S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 17, стр. 899–912 (2020) УДК 517.953

DOI 10.33048/semi.2020.17.066 MSC 35M99

ON A BOUNDARY VALUE PROBLEM FOR A HIGH ORDER MIXED TYPE EQUATION

B.YU. IRGASHEV

Abstract. In this paper, we study a Dirichlet type problem for a Lavrentiev–Bitsadze type equation of high order type in a rectangular domain. The necessary and sufficient conditions for the uniqueness of the problem solution are obtained by using the spectral method. The solution is constructed in the form of a series of eigenfunctions. When substantiating the convergence of a series, the problem of «small» denominators arises. Sufficient conditions are obtained for the separability of the «small» denominator from zero.

Keywords: Differential equation, mixed type, boundary value problem, eigenvalue, eigenfunction, determinant, uniqueness, existence, «small» denominators, series, convergence.

1. Introduction

In the rectangular region $\Omega = \{(x,y) : 0 < x < l, -a < y < a\}$, where l and a are given positive real numbers, $n \in N$, consider the partial differential equation

$$Lu \equiv D_x^{2n} u\,(x,y) + (\mathrm{sgny})\; D_y^{2n} u\,(x,y) = 0. \tag{1}$$

Suppose that $\Omega_+ = \Omega \cap (y > 0)\,,\ \Omega_- = \Omega \cap (y < 0)\,.$ We study the following problem for the equation [1], similar to the Dirichlet problem.

Problem D. Given the region Ω, find a function $u\,(x,y)$, satisfying the conditions

$$u \in C^{2n-1}\left(\overline{\Omega}\right) \cap C^{2n}\left(\Omega_+ \cup \Omega_-\right), \tag{2}$$

$$Lu\,(x,y) \equiv 0, \quad (x,y) \in \Omega_+ \cup \Omega_-, \tag{3}$$

$$D_x^{2(s-1)} u\,(0,y) = D_x^{2(s-1)} u\,(l,y) = 0, \;\; -a \le y \le a, \tag{4}$$

Irgashev, B.Yu., On a boundary value problem for a high order mixed type equation.

Received December, 14, 2019, published July, 8, 2020.

$$D_y^{2(s-1)} u(x,-a) = \varphi_s(x)\,, \quad 0 \le x \le l, \tag{5}$$

$$l_s(u(a)) = \sum_{j=1}^{2n} b_{sj} D_y^{j-1} u(x,a) = \psi_s(x)\,, \; 0 \le x \le l, \tag{6}$$

where $\varphi_s(x)\,, \psi_s(x)$ are fairly smooth functions for which the corresponding conditions are satisfied, $s = 1, ..., n,\; b_{sj} \in R$. The following condition also holds:

$$rg\, B = rg \begin{pmatrix} b_{11} & b_{12} & ... & b_{1,2n} \\ b_{21} & b_{22} & ... & b_{2,2n} \\ . & . & . & . \\ b_{n1} & b_{n2} & ... & b_{n,2n} \end{pmatrix} = n.$$

The equation (1) is insufficiently explored. For $n = 1$, the equation (1) is the well-known Lavrentiev–Bitsadze equation, for which A.V. Bitsadze showed the incorrectness of the Dirichlet problem [1]. Later, the application of various methods helped to find the conditions for the uniqueness of the solution of the Dirichlet problem, including mixed-order second-order equations and high-order equations with smooth coefficients like in [2]–[4]. Note that the monograph of B.I. Ptashnik [4] contains extensive materials regarding the mentioned subject. A.V. Bitsadze pointed out the importance of studying the equations of mixed type of high order [5]. Among the recent results related to the topic under study, we should mention the works [6]–[10].

2. Uniqueness of solution

Let $u(x, y)$ be a solution of the equation (1) with the conditions (2)–(6). Consider its Fourier coefficients

$$\begin{gathered} u_k(y) = \int_0^l u(x,y)\, X_k(x)\, dx, \;\; k = 1, 2, ... \,, \\ X_k(x) = \sqrt{\tfrac{2}{l}} \sin \tfrac{\pi k}{l} x. \end{gathered} \tag{7}$$

