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INVERSE PROBLEM FOR A SECOND-ORDER HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS FOR LOWER DERIVATIVES

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Abstract. The problem of determining the memory of a medium from a second-order equation of hyperbolic type with a constant principal part and variable coefficients for lower derivatives is considered. The method is based on the reduction of the problem to a non-linear system of Volterra equations of the second kind and uses the fundamental solution constructed by S. L. Sobolev for hyperbolic equation with variable coefficients. The theorem of global uniqueness, stability and the local theorem of existence are proved.

Keywords: inverse problem, hyperbolic integro-differential equation, Volterra integral equation, stability, delta function, kernel.

1. Introduction. Setting up problem

Let us consider the equation

$$u_{tt} - qu_t - L_0 u = \int_0^t k(\tau) L_0 u(x, t-\tau)\, d\tau + \delta(x,t) \tag{1.1}$$

with initial condition

$$u|_{t<0} = 0. \tag{1.2}$$

Here $t \in \mathbb{R}^1,\ x \in \mathbb{R}^3$, $\delta(x,t)$ is delta function, L_0 is the differential operator:

$$L_0 u = \triangle u + \sum_{i=1}^{3} b_i(x) u_{x_i} + c(x)u, \tag{1.3}$$

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$$\triangle u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right);$$

$b_i(x)$ $(i = 1,2,3)$, $c(x)$ are known smooth functions, q is a some constant. We assume that the kernel of the integral term satisfies the condition $k(0) = 0$.

We pose the following **Inverse problem:** find the function $k(t)$, $t > 0$ occurring in the integral Eq. (1.1) from the information on the solution to problem (1.1) and (1.2) $u(x,t)$ at an arbitrary time $t > 0$ at the point $x = 0$:

$$u(0,t) = f(t), \ t > 0. \tag{1.4}$$

The theory and application of hyperbolic integro-differential equations play an important role in the mathematical modelling of many fields: physical, biological phenomens and engineering sciences in which it is necessary to take into consideration the effect of real problems. In many cases PDEs of the electrodynamics and elasticity with integral convolution terms are reduced to one integro-differential equation the main part of which is second-order hyperbolic operator. For the last thirty years, there has been much work related to problems of identification of memory kernel in these equations. Here we mention some of them [1]-[12] that are close to this work and the references therein for more details.

In [3]-[5] the local in time existence and the uniqueness results for of some multidimensional inverse problems for the second-order hyperbolic integro–differential equations in the class of functions having certain smoothness in the time variable and analyticity with respect to the spatial variables were obtained. Problems of determining the spatial part of the multidimensional kernel were investigated in the works [6]-[8]. The works [9]-[12] discusses the issues of global solvability of one-dimesional memory problems.

We introduce a new function $\tilde{v}(z,t)$ by formula

$$\tilde{v}(x,t) = \left(u(x,t) + \int_0^t k(t-\tau)u(x,\tau)d\tau\right)\exp(-qt/2).$$

Then, as it is easy to see by direct calculation, the function $u(x,t)$ is expressed by the formula

$$u(x,t) = \tilde{v}(x,t)\exp(qt/2) + \int_0^t r(t-\tau)\exp(q\tau/2)\tilde{v}(x,\tau)d\tau, \tag{1.5}$$

where $r(t)$ is solution of integral equation:

$$r(t) = -k(t) - \int_0^t k(t-\tau)r(\tau)d\tau. \tag{1.6}$$

In terms of the new functions $\tilde{v}(x,t)$ and $r(t)$, Eqs. (1.1) and (1.2) take the form

$$\tilde{v}_{tt} - L\tilde{v} = \int_0^t h(\tau)\tilde{v}(x,t-\tau)\,d\tau + \delta(x,t) \tag{1.7}$$

$$\tilde{v}\big|_{t<0} = 0, \tag{1.8}$$

where $h(t) = (r'(t)q - r''(t))\exp(-qt/2)$, and the operator L is different from L_0 with the function $\hat{c}(x) = -r'(0) + \frac{q^2}{4} + c(x)$ instead of $c(x)$. The symbols $'$ and $''$ means the operation of of single and double differentiation.

The direct problem is the determining function $\tilde{v}(x,t)$ satisfying (1.7), (1.8) for given q, $b_i(x)$ $(i = 1,2,3)$, $c(x)$, $k(t)$. Then the function $u(x,t)$, (i.e. the solution of the problem (1.1), (1.2)) is found by the formula (1.5), where $r(t)$ is the solution of the integral equation (1.6). The additional condition (1.4) for the function $\tilde{v}(x,t)$ is transferred to equality

$$\tilde{v}(0,t) = \left(f(t) + \int_0^t k(t-\tau)f(\tau)d\tau\right)\exp(-qt/2), \ t>0. \tag{1.9}$$

Using the results of [13], we invert the operator of the left part (1.7) on Cauchy data (1.8):

$$\tilde{v}(x,t) = \frac{1}{4\pi}\sigma(x)\delta\left(t-|x|\right) + \frac{1}{4\pi}\int\limits_{|\xi-x|\le t}\Big[L_\xi^*\sigma(x,\xi)\tilde{v}(\xi,t-|\xi-x|)+$$

$$+\sigma(x,\xi)\int_0^{t-|\xi-x|} h(\nu)\tilde{v}(\xi,t-|\xi-x|-\nu)\,d\nu\Big]\,d\xi, \tag{1.10}$$

where

$$\sigma(x,\xi) =$$

$$= \frac{1}{|\xi-x|}\exp\left\{\frac{1}{2}\int_0^1\sum_{i=1}^3 b_i\left(\xi+\alpha(x-\xi)\right)(x_i-\xi_i)\,d\alpha\right\}; \tag{1.11}$$

$\sigma(x) := \sigma(x,0)$, L_ξ^* is operator conjugate (by Lagrange) with the operator L. For it, equality

$$L_\xi^*\left[|\xi-x|\sigma(x,\xi)\right] = |\xi-x|L_\xi^*\sigma(x,\xi), \tag{1.12}$$

is valid, which is easily checked directly.

The solution of equation (1.10) will be presented in the form

$$\tilde{v}(x,t) = \frac{\theta(t)}{4\pi}\sigma(x)\delta\left(t-|x|\right) + v(x,t), \tag{1.13}$$

where $\theta(t)$ is Heaviside function, $v(x,t)$ is regular function that has the physical meaning of a scattered wave.

Obviously, it is true the relation $\tilde{v}(x,t) = v(x,t)$ for $t>|x|$. By this, we note that the equality (1.9) for $v(x,t)$ has the same form

$$v(0,t) = \left(f(t) + \int_0^t k(t-\tau)f(\tau)d\tau\right)\exp(-qt/2), \ t>0. \tag{1.14}$$

Therefore **Inverse problem** was transferred to the problem of determining the function $k(t)$, $t>0$ from Eqs. (1.6), (1.10), (1.13) and (1.14).

Remark 1. In what follows, we see the method of study permits finding the functions $v(x,t)$ and $k(t)$ simultaneously.

2. Reduction the inverse problem to equivalent system of integral equations

Let us reduce the inverse problem (1.6), (1.10), (1.13) and (1.14) to the system of non-linear integral equations. For the new function $v(x,t)$ introduced by (1.13) from (1.10) we obtain the integral equation

$$v(x,t) = \frac{1}{16\pi^2}\int\limits_{D(x,t)}\sigma(\xi)L_\xi^*\sigma(x,\xi)\delta(t-|\xi|-|\xi-x|)\,d\xi+$$

$$+\frac{1}{16\pi^2}\int\limits_{D(x,t)}\sigma(\xi)\sigma(x,\xi)\int_0^{t-|\xi-x|}h(\nu)\delta(t-|\xi|-|\xi-x|-\nu)\,d\nu\,d\xi+$$

$$+\frac{1}{4\pi}\int\limits_{D(x,t)}\left[L_\xi^*\sigma(x,\xi)v(\xi,t-|\xi-x|)+\sigma(x,t)\int_0^{t-|\xi-x|}h(\nu)v(\xi,(t-|\xi-x|-\nu)\,d\nu\right]d\xi,$$

where $D(x,\,t)$ is the domain in the variable space ξ, bounded by a surface $S(x,\,t)=\left\{\xi\;\middle|\;|\xi-x|+|\xi|=t\right\}$. Let us write the last equality in the equivalent form using the δ-function property

$$v(x,t)=\int\limits_{S(x,t)}\frac{\Sigma_0(x,\xi)}{|\xi||\xi-x|}\frac{ds}{|\nabla_\xi(|\xi|+|\xi-x|)|}+$$

$$+\int_{|x|}^t h(t-\tau)\int\limits_{S(x,t)}\frac{\Sigma_1(x,\xi)}{|\xi||\xi-x|}\frac{ds\,d\tau}{|\nabla_\xi(|\xi|+|\xi-x|)|}+$$

$$+\int\limits_{D(x,t)}\left[\Sigma_2(x,\xi)\frac{v(\xi,t-|\xi-x|)}{|\xi-x|}+\frac{\Sigma_3(x,\xi)}{|\xi-x|}\int_0^{t-|\xi-x|}h(\nu)v(t-|\xi-x|-\nu)\,d\nu\right]d\xi. \tag{2.1}$$

