Special values for the generalized Euler-Zagier-Hurwitz type of multiple zeta functions at non-positive integers

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Abstract

The purpose of this work is to calculate the values for the generalized Euler-Zagier-Hurwitz type of multiple zeta functions at non-positive integers by using the *Raabe*'s formula and the *Bernoulli* numbers.

Mathematics Subject Classifications: 11M32; 11M41.
Key words: Euler-Zagier-Hurwitz type of multiple zeta functions; integral representation; special values; Bernoulli numbers; Raabe's formula.

Introduction and notations

Let n an positive integer, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $\underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ such that

$$\Re(\beta_j) > 0 \qquad \text{and} \qquad \Re(\alpha_j) > -\Re(\beta_j) \qquad \forall\, 1 \leq j \leq n$$

The generalized Euler-Zagier-Hurwitz type of multiple zeta function is defined for $\underline{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$, by

$$\zeta_n(\underline{\alpha}; \underline{\beta}; \underline{s}) = \sum_{\underline{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{1}{(\beta_1 m_1 + \cdots + \beta_i m_i + \alpha_i)^{s_i}} \tag{0.1}$$

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This series converges absolutely in the domain

$$\mathcal{D}_n = \{(s_1, \ldots, s_n) \in \mathbb{C}^n : \Re(s_{n-k+1+\cdots+s_n}) > k \qquad (1 \le j \le n)\} \tag{0.2}$$

and uniformly in any compact of $\mathcal{D}_n$ (see [15]). In the case where $\beta_j = 1$ for all $1 \le j \le n$, the series (0.1) coincide with the multiple Hurwitz zeta function

$$\zeta_n(\underline{\alpha}; s_1, \ldots, s_n) = \sum_{\underline{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \frac{1}{(m_1+\alpha_1)^{s_1} \ldots (m_1+\cdots+m_n+\alpha_n)^{s_n}} \tag{0.3}$$

which introduced by Akiyama and Ishikawa and proved its analytic continuation to $\mathbb{C}^n$ [2]. Matsumoto and Tanigawa proved the analytic continuation of wide class of multiple Dirichlet series and multiple Hurwitz zeta functions in [13] and [14], and the analytic continuation of the series (0.3) is a special case of [13, Theorem 1].

In the case where $\alpha_j = j$ and $\beta_j = 1$ for all $1 \le j \le n$, the series (0.1) is just the multiple zeta function.

In this paper, we consider the generalized Euler-Zagier-Hurwitz type of multiple zeta function, given by the following series

$$\zeta_n(\underline{\alpha}; \underline{\beta}; \underline{s}) = \sum_{\underline{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{1}{(\beta_1 m_1 + \cdots + \beta_i m_i + \sum_{j=1}^{i} \alpha_j)^{s_i}} \tag{0.4}$$

where, $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ verified some conditions.

Historically, Akiyama, Arakawa, Egami, Ishikawa, Kaneko, Matsumoto, Tanigawa and Zaho give the meromorphic continuation of the Euler-Zagier multiple zeta function [12]. In addition, Akiyama, Egami and Tanigawa studied multiple zeta values at non-positive integers in [2].

In recent years, many relations among zeta values were discovered by a lot of mathematicians, for example, Hoffman, Kaneko, Ohno, Zagier and the author (see Hoffman's web page for more references).

Our main result in this work is the values at non positive integers of the series (0.4). The key of this study is the use of the *Raabe* formula [7] which expresses the integral in terms of the sum.

1 Main results

For real numbers $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, such that, for all $1 \le i \le n$:

$$\sum_{j=1}^{i} \alpha_j \neq 0, -1, -2, ...,$$

$\underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}_+^n$, such that, for all $1 \le i \le n$:

$$\beta_i \neq 0$$

and $\underline{\alpha}$ and $\underline{\beta}$ verified the condition

$$\beta_1 m_1 + \cdots + \beta_i m_i + \alpha_i \neq 0, \qquad \forall\, 1 \le i \le n \tag{1.1}$$

So, for a complex $n-$tuples $\underline{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$, we define the generalized Euler-Zagier-Hurwitz type of multiple zeta function by

$$\begin{aligned} \zeta_n(\underline{\alpha}; \underline{\beta}; \underline{s}) &:= \zeta(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; s_1, \ldots, s_n) \\ &= \sum_{\underline{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{1}{(\beta_1 m_1 + \cdots + \beta_i m_i + \sum_{j=1}^{i} \alpha_j)^{s_i}} \end{aligned} \tag{1.2}$$

and the corresponding integral function associated to the generalized Euler-Zagier-Hurwitz type of multiple zeta function by

$$Y_n(\underline{\alpha}; \underline{\beta}; \underline{s}) = \int_{[0,+\infty[^n} \prod_{i=1}^{n} \frac{1}{(\beta_1 x_1 + \cdots + \beta_i x_i + \sum_{j=1}^{i} \alpha_j)^{s_i}} d\underline{x}. \tag{1.3}$$

Remark 1.1. *We remark that:*

- *If $\underline{\beta} = (1, \ldots, 1)$, then the series (1.2) reduces to the generalized multiple Hurwitz zeta function.*
- *If $\underline{\alpha} = (\alpha, \alpha, \ldots, \alpha)$ and $\underline{\beta} = (1, \ldots, 1)$, then the series (1.2) corresponding to the classical multiple Hurwitz zeta function.*
- *If $\underline{\alpha} = (1, 1, \ldots, 1)$ and $\underline{\beta} = (1, \ldots, 1)$, then the series (1.2) corresponding to the multiple zeta function.*

We recall that the series (1.2) converges absolutely in the domain

$$\{\underline{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n : \quad \Re(s_j + \cdots + s_n) > n + 1 - j \quad \forall\, j = 1, \ldots, n\}$$

and has a meromorphic continuation to $\mathbb{C}^n$ (for this, we refer the reader to [13, 15]). For the meromorphic continuation of the integral (1.3), it just apply the *Raabe* formula (see the point (1) of Proposition 4.1).

We first give well-known elementary result for the integral function.

Lemma 1.1.

