x_3) : \mathbb{D}_x(\mathbf{u}^0(\mathbf{x})) + \mathbb{F}_p(0, x_3) p^0(\mathbf{x}) + \mathcal{F}_v(0, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) \Big\} d\mathbf{x} \\
 & = \int_\Omega \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \mathbf{f}(\mathbf{x}, 0) \cdot \Phi(\mathbf{x}) d\mathbf{x} \quad \text{for a.e. } t \in [0, T].
 \end{aligned}$$

Further, differentiating (13.1) with respect to t , with the help of assertion (ii) of Proposition A.1, we establish the integral equality

$$\begin{aligned}
 (13.3) \quad & \frac{d}{dt} \int_\Omega \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{u}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) d\mathbf{x} \\
 & + \frac{d}{dt} \int_\Omega \left[\mathcal{V}(x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x} \\
 & + \int_\Omega \left[\mathcal{E}(x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \int_0^t \frac{\partial \mathcal{K}}{\partial t}(t - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \right. \\
 & + \mathcal{K}(0, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, t)) + \int_0^t \int_0^{\tau'} \frac{\partial \mathcal{Q}}{\partial t}(t - \tau', \tau' - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' \\
 & \left. + \int_0^t \mathcal{Q}(0, t - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \int_0^{\tau'} \mathcal{W}(t - \tau', \tau' - \tau, x_3) : \mathbb{D}_x(\mathbf{u}(\mathbf{x}, \tau)) d\tau d\tau' \\
 & + \partial_t \mathbb{F}_p(t, x_3) p^0(\mathbf{x}) + \partial_t \mathcal{F}_v(t, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \Big] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x} \\
 & = \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \partial_t \mathbf{f}(\mathbf{x}, t) \cdot \Phi(\mathbf{x}) d\mathbf{x}.
 \end{aligned}$$

This integral equality is understood in the sense of distributions on $[0, T]$.

Step 3. We extend all functions in (13.3) for $t > T$ by zero. On the strength of the well-known properties of the Laplace transform, applying formally the Laplace transform in t to (13.3), taking into account the initial data (12.2n), using representations (11.21), (11.24), (11.29), and (11.44), and remarking that the integrals containing the terms $\mathcal{K}(0, x_3)$ and $\mathcal{Q}_1^\theta(0)$ cancel, we arrive at the integral equality

$$\begin{aligned}
 (13.4) \quad & \int_{\Omega} \left\{ \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (s^2 \bar{\mathbf{u}}(\mathbf{x}, s) - s \mathbf{u}^0(\mathbf{x}) - \partial_t \mathbf{u}(\mathbf{x}, t)|_{t=0}) \cdot \Phi(\mathbf{x}) \right. \\
 & + \left[\mathfrak{H}^s(x_3) : \mathbb{D}_x(\bar{\mathbf{u}}(\mathbf{x}, s)) + s \bar{\mathbb{F}}_p(s, x_3) p^0(\mathbf{x}) \right. \\
 & + s \bar{\mathcal{F}}_v(s, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) - \mathcal{V}(x_3) : \mathbb{D}_x(\mathbf{u}^0(\mathbf{x})) \\
 & \left. \left. - \mathbb{F}_p(0, x_3) p^0(\mathbf{x}) - \mathcal{F}_v(0, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) \right\} d\mathbf{x} \\
 & = \int_{\Omega} \alpha_F \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (s \bar{\mathbf{f}}(\mathbf{x}, s) - \mathbf{f}(\mathbf{x}, 0)) \cdot \Phi(\mathbf{x}) d\mathbf{x}, \quad \forall s > 0.
 \end{aligned}$$

Combining (13.2) and (13.4) and slightly re-arranging terms, we obtain the integral equality

$$\begin{aligned}
 (13.5) \quad & \int_{\Omega} \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) s^2 \bar{\mathbf{u}}(\mathbf{x}, s) \cdot \Phi(\mathbf{x}) d\mathbf{x} + \int_{\Omega} \left[\mathfrak{H}^s(x_3) : \mathbb{D}_x(\bar{\mathbf{u}}(\mathbf{x}, s)) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x} \\
 & = s \int_{\Omega} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \left[\alpha_\tau \mathbf{u}^0(\mathbf{x}) + \alpha_F \bar{\mathbf{f}}(\mathbf{x}, s) \right] \cdot \Phi(\mathbf{x}) d\mathbf{x} \\
 & - s \int_{\Omega} \left[\bar{\mathbb{F}}_p(s, x_3) p^0(\mathbf{x}) + \bar{\mathcal{F}}_v(s, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x}, \quad \forall s > 0.
 \end{aligned}$$

Step 4. On this step we aim to prove that for an arbitrarily fixed $s > 0$ there exists function $\bar{\mathbf{u}}(\cdot, s) \in H^1(\Omega)^3$ satisfying the boundary condition $\bar{\mathbf{u}} = \bar{\mathbf{u}}^* \equiv \mathcal{L}[\mathbf{u}^*]$ on $\partial\Omega$ in the strong trace sense and the integral equality (13.5) for all admissible test vector-functions Φ . The same as on the fourth step of the proof of Theorem 2, we implement the Lax-Milgram theorem. The variable s in this formulation plays the role of a parameter.

We introduce the bilinear form $b^s: H_0^1(\Omega)^3 \times H_0^1(\Omega)^3 \mapsto \mathbb{R}$ and the linear functional $\Lambda^s: H_0^1(\Omega)^3 \mapsto \mathbb{R}$ by the formulas

$$\begin{aligned}
 b^s(\Psi, \Phi) & := s^2 \int_{\Omega} \alpha_\tau \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \Psi(\mathbf{x}) \cdot \Phi(\mathbf{x}) d\mathbf{x} \\
 & + \int_{\Omega} \left[\mathfrak{H}^s(x_3) : \mathbb{D}_x(\Psi(\mathbf{x})) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x}, \quad \forall \Phi, \Psi \in H_0^1(\Omega)^3,
 \end{aligned}$$

and

$$\begin{aligned} \Lambda^s(\Phi) := & s \int_{\Omega} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) (\alpha_{\tau} \mathbf{u}^0(\mathbf{x}) + \alpha_F \bar{\mathbf{f}}(\mathbf{x}, s) - \alpha_{\tau} s \bar{\mathbf{u}}^*(\mathbf{x}, s)) \cdot \Phi(\mathbf{x}) d\mathbf{x} \\ & - \int_{\Omega} \left[s \bar{\mathbb{F}}_p(s, x_3) p^0(\mathbf{x}) + s \bar{\mathcal{F}}_v(s, x_3) : \mathbb{D}_x(\mathbf{v}^0(\mathbf{x})) \right. \\ & \left. + \mathfrak{H}^s(x_3) : \mathbb{D}_x(\bar{\mathbf{u}}^*(\mathbf{x}, s)) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x}, \quad \forall \Phi \in H_0^1(\Omega)^3. \end{aligned}$$

