Explicit non-algebraic limit cycle for two classes of continuous piecewise differential systems

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Abstract

In this article we prove that the planar continuous piecewise differential systems separated by a straight line and formed by a quadratic isochronous center and an arbitrary linear differential system have at most one limit cycle. Moreover, this limit cycle if it exists is non-algebraic and analytically given.
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1 Introduction and statement of the main result

One of the most challenging open problems in the qualitative theory of real planar differential systems is the study of limit cycles. The existence, non-existence, uniqueness and other properties of limit cycles have been

studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc. (see for instance the books [1, **?**, 18]. This problem restricted to planar polynomial differential systems is the second part of the well known Hilbert's 16th problem, for more details we refer to [7]. In the last few years there has been an increasing interest in the study of piecewise differential systems, see [2, 5, 8, 9, 10, 12] for instance. This interest has been mainly motivated by their wider range of application in modeling real phenomena (e.g., control theory, biology, chemistry, engineering, physics, etc.).

For the planar continuous piecewise linear differential systems with two zones separated by a straight line, Lum and Chua [14, 15] in 1991 conjectured that such differential systems have at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [6].

For piecewise smooth quadratic systems, the authors in [3] showed that there are piecewise quadratic system with 9 small amplitude limit cycles. In [11], the authors showed that there are at least 5 limit cycles can bifurcate from quadratic isochronous centers under piecewise smooth quadratic perturbations. Recently, the author in [17] showed that the piecewise smooth quadratic isochronous systems can have at least 6 limit cycles. On the other hand, it seems intuitively clear that "most" limit cycles of continuous and discontinuous piecewise differential systems have to be non-algebraic. Nevertheless, In all these papers dedicated to study the limit cycles of piecewise differential systems, do not appear explicit non-algebraic limit cycles, they proved their existence using different methods as the first integrals, the averaging theory, the Poincaré map, the Newton-Kantorovich Theorem, the Melnikov function.

The goal of this paper is to study the limit cycles of two classes planar continuous piecewise differential systems separated by a straight line $x = 0$, and formed by an arbitrary linear differential system and a quadratic polynomial differential systems with an isochronous center at the origin. A center of a real planar polynomial differential system is called an isochronous center if there exists a neighborhood of which such that all periodic orbits in this neighborhood have the same period. We remark that the classification of all quadratic polynomial differential systems with an isochronous center at the origin can be found in [13]. Using the notation of [16], a quadratic isochronous centers has one of the following four forms:

S1) $\dot{x} = -y + x^2, \qquad \dot{y} = x + yx$, with the first integral

$$H_1(x,y) = \frac{x^2+y^2}{(1+y)^2}. \tag{1}$$

S2) $\dot{x} = -y + \frac{1}{4}x^2, \qquad \dot{y} = x + xy$, with the first integral

$$H_2(x,y) = \frac{\left(x^2+4y+8\right)^2}{1+y}. \tag{2}$$

S3) $\dot{x} = -y + \frac{1}{2}x^2 - \frac{1}{2}y^2, \qquad \dot{y} = x + yx$, with the first integral

$$H_3(x,y) = \frac{x^2+y^2}{1+y}. \tag{3}$$

S4) $\dot{x} = -y + 2x^2 - \frac{1}{2}y^2, \qquad \dot{y} = x + xy$, with the first integrl

$$H_4(x,y) = \frac{4x^2 - 2(y+1)^2 + 1}{(1+y)^2}. \tag{4}$$

We consider planar continuous piecewise differential systems with two linearity regions separated by a straight line $\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$, where we will assume that the two linearity regions in the phase plane are the left and right half planes

$$\Sigma_- = \{(x,y) \in \mathbb{R}^2 : x \leq 0\}, \;\; \Sigma_+ = \{(x,y) \in \mathbb{R}^2 : x \geq 0\}.$$

and formed by an arbitrary linear differential system and a quadratic isochronous center. We assume without loss of generality that in the half–plane Σ_+ we have an arbitrary linear differential system

$$\dot{x} = ax + by + c, \qquad \dot{y} = \alpha x + ly + d \tag{5}$$

and in the half–plane Σ_- we have one of the two quadratic isochronous differential systems (S1) or (S2).

