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MSC 60J80ON THE MAXIMAL DISPLACEMENT OF CATALYTIC  
BRANCHING RANDOM WALK

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**ABSTRACT.** We study the distribution of the maximal displacement of particle positions for the whole time of the existence of population in the model of critical and subcritical catalytic branching random walk on  $\mathbb{Z}$ . In particular, we prove that in the case of simple symmetric random walk on  $\mathbb{Z}$ , the distribution of the maximal displacement has a "heavy" tail, decreasing as a function of the power  $1/2$  or  $1$  when the branching process is critical or subcritical, respectively. These statements describe the effects which had not arisen before in related studies on the maximal displacement of critical and subcritical branching random walks on  $\mathbb{Z}$ .

**Keywords:** catalytic branching random walk, critical regime, subcritical regime, maximal displacement, "heavy" tails.

## 1. INTRODUCTION

Problems related to the rate of propagation of population (e.g., of particles, bacteria, individuals, or genes) in space have been attracting the attention of researchers for a long time. It suffices to mention, for example, the survey [22] and the paper [21], devoted to branching random walk (BRW), or the recent works [19] and [24], in which branching Brownian motion (BBM) is investigated. Among the models describing the evolution of population in space, a special place is occupied by catalytic branching processes and, in particular, catalytic branching Brownian motion (CBBM), see, for example, [2] and [26]. In this paper, we will focus on the

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BULINSKAYA, E.V., ON THE MAXIMAL DISPLACEMENT OF CATALYTIC BRANCHING RANDOM WALK.

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study of a catalytic branching random walk (CBRW). The distinctive feature of catalytic branching processes is that they furnish the space with catalysts, and that only there a particle may produce an offspring or die. In the absence of catalysts, particles can only move in space. Thus, the evolution of a particle depends on its spatial location.

Until now, the problems of propagation of particle population have been considered in the case of *supercritical* CBRW on  $\mathbb{Z}^d$ ,  $d \in \mathbb{N}$ , see, for example, papers [17], [12] and [10]. In a supercritical regime, the particle population in CBRW survives with positive probability, and in case of survival, the total and local numbers of particles grow exponentially fast over time (see [5] and [6]). While in *critical* and *subcritical* regimes the population degenerates locally with probability 1, in some cases it can survive globally with positive probability (for global and local extinction see, for example, [1]). Therefore, in supercritical CBRW, as time grows unboundedly, the rate of propagation of population is of interest, whereas in critical and subcritical regimes the main attention is paid to the maximal displacement of particles during the whole history of the existence of the population.

As it turned out, the rate of propagation of particle population in CBRW depends essentially on “heaviness” of the distribution tails of the walk jump. For this reason, in the series of papers [7], [8] and [9], we had to consider the cases of “light” tails separately, regularly varying tails and that of semi-exponential distribution of the walk jump. In the present work, we are interested in critical and subcritical CBRW on  $\mathbb{Z}$ . Thus, in the context of research on the propagation of population, the aim of the work is to study the maximal displacement of particles for the whole history of the existence of the population.

For the distribution functions of the maximal displacement of particles for the whole history of the process we derive a system of equations which has a unique solution. In such system, there arise the probabilities related to the behavior of the random walk only on the time interval from the moment the particle leaves the catalyst until the moment when the particle returns to it for the first time, or the moment when it first hits another catalyst. Such probabilities for arbitrary random walks have not been previously studied. However, in certain significant cases of a simple random walk (i.e. when the jumps of the walk are performed to the nearest-neighbor points of the lattice  $\mathbb{Z}$ ), these probabilities can be found on the basis of solution of the classical “ruin” problem. Therefore, in this case we study the asymptotic behavior of the distribution tails of the maximal displacement of particles in critical and subcritical CBRW on  $\mathbb{Z}$ . Whenever the simple random walk has a drift, the obtained results are natural and not surprising: the distribution tail of the maximal displacement either decays exponentially fast, or the random variable under consideration is extended. However, when the simple random walk is symmetric, the new results seem unexpected and radically different from the known statements for BRW studied in the papers [15] and [18].

For the studies of CBRW focusing on areas other than the estimation of the rate of population propagation, see, for example, [13], [20] and [25].