Let $\underline{\boldsymbol{N}} = (N_1, \ldots, N_n)$ *be a point of* $\mathbb{N}^n$*,*

(1) *The point* $(\underline{\boldsymbol{s}} = -\underline{\boldsymbol{N}})$ *is a polar divisor for the function* $Y_n(\underline{\alpha}; \underline{\beta}; \underline{\boldsymbol{s}})$ *if and only if there exists a* $\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1}$ *such that*

$$(s_n - 1)(s_n + s_{n-1} - 2 + k_n) ... \left(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i \right) = \prod_{j=1}^{n} \left(\sum_{i=j}^{n} s_i - n + j - 1 + \sum_{i=j+1}^{n} k_i \right) = 0. \tag{1.4}$$

(2) *If* $(\underline{\boldsymbol{s}} = -\underline{\boldsymbol{N}})$ *is not a polar divisor for the integral function, then the value of this function at this point exists and is given by*

$$Y_n(\underline{\alpha}; \underline{\beta}; \underline{\boldsymbol{s}}) = (-1)^n \left(\prod_{i=1}^n \beta_i^{-1} \right)$$
$$\sum_{\underline{k}=(k_2,\ldots,k_n) \in T(\underline{\boldsymbol{N}})} \frac{\binom{N_n+1}{k_n} \binom{N_n+N_{n-1}+2-k_n}{k_{n-1}} \cdots \binom{\sum_{i=2}^n N_i + n - \sum_{i=3}^n k_i}{k_2} \alpha_1^{\left(-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i\right)}}{\prod_{j=1}^n \left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i \right)} \prod_{j=2}^n \alpha_j^{k_j}$$

with

$$T(\underline{\boldsymbol{N}}) := \left\{ \underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \quad 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall\, 2 \le j \le n \right\}.$$

We give now a similar result for the generalized Euler-Zagier-Hurwitz type of multiple zeta function.

Theorem 1.

Let $\underline{\boldsymbol{N}} = (N_1, \ldots, N_n)$ *a point of* $\mathbb{N}^n$*, if the point* $(\underline{\boldsymbol{s}} = -\underline{\boldsymbol{N}})$ *is not a polar divisor for the integral function* $Y_n(\underline{\alpha}; \underline{\beta}; \underline{\boldsymbol{s}})$*, then the value of the generalized Euler-Zagier-Hurwitz type of multiple zeta function* $\zeta_n(\underline{\alpha}; \underline{\beta}; \underline{\boldsymbol{s}})$ *at the point* $(\underline{\boldsymbol{s}} = -\underline{\boldsymbol{N}})$ *exists and is given by*

$$\zeta_n(\underline{\alpha}; \underline{\beta}; -\underline{\boldsymbol{N}}) = (-1)^n \left(\prod_{i=1}^n \beta_i^{-1} \right)$$
$$\sum_{\underline{k}=(k_2,\ldots,k_n) T(\underline{\boldsymbol{N}})} \sum_{\substack{\underline{v}=(v_1,\ldots,v_n)\in\mathbb{N}^n \\ v_j \le k_j \ \forall\, 2 \le j \le n;\, v_1 \le \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} A(-\underline{\boldsymbol{N}})\, B_{\underline{v}} \prod_{j=1}^n \frac{1}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)} \tag{1.5}$$

with

$$A(-\underline{\boldsymbol{N}}) =$$
$$\binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \alpha_1^{\left(\sum_{i=1}^n N_i + n - v_1 - \sum_{i=2}^n k_i\right)} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j} \alpha_j^{k_j - v_j} \tag{1.6}$$

and

$$T(\underline{\boldsymbol{N}}) := \left\{ \underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \quad 0 \leq k_j \leq \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall\, 2 \leq j \leq n \right\}.$$

and

$$B_{\underline{v}} = \prod_{j=1}^{n} B_{v_j}$$

where B_{v_j} *is the* ${v_j}^{-th}$ *Bernoulli number.*

2 Proof of lemma 1.1

Let the integral function

$$Y_n(\underline{\alpha}; \underline{\beta}; \mathbf{\underline{s}}) = \int_{[0,+\infty[^n} \prod_{i=1}^{n} (\beta_1 x_1 + \cdots + \beta_i x_i + \sum_{j=1}^{i} \alpha_j)^{-s_i} \, d\underline{x}. \tag{2.1}$$

The change of variables:

$$t_i = \beta_i x_i \tag{2.2}$$

for all $1 \leq i \leq n$, gives

$$Y_n(\underline{\alpha}; \underline{\beta}; \mathbf{\underline{s}}) = \int_{[0,+\infty[^n} \prod_{i=1}^{n} \beta_i^{-1} \, (t_1 + \cdots + t_i + \sum_{j=1}^{i} \alpha_j)^{-s_i} \, d\underline{t}. \tag{2.3}$$

where, $\underline{t} = (t_1, \ldots, t_n)$.
Now, if we use the following change of variables:

$$y_i = t_i + \alpha_i \tag{2.4}$$

for all $1 \leq i \leq n$, we find

$$Y_n(\underline{\alpha}; \underline{\beta}; \mathbf{\underline{s}}) = \left(\prod_{i=1}^{n} \beta_i^{-1} \right) \int_{\prod_{i=1}^{n} [\alpha_i,+\infty[^n} \prod_{i=1}^{n} (y_1 + \cdots + y_i)^{-s_i} \, d\underline{y}. \tag{2.5}$$

thus, the following change of variables:

$$z_i = y_1 + \cdots + y_i - \sum_{j=2}^{i} \alpha_j \tag{2.6}$$

for all $1 \leq i \leq n$. gives

$$\begin{cases} y_1 = z_1 \\ y_i = z_i - z_{i-1} + \alpha_i, \quad \forall\, 2 \leq i \leq n \end{cases} \tag{2.7}$$