In order that both differential systems (5) and (S1) (resp. (5) and (S2)) define in the whole plane $\mathbb{R}^2$ a continuous differential system we must take

$$b = -1, \quad c = l = d = 0.$$

Then systems (5) become

$$\dot{x} = ax - y, \qquad \dot{y} = \alpha x. \tag{6}$$

It is easy to see that we cannot form continuous piecewise differential systems by using two differential systems (5) and (S3) (resp. (5) and (S4)).

More precisely, we consider the two continuous piecewise differential systems (6)+(S1) and (6)+(S2). We prove that these two systems they can have at most one limit cycle. Moreover this limit cycle if it exists is not algebraic and is explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

In what follows we state our main results.

Theorem 1 *The two continuous piecewise differential systems* (6)+(*S1*) *and* (6)+(*S2*)*, they can have at most one limit cycle. Moreover, this limit cycle if it exists is non-algebraic.*

Theorem 1 is proved in section 3.

The next Propositions shows that there are continuous piecewise differential systems of the form (6)+(S1) with non-algebraic limit cycle.

Proposition 2 *The continuous piecewise differential system defined by*

$$\begin{aligned} &\dot{x} = -y + x^2, \qquad \dot{y} = x + xy, \qquad \mathit{in}\, \Sigma_-, \\ &\dot{x} = 2\lambda x - y, \qquad \dot{y} = 2\lambda^2 x, \qquad \mathit{in}\, \Sigma_+, \end{aligned} \tag{7}$$

where $\lambda > 0$*, has exactly one explicit non-algebraic limit cycle given by*

$$\begin{aligned} \Gamma \;=\; & \left\{(x,y) \in \Sigma_+ : \left(2\lambda^2 x^2 - 2\lambda xy + y^2\right) e^{-2\arctan\frac{\lambda x}{\lambda x - y}} = 0.228\,86\right\} \\ & \cup \left\{(x,y) \in \Sigma_- : \frac{x^2+y^2}{(1+y)^2} = 0.841\,15\right\}. \end{aligned}$$

The next Propositions shows that there are continuous piecewise differential systems of the form (6)+(S2) with non-algebraic limit cycle.

Proposition 3 *The continuous piecewise differential system defined by*

$$\begin{array}{llll} \dot{x} = -y + \frac{1}{4}x^2, & \dot{y} = x + xy, & in\,\Sigma_-, \\ \dot{x} = 2\lambda x - y, & \dot{y} = 5\lambda^2 x, & in\,\Sigma_+, \end{array} \tag{8}$$

where $\lambda > 0$, *has exactly one explicit non-algebraic limit cycle given by*

$$\begin{aligned} \Gamma \;=\; & \left\{(x,y) \in \Sigma_+ : \left(5\lambda^2 x^2 - 2\lambda xy + y^2\right) e^{-\arctan \frac{2\lambda x}{\lambda x - y}} = \frac{\left(e^{\frac{\pi}{2}} - 1\right)^2}{e^{\pi}}\right\} \\ & \cup \left\{(x,y) \in \Sigma_- : \frac{\left(x^2 + 4y + 8\right)^2}{1 + y} = \frac{16\left(e^{-\frac{\pi}{2}} + 1\right)^2}{e^{-\frac{\pi}{2}}}\right\}. \end{aligned}$$

Theses Propositions will be proved in section 4.

Remark 4 *The assumption* $\lambda > 0$ *in Propositions 2 and 3 is a necessary condition for the existence of an orbit arc of right side systems in the region* Σ_+.

2 Preliminaries

The following normal forms for the linear differential systems in $\mathbb{R}^2$ and its first integrals will help us to prove our main result.

Lemma 5 *The linear system* (6), *can be one of the following linear differential systems:*
i) either a saddle (resp. a diagonal node) of the form

$$\dot{x} = 2rx - y, \quad \dot{y} = -\left(\rho^2 - r^2\right) x \tag{9}$$

with $0 < r^2 < \rho^2$ *(resp.* $\beta > 0$, *and* $r^2 > \rho^2 > 0.$*).*
ii) or a non-diagonal node of the form

$$\dot{x} = ax - y, \quad \dot{y} = \frac{1}{4}a^2 x \tag{10}$$

iii) or a focus (resp. a center) of the form

$$\dot{x} = 2\lambda x - y, \quad \dot{y} = \left(\lambda^2 + \omega^2\right) x, \tag{11}$$

with $\omega > 0$ *and* $\lambda \neq 0$ *(resp.* $\omega > 0$ *and* $\lambda = 0$*).*