In terms of b^s and Λ^s the question of finding $\bar{\mathbf{u}}$ satisfying (13.5) and the above stated set of requirements is formulated as the following variational problem:

$$(13.6) \quad \begin{aligned} & \text{Find } \mathbf{U}_1 \in H_0^1(\Omega)^3 \text{ satisfying the variational equation} \\ & b^s(\mathbf{U}_1, \Phi) = \Lambda^s(\Phi), \quad \forall \Phi \in H_0^1(\Omega)^3. \end{aligned}$$

Upon finding such \mathbf{U}_1 , it simply suffices to set $\bar{\mathbf{u}} := \mathbf{U}_1 + \bar{\mathbf{u}}^*$.

Let us find out the properties of b^s and Λ^s sufficient for the well-posedness of the problem (13.6). Due to (11.37)-(11.39) we have that $\mathfrak{H}^s \in L^\infty(0, 1)^{3 \times 3 \times 3 \times 3}$, $\forall s > 0$. Using this, (11.45), (9.7)₁, and the Cauchy-Bunyakovsky inequality, we construct the estimate

$$\begin{aligned} |b^s(\Psi, \Phi)| & \stackrel{(11.45)}{=} \left| s^2 \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) \Psi(\mathbf{x}) \cdot \Phi(\mathbf{x}) d\mathbf{x} \right. \\ & \left. + \int_{\Omega} \left[\mathfrak{H}^s(x_3) : \nabla_x \Psi(\mathbf{x}) \right] : \nabla_x \Phi(\mathbf{x}) d\mathbf{x} \right| \\ & \leq s^2 \alpha_{\tau} \max\{\rho_F, \rho_S\} \|\Psi\|_{L^2(\Omega)^3} \|\Phi\|_{L^2(\Omega)^3} \\ & \quad + \|\mathfrak{H}^s\|_{L^\infty(0,1)^{3 \times 3 \times 3 \times 3}} \|\nabla_x \Psi\|_{L^2(\Omega)^{3 \times 3}} \|\nabla_x \Phi\|_{L^2(\Omega)^{3 \times 3}} \\ & \leq \max\{s^2 \alpha_{\tau} \rho_F, s^2 \alpha_{\tau} \rho_S, \|\mathfrak{H}^s\|_{L^\infty(0,1)^{3 \times 3 \times 3 \times 3}}\} \times \\ & \quad \times (\|\Psi\|_{L^2(\Omega)^3} + \|\nabla_x \Psi\|_{L^2(\Omega)^{3 \times 3}}) (\|\Phi\|_{L^2(\Omega)^3} + \|\nabla_x \Phi\|_{L^2(\Omega)^{3 \times 3}}) \\ & \leq 2 \max\{s^2 \alpha_{\tau} \rho_F, s^2 \alpha_{\tau} \rho_S, \|\mathfrak{H}^s\|_{L^\infty(0,1)^{3 \times 3 \times 3 \times 3}}\} \times \\ & \quad \times \|\Psi\|_{H_0^1(\Omega)^3} \|\Phi\|_{H_0^1(\Omega)^3}, \quad \forall \Phi, \Psi \in H_0^1(\Omega)^3, \quad \forall s > 0. \end{aligned}$$

This estimate yields that

$$(13.7) \quad \text{the bilinear form } b^s \text{ is continuous on } H_0^1(\Omega)^3 \times H_0^1(\Omega)^3.$$

Next, using Lemmas 2 and 4 and the positive definiteness of tensor \mathfrak{H}^s (see (11.47)), for an arbitrary $\Phi \in H_0^1(\Omega)^3$ we establish the following estimate:

$$\begin{aligned} b^s(\Phi, \Phi) & \equiv s^2 \int_{\Omega} \alpha_{\tau} \langle \rho \rangle_{\Sigma \times \Theta}(x_3) |\Phi(\mathbf{x})|^2 d\mathbf{x} + \int_{\Omega} \left[\mathfrak{H}^s(x_3) : \mathbb{D}_x(\Phi(\mathbf{x})) \right] : \mathbb{D}_x(\Phi(\mathbf{x})) d\mathbf{x} \\ & \stackrel{(11.47)}{\geq} C_{10}^s \int_{\Omega} |\mathbb{D}_x(\Phi(\mathbf{x}))|^2 d\mathbf{x} \stackrel{(9.10)}{\geq} \frac{C_{10}^s}{2} \|\nabla_x \Phi\|_{L^2(\Omega)^{3 \times 3}}^2 \stackrel{(9.12)}{\geq} \frac{C_{10}^s}{2(1+c_{\Omega}^2)} \|\Phi\|_{H_0^1(\Omega)^3}^2. \end{aligned}$$

This estimate means that for all $s > 0$

$$(13.8)$$

the bilinear form b^s is $H_0^1(\Omega)^3$ -elliptic with the constant $C_{11}^s := \frac{C_{10}^s}{2(1+c_{\Omega}^2)}$.