Since $\underline{y} = (y_1, ..., y_n) \in \prod_{i=1}^{n} [\alpha_i, +\infty[$, this gives

$$\underline{z} \in V_n = \{\underline{z} \in \mathbb{R}^n : \quad \alpha_1 \leq z_1 \leq z_2 \leq \cdots \leq z_n\} \tag{2.8}$$

and, we find

$$Y_n(\underline{\alpha}; \underline{\beta}; \mathbf{\underline{s}}) = \left(\prod_{i=1}^{n} \beta_i^{-1} \right) \int_{V_n} \prod_{i=1}^{n} (z_i + \sum_{j=2}^{i} \alpha_j)^{-s_i} \; d\underline{z}. \tag{2.9}$$

This integral can be rewritten as follows.

$$Y_n(\underline{\alpha}; \underline{\beta}; \mathbf{\underline{s}}) = \left(\prod_{i=1}^{n} \beta_i^{-1} \right) \int_{V_{n-1}} \prod_{i=1}^{n-1} (z_i + \sum_{j=2}^{i} \alpha_j)^{-s_i} \; \left(\int_{z_{n-1}}^{+\infty} (z_n + \sum_{j=2}^{n} \alpha_j)^{-s_n} \; dz_n \right) dz_1 ... dz_{n-1} \tag{2.10}$$

with

$$\begin{aligned} \int_{z_{n-1}}^{+\infty} (z_n + \textstyle\sum_{j=2}^{n} \alpha_j)^{-s_n} \; dz_n \quad &= \quad \frac{(z_{n-1} + \sum_{j=2}^{n-1} \alpha_j)^{-s_n+1}}{s_n - 1} \left(1 + \frac{\alpha_n}{z_{n-1} + \sum_{j=2}^{n-1} \alpha_j} \right)^{-s_n+1} \\ &= \quad \textstyle\sum_{k_n \in \mathbb{N}} \binom{-s_n+1}{k_n} \frac{(z_{n-1} + \sum_{j=2}^{n-1} \alpha_j)^{-s_n+1-k_n}}{s_n - 1} \; \alpha_n^{k_n} \end{aligned} \tag{2.11}$$

if and only if $\Re(s_n) - 1 > 0$.
Inductively on n, we find

$$Y_n(\underline{\alpha}; \underline{\beta}; \mathbf{\underline{s}}) = \left(\textstyle\prod_{i=1}^{n} \beta_i^{-1} \right)$$

$$\sum_{\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1}} \frac{\binom{-s_n+1}{k_n} \binom{-s_n - s_{n-1} + 2 - k_n}{k_{n-1}} ... \binom{-\sum_{i=2}^{n} s_i + n - \sum_{i=3}^{n} k_i}{k_2} \alpha_1^{(-\sum_{i=1}^{n} s_i + n - \sum_{i=2}^{n} k_i)}}{(s_n - 1)(s_n + s_{n-1} - 2 + k_n) ... \left(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i \right)} \prod_{j=2}^{n} \alpha_j^{k_j}$$

if and only if for all $1 \leq i \leq n-1$

$$\Re\left(\sum_{i=1}^{n} s_i \right) - n + j - 1 + \sum_{i=2}^{n} k_i > 0 \tag{2.12}$$

and

$$\Re(s_n) - 1 > 0. \tag{2.13}$$

Therefore, for any point $\mathbf{\underline{N}} = (N_1, \ldots, N_n) \in \mathbb{N}^n$

1) The point ($\underline{\mathbf{s}} = -\underline{\mathbf{N}}$) is a polar divisor for the function $Y_n(\underline{\alpha};\underline{\mathbf{s}})$ if there exists a $\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1}$ such that

$$(s_n-1)(s_n+s_{n-1}-2+k_n)...\left(\sum_{i=1}^{n} s_i - n + \sum_{i=2}^{n} k_i\right) = \prod_{j=1}^{n}\left(\sum_{i=j}^{n} s_i - n + j - 1 + \sum_{i=j+1}^{n} k_i\right) = 0. \tag{2.14}$$

2) If ($\underline{\mathbf{s}} = -\underline{\mathbf{N}}$) is not a polar divisor we get

$$\binom{N_n+1}{k_n}...\binom{\sum_{i=2}^{n} N_i + n - \sum_{i=3}^{n} k_i}{k_2} = \prod_{j=2}^{n}\binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} = 0 \tag{2.15}$$

if and only if there exists an $\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1}$ and $2 \le j \le n$, such that

$$k_j > \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i.$$

Let

$$T(\underline{\mathbf{N}}) := \left\{\underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \quad 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall\, 2 \le j \le n\right\} \tag{2.16}$$

which is finite, then

$$Y_n(\underline{\alpha};\underline{\beta};\underline{\mathbf{s}}) = (-1)^n\left(\prod_{i=1}^{n}\beta_i^{-1}\right)$$

$$\sum_{\underline{k}=(k_2,...,k_n)\in T(\underline{\mathbf{N}})} \frac{\binom{N_n+1}{k_n}\binom{N_n+N_{n-1}+2-k_n}{k_{n-1}}...\binom{\sum_{i=2}^{n} N_i+n-\sum_{i=3}^{n} k_i}{k_2}\alpha_1^{\left(-\sum_{i=1}^{n} s_i+n-\sum_{i=2}^{n} k_i\right)}}{\prod_{j=1}^{n}\left(\sum_{i=j}^{n} N_i+n-j+1-\sum_{i=j+1}^{n} k_i\right)} \prod_{j=2}^{n}\alpha_j^{k_j}$$

3 Key Proposition

For $\underline{a} = (a_1, ..., a_n) \in \mathbb{R}_+^n$ and $\underline{\mathbf{s}} = (s_1, ..., s_n) \in \mathbb{C}^n$, we define the function

$$Y_{n,\underline{a}}(\underline{\alpha};\underline{\beta};\underline{\mathbf{s}}) = \left(\prod_{i=1}^{n}\beta_i^{-1}\right)\int_{\prod_{i=1}^{n}[\alpha_i,+\infty[^n}\prod_{i=1}^{n}(x_1 + ... + x_i + a_1 + ... + a_i)^{-s_i}\, d\underline{x}. \tag{3.1}$$

We prove the following useful result.