Proof. Consider a general linear differential system (6). The eigenvalues of this system are

$$\lambda_{1,2} = \frac{1}{2}\left(a \pm \sqrt{(a)^2 - 4\alpha}\right).$$

i) The linear differential system (6) has a saddle (resp. a diagonal node) if $\frac{1}{2}a = r$ and $a^2 - 4\alpha = 4\rho^2$ for some $\alpha < 0, r^2 < \rho^2$ (resp. for some $\alpha < 0$, $r^2 > \rho^2$), then

$$a = 2r \quad \alpha = r^2 - \rho^2.$$

Therefore we obtain system (9).

ii) The linear differential system (6) has a non-diagonal node if $a^2 - 4\alpha = 0$ for some $a > 0$. Substituting $\alpha = \frac{1}{4}a^2$ into (6), we obtain system (10).

iii) The linear differential system (6) has a focus (resp. a center) if $\frac{1}{2}a = \lambda$, and $a^2 - 4\alpha = -4\omega^2$ (resp. $a = 0$ and $\alpha = \omega^2$), for some $\omega > 0, \alpha > 0$ and $\lambda \in \mathbb{R}$, then

$$a = 2\lambda, \quad \alpha = \lambda^2 + \omega^2.$$

Therefore, we obtain system (11) (resp. $a = 0$ and $\alpha = \omega^2$). ■

Remark 6 *Since the two continuous piecewise differential systems* (6)+(*S1*) *and* (6)+(*S2*), *have the unique equilibrium* $(0;0)$ *located on the separation line* Σ. *Then if the linear system* (6) *is a saddle, or a node with distinct eigenvalues, or a non-diagonal node, no orbit of this system can touch* Σ *twice (at two points* $(0, y_0), y_0 < 0$ *and* $(0, y_1), y_1 > 0$ *) in the half plane* Σ_+, *i.e., there is not any crossing periodic orbit in systems* (6)+(*S1*) *and* (6)+(*S2*). *Therefore, in order that they can have some limit cycle such equilibrium point must be either a focus or center for the right linear differential system* (6).

Lemma 7 *The first integral of* (11) *is given by*

$$H_3(x,y) = \begin{cases} \left(\left(\lambda^2+\omega^2\right)x^2 - 2\lambda xy + y^2\right)e^{-\frac{2\lambda}{\omega}\arctan\frac{\omega x}{\lambda x - y}} & if\ \lambda \neq 0, \\ y^2 + \omega^2 x^2 & if\ \lambda = 0. \end{cases} \tag{12}$$

Proof. Since the unique equilibrium of system (11) is located at the origin $O(0;0)$ and is of focus type, than any orbit of system (11) crosses the straight line $x = 0$ at least in one point, namely $(0, C), C \in \mathbb{R}$, thus the general solution of (11) is given by

$$\begin{aligned} x(t) &= \frac{-1}{\omega} C e^{t\lambda} \sin t\omega, \\ y(t) &= \frac{1}{\omega} C e^{t\lambda} \left(\omega \cos t\omega - \lambda \sin t\omega\right), \end{aligned} \tag{13}$$

where $C \in \mathbb{R}$. So, from the first equation of (13), we obtain that

$$e^{t\lambda} \sin \omega t = -\frac{\omega}{C} x,$$

Substituting this last expression into the second equation, yields

$$e^{t\lambda}\cos\omega t=\frac{1}{C\beta}\left(\left(\lambda\right)x-y\right).$$

Therefore

$$\tan\omega t=\frac{\omega x}{\left(\lambda-\delta\right)x-y}.$$

From this last equation, we obtain

$$t=\frac{1}{\omega}\arctan\frac{\omega x}{\lambda x-y}.$$

Substituting the previous expressions in the first equation of (13), and simplifying we obtain

$$\left(\left(\lambda^2+\omega^2\right)x^2-2\lambda xy+y^2\right)e^{-\frac{2\lambda}{\omega}\arctan\frac{\omega x}{\lambda x-y}}=h,$$

where $h=\left(\beta C\right)^2\in\mathbb{R}$. ■

3 Proof of Theorem 1

Suppose that we have a continuous piecewise differential system (6)+(S1) (resp. (6)+(S2)). In order to investigate the limit cycles of these systems, we shall use the first integrals for the left side and the right side systems of (6)+(S1) (resp. (6)+(S2)).