Here ds is surface area element;

$$\Sigma_0(x,\xi)=\frac{1}{16\pi^2}|\xi|\sigma(\xi)|\xi-x|L_\xi^*\sigma(x,\xi),$$
$$\Sigma_1(x,\xi)=\frac{1}{16\pi^2}|\xi|\sigma(\xi)|\xi-x|\sigma(x,\xi), \tag{2.2}$$
$$\Sigma_2(x,\xi)=\frac{1}{4\pi}|\xi-x|L_\xi^*\sigma(x,\xi),\ \Sigma_3(x,\xi)=\frac{1}{4\pi}|\xi-x|\sigma(x,\xi).$$

Let us modify integrals in (2.1). With this purpose we introduce the Cartesian system η, whose beginning is placed at the point 0, and the axis η_3 is directed in a straight line linking the points $0,\ x$, towards the point x. We will use a system in which ξ and η are associated by transition formulas [14]

$$\xi=Q\eta, \tag{2.3}$$

$$Q=\begin{pmatrix}\cos\theta_0\cos\varphi_0 & -\sin\theta_0\sin\varphi_0 & \sin\theta_0\cos\varphi_0\\ \cos\theta_0\sin\varphi_0 & \sin\theta_0\cos\varphi_0 & \sin\theta_0\sin\varphi_0\\ -\sin\theta_0 & 0 & \cos\theta_0\end{pmatrix}.$$

Here $\theta_0,\ \varphi_0$ are angular coordinates of the unit vector $x/|x|$ in a spherical coordinate system associated with Cartesian coordinate system in the usual way. In addition, the spherical coordinate system $\rho,\ \theta,\ \varphi$ is associated with the coordinate system in the usual way. Then, denoting the first term in (2.1) via J, we find

$$J(x,t)=\frac{1}{|x|}\int_0^{2\pi}d\varphi\int_{\frac{t-|x|}{2}}^{\frac{t+|x|}{2}}\Sigma_0(x,\xi)\,dr.$$

Here variable ξ is associated with η by formulas (2.3). In return

$$\eta=\rho(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),$$

where $\rho = (t^2 - |x^2|)/2(t - |x|\cos\theta)$. Let us replace the integration variable r in the internal integral of J with the variable z

$$\rho = \frac{1}{2}(t + |x|z).$$

Then

$$J(x,t) = \frac{1}{2}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_0(x,\xi)\, dz,$$

a η is expressed through integration variables $\varphi,\ z$ by formulas

$$\eta_1 = \frac{1}{2}\sqrt{t^2 - |x|^2}\sqrt{1 - z^2}\cos\varphi,\ \ \eta_2 = \frac{1}{2}\sqrt{t^2 - |x|^2}\sqrt{1 - z^2}\sin\varphi,$$

$$\eta_3 = \frac{1}{2}\left(tz + |x|\right).$$

It is easy to see that the integral of $S(x,\tau)$ in the second term in (2.1) differs from J only by the integrand function. Therefore the transformation of this integral is carried out quite similarly the transformation $J(x,t)$. In order to transform the other integrals in (2.1), we use the ellipsoid equation $S(x,\tau)$ in the form

$$\rho(\theta,\tau) = \frac{\tau^2 - |x|^2}{2(\tau - |x|\cos\theta)}.$$

As a result, it is not difficult to obtain equality [14]

$$\frac{d\xi}{|\xi - x|} = \frac{2\rho^3(\theta,\ \tau)}{\tau^2 - |x|^2} d\omega d\tau,$$

where $d\omega = \sin\theta d\theta d\varphi$ is solid angle element centered at 0.

Thus, we see that equation (2.1) is reduced to a more convenient form:

$$v(x,t) =$$
$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_0(x,\xi)\, dz + \frac{1}{2}\int_{|x|}^{t} h(t-\tau)\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_1(x,\xi)\, dz\, d\tau +$$
$$+2\int_{|x|}^{t} \frac{d\tau}{\tau^2 - |x|^2} \int\limits_{S(x,\tau)} \rho^3(\theta,\tau)\Big[v(\xi, t-\tau+\rho)\Sigma_2(x,\xi) +$$
$$+\Sigma_3(x,\xi)\int_0^{t-\tau} h(\nu) v(\xi, t-\tau+\rho-\nu)\, d\nu\Big]\, d\omega. \tag{2.4}$$

Hence, the regular part of the solution of the problem (1.1), (1.2) satisfies the integral equation (2.4).

Let $G_T = \{x : |x| \le T\}$, T is arbitrary positive number. Let

$$b_i(x) \in C^2\left(G_T\right),\ i = 1,2,3,\ c(x) \in C\left(G_T\right),\ h(t) \in C\left(0,\ T\right],$$

where $C^2\left(G_T\right)$, $C\left(G_T\right)$ are classes of twice continuously differentiable and continuous functions in the domain G_T respectively. We will show that the function $v(x,t)$ are continuous in

$$D_T =: \big\{(x,t)\big| |x| \le t \le T - |x|\big\}$$

and can be determined from equations (2.4) by the method of successive approximations.

The paper [13] provides the estimate

$$\left|L_\xi^* \sigma(x,\xi)\right| \le \frac{M_1}{|x-\xi|},$$

where M_1 is positive constant that depends only on global properties of operator L^* coefficients. Using the representation (2.2) of the functions $\Sigma_i,\ i=0,1,2,3$ and the last inequality, we find

$$|\Sigma_i(x,\xi)| \leq M_2$$

for all $i=0,1,2,3$, where constant M_2 depends on T and norms of functions $b_i,\ c\ (i=1,2,3)$ respectively in functional spaces $C^2(G_T)$, $C(G_T)$.

We apply to equation (2.4) the method of successive approximations according to the following scheme:

$$v^0(x,t) =$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_0(x,\xi)\,dz + \frac{1}{2}\int_{|x|}^{t} h(t-\tau)\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_1(x,\xi)\,dz\,d\tau,$$

$$v^n(x,t) =$$

$$= v^0(x,t) + 2\int_{|x|}^{t} \frac{d\tau}{\tau^2-|x|^2} \int\limits_{S(x,\tau)} \rho^3(\theta,\tau)\Big[v^{n-1}(\xi,t-\tau+\rho)\Sigma_2(x,\xi)+$$

$$+\Sigma_3(x,\xi)\int_0^{t-\tau} h(\nu)v^{n-1}(\xi,t-\tau+\rho-\nu)\,d\nu\Big]\,d\omega,\ n=1,2,\ldots\,. \qquad (2.4')$$

Obviously, each of the functions $v^n(x,t)$ are continuous in the domain D_T. Denote by $h_0 = \max\limits_{t\in[0,T]}|h(t)|$ and estimate the functions $v^n(x,t)$ in this domain:

$$\left|v^0(x,t)\right| \leq 2\pi M_2\left[1+h_0(t-|x|)\right],$$

$$\left|v^1(x,t)\right| \leq 2\pi M_2\left[1+h_0(t-|x|)\right] + 4\pi^2 M_2^2 T\Big[t-|x|+h_0(t-|x|)^2+$$

$$+h_0^2\frac{(t-|x|)^3}{2!}\Big],\ \ldots$$

$$|v^n(x,t)| \leq 2\pi M_2 \sum_{m=0}^{n}\sum_{j=0}^{m+1}(2\pi M_2 T)^m\, h_0^j \frac{(t-|x|)^{m+j}}{m!j!}.$$

We write the Neumann series for equation (2.4) corresponding to the scheme (2.4$'$)

$$v^0(x,t) + \sum_{n=1}^{\infty}\left(v^n(x,t)-v^{n-1}(x,t)\right).$$

Its partial sum coincides with the function $v^n(x,t)$, and consequently, this series is majorized by a series

$$2\pi M_2 \sum_{n=0}^{\infty}\sum_{j=0}^{n+1}(2\pi M_2 T)^n\, h_0^j \frac{(t-|x|)^{n+j}}{n!j!},$$

which, in turn, for all $(x,t)\in D_T$ is majored by a convergent numeric series

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n+1}(2\pi M_2)^{n+1}\, h_0^j \frac{T^{2n+j}}{n!j!}.$$

So the Neumann series converges uniformly, and hence its sum is a continuous function in the domain D_T. It is not difficult to prove that the sum of the series is the solution of equation (2.4), and it is unique.