Proposition 3.1. *Let $\underline{\boldsymbol{N}} = (N_1, ..., N_n)$ a point of $\mathbb{N}^n$, then we have for $\underline{a} \in \mathbb{R}^+$*

$$Y_{n,\underline{a}}(\underline{\alpha};\underline{\beta};-\underline{\boldsymbol{N}}) = (-1)^n\left(\prod_{i=1}^{n}\beta_i^{-1}\right)\sum_{\underline{k}=(k_2,...,k_n)\in\mathbb{N}^{n-1}}$$

$$\sum_{\substack{\underline{v}=(v_1,...,v_n)\in\mathbb{N}^n \\ v_j\le k_j\ \forall\, 2\le j\le n;\, v_1\le\left(-\sum_{i=1}^{n} s_i+n-\sum_{i=2}^{n} k_i\right)}} \frac{A(-\underline{\boldsymbol{N}})\, a_1^{v_1}}{\left(\sum_{i=1}^{n} N_i+n-\sum_{i=2}^{n} k_i\right)}\prod_{j=2}^{n}\frac{a_j^{v_j}}{\left(\sum_{i=j}^{n} N_i+n-j+1-\sum_{i=j+1}^{n} k_i\right)} \tag{3.2}$$

with

$$A(-\underline{\boldsymbol{N}}) = \binom{\sum_{i=1}^n N_i+n-\sum_{i=2}^n k_i}{v_1} \alpha_1^{\left(\sum_{i=1}^n N_i+n-v_1-\sum_{i=2}^n k_i\right)} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j} \alpha_j^{k_j-v_j} \tag{3.3}$$

and

$$T(\underline{\boldsymbol{N}}) := \left\{ \underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \quad 0 \leq k_j \leq \sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i, \ \forall\, 2 \leq j \leq n \right\} \tag{3.4}$$

Proof. Let $\underline{a} \in \mathbb{R}_+^n$, such that for all $\underline{x} = (x_1, ..., x_n) \in [\alpha, +\infty[^n$ and for all $1 \leq i \leq n$

$$\frac{\alpha_i + a_i}{x_1 + ... + x_{i-1} + a_1 + ... + a_{i-1}} < 1, \tag{3.5}$$

we have

$$Y_{n,\underline{a}}(\underline{\alpha}; \underline{\beta}; \mathbf{s}) = \left(\prod_{i=1}^n \beta_i^{-1} \right) \int_{\prod_{i=1}^n [\alpha_i, +\infty[^n} \prod_{i=1}^n (x_1 + ... + x_i + a_1 + ... + a_i)^{-s_i} \, d\underline{x}. \tag{3.6}$$

This integral can be written as follows

$$\begin{aligned} Y_{n,\underline{a}}(\underline{\alpha}; \underline{\beta}; \mathbf{s}) &= \left(\prod_{i=1}^n \beta_i^{-1} \right) \int_{\prod_{i=1}^{n-1} [\alpha_i, +\infty[^{n-1}} \prod_{i=1}^{n-1} (x_1 + ... + x_i + a_1 + ... + a_i)^{-s_i} \\ &\quad \times \left(\int_{\alpha_n}^{+\infty} (x_1 + ... + x_n + a_1 + ... + a_n)^{-s_n} \, dx_n \right) dx_1 ... dx_{n-1} \end{aligned}$$

Since for $\Re(s_n) > 1$ we have

$$\int_{\alpha_n}^{+\infty} (x_1 + ... + x_n + a_1 + ... + a_n)^{-s_n} \, dx_n = \frac{(x_1 + ... + x_{n-1} + a_1 + ... a_{n-1} + \alpha_n + a_n)^{-s_n+1}}{s_n - 1} \tag{3.7}$$

condition (3.5) yields

$$\begin{aligned} &\int_{\alpha_n}^{+\infty} (x_1 + ... + x_n + a_1 + ... + a_n)^{-s_n} \, dx_n \\ &= \sum_{k_n \in \mathbb{N}} \binom{-s_n+1}{k_n} \frac{(\alpha_n + a_n)^{k_n}}{s_n - 1} (x_1 + ... + x_{n-1} + a_1 + ... a_{n-1})^{-s_n+1-k_n} \end{aligned} \tag{3.8}$$

If for $1 \leq j \leq n-1$

$$\left(\sum_{i=j}^n \Re(s_i) - n + j - 1 + \sum_{i=j+1}^n k_i \right) > 0 \tag{3.9}$$

then inductively we find

$$Y_{n,\underline{a}}(\underline{\alpha};\underline{\beta};\underline{\mathbf{s}}) = (-1)^n \left(\prod_{i=1}^n \beta_i^{-1}\right) \sum_{\underline{k}=(k_2,\dots,k_n)\in\mathbb{N}^{n-1}} \frac{(\alpha_1+a_1)^{-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i}}{\left(-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i\right)}$$
$$\times \prod_{j=2}^n \binom{-\sum_{i=j}^n s_i+n-j+1-\sum_{i=j+1}^n k_i}{k_j} \frac{(\alpha_j+a_j)^{k_j}}{\left(-\sum_{i=j}^n s_i+n-j+1-\sum_{i=j+1}^n k_i\right)} \tag{3.10}$$

But, for all $2 \leq j \leq n$ we have

$$(\alpha_j + a_j)^{k_j} = \sum_{\substack{v_j\in\mathbb{N} \\ v_j\leq k_j}} \binom{k_j}{v_j} \alpha_j^{k_j-v_j} a_j^{v_j} \tag{3.11}$$

and

$$(\alpha_1 + a_1)^{-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i} =$$
$$\sum_{\substack{v_1\in\mathbb{N} \\ v_1\leq\left(-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i\right)}} \binom{-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i}{v_1} \alpha_1^{\left(-\sum_{i=1}^n s_i+n-v_1-\sum_{i=2}^n k_i\right)} a_1^{v_1} \tag{3.12}$$