According to Remark 5, we assume that the linear differential system (6) is of a focus (resp. a center) type. By Lemma 4, we can write such a linear differential system as (11). By Lemma 6, this linear differential systems have the first integral H_3 described in (12).

Suppose that this continuous piecewise differential system has some limit cycles intersecting Σ in two points, namely, $(0,y_0)$ with $y_0<0$, and $(0,y_1)$ with $y_1>0$. Then, the first integrals H_i, and H_3 must satisfy the following two equations:

$$\begin{aligned}H_i\left(0,y_0\right)-H_i\left(0,y_1\right)&=0,\\ H_3\left(0,y_0\right)-H_3\left(0,y_1\right)&=0,\end{aligned}\tag{14}$$

where $i=1$ for (11)+(S1) (resp. $i=2$ for (11)+(S2)).

The implicit form of the orbit arc of (11) in Σ_+ which starting at the point $(0,y_0); y_0<0$ when $t=0$ is given by $H_3\left(x,y\right)-y_0^2=0$, this last orbit can be given also by the analytic curves that $(x_+(t),y_+(t))$, where

$$\begin{aligned}x_+\left(t\right)&=-\frac{1}{\omega}y_0e^{t\lambda}\sin t\omega,\\ y_+\left(t\right)&=\frac{1}{\omega}y_0e^{t\lambda}\left(\omega\cos t\omega-\lambda\sin t\omega\right).\end{aligned}$$

Denote by t_+ the minimum positive time such that $x(t_+)=x(0)=0$, then $t_+=\frac{\pi}{\omega}$. Since that the orbits starting at the point $(0,y_0)$ go into the left zone

Σ_- under the flow of the left differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ with $y_1 > 0$ after the time $t_+ = \frac{\pi}{\omega}$, then

$$y_1 = y(t_+) = -y_0 e^{\frac{\lambda\pi}{\omega}},$$

this prove that $H_3(0, y_0) - H_3(0, y_1) = 0$.

Now, it is easy to see that the existence of periodic solutions of continuous piecewise differential system (11)+(S1) (resp. (11)+(S2)) is equivalent to the existence of negative values of y_0 satisfying

$$H_i(0, y_0) - H_i\left(0, -y_0 e^{\frac{\lambda\pi}{\omega}}\right) = 0. \tag{15}$$

with $i = 1$ for (11)+(S1) (resp. $i = 2$ for (11)+(S2)).

a) - For system (11)+(S1), the equation (15) become

$$-\frac{y_0^2\left(e^{\frac{\lambda\pi}{\omega}} + 1\right)}{\left(y_0 e^{\frac{\lambda\pi}{\omega}} - 1\right)^2 (y_0 + 1)^2}\left(e^{\frac{\lambda\pi}{\omega}} + 2y_0 e^{\frac{\lambda\pi}{\omega}} - 1\right) = 0. \tag{16}$$

it is easy to check that when $\lambda = 0$ (i.e. (6) is of a center type), the unique solution of (16) is $y_0 = 0$. So, in this case, the continuous piecewise differential system (11)+(S1) has no limit cycles.
When $\lambda \neq 0$ (i.e. (6) is of a focus type), the equation (16) has two roots; $y_{01} = 0$, which, cannot contribute a limit cycle and $y_0 = \frac{1}{2}\left(e^{-\frac{\lambda\pi}{\omega}} - 1\right) \neq 0$. Moreover, we can choose the appropriate parameters λ and ω in such a way that (16) has exactly one real negative roots $y_0 = \frac{1}{2}\left(e^{-\frac{\lambda\pi}{\omega}} - 1\right)$. Obtaining in this way at most one limit cycle for the continuous piecewise differential system (11)+(S1). Using the first integrals of both differential systems and knowing that the non-algebraic periodic orbit passe through the point $(0, y_0)$ when $t = 0$ and through the point $\left(0, -y_0 e^{\frac{\lambda\pi}{\omega}}\right)$ when $t = \frac{\pi}{\omega}$ where $y_0 = \frac{1}{2}\left(e^{-\frac{\lambda\pi}{\omega}} - 1\right) < 0$. Thus this expression is :