Differentiating equality (2.4) with respect to variable t, we find

$$v_t(x,t) = \frac{t}{4\sqrt{t^2-|x|^2}} \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{1} \left\langle \nabla_\xi \Sigma_0,\ Q\eta^{'}(t,x,\varphi,z)\right\rangle dz+$$

$$+\pi h(t-|x|) \int\limits_{-1}^{1} \Sigma_1 \left(x, 0, 0, \frac{z+1}{2}|x|\right)\ dz+$$

$$+\frac{1}{4}\int\limits_{|x|}^{t} \frac{h(t-\tau)\tau\, d\tau}{\sqrt{\tau^2-|x|^2}} \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{1} \left\langle \nabla_\xi \Sigma_1,\ Q\eta^{'}(\tau,x,\varphi,z)\right\rangle dz+$$

$$+\frac{2}{t^2-|x|^2} \int\limits_{S(x,t)} \rho^3(\theta,t) v(\xi,\rho)\Sigma_2(x,\xi)\, d\omega+$$

$$+2\int\limits_{|x|}^{t} \frac{d\tau}{\tau^2-|x|^2} \int\limits_{S(x,\tau)} \rho^3(\theta,\tau)\Big\{ v_t(\xi, t-\tau+\rho)\Sigma_2(x,\xi)+$$

$$+\Sigma_3(x,\xi)\left[h(t-\tau)v(\xi,\rho) + \int\limits_0^{t-\tau} h(\nu) v_t(\xi, t-\tau+\rho-\nu)\, d\nu\right]\Bigg\}\, d\omega. \qquad (2.5)$$

Here

$$\nabla_\xi \Sigma_j(x,\xi) = \left(\Sigma_{j\xi_1},\ \Sigma_{j\xi_2},\ \Sigma_{j\xi_3}\right)(x,\xi),\ j = 0, 1,$$

$\langle .\ ,\ .\rangle$ is the scalar product in R^3;

$$\eta^{'}(t,x,z,\varphi) =$$

$$= \left(\eta_1^{'},\ \eta_2^{'},\ \eta_3^{'}\right) = \left(\sqrt{1-z^2}\cos\varphi,\ \sqrt{1-z^2}\sin\varphi,\ \frac{z}{t}\sqrt{t^2-|x|^2}\right).$$

We introduce linear integral operators by formulas

$$A[h(t), v(x,t)] = 2\int_{|x|}^{t} \frac{d\tau}{\tau^2-|x|^2} \int\limits_{S(x,\tau)} \rho^3(\theta,\tau)\Big[v(\xi, t-\tau+\rho)\Sigma_2(x,\xi)+$$

$$+\Sigma_3(x,\xi)\int_0^{t-\tau} h(\nu) v(\xi, t-\tau+\rho-\nu)\, d\nu\Big]\, d\omega,$$

$$B[h(t), v(x,t)] =$$

$$= 8\sqrt{t^2-|x|^2} \int\limits_{|x|}^{t} \frac{h(t-\tau) d\tau}{\tau^2-|x|^2} \int\limits_{S(x,\tau)} \rho^3(\theta,\tau) v(\xi,\rho)\Sigma_3(x,\xi)\, d\omega, \qquad (2.6)$$

$$E[v(x,t)] = \frac{8}{\sqrt{t^2-|x|^2}} \int\limits_{S(x,\tau)} \rho^3(\theta,t) v(\xi,\rho)\Sigma_2(x,\xi)\, d\omega.$$

Let

$$v_0(x,t) = \frac{1}{2}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_0(x,\xi)\, dz,$$

$$w_0(x,t) = t\int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{1} \left\langle \nabla_\xi \Sigma_0,\ Q\eta^{'}\right\rangle dz. \qquad (2.7)$$

Let us introduce a new function $w(x,t)$ by the formula

$$w(x,t) = 4\sqrt{t^2-|x|^2}\,v_t(x,t).$$

Then equations (2.4), (2.5) will be rewritten as

$$v(x,t) =$$

$$= v_0(x,t) + \frac{1}{2}\int_{|x|}^{t} h(t-\tau)\int_0^{2\pi} d\varphi\int_{-1}^{1}\Sigma_1(x,\xi)\,dz\,d\tau + A[h(t),v(x,t)], \tag{2.8}$$

$$w(x,t) = w_0(x,t) + 4\pi\sqrt{t^2-|x|^2}h(t-|x|)\int_{-1}^{1}\Sigma_1\left(x,0,0,\frac{z+1}{2}|x|\right)\,dz+$$

$$+\sqrt{t^2-|x|^2}\int_0^{\sqrt{t^2-|x|^2}} h\left(t-\sqrt{\beta^2-|x|^2}\right)d\beta\int_0^{2\pi}d\varphi\times$$

$$\times\int_{-1}^{1}\left\langle\nabla_\xi\Sigma_1, Q\eta'\right\rangle|_{\tau=\sqrt{\beta^2-|x|^2}}dz + B[h(t),v(x,t)]+$$

$$+E[v(x,t)] + \sqrt{t^2-|x|^2}A\left[h(t),\frac{w(x,t)}{\sqrt{t^2-|x|^2}}\right]. \tag{2.9}$$

Passing in the last equality to the limit at $x\to 0$, taking into account the condition (1.11), we obtain an equation with respect to the unknown function $h(t)$. For further research, this equation is more convenient to solve with respect to $r''(t)$

$$r''(t) =$$

$$= r'(t)q + F(t) + \int_0^t\left(k'(t-\tau) - \frac{q}{2}k(t-\tau)\right)f(\tau)d\tau$$

$$+\bar{w}_0(t) + 2\pi\exp(qt/2)\int_0^t h(t-\beta)d\beta\int_0^{2\pi}d\varphi\int_{-1}^{1}\left\langle\nabla_\xi\Sigma_1, Q\eta'\right\rangle|_{\tau=\beta,\ x=0}dz+$$

$$+\frac{2\pi}{t}\exp(qt/2)\Big\{B[h(t),v(x,t)] + E[v(x,t)]+$$

$$+\sqrt{t^2-|x|^2}A\left[h(t),\frac{w(x,t)}{\sqrt{t^2-|x|^2}}\right]\Big\}_{x=0}, \tag{2.10}$$

where

$$F(t) = -8\pi\left(f'(t) - \frac{q}{2}f(t)\right),\ \ \bar{w}_0(t) = \frac{2\pi}{t}w_0(0,t)\exp(qt/2).$$

The order of the value in curly brackets in (2.10) at $x=0$ is t^2 (this is easy to establish from the formula (2.10)).

To close the system (2.8)–(2.10) we use the following from (1.6) integral equations:

$$k(t) = -r(t) - \int_0^t r(t-\tau)k(\tau)d\tau, \tag{2.11}$$

$$k'(t) = -r'(t) - \int_0^t r'(t-\tau)k(\tau)d\tau \tag{2.12}$$

and obvious equations for the unknown functions $r(t)$ and $r'(t)$

$$r(t) = \int_0^t r'(\tau) d\tau, \tag{2.13}$$

$$r'(t) = r'(0) + \int_0^t r''(\tau) d\tau, \tag{2.14}$$

where $r'(0)$ to be determined later.

3. Existence and uniqueness theorem

The system of equations (2.8)–(2.14) is a closed system of nonlinear integral equations with respect to the functions $v(x,\ t)$, $w(x,\ t)$, $r''(t)$, $k(t)$, $k'(t)$, $r(t)$ and $r'(t)$. Write it down as an operator equation

$$\psi = U\psi, \tag{3.1}$$

where ψ – vector function of variables $x,\ t$ with components $\psi_i,\ i = 1, 2, \ldots, 7$ and $\psi_1(x,t) = v(x,t)$, $\psi_2(x,t) = w(x,t)$, $\psi_3(x,t) \equiv \psi_3(t) = qr'(t) - r''(t)$, $\psi_4(t) = k(t)+r(t)$, $\psi_5(t) = k'(t)+r'(t)$, $\psi_6(t) = r(t)$, $\psi_7(t) = r'(t)$. The operator U is defined on a set of functions $\psi \in C[D_T]$ and according to the equalities (2.8)–(2.14) has the form

$$U = (U_1, U_2, \ldots, U_7),$$

$$U_1\psi =$$

$$= v_0(x,t) + \frac{1}{2}\int_{|x|}^t \psi_3(t-\tau) \int_0^{2\pi} d\varphi \int_{-1}^1 \Sigma_1(x,\xi)\, dz\, d\tau + A\left[\psi_3(t), \psi_1(x,t)\right],$$

$$U_2\psi = w_0(x,t) + 4\pi\sqrt{t^2-|x|^2}\psi_3(t-|x|)\int\limits_{-1}^{1} \Sigma_1\left(x, 0, 0, \frac{z+1}{2}|x|\right) dz +$$

$$+\sqrt{t^2-|x|^2} \int\limits_0^{\sqrt{t^2-|x|^2}} \psi_3\left(t - \sqrt{\beta^2-|x|^2}\right) d\beta \int\limits_0^{2\pi} d\varphi \times$$

$$\times \int\limits_{-1}^{1} \left\langle \nabla_\xi \Sigma_1, Q\eta^{'} \right\rangle |_{\tau=\sqrt{\beta^2-|x|^2}} dz + B[\psi_3(t), \psi_1(x,t)] +$$

$$+E[\psi_1(x,t)] + \sqrt{t^2-|x|^2} A\left[\psi_1(t), \frac{\psi_2(x,t)}{\sqrt{t^2-|x|^2}}\right],$$

$$U_3\psi = -F(t) - \bar{w}_0(t) - \int_0^t \left(\psi_5(t-\tau) - \psi_7(t-\tau) - \frac{q}{2}(\psi_4(t-\tau) - \psi_6(t-\tau))\right) f(\tau) d\tau$$

$$-2\pi \exp(qt/2) \int\limits_0^t \psi_3(t-\beta) d\beta \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{1} \left\langle \nabla_\xi \Sigma_1, Q\eta^{'} \right\rangle |_{\tau=\beta,\ x=0} dz$$

$$-\frac{2\pi}{t}\exp(qt/2) \Big\{ B[\psi_3(t), \psi_1(x,t)] + E[\psi_(x,t)] +$$

$$+\sqrt{t^2-|x|^2} A\left[\psi_3(t), \frac{\psi_2(x,t)}{\sqrt{t^2-|x|^2}}\right] \Big\}_{x=0},$$

$$U_4\psi = -\int_0^t \psi_6(t-\tau)(\psi_4(\tau)-\psi_6(\tau))d\tau,$$
$$U_5\psi = -\int_0^t \psi_7(t-\tau)(\psi_4(\tau)-\psi_6(\tau))d\tau$$
$$U_6\psi = \int_0^t \psi_7(\tau)d\tau,$$
$$U_7\psi = r'(0)+\int_0^t (\psi_3(\tau)-q\psi_7(\tau))d\tau.$$