which yields

$$Y_{n,\underline{a}}(\underline{\alpha};\underline{\beta};\underline{\mathbf{s}}) = (-1)^n \left(\prod_{i=1}^n \beta_i^{-1}\right) \sum_{\underline{k}=(k_2,\dots,k_n)\in\mathbb{N}^{n-1}}$$
$$\sum_{\substack{\underline{v}=(v_1,\dots,v_n)\in\mathbb{N}^n \\ v_j\leq k_j\ \forall\ 2\leq j\leq n;\, v_1\leq\left(-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i\right)}} \frac{A(\underline{\mathbf{s}})\, a_1^{v_1}}{\left(-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i\right)} \prod_{j=2}^n \frac{a_j^{v_j}}{\left(-\sum_{i=j}^n s_i+n-j+1-\sum_{i=j+1}^n k_i\right)} \tag{3.13}$$

with

$$A(\underline{\mathbf{s}}) =$$
$$\binom{-\sum_{i=1}^n s_i+n-\sum_{i=2}^n k_i}{v_1} \alpha_1^{\left(-\sum_{i=1}^n s_i+n-v_1-\sum_{i=2}^n k_i\right)} \prod_{j=2}^n \binom{-\sum_{i=j}^n s_i+n-j+1-\sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j} \alpha_j^{k_j-v_j} \tag{3.14}$$

Setting $\underline{\mathbf{s}} = -\underline{\mathbf{N}} = -(N_1, ..., N_n) \in \mathbb{N}^n$ yields (3.2) and ends the proof of Proposition 3.1. □

4 Proof of Theorem 1

The proof relies on the *Raabe* formula [7], which expresses the integral in terms of the sum.

Proposition 4.1.

(1) *Raabe formula:*

for all $\underline{\boldsymbol{s}} \in \mathbb{C}^n$*, outside the possible polar divisors of* $Y_n(\underline{\alpha};\underline{\beta};\underline{\boldsymbol{s}})$*, we have:*

$$Y_n(\underline{\alpha};\underline{\beta};\underline{\boldsymbol{s}}) = \int_{\underline{\boldsymbol{t}}\in[0,1]^n} \zeta_{n,\underline{\boldsymbol{t}}}(\underline{\alpha};\underline{\beta};\underline{\boldsymbol{s}})\, d\underline{\boldsymbol{t}} \tag{4.1}$$

where:

$$\zeta_{n,\underline{\boldsymbol{t}}}(\underline{\alpha};\underline{\beta};\underline{\boldsymbol{s}}) = \sum_{\underline{m}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{1}{((\beta_1 m_1 + t_1 + \alpha_1) + \cdots + (\beta_i m_i + t_i + \alpha_i))^{s_i}}$$

and $d\underline{\boldsymbol{t}}$ *is the Lebesgue measure on* $\mathbb{R}^n$*.*

(2) For a fixed point $\underline{\boldsymbol{N}} = (N_1, ..., N_n)$ *in* $\mathbb{N}^n$ *the maps* $\underline{\boldsymbol{a}} \mapsto Y_{n,\underline{a}}(\underline{\alpha};\underline{\beta};-\underline{\boldsymbol{N}})$ *and* $\underline{\boldsymbol{a}} \mapsto \zeta_{n,\underline{\boldsymbol{t}}}(\underline{\alpha};\underline{\beta};-\underline{\boldsymbol{N}})$ *are polynomials in* $\underline{\boldsymbol{a}} = (a_1, ..., a_n) \in \mathbb{R}_+^n$*.*

Proof.

(1) Let $\underline{\mathbf{s}} \in \mathbb{C}^n$ be chosen in such a way that the integral function (1.3) and the series (1.2) are absolutely convergent.
Thus, for $\underline{\mathbf{t}} \in \mathbb{R}_+^n$, we have:

$$\begin{aligned}
&\int_{[0,1]^n} \zeta_{n,\underline{\mathbf{t}}}(\underline{\alpha};\underline{\beta};\underline{\mathbf{s}})\, d\underline{\mathbf{t}} \\
&= \int_{[0,1]^n} \sum_{\underline{\mathbf{m}}\in\mathbb{N}^n} \prod_{i=1}^{n} (t_1 + \cdots + t_i + \beta_1 m_1 + \cdots + \beta_i m_i + \sum_{j=1}^{i} \alpha_j)^{-s_i}\, d\underline{\mathbf{t}} \\
&= \sum_{\underline{\mathbf{m}}\in\mathbb{N}^n} \int_{\prod_{i=1}^n [m_i, m_i+1]} \prod_{i=1}^{n} (t_1 + \beta_1 m_1 + \cdots + t_i + \beta_i m_i + \sum_{j=1}^{i} \alpha_j)^{-s_i}\, d\underline{\mathbf{t}} \\
&= \int_{[0,+\infty[^n} \prod_{i=1}^{n} (\beta_1 x_1 + \cdots + \beta_i x_i + \alpha_i)^{-s_i}\, d\underline{\mathbf{x}} = Y_n(\underline{\alpha};\underline{\beta};\underline{\mathbf{s}}).
\end{aligned}$$

This last equality which is verified for all $\underline{\mathbf{s}} \in \mathbb{C}^n$ follows by analytic continuation outside the polar divisors.

(2) follows from (3.2) combined with the *Raabe* formula.

□

Lemma 4.1 ([6])**.**
Let P *and* Q *to be two polynomials in* n *variables linked by*

$$P(\underline{\boldsymbol{a}}) = \int_{\underline{\boldsymbol{t}}\in[0,1]^n} Q(\underline{\boldsymbol{a}} + \underline{\boldsymbol{t}})\, d\underline{\boldsymbol{t}}. \tag{4.2}$$

Write out

$$P(\underline{\boldsymbol{a}}) = P(a_1, ..., a_n) = \sum_{\underline{\boldsymbol{L}}} h_{\underline{\boldsymbol{L}}} \prod_{i=1}^{n} a_i^{L_i} \tag{4.3}$$

where $h_{\underline{\boldsymbol{L}}} \in \mathbb{C}$ *and* $\underline{\boldsymbol{L}} = (L_1, ..., L_n) \in \mathbb{N}^n$ *ranges over a finite set of multi-index. Then*

$$Q(\underline{\boldsymbol{a}}) = Q(a_1, ..., a_n) = \sum_{\underline{\boldsymbol{L}}} h_{\underline{\boldsymbol{L}}} \prod_{i=1}^{n} B_{L_i}(a_i) \tag{4.4}$$

where the $B_{L_i}(a_i)$ *are the Bernoulli polynomials [3].*
Conversely, if Q *is given by (4.4), then the relations (4.2) and (4.3) yield equivalent formulas for the polynomial* P*.*