$$\Gamma = \left\{ \begin{array}{c} (x, y) \in \Sigma_+ : \left(\left(\lambda^2 + \omega^2\right) x^2 - 2\lambda xy + y^2\right) e^{-\frac{2\lambda}{\omega} \arctan \frac{\omega x}{\lambda x - y}} \\ = \frac{1}{4}\left(e^{-\frac{\lambda\pi}{\omega}} - 1\right)^2 \end{array} \right\}$$
$$\cup \left\{ (x, y) \in \Sigma_- : \frac{x^2 + y^2}{(1 + y)^2} = \frac{\left(e^{\frac{\lambda\pi}{\omega}} - 1\right)^2}{\left(e^{\frac{\lambda\pi}{\omega}} + 1\right)^2} \right\}.$$

b) Similar to the previous case, for system (11)+(S2), the equation (15) become

$$\frac{16y_0^2\left(e^{\frac{\lambda\pi}{\omega}} + 1\right)}{\left(y_0 e^{\frac{\lambda\pi}{\omega}} - 1\right)(y_0 + 1)}\left(e^{\frac{\lambda\pi}{\omega}} + y_0 e^{\frac{\lambda\pi}{\omega}} - 1\right) = 0. \tag{17}$$

Assume that $\lambda = 0$, i.e. the equilibrium point of system (6) is a center. Then the unique solution of (17) is $y_0 = 0$. So, in this case, the continuous piecewise differential system (11)+(S2) has no limit cycles.
in the case when the equilibrium point of system (6) is a focus, i.e. $\lambda \neq 0$, the unique solution $y_0 \neq 0$ of (17) is $y_0 = e^{-\frac{\lambda\pi}{\omega}} - 1$. Obtaining in this way at most one limit cycle for the continuous piecewise differential system (11)+(S2). Using the first integrals of both differential systems and knowing that the non-algebraic periodic orbit passes through the point $(0, y_0)$ when $t = 0$ and through the point $\left(0, -y_0 e^{\frac{\lambda\pi}{\omega}}\right)$ when $t = \frac{\pi}{\omega}$ where $y_0 = e^{-\frac{\lambda\pi}{\omega}} - 1 < 0$. Thus this expression is :

$$\begin{aligned} \Gamma &= \left\{ \begin{array}{c} (x, y) \in \Sigma_+ : \left(\left(\lambda^2 + \omega^2\right) x^2 - 2\lambda xy + y^2\right) e^{-\frac{2\lambda}{\omega} \arctan \frac{\omega x}{\lambda x - y}} \\ = \left(e^{\frac{-\lambda\pi}{\omega}} - 1\right)^2 \end{array} \right\} \\ &\cup \left\{ (x, y) \in \Sigma_- : \frac{\left(x^2 + 4y + 8\right)^2}{1 + y} = 16 e^{\frac{\lambda\pi}{\omega}} \left(e^{-\frac{\lambda\pi}{\omega}} + 1\right)^2 \right\}. \end{aligned}$$

This completes the proof of Theorem 1.

4 Proof of propositions 2 and 3

4.1 Proof of proposition 2

Suppose that we have a continuous piecewise differential system (7), we shall prove that this continuous piecewise differential system has exactly one non-algebraic limit cycle. Since $(1 \pm i)\lambda, \lambda > 0$ are the eigenvalues of the matrices of the right half system of (7), then this system have their equilibria as focus type at the origin.
The two differential systems of (7) have the following first integrals

$$\begin{aligned} H_1(x, y) &= \frac{x^2 + y^2}{(1 + y)^2}, \\ H_3(x, y) &= \left(2\lambda^2 x^2 - 2\lambda xy + y^2\right) e^{-2 \arctan \frac{\lambda x}{\lambda x - y}}, \end{aligned}$$

in Σ_- and Σ_+ respectively. The solution of right half system of (7) starting at the point $(0, y_0); y_0 < 0$ when $t = 0$ is

$$\begin{aligned} x_+(t) &= -\frac{1}{\lambda} y_0 e^{t\lambda} \sin \lambda t, \\ y_+(t) &= y_0 e^{t\lambda} (\cos \lambda t - \sin \lambda t). \end{aligned}$$

Let by t_+ the minimum positive time such that $x(t_+) = x(0) = 0$, then $t_+ = \frac{\pi}{\lambda}$. Since that the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{\lambda}$, then

$$y_1 = y\left(t_+\right) = -y_0 e^{\pi}.$$

Then, for the continuous piecewise differential system (7), the equation (15) becomes

$$-\frac{y_0^2}{\left(y_0+1\right)^2\left(y_0 e^{\pi}-1\right)^2}\left(e^{\pi}+1\right)\left(e^{\pi}+2y_0 e^{\pi}-1\right)=0.$$