## 2. MAIN RESULTS

The description of CBRW model on  $\mathbb{Z}$  with  $N$  catalysts, forming the set  $W = \{w_1, \dots, w_N\} \subset \mathbb{Z}$ , can be found, for example, in [12], [5], [8]. However, for the sake of the readers’ convenience, we will recall it here. Assume that all the random

variables are defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  is a sample space consisting of outcomes  $\omega$ . Moreover, the index  $x$  of the probability  $\mathbb{P}_x$  and mathematical expectation  $\mathbb{E}_x$  denote the starting point of CBRW or of the random walk, depending on the context.

Suppose that at the moment of time  $t = 0$  there is a single particle located on the lattice at a point  $z \in \mathbb{Z}$ . When  $z \notin W$ , the movement of the particle until it hits the set of catalysts  $W$  for the first time is determined by the Markov chain  $S = \{S(t), t \geq 0\}$ . The space-homogeneous random walk  $S$  is specified by the infinitesimal matrix  $Q = (q(x, y))_{x, y \in \mathbb{Z}}$ , which is assumed irreducible and conservative, i.e.

$$(1) \quad q(x, y) = q(x - y, 0) = q(0, y - x) \quad \text{and} \quad \sum_{y \in \mathbb{Z}} q(x, y) = 0,$$

where  $q(x, y) \geq 0$  for  $x \neq y$ , and  $q(x, x) \in (-\infty, 0)$  for all  $x, y \in \mathbb{Z}$ . Whenever  $z \in W$ , or when the particle hits the set  $W$  for the first time, for instance, on a catalyst  $w_k$ , where  $k = 1, \dots, N$ , the particle spends there a random time distributed exponentially with parameter  $\beta_k > 0$ . After that, with probability  $\alpha_k \in [0, 1)$ , the particle instantly produces a random number  $\xi_k$  of offsprings which are also located at  $w_k$ , and then dies. Otherwise, the particle performs a jump to point  $y$  with probability  $-(1 - \alpha_k)q(w_k, y)/q(w_k, w_k)$ , where  $y \in \mathbb{Z}$  and  $y \neq w_k$ , and continues walking until it hits the catalysts set the next time. The new particles behave as independent copies of the parent particle.

We denote the probability generating function of an arbitrary variable  $\xi_k$  by  $f_k(s) := \mathbb{E}s^{\xi_k}$ , where  $s \in [0, 1]$ , and set  $m_k := \mathbb{E}\xi_k = f'_k(1) < \infty$ , where  $k = 1, \dots, N$ . We exclude the deterministic case when  $f_k(s) = s$ , where  $s \in [0, 1]$ , for all  $k = 1, \dots, N$ .

Paper [5] proposed a classification in which CBRW is called supercritical, critical or subcritical when the Perron root (i.e., the maximal positive eigenvalue) of the matrix

$$(2) \quad D = (\delta_{i,j}\alpha_i m_i + (1 - \alpha_i)W_j F_{w_i, w_j}(\infty))_{i,j=1}^N$$

is larger than, equal to, or less than 1, respectively. Here  $\delta_{i,j} = 1$ , if  $i = j$ , and  $\delta_{i,j} = 0$  otherwise. Suppose also that  $W_j := W \setminus \{w_j\}$ . Then  $W_j F_{w_i, w_j}(\infty)$  is a probability for a random walk  $w_j$  to hit the point  $w_j$  avoiding the set  $W_j$ , whenever the starting point is  $w_i$ , where  $i, j = 1, \dots, N$ . In [6], it was established that in supercritical CBRW only the total and local numbers of particles grow exponentially fast over time, whereas the probabilities of global and local survival are positive. The propagation rate of the population of particles in supercritical CBRW was studied in works [17], [12], [7]–[10]. Since in critical and subcritical CBRW the population of particles degenerates locally contrary to what is observed for supercritical CBRW, it makes no sense to talk about the rate of propagation of population. However, one can pose a question regarding the way remote points are visited by the particles *during the whole history of the existence of population*. Our work is devoted to answering this question in cases of critical and subcritical CBRW on  $\mathbb{Z}$ .