Finally, on the strength of the assumptions in the theorem imposed on the given data, using the standard technics we get the estimate

$$(13.9) \quad |\Lambda^s(\Phi)| \leq C_{12}^s \|\Phi\|_{H_0^1(\Omega)^3}, \quad \forall \Phi \in H_0^1(\Omega)^3, \quad s > 0,$$

where the positive constant C_{12} depends on $s, \alpha_\tau, \alpha_F, \rho_F, \rho_S, \|\mathbf{u}^0\|_{L^2(\Omega)^3}, \|p^0\|_{L^2(\Omega)}, \|\mathbf{v}^0\|_{H^1(\Omega)^3}, \|\mathbf{f}\|_{L^2(\Omega \times (0,T))^3}, \|\mathbf{u}^*\|_{C^1(\bar{\Omega} \times [0,T])^3}, \|\mathbb{F}_p\|_{L^\infty((0,T) \times (0,1))^{3 \times 3}},$ and $\|\mathcal{F}_v\|_{L^\infty((0,T) \times (0,1))^{3 \times 3 \times 3 \times 3}}$. This estimate yields that

$$(13.10) \quad \begin{aligned} &\text{the linear functional } \Lambda^s \text{ is continuous on } H_0^1(\Omega)^3 \text{ for } s > 0; \\ &\text{moreover, } \|\Lambda^s\|_{H^{-1}(\Omega)^3} \leq C_{12}^s. \end{aligned}$$

On the strength of (13.7) and (13.8)-(13.10), by the Lax-Milgram theorem we conclude that there exists the unique solution $\mathbf{U}_1 \in H_0^1(\Omega)^3$ to the variational problem (13.6), and this solution admits the estimate

$$\|\mathbf{U}_1\|_{H_0^1(\Omega)} \leq (1/C_{11}^s) \|\Lambda^s\|_{H^{-1}(\Omega)^3} \leq C_{12}^s/C_{11}^s, \quad s > 0.$$

Hence there exists the unique function $\bar{\mathbf{u}} = \mathbf{U}_1(\mathbf{x}; s) + \bar{\mathbf{u}}^*(\mathbf{x}, s)$ satisfying the integral equality (13.5), the boundary condition $\bar{\mathbf{u}}(\mathbf{x}, s) = \bar{\mathbf{u}}^*(\mathbf{x}, s)$ for $\mathbf{x} \in \partial\Omega$ and $s > 0$, and the regularity requirement $\bar{\mathbf{u}}(\cdot, s) \in H^1(\Omega)^3$.

Step 5. Analogously to the fifth step of the proof of Theorem 2, applying the inverse Laplace transform in s to the variational equation (13.6) and fulfilling the steps 1-3 of the present proof in the reverse order, we conclude that the function

$$\mathbf{u}(\mathbf{x}, t) = \mathcal{L}^{-1}[\mathbf{U}_1](\mathbf{x}, t) + \mathbf{u}^*(\mathbf{x}, t)$$

is a weak solution to Model HBS in the sense of Definition 9. This solution is unique due to one-to-oneness of \mathcal{L} and \mathcal{L}^{-1} .

Theorem 3 is proved. □

14. CONCLUDING REMARKS

The homogenization and asymptotic decomposition procedures provide us with the following algorithm of determining the effective macroscopic physical characteristics of the reciprocal motion of the fine two-level elastic bristle structure and the viscous weakly compressible fluid, starting from the microstructure.

- (i) Using the given data of the microstructure, solve Problems Z1-Z7 to find $\mathbf{Z}_{00}^{ij}, \mathbf{Z}_{10}^{ij}, \mathbf{Z}_{20},$ and \mathbf{Z}_{30}^{ij} .
- (ii) Inserting the solutions of Problems Z1-Z7 into (7.14), (7.16), (7.18), (7.20), (7.22), and (7.23), find the tensors $\mathcal{A}_0^f, \mathcal{B}_0^f, \mathcal{A}_1^f,$ and \mathcal{B}_1^f and the matrices \mathbb{F}_f^0 and \mathbb{F}_{sol}^{0ij} , respectively.
- (iii) Using the tensors $\mathcal{A}_0^f, \mathcal{B}_0^f, \mathcal{A}_1^f,$ and \mathcal{B}_1^f and the matrices \mathbb{F}_f^0 and \mathbb{F}_{sol}^{0ij} obtained on the previous step, solve Problems Y1-Y7 to find $\mathbf{Y}_0^{ij}, \mathbf{Y}_1^{ij}, \mathbf{Y}_2,$ and \mathbf{Y}_3^{ij} .
- (iv) Inserting the solutions of Problems Y1-Y7 into (11.16), (11.17), (11.19), (11.20), (11.22), (11.23), (11.25)-(11.28), (11.30), (11.32), (11.33), (11.35), and (11.36), calculate the homogenized macroscopic tensors and matrices $\mathcal{V}_{\text{corr}}^\sigma, \mathcal{V}_{\text{corr}}^\theta, \mathcal{E}_{\text{corr}}^\sigma, \mathcal{E}_{\text{corr}}^\theta, \mathcal{K}^\sigma, \mathcal{K}^\theta, \mathcal{Q}^\sigma, \mathcal{Q}_0^\theta, \mathcal{Q}_1^\theta, \mathbb{Q}^{pq}, \mathcal{W}_1^\theta, \mathcal{F}_{\text{v-corr}}^\sigma,$ $\mathcal{F}_{\text{v-corr}}^\theta, \mathbb{F}_{\text{p-corr}}^\sigma,$ and $\mathbb{F}_{\text{p-corr}}^\theta,$ respectively.
- (v) Provided with the data obtained on the previous step, solve Problem HBS to find the macroscopic velocity distribution \mathbf{u} .

This five-step algorithm is quite possible to implement either numerically, or, for the most simple forms of Θ_S and Σ_S and given boundary data, even analytically. In contrast, Model A_ε with a small ε is inaccessible for practical analysis due to the enormous amount of necessary calculations.

The interim homogenized Models H-3sc and H-2sc also worth consideration in line with a possible numerical analysis.

Notice that discarding the layer $\Omega^{\delta*}$ in Model HBS followed by the matching of the interfacial conditions, we arrive exactly at the Hoffmann-Botkin-Starovoitov equations [12], which should have been expected. The difference between the approaches in the present article and in [12] is that we formulate the cell problems (Problems Z1-Z7 and Y1-Y7) using the well-known old-fashioned straightforward asymptotic decomposition routine, which ascends to the classical works [14, 23], in turn, in [12], in order to solve the cell problem, the authors construct and use a new rather subtle method based on the semigroup theory.