The unique solution $y_0 \neq 0$ of this last equation is

$$y_0 = \frac{1}{2}\left(e^{-\pi}-1\right).$$

From this value of y_0 we get the value of $y_1 = \frac{1}{2}\left(e^{\pi}-1\right)$.
Straightforward computations show that the solution passing through the crossing points $(0, y_0)$ and $(0, y_1)$ correspond to

$$\begin{aligned}\Gamma &= \left\{(x,y)\in\Sigma_+ : \left(2\lambda^2x^2-2\lambda xy+y^2\right)e^{-2\arctan\frac{\lambda x}{\lambda x-y}} = 0.228\,86\right\}\\ &\cup\left\{(x,y)\in\Sigma_- : \frac{x^2+y^2}{\left(1+y\right)^2} = 0.841\,15\right\}.\end{aligned}$$

Moreover, Γ is non-algebraic limit cycle and is traveled in counterclockwise sense, around the origin.

Example 8 *Consider a system* (7) *with* $\lambda = 1$

$$\begin{aligned}&\dot{x} = 2x-y, && \dot{y} = 2x, && in\,\Sigma_+,\\ &\dot{x} = -y+x^2, && \dot{y} = x+yx, && in\,\Sigma_-.\end{aligned} \tag{18}$$

then, this system has exactly one explicit non-algebraic limit cycle Γ. *This limit cycle intersects the switching line* Σ *at the two points*

$$y_0 = -0.478\,39, \qquad y_1 = 11.\,07$$

and is given by

$$\begin{aligned}\Gamma &= \left\{(x,y)\in\Sigma_+ : \left(2x^2-2xy+y^2\right)e^{-2\arctan\frac{x}{x-y}} = 0.228\,86\right\}\\ &\cup\left\{(x,y)\in\Sigma_- : \frac{x^2+y^2}{\left(1+y\right)^2} = 0.841\,15\right\}.\end{aligned}$$

see Figure 1.

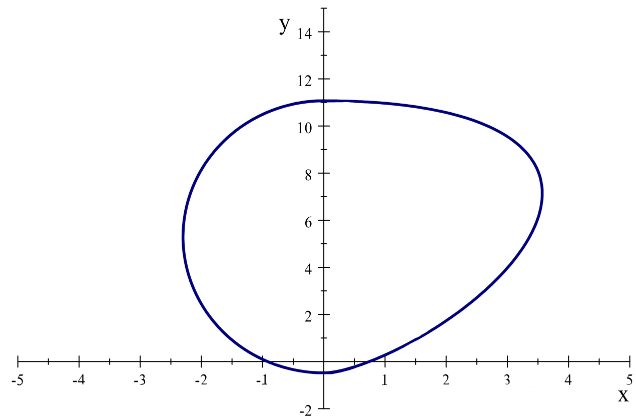

Figure 1: The unique non-algebraic limit cycle of system (18)

4.2 Proof of proposition 3

We consider planar piecewise differential system (8), for this system it is easy to check that the right linear differential system is a focus with eigenvalues $(1 \pm 2i)\lambda; \lambda > 0$. In order to prove that the discontinuous piecewise differential system (8) has exactly one non-algebraic limit cycle, we shall use the first integrals for the right and the left side systems of (8).

The two differential systems of (8) have the following first integrals

$$\begin{aligned} H(x,y) &= \frac{\left(x^2+4y+8\right)^2}{1+y}, \\ H_1(x,y) &= \left(5\lambda^2x^2 - 2\lambda xy + y^2\right) e^{-\arctan \frac{2\lambda x}{\lambda x - y}}, \end{aligned}$$

in Σ_- and Σ_+ respectively. The solution of right half system of (8) starting at the point $(0, y_0); y_0 < 0$ when $t = 0$ is

$$\begin{aligned} x_+(t) &= -\frac{1}{2\lambda} y_0 e^{t\lambda} \sin 2\lambda t, \\ y_+(t) &= \frac{1}{2\lambda} y_0 e^{t\lambda} \left(2\lambda \cos 2\lambda t - \lambda \sin 2\lambda t\right). \end{aligned}$$