We make the following new assumptions about the functions $f(t)$, $b_i(x)$, $i=1,2,3$, $c(x)$:

$$f(t)\in C^1[0,T],\ b_i(x)\in C^3\left[G_T\right],\ c(x)\in C^1\left[G_T\right].$$

Let

$$\|\psi\| = \max_{1\le i\le 7}\left[\max_{(x,t)\in D_T}|\psi_i(x,t)|\right],$$
$$\psi_0(x,t) = [\psi_{01},\ \psi_{02},\ \psi_{03},\ \psi_{04},\ \psi_{05},\ \psi_{06},\ \psi_{07}]$$
$$= [v_0(x,t),\ w_0(x,t),\ -F(t)-\bar{w}_0(t),\ 0,\ 0,\ 0,\ r'(0)],$$
$$f_0 = \|f(t)\|_{C^1[0,\ T]},\ b_0 = \max_{1\le i\le 3}\left\{\|b_i\|_{C^3[G_T]}\right\},\ c_0 = \|\hat{c}\|_{C^1[G_T]}.$$

For further research we will need estimates for the functions

$$\Sigma_i(x,\ \xi),\ i=0,1,2,3,\ \left\langle \nabla_\xi \Sigma_j(x,\xi),\ Q\eta^{'}(t,x,\varphi,z)\right\rangle,\ j=0,1,$$
$$(x,\xi)\in G_T\times G_T,\ t\in[0,\ T],\ \varphi\in[0,\ 2\pi],\ z\in[-1,\ 1].$$

Therefore, we perform some calculations characterizing the dependence of the functions $\Sigma_i(x,\ \xi)$, $i=0,1,2,3$ on $b_i,\ \hat{c}, i=0,1,2$. Using (1.11), (1.12), (2.2), we find

$$|\xi-x|L_\xi^*\sigma(x,\xi)$$
$$= L_\xi^*\exp\left\{\frac{1}{2}\int_0^1\sum_{i=1}^3 b_i\left[\xi+\alpha(x-\xi)\right](x_i-\xi_i)\,d\alpha\right\} =$$
$$= \triangle_\xi \exp\left\{\frac{1}{2}\int_0^1\sum_{i=1}^3 b_i\left[\xi+\alpha(x-\xi)\right](x_i-\xi_i)\,d\alpha\right\} -$$
$$-\sum_{i=1}^3\frac{\partial}{\partial\xi_i}\left\{b_i(\xi)\exp\left\{\frac{1}{2}\int_0^1\sum_{i=1}^3 b_i\left[\xi+\alpha(x-\xi)\right](x_i-\xi_i)\,d\alpha\right\}\right\}$$
$$+c(\xi)\exp\left\{\frac{1}{2}\int_0^1\sum_{i=1}^3 b_i\left[\xi+\alpha(x-\xi)\right](x_i-\xi_i)\,d\alpha\right\} :=$$
$$:= \exp\left\{\frac{1}{2}\int_0^1\sum_{i=1}^3 b_i\left[\xi+\alpha(x-\xi)\right](x_i-\xi_i)\,d\alpha\right\}\Psi(x,\ \xi),$$

where

$$\Psi(x,\ \xi) =$$
$$= \frac{1}{4}\bigg\{\sum_{i=1}^3\int_0^1\left[\sum_{j=1}^3 b_{j\xi_i}[x+\alpha(\xi-x)](x_j-\xi_j)\alpha-\right.$$

$$-b_i\left[x+\alpha(\xi-x)\right]\Bigg]d\alpha\Bigg\}^2+$$

$$+\frac{1}{2}\int\limits_0^1\left\{\sum_{i=1}^{3}\triangle_\xi b_i\left[x+\alpha(\xi-x)\right](x_i-\xi_i)\alpha^2-2div_\xi b\left[x+\alpha(\xi-x)\right]\right\}d\alpha-$$

$$-\frac{1}{2}\sum_{i=1}^{3}b_i(\xi)\int_0^1\left[\sum_{j=1}^{3}b_{j\xi_i}\left[x+\alpha(\xi-x)\right](x_j-\xi_j)\alpha-b_i\left[x+\alpha(\xi-x)\right]\right]d\alpha-$$

$$-\sum_{i=1}^{3}b_{j\xi_i}(\xi)+\hat{c}(\xi). \tag{3.2}$$

Here the subscript ξ of operators $\triangle$, div indicates that they are applied by the variable ξ.

Remark 2. From equalities (1.11), (2.4), (3.2) will follow that

$$-r'(0)+\frac{q^2}{4}+c(0)+\sum_{i=1}^{3}\left[\frac{3}{4}b_i^2(0)-2b_{ix_i(0)}\right]=8\pi f(0),$$

from which $r'(0)$ is easily determined. Further, this value will be considered known.

It follows from (3.2) that

$$\Sigma_0(x,\xi)=\Sigma_1(x,\xi)\Psi(x,\xi),\ \Sigma_2(x,\xi)=\Sigma_3(x,\xi)\Psi(x,\xi). \tag{3.3}$$

Further,

$$\frac{\partial}{\partial\xi_\nu}\Sigma_0(x,\xi)=$$

$$=\frac{1}{2}\Sigma_0(x,\xi)\int\limits_0^1\left\{\sum_{i=1}^{3}\left[b_{i\xi_\nu}(\alpha\xi)\alpha\xi_i+b_{i\xi_\nu}\left[x+\alpha(\xi-x)\right]\alpha(x_i-\xi_i)\right]+\right.$$

$$\left.+b_\nu(\alpha\xi)-b_\nu\left[x+\alpha(\xi-x)\right]\right\}d\alpha+\Sigma_1(x,\xi)\frac{\partial}{\partial\xi_\nu}\Psi(x,\ \xi),\ \nu=1,2,3,$$

$$\frac{\partial}{\partial\xi_\nu}\Psi(x,\ \xi)=$$

$$=\frac{1}{4}\sum_{i=1}^{3}\int\limits_0^1\left\{\sum_{j=1}^{3}b_{j\xi_i}\left[x+\alpha(\xi-x)\right](x_j-\xi_j)-b_i\left[x+\alpha(\xi-x)\right]\right\}d\alpha\times$$

$$\times\sum_{i=1}^{3}\int\limits_0^1\left\{\sum_{j=1}^{3}b_{j\xi_i\xi_\nu}\left[x+\alpha(\xi-x)\right](x_j-\xi_j)-b_{\nu\xi_i}\left[x+\alpha(\xi-x)\right]\alpha-\right.$$

$$\left.-b_{i\xi_\nu}\left[x+\alpha(\xi-x)\right]\alpha\right\}d\alpha+$$

$$+\frac{1}{2}\int\left\{\triangle_\xi b_{i\xi_\nu}\left[x+\alpha(\xi-x)\right]\alpha^3-\triangle_\xi b_\nu\left[x+\alpha(\xi-x)\right]\alpha^2-\right.$$

$$\left.-2div_\xi b_{\xi_\nu}\left[x+\alpha(\xi-x)\right]\right\}d\alpha-$$

$$-\frac{1}{2}\sum_{i=1}^{3} b_{i\xi_\nu}(\xi)\int\limits_0^1\left\{\sum_{j=1}^{3} b_{j\xi_i\xi_\nu}\left[x+\alpha(\xi-x)\right](x_j-\xi_j)\alpha^2-\right.$$

$$\left.-b_{\nu\xi_i}\left[x+\alpha(\xi-x)\right]\alpha-b_{i\xi_\nu}\left[x+\alpha(\xi-x)\right]\alpha\right\}d\alpha-$$

$$-\sum_{i=1}^{3} b_{j\xi_i\xi_\nu}(\xi)+\hat{c}_{\xi_\nu}(\xi). \tag{3.4}$$

$$\Sigma_{1\xi_\nu}(x,\xi)=\Sigma_1(x,\xi)\int\limits_0^1\left\{\sum_{i=1}^{3}\left[b_{i\xi_\nu}(\alpha\xi)\alpha\xi_i+\right.\right.$$

$$\left.\left.+b_{i\xi_\nu}\left[x+\alpha(\xi-x)\right]\alpha(x_i-\xi_i)\right]+b_\nu(\alpha\xi)+b_\nu\left[x+\alpha(\xi-x)\right]\right\}d\alpha.$$

From (3.2), (3.4) we find

$$|\Psi(x,\xi)|\leq$$

$$\leq\frac{3b_0^2}{4}\left(\frac{3}{2}T+1\right)(\frac{3}{2}T+3)+\frac{3b_0}{2}(T+4)+c_0:=\chi_0,(x,\xi)\in G_T\times G_T,$$

$$\left|\frac{\partial}{\partial\xi_\nu}\Psi(x,\xi)\right|\leq\frac{3b_0^2}{4}(3T+2)(9T+4)+\frac{3b_0^2}{2}(T+2)+\frac{13b_0}{3}+c_0:=\chi_1,$$

$$\nu=1,2,3,\ (x,\xi)\in G_T\times G_T.$$

Taking into account these inequalities, we find the following estimates:

$$|\Sigma_1(x,\xi)|\leq\frac{1}{16\pi^2}\exp\left(3b_0T\right):=\alpha_1,$$

$$|\Sigma_0(x,\xi)|\leq|\Sigma_1(x,\xi)|\,|\Psi(x,\ t)|\leq\alpha_1\chi_0:=\alpha_0,$$

$$|\Sigma_3(x,\xi)|\leq\frac{1}{4\pi}\exp\left(\frac{3}{2}b_0T\right):=\alpha_3,$$

$$|\Sigma_2(x,\xi)|\leq|\Sigma_3(x,\xi)|\,|\Psi(x,\ t)|\leq\alpha_3\chi_0:=\alpha_2, \tag{3.5}$$