Finalizing this research, let us expose some ideas about possible applications of Model HBS in biology, biotechnology, and bionics. Notice that Model A has a rather long history of applications in modeling of biological systems. Therefore its modifications to specific topics in biology and biotechnology are in demand, as well (see, [5, 15] and references therein). Model HBS can be applied to a study of aerodynamics in a neighborhood of a plant leaf with trichomes being taken into account. Trichomes are bristles (fuzz) on a leaf epithelium. Clearly, this question lays in the field of continuum mechanics. More precisely, it is natural to simulate a leaf with trichomes as an either elastic or absolutely rigid plate with pins attached to it immersed into an air flux. In this framework, a rather simple model proposed in S. Goldstein's monograph [11, Secs. 53 and 145] is still in demand. In this model, a laminar flow around a thin plate with one perpendicularly welded pin is considered. The air blows in parallel to the plate. The conditions for a flow to stay laminar after passing by the pin are found. Originally, the model in [11, Secs. 53 and 145] arose in the wing theory and did not belong to biology. Nevertheless, in a number of works (for example, see [25]), this model was successfully adapted for airflow by a leaf with trichomes of the same length. In [25], numerical simulations for some plants were conducted. These simulations were very simple and consisted of explicit algebraic calculations with a help of an ordinary scientific calculator. They allowed to find critical lengths of trichomes and wind speeds, for which the airflow sustains a laminar regime. Worth to remark that it is impossible to study turbulent flows within this framework. In the general case, trichomes on the same leaf of a plant may belong to different types depending on length and form. In a number of works (for example, see [8]), it was noticed that the quantities of trichomes of different lengths have different orders — there are considerably many shorter trichomes per just one long trichome. In this view, the approach elaborated in [25] is not suitable, in general. At the same time, the proposed in the present article Model HBS covers the two-lengths case well enough. Besides, since Model HBS is derived on the base of rather general physical hypotheses, it is applicable for both laminar and turbulent regimes of airflow.

Also Model HBS somewhat extends mathematical modeling of biosensors. Matter is that the design of the original Hoffmann-Botkin-Starovoitov system [12, Sec. 5.1] *'was motivated by modeling a surface acoustic wave sensor based on the generation and detection of horizontally polarized shear waves (see [3]). Acoustic shear*

waves are excited through an alternate voltage applied to electrodes deposited on a quartz crystal substrate. The waves are transmitted into a thin isotropic guiding layer covered by a thin gold film that contacts a liquid containing a protein to be detected. The protein adheres to a specific receptor (aptamer) placed on the surface of the gold film. The arising mass loading causes a phase shift in the electric signal to be measured by an electronic circuit. One can impress the aptamer-protein layer as a periodic bristle or pin structure on the top of the gold film contacting with the liquid. The thickness of the aptamer-protein layer is about 4 nm, and the number of bristles per surface unit is enormous large. Therefore, the direct numerical modeling of such a structure using fluid-solid interface conditions is impossible. Proper models can be derived using the homogenization technique along with the strict treatment of the solid-fluid interface' (citation from [12, Introduction]).

Finally, let us mention a possible use of Model HBS in bionics (or biomimicry). Biomimicry describes processes in which the ideas and concepts developed by nature are implemented into technology. In line with the observations in [16], bristles in general have a strong influence on the plate wettability: the plate surfaces may be super-hydrophobic, self-cleaning (super-oleophobic), and low adhesive, which often are very advantageous features of materials. Two-level (hierarchical) roughness structures are typical for super-hydrophobic surfaces in nature. In [2, Sec. 42.4.3], the question about how to generate a two-level super-hydrophobic surface is discussed in detail. Highly likely, Model HBS may be helpful in such design.

APPENDIX A. NOTIONS AND SOME PROPERTIES OF THE CONVOLUTION INTEGRALS AND THE LAPLACE TRANSFORM

In this appendix, for convenience of reading, we recall the definitions and some properties of convolution integrals and the Laplace transform, which are systematically used throughout the article. We introduce the notion of convolution integral with the truncated integration limit, which is the important particular case of the general definition [13, Ch. 1, Sec. 1.3].

Definition A.1. For functions $\Phi = \Phi(t)$, $\Psi = \Psi(t)$ supported on $[0, +\infty)$, their convolution integral is

$$(A.1) \quad (\Phi * \Psi)(t) = \int_0^t \Phi(t - \tau)\Psi(\tau)d\tau.$$

The convolution integral of Φ and Ψ exists if $\Phi, \Psi \in L^1(\mathbb{R}_+)$, and in this case $\Phi * \Psi \in L^1(\mathbb{R}_+)$ [26, Ch. 1, Th. 1.3]. In particular, $\Phi * \Psi$ is well defined for piece-wise continuous Φ and Ψ with bounded supports. The following properties of the convolution integral are valid for sufficiently regular functions $\Phi, \Psi: \mathbb{R}_+ \mapsto \mathbb{R}$.

Proposition A.1. (i) (Commutativity.) $(\Phi * \Psi)(t) = (\Psi * \Phi)(t)$ or, equivalently,

$$(A.2) \quad \int_0^t \Phi(t - \tau)\Psi(\tau)d\tau = \int_0^t \Phi(\tau)\Psi(t - \tau)d\tau.$$

(ii) (Relationship with differentiation.) Assume either Φ , or Ψ , or both Φ and Ψ are differentiable; then

$$(A.3) \quad \frac{d}{dt} \int_0^t \Phi(t - \tau)\Psi(\tau)d\tau = \int_0^t \Phi(t - \tau) \frac{d\Psi(\tau)}{d\tau} d\tau + \Phi(t)\Psi(0)$$

and (or)

$$(A.4) \quad \frac{d}{dt} \int_0^t \Phi(t-\tau)\Psi(\tau)d\tau = \int_0^t \frac{\partial\Phi(t-\tau)}{\partial t} \Psi(\tau)d\tau + \Phi(0)\Psi(t).$$

(iii) (Relationship with integration.)

$$(A.5) \quad \int_0^t \int_0^{t'} \Phi(t'-\tau)\Psi(\tau)d\tau dt' = \int_0^t \Phi(t-\tau)(J_\tau\Psi)(\tau)d\tau,$$

$$(A.6) \quad \int_0^t \Phi(t-\tau)(J_\tau\Psi)(\tau)d\tau = \int_0^t (J_{t-\tau}\Phi)(t-\tau)\Psi(\tau)d\tau.$$

Here J_τ is the Volterra operator, see (2.6).