Based on (7), we introduce the functions

$$u_{k,\varepsilon}(y) = \int_{\varepsilon}^{l-\varepsilon} u(x,y)\, X_k(x)\, dx, \tag{8}$$

where ε is a fairly small number. Differentiating the equality (8) with respect to y under the integral sign $2n$ times and taking into account the equation (1), for $y > 0$ and $y < 0$ we obtain

$$u_{k,\varepsilon}^{(2n)}(y) = \int_{\varepsilon}^{l-\varepsilon} D_y^{2n} u(x,y)\, X_k(x)\, x dx = -\int_{\varepsilon}^{l-\varepsilon} D_x^{2n} u(x,y)\, X_k(x)\, dx, \; (y > 0)\,, \tag{9}$$

$$u_{k,\varepsilon}^{(2n)}(y) = \int_{\varepsilon}^{l-\varepsilon} D_y^{2n} u(x,y)\, X_k(x)\, dx = \int_{\varepsilon}^{l-\varepsilon} D_x^{2n} u(x,y)\, X_k(x)\, dx, \; (y < 0)\,. \tag{10}$$

In the integrals on the right-hand sides of the equalities (9) and (10), integrating by parts $2n$ times, passing to the limit as $\varepsilon \to +0$, and taking the boundary conditions

(4) into account, we obtain the following ordinary differential equation:

$$u_k^{(2n)}(y) + (-1)^n (\text{sgn}y) \left(\frac{\pi k}{l}\right)^{2n} u_k(y) = 0.$$

We consider separately the cases of even and odd n. Let the order of the equation be $4m$ (the case when the order of the equation is $4m+2$ is considered similarly). Given the continuity of the function $u(x,y)$ and its derivatives, on the line $y=0$ we obtain the following problem:

$$\begin{cases} u_k^{(4m)}(y) + \text{sgn}y\left(\frac{\pi k}{l}\right)^{4m} u_k(y) = 0, \\ u_k^{(2s-2)}(-a) = \varphi_{sk}, \\ l_s(u_k(a)) = \psi_{sk}, \\ u_k^{(p)}(+0) = u_k^{(p)}(-0), \ s = \overline{1, 2m}, \ p = \overline{0, (4m-1)}, \end{cases} \tag{11}$$

where

$$\varphi_{sk} = \int_0^l \varphi_s(x) X_k(x)\, dx, \ \psi_{sk} = \int_0^l \psi_s(x) X_k(x)\, dx\,.$$

For $y > 0$, the general solution of the equation (11) has the form

$$u_k(y) = \sum_{p=0}^{2m-1} \left(c_p^1 Y_{1p}^{+}(y) + c_p^2 Y_{2p}^{+}(y)\right),$$

where

$$Y_{1p}^{+}(y) = e^{\alpha_p y}\cos\beta_p y, \ Y_{2p}^{+}(y) = e^{\alpha_p y}\sin\beta_p y,$$

$$\begin{aligned} \left(Y_{1p}^{+}(y)\right)^{(q)} &= \left(\tfrac{\pi k}{l}\right)^q e^{\alpha_p y}\cos(\beta_p y + q\theta_p), \\ \left(Y_{2p}^{+}(y)\right)^{(q)} &= \left(\tfrac{\pi k}{l}\right)^q e^{\alpha_p y}\sin(\beta_p y + q\theta_p), \end{aligned}$$

$$\begin{gathered} \alpha_p = \tfrac{\pi k}{l}\cos\theta_p, \beta_p = \tfrac{\pi k}{l}\sin\theta_p, \ \theta_p = \tfrac{\pi}{4m}(1+2p), \\ p = \overline{0, (2m-1)}, \ q = \overline{0, 4m-1}, \ \alpha_p > 0, \ p = \overline{0, (m-1)}. \end{gathered}$$

For $y < 0$,

$$u_k(y) = d_0 e^{\frac{\pi k}{l}y} + \sum_{s=1}^{2m-1} e^{\mu_s y}\left(d_s^1\cos\nu_s y + d_s^2\sin\nu_s y\right) + d_{2m}e^{-\frac{\pi k}{l}y},$$

$$\begin{aligned} u_k^{(j)}(y) = \left(\tfrac{\pi k}{l}\right)^j \Bigg(& d_0 e^{\frac{\pi k y}{l}} + \sum_{s=1}^{2m-1} e^{\mu_s y}\left(d_s^1\cos(\nu_s y + j\sigma_s) + d_s^2\sin(\nu_s y + j\sigma_s)\right) + \\ & + (-1)^j d_{2m} e^{-\frac{\pi k y}{l}}\Bigg), \end{aligned}$$

where

$$\begin{gathered} \mu_s = \tfrac{\pi k}{l}\cos\sigma_s, \nu_s = \tfrac{\pi k}{l}\sin\sigma_s, \ \sigma_s = \tfrac{\pi s}{2m}, \\ s = \overline{0, (2m-1)}, \mu_s > 0, \ s = \overline{0, (m-1)}, \ \mu_m = 0. \end{gathered}$$