From (3.4) we find

$$\left|\left\langle\nabla_\xi\Sigma_0(x,\xi),\ Q\eta^{'}\right\rangle\right|\leq\max_{1\leq i\leq 3}\{\|\Sigma_{0\xi_i}\|\}\max_{1\leq i\leq 3}\left\{1,\ \left\|\eta_i^{'}\right\|\right\}\leq$$

$$\leq\frac{3}{2}\alpha_0b_0T+\alpha_1\chi_1:=\beta_0,$$

$$\left|\left\langle\nabla_\xi\Sigma_1(x,\xi),\ Q\eta^{'}\right\rangle\right|\leq\max_{1\leq i\leq 3}\{\|\Sigma_{1\xi_i}\|\}\max_{1\leq i\leq 3}\left\{1,\ \left\|\eta_i^{'}\right\|\right\}\leq$$

$$\leq 4\alpha_1b_0T:=\beta_1,$$

$$(x,\xi)\in G_T\times G_T,\ t\in[0,\ T],\ \varphi\in[0,\ 2\pi],\ z\in[-1,\ 1].$$

Let $M>0$ be fixed and consider the ball

$$\Omega(\psi_0,\ M)=\left\{g(x,t)\in C\left[D_T\right]\Big|\|\psi_0-g\|\leq M\right\}.$$

It is easy to see that for $\psi\in\Omega\left(\psi_0,\ M\right)$ the estimate is fair

$$\|\psi\|\leq\|\psi_0\|+M\equiv N. \tag{3.6}$$

Let

$$T^*=\min_{1\leq i\leq 12}\left\{\gamma_i\right\},$$

where $\gamma_i,\ i=1,2,\dots,12$ are the roots of the following equations correspondingly:

$$\begin{gathered}
\gamma\left(\gamma^2+\frac{2\alpha_2}{\alpha_3 N}\gamma+\frac{4\alpha_1}{\alpha_3 N}\right)=\frac{2M}{\pi\alpha_3 N^2},\\
\gamma\left[\left(\frac{4}{3}+\frac{\ln\gamma}{6\alpha_3}\right)\gamma^2+\frac{8\alpha_2+6\beta_1}{3\alpha_3 N}\gamma+\frac{4\alpha_1}{\alpha_3 N}\right]=\frac{M}{2\pi\alpha_3 N^2},\\
\gamma\left(\frac{4\pi^2(2+|q|)f_0\alpha_3}{N}+\left[\frac{4}{3}\gamma+\frac{8\alpha_2}{3\alpha_3 N}+\frac{\beta_0}{\pi\alpha_3 N^2}+\frac{2\beta_1}{\alpha_3 N}\right]\exp(q\gamma/2)\right)=\frac{M}{4\pi^2\alpha_3 N^2},\\
\gamma=\frac{M}{2N^2},\quad \gamma=\frac{M}{N},\quad \gamma=\frac{M}{N(1+|q|)},
\end{gathered}\tag{3.7}$$

$$\begin{gathered}
\gamma\left(\gamma^2+\frac{3\alpha_2}{4\alpha_3 N}\gamma+\frac{3\alpha_1}{\alpha_3 N}\right)=\frac{3}{2\pi\alpha_3 N},\\
\gamma\left[\left(1+\frac{N}{3}+\frac{N}{6\alpha_3}\ln\gamma\right)\gamma^2+\frac{\alpha_2+\beta_1}{\alpha_3 N}\gamma+\frac{\alpha_1}{\alpha_3 N}\right]=\frac{1}{4\pi\alpha_3 N},\\
\gamma\left(\frac{8\pi^2 f_0(1+|q|/2)\alpha_2}{N}+\left[4\gamma+\frac{4\alpha_2+3\beta_1}{\alpha_3 N}\right]\exp(q\gamma/2)\right)=\frac{3}{8\pi^2\alpha_2 N},\\
\gamma=\frac{1}{4N},\quad \gamma=1,\quad \gamma=\frac{1}{1+|q|}.
\end{gathered}\tag{3.8}$$

Theorem 1. *Let performed assumption about functions $f(t),\ b_i(x),\ c(x), i=1,2,3$, made above. Then in the domain $D_T,\ T\in(0,T^*)$, there is a unique continuous solution of the equation (3.1), belonging to the ball $\Omega(\psi_0,M)$.*

Доказательство. Note that the following inequalities are true

$$\begin{gathered}
|A[1,1]|\le\frac{\pi}{6}(t-|x|)\left[3\alpha_2(t+|x|)+\alpha_3(t-|x|)(t+2|x|)\right]\\
|B[1,\ 1]|\le 2\pi\alpha_3\sqrt{(t^2-|x|^2)^3},\\
|E[1]|\le 4\pi\alpha_2 t\sqrt{t^2-|x|^2},\\
\left|A\left[\psi_3(t),\ \frac{1}{\sqrt{t^2-|x|^2}}\right]\right|\le\frac{3\pi}{4}\left[\sqrt{t^2-|x|^2}\left(\alpha_2+\frac{\alpha_3\,\|\psi_3\|\,t}{2}\right)+\right.\\
\left.+\frac{\|\psi_3\|\,x^2}{2}\ln\frac{|x|}{t+\sqrt{t^2-|x|^2}}\right],
\end{gathered}\tag{3.9}$$

which are easily derived from formulas (2.6) for all $(x,t)\in D_T$.

For example, we prove the validity of the first of the inequalities (3.9). By definition

$$A[1,1]=2\int_{|x|}^{t}\frac{d\tau}{\tau^2-|x|^2}\int\limits_{S(x,\tau)}r^3(\theta,\tau)\Big[\Sigma_2(x,\xi)+\Sigma_3(x,\xi)(t-\tau)\Big]\,d\omega.$$

Let's use the formula

$$\rho^2(\theta,\tau)\sin\theta d\theta=-\frac{\tau^2-|x|^2}{2|x|}dr,$$

that follows from equation confocal ellipsoids $S(x, \tau)$ in polar coordinates:

$$\rho(\theta, \tau) = \frac{\tau^2 - |x|^2}{2(\tau - |x|\cos\theta)}.$$

Change the integration variable to r and, given the estimates (3.5) for Σ_2, Σ_3, we find

$$|A[1,1]| \leq \frac{2\pi}{|x|} \int\limits_{|x|}^{t} (\alpha_2 + \alpha_3(t-\tau))\, d\tau \int\limits_{\frac{\tau-|x|}{2}}^{\frac{\tau+|x|}{2}} \rho d\rho =$$

$$= \frac{\pi}{6}(t - |x|)\left[3\alpha_2(t+|x|) + \alpha_3(t-|x|)(t+2|x|)\right].$$

We show that for any $T \in (0, T^*)$ operator U maps the ball $\Omega(\psi_0,\ M)$ in itself. Composing the norm of differences and using the equations of the system (3.1), inequalities (3.5), (3.6), (3.8), we find

$$\|U_1\psi - \psi_{01}\| \leq 2\pi\alpha_1 \|\psi_3\| (t - |x|) + \|\psi_1\| \left|A\left[\|\psi_3\|,\ 1\right]\right| \leq$$
$$\leq \frac{\pi\alpha_3}{2}N^2T^3 + \pi\alpha_2 NT^2 + 2\pi\alpha_1 NT,$$

$$\|U_2\psi - \psi_{02}\| \leq 8\pi\alpha_1 \|\psi_3\| \sqrt{t^2 - |x|^2} + 4\pi\beta_1 \|\psi_3\| \left(t^2 - |x|^2\right) +$$
$$+ \|\psi_1\| \|\psi_3\| B[1,\ 1]| + \|\psi_1\| |E[1]| +$$
$$+ \|\psi_2\| \left| A\left[\|\psi_3\|, \frac{1}{\sqrt{t^2 - |x|^2}}\right]\right| \sqrt{t^2 - |x|^2} \leq$$
$$\leq 2\pi\alpha_3 N^2T^3 + \frac{4\pi NT}{3}\left[T\left(\alpha_2 + \frac{\alpha_3 NT}{2}\right) + \frac{NT^2}{4}\ln T\right] +$$
$$+ 4\pi\alpha_2 NT^2 + 4\pi\beta_1 NT^2 + 8\pi\alpha_1 NT,$$

$$\|U_3\psi - \psi_{03}\| \leq t f_0 \left(\|\psi_5\| + \|\psi_7\| + \frac{|q|}{2}(\|\psi_4\| + \|\psi_6\|)\right)$$
$$+ \exp(|q|t)/2)\left[8\pi^2\beta_1 \|\psi_3\| t + \frac{2\pi}{t}\left\{\|\psi_1\| \|\psi_3\| B[1,\ 1]|\right.\right.$$
$$\left.\left. + \|\psi_1\| |E[1]| + \|\psi_2\| \left|A\left[\|\psi_3\|, \frac{1}{\sqrt{t^2 - |x|^2}}\right]\right| \sqrt{t^2 - |x|^2}\right\}_{x=0}\right] \leq$$
$$\leq TNf_0(2 + |q|) + \exp(|q|T)/2)\times$$
$$\times\left[4\pi^2\alpha_3 N^2T^2 + \frac{8\pi^2 NT}{3}\left(\alpha_2 + \frac{\alpha_3 N}{2}\right) + 8\pi^2\alpha_2 NT + 8\pi^2\beta_1 NT\right],$$

$$\|U_4\psi - \psi_{04}\| \leq t\|\psi_6\|(\|\psi_4\| + \|\psi_6\|) \leq 2N^2T,$$

$$\|U_5\psi - \psi_{05}\| \leq t\|\psi_7\|(\|\psi_4\| + \|\psi_6\|) \leq 2N^2T,$$

$$\|U_6\psi - \psi_{06}\| \leq t\|\psi_7\| \leq NT,$$

$$\|U_7\psi - \psi_{07}\| \leq t(\|\psi_3\| + |q|\|\psi_7\|) \leq NT(1 + |q|).$$

From these estimates follows (taking into account (3.7) for all $T \in (0, T^*)$ $\|U\psi - \psi_0\| \le M$, that is $U\psi \in \Omega(\psi_0,\ M)$.