The time t_+ that the solution $(x_+(t), y_+(t))$ contained in Σ_+ needs to reach the point $(0, y_1)$ is $t_+ = \frac{\pi}{2\lambda}$. Since that the orbits starting at the point $(0, y_0)$ go into the left zone Σ_- under the flow of the left differential systems and since these orbits can reach Σ again at some point $(0, y_1)$ after the time $t_+ = \frac{\pi}{2\lambda}$, then

$$y_1 = y(t_+) = -y_0 e^{\frac{\pi}{2}}.$$

Then, for the continuous piecewise differential system (8), the equation (15) becomes

$$\frac{16 y_0^2 \left(e^{\frac{\pi}{2}} + 1\right)}{\left(y_0 e^{\frac{\pi}{2}} - 1\right)(y_0 + 1)} \left(e^{\frac{\pi}{2}} + y_0 e^{\frac{\pi}{2}} - 1\right) = 0.$$

The unique solution $y_0 \neq 0$ of this last equation is

$$y_0 = e^{\frac{-1}{2}\pi} - 1.$$

From this value of y_0 we get the value of $y_1 = \left(e^{\frac{1}{2}\pi} - 1\right)$.
Therefore, the solution passing through the crossing points $(0, y_0)$ and $(0, y_1)$ it writes as

$$\begin{aligned} \Gamma \quad = \quad & \left\{(x,y) \in \Sigma_+ : \left(5\lambda^2 x^2 - 2\lambda xy + y^2\right) e^{-\arctan \frac{2\lambda x}{\lambda x - y}} = \left(e^{\frac{-1}{2}\pi} - 1\right)^2\right\} \\ & \cup \left\{(x,y) \in \Sigma_- : \frac{\left(x^2 + 4y + 8\right)^2}{1+y} = \frac{16\left(e^{-\frac{\pi}{2}} + 1\right)^2}{e^{-\frac{\pi}{2}}}\right\}. \end{aligned}$$

Moreover, Γ is non-algebraic limit cycle and is traveled in counterclockwise sense, around the origin.

Example 9 *When $\lambda = 1$, system (8) reads*

$$\begin{array}{lll} \dot{x} = -y + \frac{1}{4}x^2, & \dot{y} = x + xy, & in\,\Sigma_-, \\ \dot{x} = 2x - y, & \dot{y} = 5x, & in\,\Sigma_+, \end{array} \tag{19}$$

then, this system has exactly one explicit non-algebraic limit cycle Γ. This limit cycle intersects the switching line Σ at the two points

$$y_0 = -0.792\,12, \qquad y_1 = 3.\,810\,5,$$

and is given by

$$\begin{aligned} \Gamma \quad = \quad & \left\{(x,y) \in \Sigma_+ : \left(5x^2 - 2xy + y^2\right) e^{-\arctan \frac{2x}{x - y}} = \left(e^{\frac{-1}{2}\pi} - 1\right)^2\right\} \\ & \cup \left\{(x,y) \in \Sigma_- : \frac{\left(x^2 + 4y + 8\right)^2}{1+y} = \frac{16\left(e^{-\frac{\pi}{2}} + 1\right)^2}{e^{-\frac{\pi}{2}}}\right\}. \end{aligned}$$

see Figure 2.

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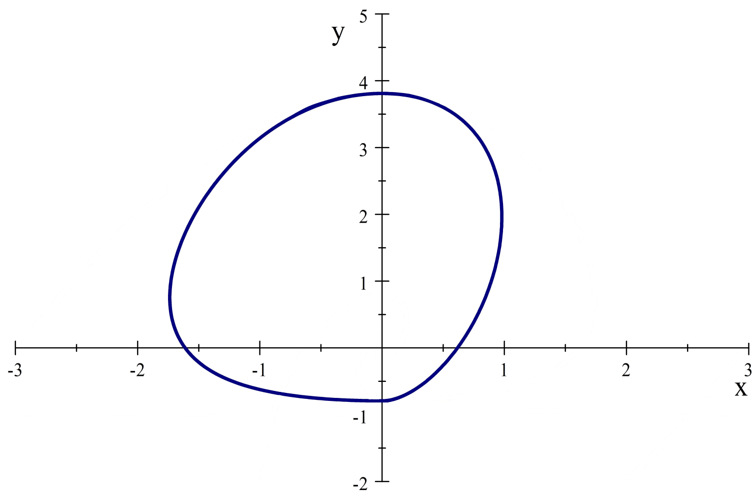

Figure 2: The unique non-algebraic limit cycle of system (19)

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