With the boundary conditions for the problem (11), we obtain a system of algebraic equations, as follows:

$$\left\{\begin{array}{l}\sum\limits_{s=0}^{2m-1} e^{-a\mu_s}\left(d_s^1\cos\left(-\nu_s a+2j\sigma_s\right)+d_s^2\sin\left(-\nu_s a+2j\sigma_s\right)\right)=\\ =\left(\frac{l}{\pi k}\right)^{2j}\varphi_{jk},\\ \sum\limits_{p=0}^{2m-1}\left(c_p^1 l_j\left(Y_{1p}^{+}(a)\right)+c_p^2 l_j\left(Y_{2p}^{+}(a)\right)\right)=\psi_{jk},\\ \sum\limits_{p=0}^{2m-1}\left(c_p^1\cos\left(t\theta_p\right)+c_p^2\sin\left(t\theta_p\right)\right)=\\ =d_0+\sum\limits_{p=1}^{2m-1}\left(d_p^1\cos\left(t\sigma_p\right)+d_p^2\sin\left(t\sigma_p\right)\right)+(-1)^t d_{2m},\\ j=\overline{1,2m},\quad t=\overline{0,(4m-1)}.\end{array}\right. \tag{12}$$

We introduce the following notation:

$$\tau_{j,s}=-\nu_s a+2j\sigma_s,$$

$$A_{2m,2m}^{+}=\begin{pmatrix} l_1^2\left(Y_{10}^{+}(a)\right) & l_1^2\left(Y_{20}^{+}(a)\right) & . & . & l_1^2\left(Y_{2(n-1)}^{+}(a)\right)\\ l_2^2\left(Y_{10}^{+}(a)\right) & l_2^2\left(Y_{20}^{+}(a)\right) & . & . & l_2^2\left(Y_{2(n-1)}^{+}(a)\right)\\ . & . & . & . & .\\ l_{2m}^2\left(Y_{10}^{+}(a)\right) & l_{2m}^2\left(Y_{20}^{+}(a)\right) & . & . & l_{2m}^2\left(Y_{2(n-1)}^{+}(a)\right)\end{pmatrix},$$

$$A_{2m,2m}^{-}=\begin{pmatrix} l_1^2\left(Y_{1m}^{+}(a)\right) & l_1^2\left(Y_{2m}^{+}(a)\right) & . & . & l_1^2\left(Y_{2(2m-1)}^{+}(a)\right)\\ l_2^2\left(Y_{1m}^{+}(a)\right) & l_2^2\left(Y_{2m}^{+}(a)\right) & . & . & l_2^2\left(Y_{2(2m-1)}^{+}(a)\right)\\ . & . & . & . & .\\ l_{2m}^2\left(Y_{1m}^{+}(a)\right) & l_{2m}^2\left(Y_{2m}^{+}(a)\right) & . & . & l_{2m}^2\left(Y_{2(2m-1)}^{+}(a)\right)\end{pmatrix},$$

$$B_{2m,2m-1}^{+}=\begin{pmatrix} e^{-\mu_{m+1}a}\cos\tau_{0,m+1} & . & . & e^{-\mu_{2m-1}a}\sin\tau_{0,2m-1} & e^{\frac{\pi k}{l}a}\\ e^{-\mu_{m+1}a}\cos\tau_{1,m+1} & . & . & e^{-\mu_{2m-1}a}\sin\tau_{1,2m-1} & e^{\frac{\pi k}{l}a}\\ . & . & . & . & .\\ e^{-\mu_{m+1}a}\cos\tau_{2m-1,m+1} & . & . & e^{-\mu_{2m-1}a}\sin\tau_{2m-1,2m-1} & e^{\frac{\pi k}{l}a}\end{pmatrix},$$

or

$$B_{2m,2m-1}^{+}=\left(e^{-\mu_s a}\cos\tau_{j,s},e^{-\mu_s a}\sin\tau_{j,s},e^{\frac{\pi k}{l}a}\right)_{s=\overline{m+1,2m-1}}^{j=\overline{0,2m-1}},$$

$$B_{2m,2m-1}^{-}=\left(e^{-\frac{\pi k}{l}a},e^{-\mu_s a}\cos\tau_{j,s},e^{-\mu_s a}\sin\tau_{j,s}\right)_{s=\overline{1,m-1}}^{j=\overline{0,2m-1}},$$