Now let $\psi^{(1)}$, $\psi^{(2)}$ be two arbitrary elements of the ball $\Omega(\psi_0, M)$, $T < T^*$. Then, by using auxiliary inequalities

$$\left|\psi_i^{(1)}\psi_j^{(1)} - \psi_i^{(2)}\psi_j^{(2)}\right| \le \left|\psi_i^{(1)}\right|\left|\psi_j^{(1)} - \psi_j^{(2)}\right| +$$
$$+\left|\psi_j^{(2)}\right|\left|\psi_i^{(1)} - \psi_i^{(2)}\right| \le 2N\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

we obtain the relations

$$\left\|U_1\psi^{(1)} - U_1\psi^{(2)}\right\| \le 2\pi\alpha_1(t-|x|)\left\|\psi_3^{(1)} - \psi_3^{(2)}\right\| +$$
$$+\left|A\left[\psi_3^{(1)},\ \psi_1^{(1)}\right] - A\left[\psi_3^{(2)},\ \psi_1^{(2)}\right]\right| \le$$
$$\le \left(2\pi\alpha_1 T + \frac{\pi}{2}\alpha_2 T^2 + \frac{2\pi}{3}\alpha_3 N T^3\right)\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

$$\left\|U_2\psi^{(1)} - U_2\psi^{(2)}\right\| \le 8\pi\alpha_1\sqrt{t^2-|x|^2}\left\|\psi_3^{(1)} - \psi_3^{(2)}\right\| +$$
$$+4\pi\beta_1(t^2-|x|^2)\left\|\psi_3^{(1)} - \psi_3^{(2)}\right\| + \left|\psi_1^{(1)}\psi_3^{(1)} - \psi_1^{(2)}\psi_3^{(2)}\right||B[1,\ 1]|+$$
$$+\left\|\psi_1^{(1)} - \psi_1^{(2)}\right\||E[1]| + \sqrt{t^2-|x|^2}\,||\left|A\left[\psi_3^{(1)},\ \frac{\psi_2^{(1)}}{\sqrt{t^2-|x|^2}}\right] -\right.$$
$$\left.-A\left[\psi_3^{(2)},\ \frac{\psi_2^{(2)}}{\sqrt{t^2-|x|^2}}\right]\right| \le \left[8\pi\alpha_1 T + 4\pi\beta_1 T^2 + 4\pi\alpha_2 T^2 +\right.$$
$$\left.+4\pi\alpha_3 N T^3 + \frac{8\pi N^2 T^2}{3}\left(\alpha_2 + \frac{\alpha_3 T}{2} + \frac{T}{4}\ln T\right)\right]\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

$$\left\|U_3\psi^{(1)} - U_3\psi^{(2)}\right\| \le t f_0 \times$$
$$\times\left[\left\|\psi_5^{(1)} - \psi_5^{(2)}\right\| + \left\|\psi_7^{(1)} - \psi_7^{(2)}\right\| + \frac{|q|}{2}\left(\left\|\psi_4^{(1)} - \psi_4^{(2)}\right\| + \left\|\psi_6^{(1)} - \psi_6^{(2)}\right\|\right)\right]$$
$$+\exp(|q|t)/2)\left[8\pi^2\beta_1 t\left\|\psi_3^{(1)} - \psi_3^{(2)}\right\| +\right.$$
$$+\frac{2\pi}{t}\left\{\left|\psi_1^{(1)}\psi_3^{(1)} - \psi_1^{(2)}\psi_3^{(2)}\right||B[1,\ 1]| + \left\|\psi_1^{(1)} - \psi_1^{(2)}\right\||E[1]|+\right.$$
$$\left.\left.+\sqrt{t^2-|x|^2}\,||\left|A\left[\psi_3^{(1)},\ \frac{\psi_2^{(1)}}{\sqrt{t^2-|x|^2}}\right] - A\left[\psi_3^{(2)},\ \frac{\psi_2^{(2)}}{\sqrt{t^2-|x|^2}}\right]\right|\right\}_{x=0}\right] \le$$
$$\le\left[8\pi^2\alpha_2 T + 8\pi^2\beta_1 T + 8\pi^2\alpha_3 N T^2 + \frac{8\pi^2 T}{3}(\alpha_2 + \alpha_3 N T)\right]\exp(|q|T)/2)\times$$
$$\times\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

$$\left\|U_4\psi^{(1)} - U_4\psi^{(2)}\right\| \le t\left(\left|\psi_6^{(1)}\psi_4^{(1)} - \psi_6^{(2)}\psi_4^{(2)}\right| + \left|\psi_6^{(1)}\psi_6^{(1)} - \psi_6^{(2)}\psi_6^{(2)}\right|\right)$$
$$\le 4NT\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

$$\left\|U_5\psi^{(1)} - U_5\psi^{(2)}\right\| \le t\left(\left|\psi_7^{(1)}\psi_4^{(1)} - \psi_7^{(2)}\psi_4^{(2)}\right| + \left|\psi_7^{(1)}\psi_6^{(1)} - \psi_7^{(2)}\psi_6^{(2)}\right|\right)$$
$$\le 4NT\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$
$$\left\|U_6\psi^{(1)} - U_6\psi^{(2)}\right\| \le T\left\|\psi^{(1)} - \psi^{(2)}\right\|,$$
$$\left\|U_4\psi^{(1)} - U_4\psi^{(2)}\right\| \le T(1+|q|)\left\|\psi^{(1)} - \psi^{(2)}\right\|.$$

Taking into account (3.8) from the last inequality it follows the estimate

$$\left\|U\psi^{(1)} - U\psi^{(2)}\right\| \le \frac{T}{T^*}\left\|\psi^{(1)} - \psi^{(2)}\right\|.$$

Hence, for any $T \in (0,\ T^*)$ operator U is a contraction operator on the ball $\Omega(\psi_0,\ M)$. Then, according to Banach's theorem, there is a unique continuous solution of equation (3.1) in the ball $\Omega(\psi_0,\ M)$. Therefore, solving the system of equations (2.8)–(2.11) by the method of successive approximations, we uniquely construct in the domain $[D_T]$ for $T \in (0,\ T^*)$ functions $v(x,\ t)$, $w(x,\ t)$, $r''(t)$, $k(t)$, $k'(t)$, $r(t)$, $r'(t)$. □

Theorem 2. *Under the conditions of Theorem 1, the function $k(t),\ t \in [0,T]$ is uniquely determined by the information (1.4).*

Доказательство. Suppose the opposite statement. Let there be two solutions $k_1,\ k_2$. Then according to Theorem 1 there is such $T \in (0,\ T^*)$ that

$$k_1(t) = k_2(t),\ t \in [0,T]. \tag{3.10}$$

Let T_0 be the supremum of all possible numbers $T \in (0,T^*)$, for which equality (3.8) is satisfied. Let us now consider a problem similar to the problem (1.1), (1.2), (1.4), in which Cauchy data is given on a plane $t = T_0$. Since

$$u_1\Big|_{t<T_0} \equiv u_2\Big|_{t<T_0} \equiv 0,\ u_1\Big|_{x=0} = u_2\Big|_{x=0} = f(t),$$

then according to Theorem 1, there is such a number $T^{**} \in (T_0,T)$, that $k_1(t) = k_2(t)$. This fact contradicts the definition of T_0. This contradiction proves the validity of the Theorem 2. □

4. Stability theorem

Let T^* be the number from the statement of Theorem 1. Let $\Phi(T)$, $T \in (0,T^*)$ is the set of functions $(b,\ c,\ f)$ for which $b(x) = (b_1,\ b_2,\ b_3) \in C^3[G(T)]$, $c(x) \in C^1[G(T)]$, $f(t) \in C^1[0,T]$.