Let  $Z(t)$  be a random set of particles existing in CBRW at time  $t \geq 0$ . For a particle  $v \in Z(t)$ , we denote by  $X^v(t)$  its position at time  $t \geq 0$ . Let  $M_t := \max\{X^v(t), v \in Z(t)\}$  be the maximum of CBRW at time  $t \geq 0$ , i.e. the position of the right-most particle existing in CBRW at time  $t$ . We are interested in a random variable  $M := \max\{M_t, t \geq 0\}$ , which is the maximal displacement (to the right

from the origin) of CBRW for the whole history of the population of the particle. Clearly,  $M \geq z$ .

In statements of Theorems 1–4, we consider a simple random walk  $S$  on the lattice  $\mathbb{Z}$ . It means that

$$\frac{q(x, x+1)}{-q(x, x)} = p, \quad \frac{q(x, x-1)}{-q(x, x)} = q, \quad q(x, y) = 0, \quad \text{for } |x-y| \geq 2,$$

where  $p+q=1$  and  $p, q \in (0, 1)$ . Such random walk is called symmetric when  $p=q$ , and asymmetric otherwise. In other words, during a single jump, a particle performing a simple random walk on  $\mathbb{Z}$  moves to the nearest point to the right with probability  $p$  and to the nearest point to the left with probability  $q$ . A simple random walk on  $\mathbb{Z}$  is recurrent if and only if it is symmetric (see, for example, [3], Theorem 13.3.1).

To prove Theorems 1–4 we derive equations (9)–(12) for the probabilities under consideration. These equations are valid for an arbitrary number of catalysts and for any random walk satisfying condition (1) (not only for a simple random walk). However, for the subsequent study of solutions of the equations we have to know the properties of the random walks which can be established easily in the case of a simple random walk and constitutes a topic of a separate study in the other case. Therefore, in the present work, our main results are based on the assumption of the random walk simplicity.

Moreover, for Theorems 1–4, we assume that the set  $W$  consists of a single catalyst located at the origin 0, and the starting point is also positioned at 0. The asymptotic results in Theorems 1–4 hold true under more general assumptions of any finite number of catalysts and an arbitrary starting point. The difference consists in constants arising in the asymptotic estimations. However, the form of these constants depends essentially on the relative location of both the starting point and the catalysts as well as the distances between them. That is why the corresponding bulky results are not reproduced here.

In the next theorem, we establish the asymptotic behavior of the distribution tail of the random variable  $M$  for a *critical* CBRW on  $\mathbb{Z}$ , in which the random walk is simple and symmetric. From here on, whenever we mention a single catalyst, we assume that, without loss of generality, it is located at 0, and omit the index 1 of the symbols  $\alpha_1$ ,  $\xi_1$ ,  $f_1$ , and  $m_1$ . As mentioned above, a simple symmetric random walk is recurrent, hence the probability of return from 0 to 0 previously denoted as  ${}_0F_{0,0}(\infty)$  equals 1. Thus, from the definition of a critical CBRW (see formula (2)) we obtain the equality  $\alpha m + (1-\alpha) {}_0F_{0,0}(\infty) = 1$  which is equivalent to  $m=1$ . In other words, for a recurrent random walk, CBRW with a single catalyst is critical if and only if the Galton-Watson branching process with an offspring number  $\xi_1$  is critical.

**Theorem 1.** *Suppose that  $f'(1) = 1$  and  $f''(1) = \sigma^2 \in (0, \infty)$  for CBRW on  $\mathbb{Z}$ , in which the random walk  $S$  is simple and symmetric. Then we have*

$$(3) \quad \mathbb{P}_0(M > x) \sim \frac{\sqrt{1-\alpha}}{\sqrt{\alpha\sigma^2}\sqrt{x}}, \quad x \rightarrow \infty.$$

The result of Theorem 1 is a counterpart of the main result of the paper [15], derived for the model of a critical BRW on  $\mathbb{Z}$ . However, in the latter model the decay rate of the probability  $\mathbb{P}_0(M > x)$  has an order  $1/x^2$  as  $x \rightarrow \infty$ . Therefore, the particles in the critical CBRW manage to go further away from the origin

before returning back and, possibly, dying, than in the model of BRW, in which the particles may die at any point.

Theorem 2 gives the solution to the same problem as in Theorem 1. The only difference is that now we consider a *subcritical* CBRW on  $\mathbb{Z}$ .