$$C_{4m,2m}^{+}=\left(\cos j\theta_s,\sin j\theta_s\right)_{s=\overline{0,m-1}}^{j=\overline{0,4m-1}},$$
$$C_{4m,2m}^{-}=\left(\cos j\theta_s,\sin j\theta_s\right)_{s=\overline{m,2m-1}}^{j=\overline{0,4m-1}},$$

$$D_{4m,2m-1}^{+}=\left(-\cos j\sigma_s,-\sin j\sigma_s,(-1)^j\right)_{s=\overline{m+1,2m-1}}^{j=\overline{0,4m-1}},$$
$$D_{4m,2m-1}^{-}=\left(1,-\cos j\sigma_s,-\sin j\sigma_s\right)_{s=\overline{1,m-1}}^{j=\overline{0,4m-1}},$$

formally

$$B^0_{2m,2}=\begin{pmatrix}\cos\tau_{0,m} & \sin\tau_{0,m}\\ \cos\tau_{1,m} & \sin\tau_{1,m}\\ . & .\\ \cos\tau_{2m-1,m} & \sin\tau_{2m-1,m}\end{pmatrix}=$$

$$=\frac{i}{2}\begin{pmatrix} e^{i\tau_{0,m}} & e^{-i\tau_{0,m}}\\ . & .\\ e^{i\tau_{2m-1,m}} & e^{-i\tau_{2m-1,m}}\end{pmatrix}=\frac{i}{2}\begin{pmatrix} B^+_{2m,1} & B^-_{2m,1}\end{pmatrix},$$

$$D^0_{4m,2}=\begin{pmatrix}\cos 0 & \sin 0\\ \cos\frac{\pi}{2} & \sin\frac{\pi}{2}\\ . & .\\ \cos(4m-1)\frac{\pi}{2} & \sin(4m-1)\frac{\pi}{2}\end{pmatrix}=$$

$$=\frac{i}{2}\begin{pmatrix} e^{i0\cdot\frac{\pi}{2}} & e^{-i0\cdot\frac{\pi}{2}}\\ . & .\\ e^{i(4m-1)\frac{\pi}{2}} & e^{-i(4m-1)\frac{\pi}{2}}\end{pmatrix}=\frac{i}{2}\begin{pmatrix} D^+_{4m,1} & D^-_{4m,1}\end{pmatrix}.$$

Then the main determinant of the system (12) has the form

$$\Delta=\frac{i}{2}\det\begin{pmatrix} A^+_{2m,2m} & A^-_{2m,2m} & 0 & 0 & 0 & 0\\ 0 & 0 & B^-_{2m,2m-1} & B^+_{2m,2m-1} & B^+_{2m,1} & B^-_{2m,1}\\ C^+_{4m,2m} & C^-_{4m,2m} & D^-_{4m,2m-1} & D^+_{4m,2m-1} & D^+_{4m,1} & D^-_{4m,1}\end{pmatrix}.$$

When calculating the determinant Δ, we obtain the sum in which each term contains a coefficient multiplied by an exponential with some exponent. We find the asymptotic behavior of the determinant Δ for large values of k, calculating the term which includes the exponent with the largest positive exponent. Up to a sign, it has the form

$$\frac{i}{2}\left|A^+_{2m,2m}\right|\left(\left|B^+_{2m,2m-1}B^+_{2m,1}\right|\cdot\left|C^-_{4m,2m}D^-_{4m,2m-1}D^-_{4m,1}\right|-\right.$$
$$\left.-\left|B^+_{2m,2m-1}B^-_{2m,1}\right|\cdot\left|C^-_{4m,2m}D^-_{4m,2m-1}D^+_{4m,1}\right|\right).$$

Next, we calculate each factor in the last expression

$$\left|A^+_{2m,2m}\right|=\left(\frac{i}{2}\right)^m e^{2\alpha a}\begin{vmatrix}\sum\limits_{j=0}^{4m-1} b_{1j+1}\left(\frac{\pi k}{l}e^{i\theta_0}\right)^j & . \;.\; . & \sum\limits_{j=0}^{4m-1} b_{1j+1}\left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^j\\ \sum\limits_{j=0}^{4m-1} b_{2j+1}\left(\frac{\pi k}{l}e^{i\theta_0}\right)^j & . \;.\; . & \sum\limits_{j=0}^{4m-1} b_{2j+1}\left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^j\\ . & . \;.\; . & .\\ \sum\limits_{j=0}^{4m-1} b_{2mj+1}\left(\frac{\pi k}{l}e^{i\theta_0}\right)^j & . \;.\; . & \sum\limits_{j=0}^{4m-1} b_{2mj+1}\left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^j\end{vmatrix},$$