Proof. (i) Changing the variable of integration to $\tau' = t - \tau$ the result holds.

(ii) We recall the well-known formula from integral calculus [7, Ch. 14, § 1, Th. 6]:

$$(A.7) \quad \frac{d}{dt} \int_{a(t)}^{b(t)} W(t, \tau) d\tau = \int_{a(t)}^{b(t)} \frac{\partial W(t, \tau)}{\partial t} d\tau + b'(t)W(t, b(t)) - a'(t)W(t, a(t)),$$

where a and b are differentiable on an interval $[0, T]$ and W and $\partial_t W$ are integrable on $(0, T) \times (c_-, c_+)$,

$$c_- = \min \left\{ \min_{t \in [0, T]} a(t), \min_{t \in [0, T]} b(t) \right\}, \quad c_+ = \max \left\{ \max_{t \in [0, T]} a(t), \max_{t \in [0, T]} b(t) \right\}.$$

Taking $a(t) = 0$, $b(t) = t$, and $W(t, \tau) = \Phi(t - \tau)\Psi(\tau)$ from (A.7) we immediately deduce (A.4). Taking $a(t) = 0$, $b(t) = t$, and $W(t, \tau) = \Phi(\tau)\Psi(t - \tau)$ followed by using the commutativity property (i), from (A.7) we deduce (A.3), which completes the proof of assertion (ii).

(iii) In (A.3) we substitute $J_\tau\Psi$ for Ψ and t' for t . Next, we notice that $(J_\tau\Psi)(0) = 0$ and integrate the resulting equality in t' from 0 to t , which immediately yields (A.5). Next, in the left hand side of (A.6) set $\phi = J_t\Phi$ so that $\Phi = d\phi/dt$ and integrate by parts to get the desired identity. \square

In this article, we use the Laplace transform in the time variable $t \in \mathbb{R}_+$ only.

Definition A.2. ([17, Sec. 6.1], [6, Ch. XVI, § 1, formula (1.18)].) *If $\varphi(t)$ is a function defined for all $t \geq 0$, its Laplace transform is the integral*

$$(A.8) \quad \bar{\varphi}(s) = \mathcal{L}[\varphi](s) = \int_0^{+\infty} \varphi(t)e^{-st}dt, \quad s > 0.$$

Proposition A.2. ([17, Sec. 6.1]) *The Laplace transform exists for any function, which is integrable over every finite interval $0 < t < l$, and which, as $t \nearrow +\infty$, does not grow faster than some exponential function. This last condition is more precisely stated by*

$$(A.9) \quad |\varphi(t)| < Me^{\beta t} \quad \text{as } t \nearrow +\infty,$$

where M and β are some real positive numbers.

For functions $F = F(s)$ from a rather wide class (details see in [6, Ch. XVI, § 1, Th. 2]), there exists the inverse Laplace transform $\mathcal{L}^{-1}[F](t) = \varphi(t)$. It is given by the Bromwich integral [6, Ch. XVI, § 1, formula (1.30)] (or, [18, Ch. 4, Sec. 7]):

$$(A.10) \quad \mathcal{L}^{-1}[F](t) = \frac{1}{2\pi i} \int_{s_1 - i\infty}^{s_1 + i\infty} F(s)e^{st}ds,$$

which is a contour integral on the complex plane, where $s_1 > 0$ is an arbitrarily fixed real value. Both \mathcal{L} and \mathcal{L}^{-1} are one-to-one mappings, and we have

$$(A.11) \quad \mathcal{L}^{-1}[\mathcal{L}[\varphi]](t) = \varphi(t), \quad \mathcal{L}[\mathcal{L}^{-1}[F]](s) = F(s) \quad \text{for admissible } \varphi \text{ and } F.$$

The Laplace transform and the inverse Laplace transform possess many remarkable properties. We state here just those of them that are relevant to our work.

Proposition A.3. (i) (Linearity.) *Both \mathcal{L} and \mathcal{L}^{-1} are linear mappings.*

(ii) (Relation with differentiation.) *Assume function $\varphi = \varphi(t)$ is differentiable and $\bar{\varphi}(s) = \mathcal{L}[\varphi](s)$; then*

$$(A.12) \quad \mathcal{L}\left[\frac{d\varphi}{dt}\right](s) = s\bar{\varphi}(s) - \varphi(0), \quad \mathcal{L}\left[\frac{d^2\varphi}{dt^2}\right](s) = s^2\bar{\varphi}(s) - s\varphi(0) - \frac{d\varphi}{dt}(0).$$

(iii) (Relation with the convolution integral.) *Let $\Phi = \Phi(t)$ and $\Psi = \Psi(t)$ be two arbitrary functions each possessing a Laplace transform so that $\bar{\Phi}(s) = \mathcal{L}[\Phi](s)$ and $\bar{\Psi}(s) = \mathcal{L}[\Psi](s)$. Then, under certain conditions,*

$$(A.13) \quad \mathcal{L}\left[\int_0^t \Phi(t-\tau)\Psi(\tau)d\tau\right](s) = \bar{\Phi}(s)\bar{\Psi}(s),$$

whence

$$(A.14) \quad \int_0^t \Phi(t-\tau)\Psi(\tau)d\tau = \mathcal{L}^{-1}[\bar{\Phi}\bar{\Psi}](t).$$

(iv) (Transforms of unity and the Dirac delta-function.)

$$(A.15) \quad \mathcal{L}[1](s) = \frac{1}{s}, \quad s > 0, \quad \text{whence } \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) = 1,$$

$$(A.16) \quad \mathcal{L}[\delta](s) = 1, \quad s > 0, \quad \text{whence } \mathcal{L}^{-1}[1](t) = \delta(t).$$

In (A.16) both latter and former equalities are understood in the distributions sense.

Proof. Assertions (i)-(iv) are well-known in the theory of the Laplace transform, see, for example, [6, Ch. XVI, § 1], [17, Sec. 6.1]. □

APPENDIX B. NOMENCLATURE

In this appendix, we put the list of almost all used notations in the article.