**Theorem 2.** *Suppose that  $m = f'(1) < 1$  for a CBRW on  $\mathbb{Z}$ , in which the random walk  $S$  is simple and symmetric. Then*

$$(4) \quad P_0(M > x) \sim \frac{1 - \alpha}{2\alpha(1 - m)x}, \quad x \rightarrow \infty.$$

The result of Theorem 2 is a counterpart of the main result of the paper [18], devoted to a subcritical BRW on  $\mathbb{Z}$ . However, in the latter case the probability  $P_0(M > x)$  decays exponentially fast, which significantly differs from our result. Again, this difference is related to the possibility for the particles to die at any point of the lattice in the BRW model.

Theorems 1 and 2 focus on the case of a simple *symmetric* random walk on  $\mathbb{Z}$ . The two following theorems are devoted to the investigation of critical and subcritical CBRW, in which the random walk is simple and *asymmetric*, i.e. it has a drift to the right when  $p > q$ , or to the left when  $p < q$ . Because of a drift, the random walk no longer remains recurrent. Therefore, the criticality condition of CBRW changes as well. Now we have  $r := 1 - \varnothing F_{0,0}(\infty) \in (0, 1)$ , and according to (2), the criticality of CBRW implies that  $\alpha m + (1 - \alpha)(1 - r) = 1$ , which is equivalent to  $m = 1 + r\alpha^{-1}(1 - \alpha)$ .

In the next theorem, we estimate the distribution tail of the random variable  $M$  for a *critical* CBRW on  $\mathbb{Z}$ , in which the underlying random walk is *simple and asymmetric*.

**Theorem 3.** *Suppose that  $m = 1 + r\alpha^{-1}(1 - \alpha)$  and  $f''(1) = \sigma^2 \in (0, \infty)$  for a CBRW on  $\mathbb{Z}$ , in which the random walk is simple and asymmetric. Then the following relations hold:*

$$(5) \quad P_0(M > x) \sim \frac{\sqrt{2(1 - \alpha)(q - p)}}{\sqrt{\alpha\sigma^2}} \left(\frac{p}{q}\right)^{\frac{x+1}{2}}, \quad \text{for } p < q,$$

$$(6) \quad P_0(M > x) \rightarrow s_0, \quad \text{for } p > q,$$

as  $x \rightarrow \infty$ , where  $s_0 \in (0, 1)$  is a unique solution to the equation  $\alpha(1 - f(1 - s)) + (2q(1 - \alpha) - 1)s + (1 - \alpha)(p - q) = 0$  with respect to the unknown variable  $s$ ,  $s \in [0, 1]$ .

The following result contains a solution to the problem that is the subject of Theorem 3, but now we take *subcritical* CBRW on  $\mathbb{Z}$ .

**Theorem 4.** *Suppose that  $m < 1 + r\alpha^{-1}(1 - \alpha)$  for CBRW on  $\mathbb{Z}$ , in which the random walk is simple and asymmetric. Then*

$$(7) \quad P_0(M > x) \sim \frac{(1 - \alpha)(q - p)}{1 - 2p(1 - \alpha) - \alpha m} \left(\frac{p}{q}\right)^{x+1}, \quad \text{when } p < q,$$

$$(8) \quad P_0(M > x) \rightarrow s_0, \quad \text{when } p > q,$$

as  $x \rightarrow \infty$ , where  $s_0 \in (0, 1)$  is a unique root of the equation  $\alpha(1 - f(1 - s)) + (2q(1 - \alpha) - 1)s + (1 - \alpha)(p - q) = 0$  with respect to the unknown variable  $s$ ,  $s \in [0, 1]$ .

The results of Theorems 3 and 4 are natural and expected. Namely, if the random walk  $S$  has a drift to the left ( $p < q$ ), then the particles in CBRW do not manage to get far away to the right, since they drift to the left. Conversely, if the random walk  $S$  has a drift to the right ( $p > q$ ), then there are particles in CBRW which will go away to the right to “infinity”, and therefore we will have  $M = \infty$  with positive probability  $s_0$ .