Theorem 3. *Let $(b,\ c,\ f) \in \Phi(T)$, $(\bar{b},\ \bar{c},\ \bar{f}) \in \Phi(T)$. Then for the corresponding solutions of $k(t)$, $\bar{k}(t)$ of the inverse problem (1.1), (1.2), (1.4) fair estimate*

$$\|k(t) - \bar{k}(t)\|_{C[0,T]} \le \tilde{C}d, \tag{4.1}$$
$$d = \|b - \bar{b}\|_{C^3[G(T)]} + \|c - \bar{c}\|_{C^1[G(T)]} + \|f - \bar{f}\|_{C^1[0,T]},$$

and the constant $\tilde{C}$ depends on T, q, M_0, where

$$M_0=\max_{1\le i\le 3}\Big\{\|b_i\|_{C^3[G(T)]},\ \|\bar b_i\|_{C^3[G(T)]},\ \|c\|_{C^1[G(T)]},\ \|\bar c\|_{C^1[G(T)]},$$
$$\|h\|_{C[0,T]},\ \|\bar h\|_{C[0,T]},\|r\|_{C^1[0,T]},\ \|\bar r\|_{C^1[0,T]}\Big\}.$$

Доказательство. Let ψ, $\bar\psi$ be the solutions of equation (3.1), corresponding to functions (b,c,f), $(\bar b,\bar c,\bar f)$. We denote the difference between the two corresponding vector functions by the symbol $\sim$: $\tilde\psi=\psi-\bar\psi$. Then from equations of (3.1) it is not difficult to obtain the following system of equalities:

$$\tilde\psi_1(x,t)=\tilde v_0(x,t)+\frac12\int_{|x|}^t\bar\psi_3(t-\tau)\int_0^{2\pi}d\varphi\int_{-1}^1\tilde\Sigma_1(x,\xi)\,dz\,d\tau+$$
$$+\frac12\int_{|x|}^t\tilde\psi_3(t-\tau)\int_0^{2\pi}d\varphi\int_{-1}^1\Sigma_1(x,\xi)\,dz\,d\tau+$$
$$+2\int_{|x|}^t\frac{d\tau}{\tau^2-|x|^2}\int\limits_{S(x,\tau)}\rho^3(\theta,\tau)\Sigma_3(x,\xi)\int_0^{t-\tau}\tilde\psi_3(\nu)\psi_1(\xi,t-\tau+\rho-\nu)\,d\nu\,d\omega+$$
$$+A[\bar\psi_3(t),\ \tilde\psi_1(x,t)]+A^{[\tilde\Sigma_2,\tilde\Sigma_3]}[\bar\psi_3(t),\ \bar\psi_1(x,t)],\tag{4.2}$$

$$\tilde\psi_2(x,t)=\tilde w_0(x,t)+4\pi\sqrt{t^2-|x|^2}\tilde\psi_3(t-|x|)\int\limits_{-1}^{1}\Sigma_1\left(x,0,0,\frac{z+1}{2}|x|\right)\,dz+$$
$$+4\pi\sqrt{t^2-|x|^2}\bar\psi_3(t-|x|)\int\limits_{-1}^{1}\tilde\Sigma_1\left(x,0,0,\frac{z+1}{2}|x|\right)dz+$$
$$+\sqrt{t^2-|x|^2}\int\limits_0^{\sqrt{t^2-|x|^2}}\tilde\psi_3\left(t-\sqrt{\beta^2+|x|^2}\right)d\beta\int\limits_0^{2\pi}d\varphi\int\limits_{-1}^{1}\left\langle\nabla_\xi\Sigma_1,Q\eta'\right\rangle\Big|_{\tau=\sqrt{\beta^2-|x|^2}}dz$$
$$+\sqrt{t^2-|x|^2}\int\limits_0^{\sqrt{t^2-|x|^2}}\bar\psi_3\left(t-\sqrt{\beta^2-|x|^2}\right)d\beta\int\limits_0^{2\pi}d\varphi\int\limits_{-1}^{1}\left\langle\nabla_\xi\tilde\Sigma_1,Q\eta'\right\rangle\Big|_{\tau=\sqrt{\beta^2-|x|^2}}dz$$
$$+2\int_{|x|}^t\frac{d\tau}{\tau^2-|x|^2}\int\limits_{S(x,\tau)}\rho^3(\theta,\tau)\Sigma_3(x,\xi)\int_0^{t-\tau}\tilde\psi_3(\nu)\frac{\psi_2(\xi,\ t-\tau+\rho-\nu)\,d\nu\,d\omega}{\sqrt{(t-\tau+\rho-\nu)^2-\rho^2}}$$
$$+B\left[\tilde\psi_3(t),\ \psi_1(x,t)\right]+B\left[\bar\psi_3(t),\ \tilde\psi_1(x,t)\right]+B^{\tilde\Sigma_3}\left[\bar\psi_3(t),\ \bar\psi_1(x,t)\right]$$
$$+E\left[\tilde\psi_1(x,t)\right]+E^{\tilde\Sigma_2}\left[\tilde\psi_1(x,t)\right]+\sqrt{t^2-|x|^2}A\left[\bar\psi_3(t),\ \frac{\tilde\psi_2(x,t)}{\sqrt{t^2-|x|^2}}\right]+$$
$$+\sqrt{t^2-|x|^2}A^{(\tilde\Sigma_2,\tilde\Sigma_3)}\left[\bar\psi_3(t),\ \frac{\bar\psi_2(x,t)}{\sqrt{t^2-|x|^2}}\right].\tag{4.3}$$
$$\tilde\psi_3(t)=-\tilde F(t)-\tilde{\bar w}_0(t)$$

$$-\int\limits_0^t \Big(\tilde{\psi}_5(t-\tau) - \tilde{\psi}_7(t-\tau) - \frac{q}{2}(\tilde{\psi}_4(t-\tau) - \tilde{\psi}_6(t-\tau))f(\tau)\Big)\,d\tau$$

$$-\int\limits_0^t \Big(\bar{\psi}_5(t-\tau) - \bar{\psi}_7(t-\tau) - \frac{q}{2}(\bar{\psi}_4(t-\tau) - \bar{\psi}_6(t-\tau))\tilde{f}(\tau)\Big)\,d\tau$$

$$-2\pi\exp(qt/2)\int\limits_0^t \tilde{\psi}_3(t-\beta)d\beta\int\limits_0^{2\pi} d\varphi\int\limits_{-1}^{1}\Big\langle \nabla_\xi \Sigma_1,\ Q\eta^{'}\Big\rangle|_{\tau=\beta,\ x=0}dz$$

$$-2\pi\exp(qt/2)\int\limits_0^t \bar{\psi}_3(t-\beta)d\beta\int\limits_0^{2\pi} d\varphi\int\limits_{-1}^{1}\Big\langle \nabla_\xi \tilde{\Sigma}_1,\ Q\eta^{'}\Big\rangle|_{\tau=\beta,\ x=0}dz$$

$$-\exp(qt/2)\int_0^t \tau d\tau\int\limits_0^{2\pi}\int\limits_0^{\pi}\int_0^{t-\tau}\tilde{\psi}_3(\nu)\left[\frac{\Sigma_3(x,\xi)\psi_2(\xi,\ t-\tau+\rho-\nu)\,d\nu\,d\omega}{\sqrt{(t-\tau+\rho-\nu)^2-\rho^2}}\right]_{x=0}$$

$$-\frac{2\pi}{t}\exp(qt/2)\Big\{B\Big[\tilde{\psi}_3(t),\ v(x,t)\Big] + B\Big[\bar{\psi}_3(t),\ \tilde{\psi}_1(x,t)\Big] + B^{\tilde{\Sigma}_3}\left[\bar{\psi}_3(t),\ \bar{\psi}_1(x,t)\right]$$

$$+E\Big[\tilde{\psi}_1(x,t)\Big] + E^{\tilde{\Sigma}_2}\left[\bar{\psi}_1(x,t)\right]$$

$$+\sqrt{t^2-|x|^2}\left(A\left[\tilde{\psi}_3(t),\ \frac{\tilde{\psi}_2(x,t)}{\sqrt{t^2-|x|^2}}\right] + A^{(\tilde{\Sigma}_2,\tilde{\Sigma}_3)}\left[\bar{\psi}_3(t),\ \frac{\bar{\psi}_2(x,t)}{\sqrt{t^2-|x|^2}}\right]\right)\Big\}_{x=0}, \tag{4.4}$$

$$\tilde{\psi}_4(t) = -\int_0^t\Big(\tilde{\psi}_6(t-\tau)\psi_4(\tau) + \bar{\psi}_6(t-\tau)\tilde{\psi}_4(\tau)$$
$$-\tilde{\psi}_6(t-\tau)\psi_6(\tau) - \bar{\psi}_6(t-\tau)\tilde{\psi}_6(\tau)\Big)d\tau, \tag{4.5}$$

$$\tilde{\psi}_5(t) = -\int_0^t\Big(\tilde{\psi}_7(t-\tau)\psi_4(\tau) + \bar{\psi}_7(t-\tau)\tilde{\psi}_4(\tau)$$
$$-\tilde{\psi}_7(t-\tau)\psi_6(\tau) - \bar{\psi}_7(t-\tau)\tilde{\psi}_6(\tau)\Big)d\tau, \tag{4.6}$$

$$\tilde{\psi}_6(t) = \int_0^t \tilde{\psi}_7(\tau)d\tau, \tag{4.7}$$

$$\tilde{\psi}_7(t) = \int_0^t\Big(\tilde{\psi}_3(\tau) - q\tilde{\psi}_7(\tau)\Big)\,d\tau. \tag{4.8}$$

Here, the upper indices of the operators A, B, E mean, that the functions Σ_i, $i = 1, 2, 3$, included in the definition of these operators are replaced by upper index functions. For example, $A^{(\tilde{\Sigma}_2,\tilde{\Sigma}_3)}[.\ ,\ .]$ is the operator, defined in (2.6), in which the functions Σ_2 and Σ_3 are replaced by $\tilde{\Sigma}_2$ и $\tilde{\Sigma}_3$.