Roman Symbols		
Notation	Description	Introduced in
$\mathcal{A}_0, \mathcal{A}_1$	effective two-scale tensors	(7.3), (7.5)
$\mathcal{A}_0^f, \mathcal{A}_1^f$	tensors derived from microstructure	(7.14), (7.18)
$\mathcal{B}_0, \mathcal{B}_1$	effective two-scale tensors	(7.4), (7.6)
$\mathcal{B}_0^f, \mathcal{B}_1^f$	tensors derived from microstructure	(7.16), (7.20)
\mathbb{D}_x	symmetric part of the gradient ∇_x	Section 1
$\mathbb{D}_{\hat{y}}, \mathbb{D}_{\hat{z}}$	symmetric parts of $\nabla_{\hat{y}}$ and $\nabla_{\hat{z}}$, resp.	(5.6)
\mathcal{E}	effective instantaneous elasticity tensor	(11.3)
$\mathcal{E}_{\text{corr}}^\theta$	instantaneous elasticity corrector term on $[\Delta, \Delta + \delta_*]$	(11.20)
$\mathcal{E}_{\text{corr}}^\sigma$	instantaneous elasticity corrector term on $(\Delta + \delta_*, \Delta + \delta^*)$	(11.19)
e_m	Cartesian basis vector in \mathbb{R}^3	Section 1

$\mathcal{F}_v, \mathcal{F}_v^0$	tensors derived from the meso- and microstructures	(11.7), (11.8)
$\mathcal{F}_{v\text{-corr}}^\theta$	tensor derived from the meso- and microstructures	(11.33)
$\mathcal{F}_{v\text{-corr}}^\sigma$	tensor derived from the meso- and microstructures	(11.32)
$\mathbb{F}_p, \mathbb{F}_p^0$	matrices derived from the meso- and microstructures	(11.9), (11.10)
\mathbb{F}^0	effective two-scale matrix	(7.7)
$\mathbb{F}_f^0, \mathbb{F}_{sol}^{0mn}$	matrices derived from microstructure	(7.22), (7.23)
$\mathbb{F}_{p\text{-corr}}^\theta$	tensor derived from the meso- and microstructures	(11.36)
$\mathbb{F}_{p\text{-corr}}^\sigma$	tensor derived from the meso- and microstructures	(11.35)
f	distributed mass force	(1.1a), (2.5a)
\mathcal{G}	elastic stiffness tensor	(1.1c), (2.5d)
\mathcal{G}^θ	effective elastic stiffness tensor in Ω_{δ_*}	(12.2i)
\mathcal{G}^σ	effective elastic stiffness tensor in Ω_{δ^*}	(12.2d)
\mathcal{H}^s	Laplace image of the principle two-scale stress tensor multiplied by s	(7.31)
\mathfrak{H}^s	Laplace image of the principle effective stress tensor multiplied by s	(11.44)
\mathfrak{H}_θ^s	restriction of \mathfrak{H}^s to $[\Delta, \Delta + \delta_*]$	(11.63)
\mathfrak{H}_σ^s	restriction of \mathfrak{H}^s to $(\Delta + \delta_*, \Delta + \delta^*]$	(11.62)
\mathbb{I}	unit 3×3 -matrix	Section 1
J_t, J_τ, J_κ	Volterra operator (primitive of function)	(2.6)
\mathbb{J}^{mn}	3×3 -matrix $(1/2)(e_m \otimes e_n + e_n \otimes e_m)$	Section 1
\mathcal{K}	effective relaxation tensor	(11.4)
\mathcal{K}^θ	restriction of \mathcal{K} to $[\delta_*, \Delta + \delta_*]$	(11.23)
\mathcal{K}^σ	restriction of \mathcal{K} to $(\Delta + \delta_*, \Delta + \delta^*]$	(11.22)
\mathcal{L}	Laplace transform	(A.8)
\mathcal{L}^{-1}	inverse Laplace transform	(A.10)
\mathcal{M}^t	homogenized three-scale uniform stress tensor	after (5.5)
$\mathcal{M}^{t\varepsilon}$	uniform stress tensor in microscopic description	(2.9)
\mathbb{M}^0	homogenized three-scale partial initial data for stress	(5.5)
$\mathbb{M}^{0\varepsilon}$	partial initial data for stress in microscopic description	(2.11)
n	unit normal to the fluid-structure interface	(1.1e), (2.5f)
n_Σ	unit outward normal to $\partial\Sigma_S$	(10.6)
\mathcal{P}	dimensionless viscous stress tensor	(2.4), (2.5b)
\mathcal{P}^θ	instantaneous viscous stress tensor in Ω_{δ_*}	(12.2h)
\mathcal{P}^σ	instantaneous viscous stress tensor in Ω_{δ^*}	(12.2c)
p, p_ε	pressure	Introduction
p^0	initial pressure distribution	(1.1f), (2.5g)
\mathcal{Q}	effective relaxation tensor	(11.5)