Thus, in the case of a simple random walk, we find the asymptotic behavior of probability  $P_0(M > x)$  as  $x \rightarrow \infty$  in critical and subcritical CBRW on  $\mathbb{Z}$  with a single catalyst at 0. The Theorems 3 and 4 in the case of asymmetric simple random walk are not surprising and are presented for the sake of completeness. The results of Theorems 1 and 2 describe new effects and are of the main interest. Indeed, they are radically different from the corresponding statements for BRW on  $\mathbb{Z}$  studied in [15] and [18]. The results which we obtained are pioneering in description of propagation of population in critical and subcritical CBRW. It is worth mentioning that visible differences in the propagation of particle population in supercritical CBRW and supercritical BRW were revealed only in the second term of the asymptotic expansions for the corresponding maximums (see, for example, [11], [12] and [16]). Meanwhile, our research has shown that in critical and subcritical CBRWs, and the corresponding critical and subcritical BRWs, the differences becomes noticeable even in the first asymptotic approximation of the probability  $P_0(M > x)$  as  $x \rightarrow \infty$ .

### 3. PROOFS

First of all, we recall the definition (see, for example, [4]) of hitting times under taboo, which we need for deriving equations with respect to the probability under consideration, denoted by  $P_z(M > x)$ , where  $z \in \mathbb{Z}$ . Set

$$\tau_x := \inf\{t > 0 : S(t) \neq S(0)\} \mathbb{I}(S(0) = x),$$

i.e. we introduce the exit moment of the random walk  $S$  from the starting point  $x \in \mathbb{Z}$ . As usual,  $\mathbb{I}(A)$  is an indicator of the event  $A \in \mathcal{F}$ . We denote by

$${}_H\tau_{x,y} := \inf\{t \geq \tau_x : S(t) = y, S(u) \notin H, \tau \leq u \leq t\} \mathbb{I}(S(0) = x)$$

the time when the random walk  $S$  (first) hits the point  $y \in \mathbb{Z}$  under taboo on the visit of the set  $H \subset \mathbb{Z}$ , where  $y \notin H$  and the walk starts at the point  $x \in \mathbb{Z}$ . If after the start at the point  $x$ , the trajectory of the random walk  $S(\cdot, \omega)$  visits the set  $H$  before it hits the point  $y$ , then we naturally set  ${}_H\tau_{x,y}(\omega) = \infty$ . Note that  $W_j F_{w_i, w_j}(\infty) = P_{w_i}(W_j \tau_{w_i, w_j} < \infty)$ .

**Lemma 1.** *The following system of equations holds true with respect to probabilities  $P_{w_i}(M > x)$ ,  $x \in \mathbb{Z}$ ,  $i = 1, \dots, N$ :*

$$\begin{aligned} (9) \quad & P_{w_i}(M > x) = \alpha_i (1 - f_i (1 - P_{w_i}(M > x))) \\ & + (1 - \alpha_i) \sum_{j=1}^N P_{w_i}(\max\{S(t), 0 \leq t \leq W_j \tau_{w_i, w_j}\} \leq x, W_j \tau_{w_i, w_j} < \infty) P_{w_j}(M > x) \\ & + (1 - \alpha_i) P_{w_i}\left(\max\left\{S(t), 0 \leq t \leq \min_{j=1, \dots, N} W_j \tau_{w_i, w_j}\right\} > x\right), \end{aligned}$$

where  $W_j := W \setminus \{w_j\}$ ,  $j = 1, \dots, N$ .

The case when CBRW starts at an arbitrary point  $z \in \mathbb{Z} \setminus W$  is reduced to the previous one:

$$(10) \quad \begin{aligned} \mathbb{P}_z(M > x) &= \mathbb{P}_z \left( \max \left\{ S(t), 0 \leq t \leq \min_{i=1, \dots, N} W_i \tau_{z, w_i} \right\} > x \right) \\ &+ \sum_{i=1}^N \mathbb{P}_z(\max \{S(t), 0 \leq t \leq W_i \tau_{z, w_i}\} \leq x, W_i \tau_{z, w_i} < \infty) \mathbb{P}_{w_i}(M > x), \end{aligned}$$

where, obviously,  $\mathbb{P}_z(M > x) = 1$  for  $x < z$ .