In solving the direct problem (1.7), (1.8), an estimate was obtained

$$|\psi_1(x,t)| \le \sum_{n=0}^{\infty}\sum_{j=0}^{n+1}(2\pi M_2)^{n+1}\,h_0^j\frac{T^{2n+j}}{n!j!} := \mu_0, \tag{4.9}$$

where $M_2 = \max\limits_{0\le i\le 3} \alpha_i$. Since $\left(\bar b,\ \bar c,\ \bar f\right) \in \Phi(T)$, this estimate holds for $\left|\bar\psi_1(x,t)\right|$:

$$\left|\bar\psi_1(x,t)\right| \le \mu_0,\ (x,t) \in D_T.$$

Estimating (2.9) similarly to the estimation (4.6), we obtain

$$|\psi_2(x,t)| \le \mu_1,\ \left|\bar\psi_2(x,t)\right| \le \mu_1,\ (x,t) \in D_T,$$

where the constant μ_1 depends on $T,\ M_0$. The domain D_T has an equivalent description

$$D_T = \left\{(x,t)\middle| 0 \le |x| \le \frac{T}{2} - \left|t - \frac{T}{2}\right|,\ 0 \le t \le T\right\}.$$

Let

$$\omega(t) = \max\Bigg[\max_{0\le|x|\le T/2-|t-T/2|} |\tilde\psi_1(x,t)|, \max_{0\le|x|\le T/2-|t-T/2|} |\tilde\psi_2(x,t)|,$$
$$\max|\tilde\psi_j(t)|, j = 3,4,5,6,7\Bigg],\ t \in [0,T].$$

Let us evaluate the unknown functions of the system (4.2)–(4.8) in the domain D_T by the value d, defined in Theorem 3. We estimate the first term in equation (4.2). By virtue of the first equality of (2.7), we have

$$|\tilde v_0(x,t)| \le \frac{1}{2} \max_{(x,t)\in D_T} \left|\int_0^{2\pi}\int_{-1}^{1} \tilde\Sigma_0(x,\xi)dz\right| \le 2\pi \max_{(x,\xi)\in[G_T\times G_T]} \left|\tilde\Sigma_0(x,\xi)\right|, \tag{4.10}$$

where

$$\tilde\Sigma_0(x,\xi) = \Sigma_0(x,\xi) - \bar\Sigma_0(x,\xi);$$

$\Sigma_0(x,\xi)$ is defined by the first term (2.2) and

$$\bar\Sigma_0(x,\xi) = \frac{1}{16\pi^2}|\xi|\bar\sigma(\xi)|\xi - x|\bar L^*_\xi \bar\sigma(x,\xi),$$

function $\bar\sigma(x,\xi)$ has the form (1.11) with $\bar b_i$, $\bar\sigma(x) = \bar\sigma(x,0)$, $\bar L^*_\xi$ is the operator L^*_ξ, with coefficients $\bar b_i,\ \bar{\hat c},\ i = 1,2,3$. By virtue of the equalities (1.11), (3.3) we have

$$\tilde\Sigma_0(x,\xi) =$$
$$= \Sigma_0(x,\xi) - \bar\Sigma_0(x,\xi) = \Sigma_1(x,\xi)\Psi(x,\ \xi) - \bar\Sigma_1(x,\xi)\bar\Psi(x,\ \xi) =$$
$$= \tilde\Sigma_1(x,\xi)\Psi(x,\ \xi) + \bar\Sigma_1(x,\xi)\tilde\Psi(x,\ \xi), \tag{4.11}$$

where $\bar\Sigma_1(x,\xi)$, $\bar\Psi(x,\ \xi)$ is defined by equalities (2.2), (3.2) respectively, in which functions $b_i,\ \hat c,\ i = 1,2,3$, are replaced by $\bar b_i,\ \bar{\hat c},\ i = 1,2,3$. Note that $\hat c - \bar{\hat c} = c - \bar c$. Function estimates for $\tilde\Sigma_1(x,\xi)$, $\tilde\Psi(x,\ \xi)$ are done as follows

$$\left|\tilde\Sigma_1(x,\xi)\right| =$$
$$= \frac{1}{16\pi^2}\left|\exp\left\{\frac{1}{2}\int_0^1 \sum_{i=1}^3 \left[b_i(\alpha\xi)\xi_i + b_i\left(\xi + \alpha(x-\xi)\right)(x_i - \xi_i)\right] d\alpha\right\} -\right.$$
$$\left. - \exp\left\{\frac{1}{2}\int_0^1 \sum_{i=1}^3 \left[\bar b_i(\alpha\xi)\xi_i + \bar b_i\left(\xi + \alpha(x-\xi)\right)(x_i - \xi_i)\right] d\alpha\right\}\right| \le$$
$$\le \frac{3T}{32\pi^2} \exp\left[3TM_0\right] \left\|b - \bar b\right\|,\ (x,\xi) \in G_T \times G_T.$$

Using (3.2), similar to the previous one, we find

$$\left|\tilde{\Psi}(x,\xi)\right| =$$

$$= \left|\Psi(x,\xi) - \bar{\Psi}(x,\xi)\right| \leq \mu_2 \left(\|b-\bar{b}\| + \|c-\bar{c}\|\right), \ (x,\xi) \in G_T \times G_T,$$

where the constant μ_2 depends on T, M_0. Thus, from (4.10) and (4.11) follows the estimate of the first term on the right side of the equation (4.2)

$$|\tilde{v}_0(x,t)| \leq \mu_3(T,M_0)d, \ (x,t) \in D_T. \tag{4.12}$$

It is quite easy to obtain estimates of the second and last terms in (4.2) by the value d

$$\left|\frac{1}{2}\int_{|x|}^{t} \tilde{\psi}_3(t-\tau)\int_0^{2\pi} d\varphi \int_{-1}^{1} \Sigma_1(x,\xi)\, dz\, d\tau\right| \leq \mu_4(T,M_0)d, \ (x,t) \in D_T, \tag{4.13}$$

$$\left|A^{[\tilde{\Sigma}_2,\tilde{\Sigma}_3]}\left[\bar{\psi}_3(t), \ \bar{\psi}_1(x,t)\right]\right| \leq \mu_5(T,M_0)d, \ (x,t) \in D_T. \tag{4.14}$$

The remaining terms in (4.2) are evaluated by the following integral expression:

$$\mu_6(T,M_0)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0, \ T].$$

So, from the above reasoning follows, that the function $\psi_1(x,t)$ admits an estimate

$$\left|\tilde{\psi_1}(x, \ t)\right| \leq \mu_7(T,M_0)d + \mu_6(T,M_0)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T]. \tag{4.15}$$

Using similar reasoning for equations (4.3)–(4.8), we find for $\tilde{\psi_j}, j = 2, \ldots, 7:$

$$\left|\tilde{\psi_2}(x, \ t)\right| \leq \mu_8(T,M_0)d + \mu_9(T,M_0)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T], \tag{4.16}$$

$$\left|\tilde{\psi_3}(t)\right| \leq \mu_{10}(T,q,M_0)d + \mu_{11}(T,q,M_0)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T], \tag{4.17}$$

$$\left|\tilde{\psi_4}(t)\right| \leq \mu_{12}(M_0)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T], \tag{4.18}$$

$$\left|\tilde{\psi_5}(t)\right| \leq \mu_{13}(M_0)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T], \tag{4.19}$$

$$\left|\tilde{\psi_6}(t)\right| \leq \int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T], \tag{4.20}$$

$$\left|\tilde{\psi_7}(t)\right| \leq \mu_{14}(q)\int\limits_0^t \omega(\tau)d\tau, \ t \in [0,T], \tag{4.21}$$

From the relations (4.15)–(4.21) it follows that $\omega(t)$ satisfies the integral inequality

$$\omega(t) \leq \mu_{00}d + \tilde{\mu}\int_0^t \omega(\tau)d\tau,\ t \in [0,\ T],$$

in which $\mu_{00} = \max(\mu_7, \mu_8, \mu_{10})$, $\tilde{\mu} = \max(1, \mu_6, \mu_9, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{14})$. Using the Gronwall inequality, we obtain an estimate

$$\omega(t) \leq \mu_{00}\exp(\tilde{\mu}t)d,\ t \in [0, T],$$

hence,

$$\|\tilde{r}(t)\|_{C^1[0,T]} \leq \mu_{00}\exp(\tilde{\mu}t)d,\ t \in [0, T]. \tag{4.22}$$

Using the Gronwall inequality, from the integral equation of the form (1.6) for $\tilde{k}(t)$ we obtain

$$\|\tilde{k}(t)\|_{C^1[0,T]} \leq \|\tilde{r}(t)\|_{C^1[0,T]}\exp(t\|\tilde{r}\|_{C^1[0,T]}),\ t \in [0, T]. \tag{4.23}$$

From the inequalities (4.22), (4.23) an estimate (4.1) follows with $\tilde{C} = \mu_{00}\exp(T(\tilde{\mu} + \|\tilde{r}\|_{C^1[0,T]}))$. □

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