\mathcal{Q}_0^θ	tensor derived from the meso- and microstructures	(11.26)
\mathcal{Q}_1^θ	tensor derived from the meso- and microstructures	(11.27)
\mathcal{Q}^σ	restriction of \mathcal{Q} to $(\Delta + \delta_*, \Delta + \delta^*)$	(11.25)
\mathbb{Q}^{pq}	matrix derived from the meso- and microstructures	(11.28)
\mathcal{R}_1^θ	first viscoelastic relaxation tensor in Ω_{δ_*}	(12.2j)
\mathcal{R}_2^θ	second viscoelastic relaxation tensor in Ω_{δ_*}	(12.2k)
\mathcal{R}^σ	viscoelastic relaxation tensor in Ω_{δ^*}	(12.2e)
\mathbb{S}_0^θ	homogenized partial stress in Ω_{δ_*}	(12.2l)
\mathbb{S}_0^σ	homogenized partial stress in Ω_{δ^*}	(12.2f)
s	argument of the Laplace transform image	(7.35), (A.8)
tr	trace of a matrix	Notation 1
$\mathbf{u}, \mathbf{u}_\varepsilon$	macroscopic velocity vector	Introduction
\mathbf{u}^0	initial macroscopic velocity vector	(1.1f), (2.5g)
$\mathbf{u}^{(1)}$	mesoscopic velocity vector	Introduction
$\mathbf{u}^{(2)}$	microscopic velocity vector	Introduction
\mathbf{u}^*	boundary macroscopic velocity distribution	(2.14)
\mathcal{V}	effective instantaneous viscosity tensor	(11.2)
$\mathcal{V}_{\text{corr}}^\theta$	instantaneous viscosity corrector term on $[\Delta, \Delta + \delta_*]$	(11.17)
$\mathcal{V}_{\text{corr}}^\sigma$	instantaneous viscosity corrector term on $(\Delta + \delta_*, \Delta + \delta^*)$	(11.16)
$\mathbf{v}, \mathbf{v}_\varepsilon$	macroscopic displacement vector	Introduction
\mathbf{v}^0	initial macroscopic displacement vector	(1.1g), (2.5h)
\mathcal{W}	effective relaxation tensor	(11.6)
\mathcal{W}_1^θ	tensor derived from the meso- and microstructures	(11.30)
\mathbf{x}	macroscopic position vector	Introduction
$\widehat{\mathbf{x}}$	vector (x_1, x_2)	Section 3
$\mathbf{Y}_0^{ij}, \mathbf{Y}_1^{ij}, \mathbf{Y}_2, \mathbf{Y}_3^{ij}$	solutions of the mesoscopic cell problems	(10.1), Probl. Y1-Y7
$\widehat{\mathbf{y}}$	mesoscopic position vector	Section 3
$\mathbf{Z}_{00}^{ij}, \mathbf{Z}_{10}^{ij}, \mathbf{Z}_{20}, \mathbf{Z}_{30}^{ij}$	solutions of the microscopic cell problems	(6.2)-(6.5), Probl. Z1-Z7
$\widehat{\mathbf{z}}$	microscopic position vector	Section 3

Greek Symbols

Notation	Description	Introduced in
$\alpha_\tau, \alpha_F, \alpha_\lambda,$	positive dimensionless ratios	(2.3)
$\alpha_\mu, \alpha_\gamma$		
$\Gamma, \Gamma^\varepsilon$	fluid-structure interface	Section 1
Δ	thickness of elastic plate Ω_{p1}	Introduction
δ_{ij}	Kronecker's symbol	Section 1
δ_*	height of a shorter bristle	Introduction
δ^*	height of a taller bristle	Introduction

ε	small characteristic parameter of the periodic structure	Introduction
ζ	extension of $\widehat{\zeta}$	(3.3), (4.3)
$\widehat{\zeta}$	characteristic function of Σ_F	(3.1) ₁
Θ	pattern microscopic cell	Section 3
Θ_F	liquid part of Θ	Section 3
Θ_S	elastic part of Θ	Section 3
θ	mean value of $\widehat{\psi}$ on Θ , i.e., $\sigma\theta$ is the volume fracture of the fluid in Ω_{δ_*}	(5.17)
ρ	homogenized three-scale density	(5.8)
ρ_F	fluid density	Section 1
ρ_S	density of the elastic body	Section 1
ρ_θ	mean density in Ω_{δ_*}	(5.18)
ρ_σ	mean density in Ω_{δ^*}	(5.18)
ρ_*^ε	uniform notation for density in Ω	(2.10)
$\langle \rho \rangle_{\Sigma \times \Theta}$	mean value of ρ on $\Sigma \times \Theta$	(5.16)
Σ	pattern mesoscopic cell	Section 3
Σ_F	liquid part of Σ	Section 3
Σ_S	elastic part of Σ	Section 3
σ	mean value of $\widehat{\zeta}$ on Σ , i.e., volume fracture of the fluid in Ω_{δ^*}	(5.17)
χ	homogenized three-scale characteristic function of the fluid domain	(4.2)
χ^ε	characteristic function of the fluid domain in microscopic description	(2.8), (3.2)
ψ	extension of $\widehat{\psi}$	(3.4), (4.4)
$\widehat{\psi}$	characteristic function of Θ_F	(3.1) ₂
$\Omega = (0, 1)^3$	domain of dimensionless macroscopic positions	Introduction
$\Omega_F, \Omega_F^\varepsilon$	fluid domain	Section 1
$\Omega_S, \Omega_S^\varepsilon$	elastic body	Section 1
Ω_{pl}	elastic plate without bristles	Introduction
Ω_{δ_*}	spatial layer, where the shorter bristles locate	Introduction
Ω_{δ^*}	spatial layer, where the taller bristles locate	Introduction

Some operators and binary operations

Notation	Description	Introduced in
$\operatorname{div}_{\widehat{\mathbf{y}}}, \operatorname{div}_{\widehat{\mathbf{z}}}$	divergence operators	(5.7)
:	inner product (convolution)	Notation 1
\otimes	dyad	Notation 1
$\nabla_{\widehat{\mathbf{y}}}, \nabla_{\widehat{\mathbf{z}}}$	gradient operators	(4.5)
$\bar{\varphi}$	Laplace transform of a function φ : $\bar{\varphi} = \mathcal{L}[\varphi]$	(7.35)
$\langle \varphi \rangle_\Theta$	mean value of a function $\varphi = \varphi(\widehat{\mathbf{z}})$ on Θ : $\langle \varphi \rangle_\Theta = \int_\Theta \varphi(\widehat{\mathbf{z}}) d\widehat{\mathbf{z}}$	(5.17)

$\langle \varphi \rangle_{\Sigma}$ mean value of a function $\varphi = \varphi(\hat{\mathbf{y}})$ on Σ : (5.17)

$$\langle \varphi \rangle_{\Sigma} = \int_{\Theta} \varphi(\hat{\mathbf{y}}) d\hat{\mathbf{y}}$$

$\langle \varphi \rangle_{\Theta \times \Sigma}$ mean value of a function $\varphi = \varphi(\hat{\mathbf{y}}, \hat{\mathbf{z}})$ on $\Sigma \times \Theta$: (5.16)

$$\langle \varphi \rangle_{\Sigma \times \Theta} = \int_{\Sigma \times \Theta} \varphi(\hat{\mathbf{y}}, \hat{\mathbf{z}}) d\hat{\mathbf{y}} d\hat{\mathbf{z}}$$