In particular, when  $W = \{0\}$ , the system of equations (9) transforms into the following equation with respect to  $\mathbb{P}_0(M > x)$ :

$$(11) \quad \begin{aligned} \mathbb{P}_0(M > x) &= \alpha(1 - f(1 - \mathbb{P}_0(M > x))) \\ &+ (1 - \alpha)\mathbb{P}_0(\max \{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) \mathbb{P}_0(M > x) \\ &+ (1 - \alpha)\mathbb{P}_0(\max \{S(t), 0 \leq t \leq \tau_{0,0}\} > x). \end{aligned}$$

The case when the start is at a point  $z \neq 0$ ,  $z \in \mathbb{Z}$ , is reduced to the previous one as well:

$$(12) \quad \begin{aligned} \mathbb{P}_z(M > x) &= \mathbb{P}_z(\max \{S(t), 0 \leq t \leq \tau_{z,0}\} > x) \\ &+ \mathbb{P}_z(\max \{S(t), 0 \leq t \leq \tau_{z,0}\} \leq x, \tau_{z,0} < \infty) \mathbb{P}_0(M > x). \end{aligned}$$

System (9) and, in particular, equation (11) have a unique solution lying on the intervals  $[0, 1]^N$  and  $[0, 1]$ , respectively.

*Proof.* To reduce the amount of the work, we consider the most illustrative case when  $W = \{0\}$  and  $z = 0$ . The rest of the proof for Lemma 1 is conducted on the basis of the ideas similar to that for the main case. By the formula of total probability and according to the description of the CBRW model, we have

$$\begin{aligned} \mathbb{P}_0(M \leq x) &= \alpha \sum_{k=0}^{\infty} \mathbb{P}(\xi = k) (\mathbb{P}_0(M \leq x))^k \\ &+ (1 - \alpha)\mathbb{P}_0(\max \{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) \mathbb{P}_0(M \leq x) \\ &+ (1 - \alpha)\mathbb{P}_0(\tau_{0,0} = \infty, S(t) \leq x, t \geq 0), \end{aligned}$$

which is equivalent to (11).

The solution of equation (11) with respect to  $\mathbb{P}_0(M > x)$  always exists and is unique, since the solution of the equation

$$\alpha(1 - f(1 - s)) = s(1 - (1 - \alpha)p_1) - (1 - \alpha)p_2$$

exists and is unique for  $s \in [0, 1]$ , where

$$\begin{aligned} p_1 &:= \mathbb{P}_0(\max \{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty), \\ p_2 &:= \mathbb{P}_0(\max \{S(t), 0 \leq t \leq \tau_{0,0}\} > x), \end{aligned}$$

and, obviously,  $p_1 + p_2 \leq 1$ . Indeed, we have  $0 = \alpha(1 - f(1)) > -(1 - \alpha)p_2$  for  $s = 0$ , and  $\alpha(1 - f(0)) \leq \alpha \leq 1 - (1 - \alpha)p_1 - (1 - \alpha)p_2$  for  $s = 1$ . Therefore, whenever at least one inequality in the latter relation is strict, the graphs of the functions  $\alpha(1 - f(1 - s))$  and  $s(1 - (1 - \alpha)p_1) - (1 - \alpha)p_2$  for  $s \in [0, 1]$  have a single (by the convexity of function  $f$ ) intersection point on the interval  $(0, 1)$ . For  $\alpha(1 - f(0)) = \alpha = 1 - (1 - \alpha)p_1 - (1 - \alpha)p_2$  (it is possible in the case of a recurrent random walk and the null probability for a particle to die without giving an offspring), the intersection

of the mentioned graphs is at point  $s = 1$  and there are no other intersection points, since  $\frac{d}{ds}(\alpha(1 - f(1 - s)))_{s=1} < \alpha < \frac{d}{ds}(s(1 - (1 - \alpha)p_1) - (1 - \alpha)p_2)_{s=1}$ .  $\square$

From (11), it follows that the asymptotic behavior of probability  $P_0(M > x)$  as  $x \rightarrow \infty$  is determined by that of  $P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty)$  and  $P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x)$ . Under the general assumptions on the random walk, these probabilities have not been studied. However, for a specific and significant case of the simple random walk, the study of such probabilities can be reduced to the classic “ruin problem” that has been already solved. In the following two lemmas, the formulae for the mentioned probabilities are derived separately for the cases of a simple symmetric and a simple asymmetric random walks.