Proof. Let $V := V_{m,n}$ be the finite-dimensional complex space of polynomials in n variables $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ with complex coefficients and having degree at most m. Note that both $\{\underline{\mathbf{a}}^{\underline{\mathbf{L}}}\}_{\underline{\mathbf{L}}}$ and $\{B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})\}_{\underline{\mathbf{L}}}$ are $\mathbb{C}-$bases of V. Here $\underline{\mathbf{L}} = (L_1, \ldots, L_n)$ ranges over all multi-indices with $|\underline{\mathbf{L}}| := \sum_{i=1}^{n} L_i \leq m$ and $\underline{\mathbf{a}}^{\underline{\mathbf{L}}} := \prod_{i=1}^{n} a_i^{L_i}$. That $\{B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})\}_{\underline{\mathbf{L}}}$ is a basis of $V := V_{m,n}$ can be proved by induction on m, since $\underline{\mathbf{a}}^{\underline{\mathbf{L}}} - B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})$ has degree strictly less than $|\underline{\mathbf{L}}|$. Let $f : V \longrightarrow V$ be the $\mathbb{C}-$map taking $Q = Q(\underline{\mathbf{a}}) \in V$ to

$$f(Q)(\underline{\mathbf{a}}) := \int_{\underline{\mathbf{t}} \in [0,1]^n} Q(\underline{\mathbf{a}} + \underline{\mathbf{t}}) d\underline{\mathbf{t}}$$

We can restated the Lemma as saying that the inverse map to f exists and takes $\underline{\mathbf{a}}^{\underline{\mathbf{L}}}$ to $B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})$.
Hence, it will suffice to show that $f(B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})) = \underline{\mathbf{a}}^{\underline{\mathbf{L}}}$, for then f is an isomorphism (it takes one basis to another). Using [4, p.4] and [9, pp.66-67]

$$\frac{d}{dx} B_{j+1}(x) = (j+1) B_j(x) \qquad \text{and} \qquad B_j(x+1) - B_j(x) = j x^{j-1}$$

we calculate

$$\begin{aligned} f(B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}})) &= \int_{\underline{\mathbf{t}} \in [0,1]^n} B_{\underline{\mathbf{L}}}(\underline{\mathbf{a}} + \underline{\mathbf{t}}) d\underline{\mathbf{t}} = \prod_{i=1}^{n} \int_0^1 B_{L_i}(a_i + t_i) dt_i \\ &= \prod_{i=1}^{n} \frac{1}{L_i + 1} \left(B_{L_i+1}(a_i + 1) - B_{L_i+1}(a_i) \right) = \prod_{i=1}^{n} a_i^{L_i} \end{aligned}$$

Which concludes the proof of the Lemma. □

Proposition 4.2. *If we write out the polynomial* $Y_{\underline{\boldsymbol{a}}}(\underline{\alpha}; \underline{\beta}; -\underline{\boldsymbol{N}})$ *as a sum of monomials,*

$$Y_{\underline{\boldsymbol{a}}}(\underline{\alpha}; \underline{\beta}; -\underline{\boldsymbol{N}}) = \sum_{\underline{\boldsymbol{L}}} C_{\underline{\boldsymbol{L}}}\, \underline{\boldsymbol{a}}^{\underline{\boldsymbol{L}}}$$

with $\underline{\boldsymbol{a}}^{\boldsymbol{L}} = \prod_{i=1}^{n} a_i^{L_i}$ *and* $C_{\underline{\boldsymbol{L}}} = C_{\underline{\boldsymbol{L}}}(\underline{\boldsymbol{N}}) \in \mathbb{C}$.
Then

$$\zeta_n(\underline{\alpha}; \underline{\beta}; -\underline{\boldsymbol{N}}) = \sum_{\underline{\boldsymbol{L}}} C_{\underline{\boldsymbol{L}}}\, B_{\underline{\boldsymbol{L}}}$$

where $B_{\underline{\boldsymbol{L}}} = \prod_{i=1}^{n} B_{L_i}$ *is a product of Bernoulli numbers.*
More generally, for $\underline{\boldsymbol{a}} = (a_1, ..., a_n) \in \mathbb{R}_+^n$, *we have:*

$$\zeta_{n,\underline{\boldsymbol{a}}}(\underline{\alpha}; \underline{\beta}; -\underline{\boldsymbol{N}}) = \sum_{\underline{\boldsymbol{L}}} C_{\underline{\boldsymbol{L}}}\, B_{\underline{\boldsymbol{L}}}(\underline{\boldsymbol{a}})$$

where $B_{\underline{\boldsymbol{L}}}(\underline{\boldsymbol{a}}) = \prod_{i=1}^{n} B_{L_i}(a_i)$ *is a product of Bernoulli numbers.*

Proof. It follows from the above lemma, with $P(\underline{a}) = Y_{n,\underline{a}}(\underline{\alpha}; \underline{\beta}; -\underline{\mathbf{N}})$ and $Q(\underline{a}) = \zeta_{n,\underline{a}}(\underline{\alpha}; \underline{\beta}; -\underline{\mathbf{N}})$. □