REFERENCES

- [1] G. Allaire, M. Briane, *Multiscale convergence and reiterated homogenization*, Proc. R. Soc. Edinb., **126A**:2 (1996), 297–342. Zbl 0866.35017
- [2] B. Bhushan, Y.C. Jung, M. Nosonovsky, *Lotus effect: surfaces with roughness-induced superhydrophobicity, self-cleaning and low adhesion*, Springer Handbook of Nanotechnology, Part F: Biomimetics, (B. Bhushan, ed.) 2010, Springer, Berlin, Heidelberg, 1437–1524.
- [3] N. Botkin, M. Schlensog, M. Tewes, V. Turova, *A mathematical model of a biosensor*, 2001 International Conference on Modeling and Simulation of Microsystems, MSM 2001, March 19–21, Hilton Island, SC, USA, Cambridge, MA: Computational Publ., (2001), 231–234.
- [4] D. Cioranescu, P. Donato, *An Introduction to Homogenization*, Oxford University Press, Oxford, 1999. Zbl 0939.35001
- [5] H. Cohen, S.I. Rubinow, *Some mathematical topics in biology*, Proc. Symp. on System theory, Polytechnic Press, New York, 1965, 321–337.
- [6] R. Dautray, J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 5: Evolution problems I*, Springer, Berlin, 2000. Zbl 0956.35003
- [7] G.M. Fichtenholz, *Course of Differential and Integral Calculus, vol. 2*, Nauka, Moscow, 1969. (in Russian).
- [8] M.A. Genaev, A.V. Doroshkov, T.A. Pshenichnikova, N.I. Kolchanov, D.A. Afonnikov, *Extraction of quantitative characteristics describing wheat leaf pubescence with a novel image-processing technique*, Planta, **236** (2012), 1943–1954.
- [9] P. Germain, *Cours de mécanique des milieux continus, Tome I: Théorie générale*, Masson et Cie, Paris, 1973. Zbl 0254.73001
- [10] R.P. Gilbert, A. Mikelić, *Homogenizing the acoustic properties of the seabed: part I*, Nonlinear Anal., **40**:1–8 (2000), 185–212. Zbl 0958.35108
- [11] S. Goldstein (ed.), *Modern Developments in Fluid Dynamics. An Account of Theory and Experiment Relating to Boundary Layers, Turbulent Motion and Wakes. Vol. I*, Clarendon Press, Oxford, 1938. JFM 64.1451.01
- [12] K.-H. Hoffmann, N.D. Botkin, V.N. Starovoitov, *Homogenization of interfaces between rapidly oscillating fine elastic structures and fluids*, SIAM J. Appl. Math., **65**:3 (2005), 983–1005. Zbl 1078.74040
- [13] L. Hörmander, *The analysis of linear partial differential operators I*, Grundle. Math. Wissenschaft., **256**, Springer, Berlin, Heidelberg, 1983. Zbl 0521.35001
- [14] U. Hornung (ed.), *Homogenization and porous media*, Springer-Verlag, New York, 1997. Zbl 0872.35002
- [15] W.-X. Huang, S. Alben, *Fluid-structure interactions with applications to biology*, Acta Mech. Sin., **32**:6 (2016), 977–979. Zbl 1361.74002
- [16] K. Koch, B. Bhushan, W. Barthlott, *Multifunctional plant surfaces and smart materials*, Springer Handbook of Nanotechnology, Part F: Biomimetics, (B. Bhushan, ed.) 2010, Springer, Berlin, Heidelberg, 1399–1436.
- [17] E. Kreyszig, *Advanced Engineering Mathematics*, 10th edition, John Wiley & Sons Inc., Hoboken, 2011. Zbl 1229.00004
- [18] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, 1985. Zbl 0588.35003
- [19] O.A. Ladyzhenskaja, V.A. Solonnikov, N.N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, AMS, Providence, 1968. Zbl 0174.15403
- [20] J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaire*, Dunod Gauthier-Villars, Paris, 1969. Zbl 0189.40603

- [21] D. Lukkassen, G. Nguetseng, P. Wall, *Two-scale convergence*, Int. J. Pure Appl. Math. **2**:1 (2002), 35–86. Zbl 1061.35015
- [22] O.A. Oleinik, A.S. Shamaev, G.A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992. Zbl 0768.73003
- [23] E. Sanchez-Palencia, *Non-homogeneous media and vibration theory*, Lecture Notes in Physics, **127**, Springer, Berlin, 1980. Zbl 0432.70002
- [24] S.A. Sazhenkov, *Effective thermoviscoelasticity of a saturated porous ground*, Bulletin (Vestnik) of Novosibirsk State University. Mathematics, Mechanics and Informatics Series. **8**:2 (2008), 106–130.
- [25] M.D.J. Schreuder, C.A. Brewer, C. Heine, *Modelled influences of non-exchanging trichomes on leaf boundary layers and gas exchange*, J. Theor. Biol., **210** (2001), 23–32.
- [26] E.M. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, **32**, Princeton University Press, Princeton, 1971. Zbl 0232.42007
- [27] T. Tao, *An Introduction to Measure Theory*, Graduate Studies in Mathematics, **126**, AMS, Providence, 2011. Zbl 1231.28001
- [28] R. Temam, A. Miranville, *Mathematical Modeling in Continuum Mechanics*, Cambridge University Press, Cambridge, 2005. Zbl 1077.76001

SERGEY A. SAZHENKOV
LAVRENTYEV INSTITUTE OF HYDRODYNAMICS,
SIBERIAN DIVISION OF THE RUSSIAN ACADEMY OF SCIENCES,
15, ACAD. LAVRENTYEV AVE.,
NOVOSIBIRSK, 630090, RUSSIA
LABORATORY FOR MATHEMATICAL AND COMPUTER MODELING
IN NATURAL AND INDUSTRIAL SYSTEMS,
FACULTY OF MATHEMATICS & INFORMATION TECHNOLOGIES,
ALTAI STATE UNIVERSITY,
61, LENINA AVE.,
BARNAUL, 656049, RUSSIA
Email address: sazhenkovs@yandex.ru

ELENA V. SAZHENKOVA
CHAIR OF MATHEMATICS AND NATURAL SCIENCES,
NOVOSIBIRSK STATE UNIVERSITY OF ECONOMICS AND MANAGEMENT,
56, KAMENSKAYA STR.,
NOVOSIBIRSK, 630099, RUSSIA
Email address: elsazh1977@gmail.com