**Lemma 2.** *For a simple symmetric random walk  $S$  on  $\mathbb{Z}$  and  $x \in \mathbb{N}$ , the following equalities hold:*

$$(13) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) = \frac{2x + 1}{2(x + 1)},$$

$$(14) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) = \frac{1}{2(x + 1)}.$$

*Proof.* A simple symmetric random walk on  $\mathbb{Z}$  is recurrent (see, for example, [3], Theorem 13.3.1). It means that  $\tau_{0,0} = \infty$  with probability 0 and

$$P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) = 1 - P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x).$$

Now we derive a formula for  $P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x)$  in the case of a simple symmetric random walk. Since in a random walk the jumps occur to the adjacent points, in a random event  $\{\omega : \max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x\}$  there are only such trajectories of  $S$  that start at point 0, then pass to point 1 and hit the point  $x + 1$  before returning to point 0. Thus, taking into account the results of the classic “ruin problem” (see, for example, [23], Ch. 1, §9, formula (14)) we come to relation (14) and, hence, to relation (13). Lemma 2 is proved completely.  $\square$

**Lemma 3.** *For a simple asymmetric random walk on  $\mathbb{Z}$ , the following formulae are valid:*

$$(15) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) = p \frac{(q/p)^{x+1} - (q/p)}{(q/p)^{x+1} - 1} + \min\{p, q\},$$

$$(16) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) = \frac{q - p}{(q/p)^{x+1} - 1}$$

for every  $x \in \mathbb{N}$ .

*Proof.* According to the total probability formula, we have

$$(17) \quad \begin{aligned} & P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) = P_0(S(t) < 0, \tau_0 \leq t < \tau_{0,0}, \tau_{0,0} < \infty) \\ & + P_0(S(t) \in (0, x], \tau_0 \leq t < \tau_{0,0}, \tau_{0,0} < \infty) \\ & = q P_{-1}(\exists t_2 : S(t_2) = 0, S(t) \neq 0, 0 \leq t < t_2) \\ & + p P_1(\exists t_1 : S(t_1) = 0, S(t) \neq 0, S(t) \neq x + 1, 0 \leq t < t_1). \end{aligned}$$

Here  $P_1(\exists t_1 : S(t_1) = 0, S(t) \neq 0, S(t) \neq x + 1, 0 \leq t < t_1)$  is a probability for the random walk  $S$  to exit the stripe  $(0, x + 1)$  through the lower boundary, when the starting point is located at 1, see [23], Ch. 1, §9, formula (13). Similarly, in the case



when the starting point is positioned at  $-1$ ,  $P_{-1}(\exists t_2 : S(t_2) = 0, S(t) \neq 0, 0 \leq t < t_2)$  is a probability that the random walk  $S$  exits a stripe  $(-\infty, 0)$  through the upper boundary. The latter probability can be found based on formula (13) from [23], Ch. 1, §9 as well, but in this case the probabilities  $p$  and  $q$  should be swapped, and we suppose that the upper boundary  $B$  tends to infinity. Returning to representation (17) and substituting the expressions obtained for the probabilities, we get

$$P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) = p \frac{(q/p)^{x+1} - (q/p)}{(q/p)^{x+1} - 1} + q \min\left\{\frac{p}{q}, 1\right\},$$

which coincides with relation (15).

In a similar way, with the help of formula (10) from [23], Ch. 1, §9, we obtain

$$\begin{aligned} &P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) \\ &= p P_1(\exists t_3 : S(t_3) = x + 1, S(t) \neq 0, S(t) \neq x + 1, 0 \leq t < t_3) = p \frac{(q/p) - 1}{(q/p)^{x+1} - 1}. \end{aligned}$$

Lemma 3 is proved completely. □

Now we turn to the proof of Theorem 1.

*Proof.* From equation (11), Lemma 2, and equality  $1 - f(1 - s) = f'(c)s$ , valid for  $s \in [0, 1]$  and some  $c \in (1 - s, 1)$ , it follows that

$$P_0(M > x) \leq (\alpha f'(c) + (1 - \alpha)) P_0(M > x) + P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x).$$

If  $x \in \mathbb{Z}$  is large enough, the latter inequality is possible only in the case when  $P_0(M > x) \rightarrow 0$  as  $x \rightarrow \infty$ . Then, according to Taylor's formula, we have

$$\begin{aligned} (18) \quad &1 - f(1 - P_0(M > x)) \\ &= f'(1)P_0(M > x) - \frac{f''(1)}{2} (P_0(M > x))^2 + o((P_0(M > x))^2). \end{aligned}$$

Hence, from equation (11) it follows that

$$\begin{aligned} (19) \quad &\frac{\alpha\sigma^2}{2} (P_0(M > x))^2 (1 + o(1)) \\ &= (1 - \alpha)P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) (1 + o(1)) \end{aligned}$$

as  $x \rightarrow \infty$ .