4.1 Proof of Theorem 1:

Relation (3.2) shows that for all $\underline{a} \in \mathbb{R}_+^n$

$$Y_{n,\underline{a}}(\underline{\alpha}; \underline{\beta}; -\underline{\mathbf{N}}) = (-1)^n \left(\prod_{i=1}^{n} \beta_i^{-1}\right) \sum_{\underline{k}=(k_2,...,k_n)\in\mathbb{N}^{n-1}}$$
$$\sum_{\substack{\underline{v}=(v_1,...,v_n)\in\mathbb{N}^n \\ v_j \le k_j\ \forall\ 2\le j\le n; v_1 \le \left(-\sum_{i=1}^{n} s_i + n - \sum_{i=2}^{n} k_i\right)}} \frac{A(-\underline{\mathbf{N}})\, a_1^{v_1}}{\left(\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i\right)} \prod_{j=2}^{n} \frac{a_j^{v_j}}{\left(\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i\right)} \tag{4.5}$$

with

$$A(-\underline{\mathbf{N}}) =$$
$$\binom{\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i}{v_1} \alpha_1^{\left(\sum_{i=1}^{n} N_i + n - v_1 - \sum_{i=2}^{n} k_i\right)} \prod_{j=2}^{n} \binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} \binom{k_j}{v_j} \alpha_j^{k_j - v_j} \tag{4.6}$$

and

$$T(\underline{\mathbf{N}}) := \left\{ \underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \quad 0 \le k_j \le \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i,\ \forall\ 2 \le j \le n \right\}. \tag{4.7}$$

Setting,

$$\underline{a}^{\underline{v}} = \prod_{j=1}^{n} a_j^{v_j} \tag{4.8}$$

this gives

$$Y_{n,\underline{a}}(\underline{\alpha};\underline{\beta};-\mathbf{\underline{N}})=(-1)^n\left(\prod_{i=1}^n\beta_i^{-1}\right)$$
$$\sum_{\underline{k}=(k_2,\ldots,k_n)\in T(\mathbf{\underline{N}})}\sum_{\substack{\underline{v}=(v_1,\ldots,v_n)\in\mathbb{N}^n\\ v_j\leq k_j\ \forall\ 2\leq j\leq n; v_1\leq\left(\sum_{i=1}^n N_i+n-\sum_{i=2}^n k_i\right)}} A(-\mathbf{\underline{N}})\ \underline{a}^{\underline{v}}\prod_{j=1}^n\frac{1}{\left(\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i\right)}. \tag{4.9}$$

It follows from Proposition 4.2 that

$$\zeta_n(\underline{\alpha};\underline{\beta};-\mathbf{\underline{N}})=(-1)^n\left(\prod_{i=1}^n\beta_i^{-1}\right)$$
$$\sum_{\underline{k}=(k_2,\ldots,k_n)\in T(\mathbf{\underline{N}})}\sum_{\substack{\underline{v}=(v_1,\ldots,v_n)\in\mathbb{N}^n\\ v_j\leq k_j\ \forall\ 2\leq j\leq n; v_1\leq\left(\sum_{i=1}^n N_i+n-\sum_{i=2}^n k_i\right)}} A(-\mathbf{\underline{N}})\ B_{\underline{v}}\prod_{j=1}^n\frac{1}{\left(\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i\right)} \tag{4.10}$$

with

$$B_{\underline{v}}=\prod_{j=1}^n B_{v_j}$$

and B_{v_j} is the $v_j{}^{-th}$ Bernoulli number, which ends the proof of Theorem 1.

5 Generalized Euler-Zagier-Hurwitz type of double zeta values

As an application of Theorem 1, we find

$$\zeta_2(\underline{\alpha};\underline{\beta};-\mathbf{\underline{N}})=$$
$$\beta_1^{-1}\beta_2^{-1}\ \sum_{k=0}^{N_2+1}\sum_{v_1=0}^{N_1+N_2+2-k}\sum_{v_2=0}^{k}\frac{\binom{N_2+1}{k}\binom{N_1+N_2+2-k}{v_1}\binom{k}{v_2}\alpha_1^{N_1+N_2+2-k-v_1}\ \alpha_2^{k-v_2}B_{v_1}B_{v_2}}{(N_1+N_2+2-k)(N_2+1)}$$

with B_{v_i} is the v_i^{-th} Bernoulli number.
So, we give some values in the table below

$\underline{\alpha}=(\alpha_1,\alpha_2)$	$\underline{\beta}=(\beta_1,\beta_2)$	$\underline{N}=(N_1,N_2)$	$\zeta_2(\underline{\alpha};\underline{\beta};-\mathbf{N})$	$\underline{N}=(N_1,N_2)$	$\zeta_2(\underline{\alpha};\underline{\beta};-\mathbf{N})$
$(1,1)$	$(1,1)$	$(0,0)$	$\frac{1}{3}$	$(1,0)$	$\frac{1}{24}$
		$(0,1)$	$\frac{1}{12}$	$(1,1)$	$\frac{1}{360}$
		$(0,2)$	$\frac{1}{90}$	$(1,2)$	$\frac{-1}{240}$
		$(0,3)$	$\frac{-1}{120}$	$(1,3)$	$\frac{-1}{560}$
$(1,2)$	$(1,2)$	$(0,4)$	$\frac{167}{420}$	$(1,4)$	$\frac{11}{560}$
		$(0,5)$	$\frac{53}{126}$	$(1,5)$	$\frac{11}{1008}$
		$(3,0)$	$\frac{-1}{160}$	$(4,0)$	$\frac{1}{504}$
		$(3,1)$	$\frac{-71}{20160}$	$(4,1)$	$\frac{1}{336}$
$(3,4)$	$(-1,-2)$	$(3,2)$	$\frac{880}{567}$	$(4,2)$	$\frac{212029}{136080}$
		$(3,3)$	$\frac{1812001}{453600}$	$(4,3)$	$\frac{-121097}{30240}$
		$(3,4)$	$\frac{-109723}{10080}$	$(4,4)$	$\frac{16314803}{1496880}$
		$(3,5)$	$\frac{183632431}{5987520}$	$(4,5)$	$\frac{-765493}{24948}$

6 Values of some related multiple series

In this section, we give the values of some particular case of the generalized Euler-Zagier-Hurwitz type of multiple zeta function.