Relations (14) and (19) imply the statement of Theorem 1. □

Now we proceed to the proof of Theorem 2.

*Proof.* Using the same arguments as in the proof of Theorem 1, we conclude that  $P_0(M > x) \rightarrow 0$  as  $x \rightarrow \infty$ . However, for a subcritical case, we write Taylor's formula in the form

$$(20) \quad 1 - f(1 - P_0(M > x)) = f'(1)P_0(M > x) + o(P_0(M > x))$$

and, reasoning in the same manner as in the proof of Theorem 1, we get

$$\alpha(1 - m) P_0(M > x) (1 + o(1)) = (1 - \alpha)P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) (1 + o(1))$$

as  $x \rightarrow \infty$ . From here, the statement of Theorem 2 follows. □

Recall that  $r$  is a probability that the random walk  $S$  starting from 0 will not return to point 0. We will now prove Theorem 3.

*Proof.* If  $p < q$ , then according to Lemma 3, the following relations hold:

$$(21) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) \rightarrow 2p,$$

$$(22) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) \sim (q - p) \left(\frac{p}{q}\right)^{x+1}$$

as  $x \rightarrow \infty$ . Moreover,

$$P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) \rightarrow P_0(\tau_{0,0} < \infty) = 1 - r, \quad x \rightarrow \infty.$$

Consequently,  $r = 1 - 2p$  for  $p < q$ . By virtue of equation (11), formulae (21), (22), and equality  $1 - f(1 - s) = f'(c)s$  that is valid for  $s \in [0, 1]$  and some  $c \in (1 - s, 1)$ , we get

$$P_0(M > x) \leq (\alpha f'(c) + (1 - \alpha)(1 - r)) P_0(M > x) + P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x).$$

It follows that  $P_0(M > x) \rightarrow 0$  as  $x \rightarrow \infty$ . Using equation (11), relations (21), (22), and Taylor's formula in the form (18) once again, we come to statement (5).

If  $p > q$ , then Lemma 3 implies that

$$(23) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} \leq x, \tau_{0,0} < \infty) \rightarrow 2q$$

and

$$(24) \quad P_0(\max\{S(t), 0 \leq t \leq \tau_{0,0}\} > x) \rightarrow p - q$$

as  $x \rightarrow \infty$ . Then statement (6) follows from equation (11) and the reasoning on the existence and uniqueness of solution to this equation in the proof of Lemma 1. Theorem 3 is proved completely.  $\square$

It only remains to give the proof of Theorem 4.

*Proof.* Employing the same arguments as in the beginning of the proof of Theorem 3, we come to a conclusion that  $P_0(M > x) \rightarrow 0$  as  $x \rightarrow \infty$ . Then applying relations (11), (20), (21), and (22), we get formula (7).

Statement (8) follows from relations (11), (23), (24) and the reasoning on the existence and uniqueness of solution to equation (11), in the proof of Lemma 1. Theorem 4 is proved completely.  $\square$

In conclusion, we want to make a remark regarding the general case of an arbitrary finite number of catalysts in the critical CBRW on  $\mathbb{Z}$ . To investigate the asymptotic behavior of the solution to the system of equations (9), we have to implement equivalent transformations of the system according to Cramer's rule (see, for example, [14], Ch. 1, §7), which results in that the coefficient before  $P_{w_i}(M > x)$  for every  $i = 1, \dots, N$  equals the determinant of the matrix  $D - I$ , where the matrix  $D$  is specified in the definition of the critical regime and  $I$  is the identity matrix. However, in the critical case  $\det(D - I) = 0$ . Therefore, similarly to the case of a single catalyst, all linear terms are reduced and only quadratic terms are left, including  $(P_{w_i}(M > x))^2$ . Other differences in the study of the solutions to equation (11) and system of equations (9) are not significant and hence we do not discuss them.

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