6.1 Values of generalized multiple Hurwitz zeta function at non positive integers

For $\underline{\beta} := \underline{\mathbf{1}} = (1,\ldots,1)$, we find

$$\zeta_n(\underline{\alpha};\underline{\beta};\underline{s}) := \zeta(\alpha_1,\ldots,\alpha_n;1,\ldots,1;s_1,\ldots,s_n) = \zeta_n(\underline{\alpha};\underline{s}) \tag{6.1}$$

$$= \sum_{\underline{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{1}{(m_1+\cdots+m_i+\sum_{j=1}^{i}\alpha_j)^{s_i}} \tag{6.2}$$

which is the generalized multiple Hurwitz zeta function.
Thus, if we apply Theorem 1, we find the result

Corollary 6.1.
Let $\underline{\boldsymbol{N}} = (N_1,\ldots,N_n)$ a point of $\mathbb{N}^n$, if the point $(\underline{\boldsymbol{s}} = -\underline{\boldsymbol{N}})$ is not a polar divisor for the integral function $Y_n(\underline{\alpha};\underline{\boldsymbol{1}};\underline{\boldsymbol{s}})$, then the value of the multiple Hurwitz

zeta function $\zeta_n(\underline{\alpha};\underline{\boldsymbol{1}};\underline{\boldsymbol{s}})$ *at the point* $(\underline{\boldsymbol{s}}=-\underline{\boldsymbol{N}})$ *exists and is given by*

$$\zeta_n(\underline{\alpha};\underline{\boldsymbol{1}};-\underline{\boldsymbol{N}}) = (-1)^n \sum_{\underline{k}=(k_2,\dots,k_n)\in T(\underline{\boldsymbol{N}})} \sum_{\substack{\underline{v}=(v_1,\dots,v_n)\in\mathbb{N}^n \\ v_j\le k_j\ \forall\ 2\le j\le n; v_1\le\left(\sum_{i=1}^n N_i+n-\sum_{i=2}^n k_i\right)}} A(-\underline{\boldsymbol{N}})\ B_{\underline{v}} \prod_{j=1}^n \frac{1}{\left(\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i\right)} \tag{6.3}$$

with

$$A(-\underline{\boldsymbol{N}}) = \binom{\sum_{i=1}^n N_i+n-\sum_{i=2}^n k_i}{v_1} \alpha_1^{\left(\sum_{i=1}^n N_i+n-v_1-\sum_{i=2}^n k_i\right)} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i}{k_j}\binom{k_j}{v_j}\alpha_j^{k_j-v_j}$$

and

$$T(\underline{\boldsymbol{N}}) := \left\{\underline{k}=(k_2,...,k_n)\in\mathbb{N}^{n-1}:\quad 0\le k_j\le\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i,\ \forall\ 2\le j\le n\right\}.$$

and

$$B_{\underline{v}} = \prod_{j=1}^n B_{v_j}$$

where B_{v_j} *is the* $v_j{}^{-th}$ *Bernoulli number.*

6.2 Multiple zeta values at non positive integers

Now, for $\underline{\beta} := \underline{\mathbf{1}} = (1,\dots,1)$ and $\underline{\alpha} := \underline{\mathbf{1}} = (1,\dots,1)$, we find

$$\begin{aligned}\zeta_n(\underline{\alpha};\underline{\beta};\mathbf{s}) &= \zeta_n(\underline{\mathbf{1}};\underline{\mathbf{1}};s_1,\dots,s_n) = \zeta_n(\underline{\mathbf{s}}) = \sum_{\underline{m}=(m_1,\dots,m_n)\in\mathbb{N}^n}\prod_{i=1}^n\frac{1}{(m_1+\cdots+m_i+i)^{s_i}} \\ &= \sum_{\underline{m}=(m_1,\dots,m_n)\in\mathbb{N}^{*n}}\prod_{i=1}^n\frac{1}{(m_1+\cdots+m_i)^{s_i}}\end{aligned}$$

which is the multiple zeta function.

Thus, if we apply Theorem 1, we find the same result given in [16].

Corollary 6.2.

Let $\underline{\boldsymbol{N}} = (N_1,\dots,N_n)$ *a point of* $\mathbb{N}^n$*, if the point* $(\underline{\boldsymbol{s}}=-\underline{\boldsymbol{N}})$ *is not a polar divisor for the integral function* $Y_n(\underline{\boldsymbol{1}};\underline{\boldsymbol{1}};\underline{\boldsymbol{s}})$*, then the value of the multiple zeta function* $\zeta_n(\underline{\boldsymbol{1}};\underline{\boldsymbol{1}};\underline{\boldsymbol{s}})$ *at the point* $(\underline{\boldsymbol{s}}=-\underline{\boldsymbol{N}})$ *exists and is given by*

$$\zeta_n(-\underline{\boldsymbol{N}}) := \zeta_n(\underline{\boldsymbol{1}};\underline{\boldsymbol{1}};-\underline{\boldsymbol{N}}) = (-1)^n \sum_{\underline{k}=(k_2,\dots,k_n)\in T(\underline{\boldsymbol{N}})} \sum_{\substack{\underline{v}=(v_1,\dots,v_n)\in\mathbb{N}^n \\ v_j\le k_j\ \forall\ 2\le j\le n; v_1\le\left(\sum_{i=1}^n N_i+n-\sum_{i=2}^n k_i\right)}} A(-\underline{\boldsymbol{N}})\ B_{\underline{v}} \prod_{j=1}^n \frac{1}{\left(\sum_{i=j}^n N_i+n-j+1-\sum_{i=j+1}^n k_i\right)} \tag{6.4}$$

with

$$A(-\underline{\boldsymbol{N}}) = \binom{\sum_{i=1}^{n} N_i + n - \sum_{i=2}^{n} k_i}{v_1} \prod_{j=2}^{n} \binom{\sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i}{k_j} \binom{k_j}{v_j}$$

$$T(\underline{\boldsymbol{N}}) := \left\{ \underline{k} = (k_2, ..., k_n) \in \mathbb{N}^{n-1} : \quad 0 \leq k_j \leq \sum_{i=j}^{n} N_i + n - j + 1 - \sum_{i=j+1}^{n} k_i, \ \forall\, 2 \leq j \leq n \right\}.$$

and

$$B_{\underline{v}} = \prod_{j=1}^{n} B_{v_j}$$

where B_{v_j} *is the* $v_j{}^{-th}$ *Bernoulli number.*

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