here

$$\alpha=\alpha_0+\alpha_1+...+\alpha_{n-1}.$$

Notice that the following holds:

$$T_{2m\times 2m}=\begin{pmatrix}\sum\limits_{j=0}^{4m-1} b_{1j+1}\left(\frac{\pi k}{l}e^{i\theta_0}\right)^j & . \;.\; . & \sum\limits_{j=0}^{4m-1} b_{1j+1}\left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^j\\ \sum\limits_{j=0}^{4m-1} b_{2j+1}\left(\frac{\pi k}{l}e^{i\theta_0}\right)^j & . \;.\; . & \sum\limits_{j=0}^{4m-1} b_{2j+1}\left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^j\\ . & . \;.\; . & .\\ \sum\limits_{j=0}^{4m-1} b_{2mj+1}\left(\frac{\pi k}{l}e^{i\theta_0}\right)^j & . \;.\; . & \sum\limits_{j=0}^{4m-1} b_{2mj+1}\left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^j\end{pmatrix}=$$

$$=\begin{pmatrix} b_{11} & b_{12} & ... & b_{14m} \\ b_{21} & b_{22} & ... & b_{24m} \\ . & . & . & . \\ b_{2m1} & b_{2m2} & ... & b_{2m4m} \end{pmatrix}_{2m\times 4m} .$$

$$\cdot\begin{pmatrix} 1 & . & . & 1 \\ \left(\frac{\pi k}{l}e^{i\theta_0}\right) & . & . & \left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right) \\ . & . & . & . \\ \left(\frac{\pi k}{l}e^{i\theta_0}\right)^{4m-1} & . & . & \left(\frac{\pi k}{l}e^{-i\theta_{m-1}}\right)^{4m-1} \end{pmatrix}_{4m\times 2m} .$$

Designating the first matrix by B, and the second one by L, we have

$$T = BL.$$

It is known that

$$rangT \leq min\left(rangB, rangL\right) = 2m,$$

thus, further we assume that we have the condition

$$rangT = \min\left(rangB, rangL\right) = 2m. \tag{13}$$

For example, we can take a diagonal matrix with a rank of $2m$ as a matrix B, hence, when (13) holds, the following is true:

$$\left|A^{+}_{2m,2m}\right| \neq 0.$$

Now,

$$\det\left(B^{+}_{2m,2m-1}, B^{+}_{2m,1}\right) =$$
$$= \det\left(e^{-\mu_s a}\cos\tau_{j,s}, e^{-\mu_s a}\sin\tau_{j,s}, e^{\frac{\pi k}{l}a}, e^{i\left(-\frac{\pi k}{l}a+j\pi\right)}\right)^{j=\overline{0,2m-1}}_{s=\overline{m+1,2m-1}} =$$
$$= e^{-2a\sum\limits_{s=n+1}^{2m-1}\mu_s+\frac{\pi k}{l}a} e^{-i\frac{\pi k a}{l}} \det\left(\cos\tau_{j,s}, \sin\tau_{j,s}, 1, e^{ij\pi}\right)^{j=\overline{0,2m-1}}_{s=\overline{m+1,2m-1}} =$$
$$= e^{-2a\sum\limits_{s=n+1}^{2m-1}\mu_s+\frac{\pi k}{l}a} e^{-i\frac{\pi k a}{l}} \left(\tfrac{i}{2}\right)^{n-1}\cdot$$
$$\cdot \det\left(e^{i(2j\sigma_s)}, e^{-i(2j\sigma_s)}, 1, e^{ij\pi}\right)^{j=\overline{0,2m-1}}_{s=\overline{m+1,2m-1}} =$$
$$= M_2 e^{-2a\sum\limits_{j=n+1}^{2n-1}\mu_j} \cdot e^{\frac{\pi k}{l}a} e^{-i\frac{\pi k a}{l}} \neq 0,$$

where

$$M_2 = -2\prod_{j=m+1}^{2m-1} 4\sin^3 2\sigma_j \prod_{j=m+2}^{2m-1}\prod_{k=m+1}^{j-1} 16\sin^2\left(\sigma_j+\sigma_k\right)\sin^2\left(\sigma_j-\sigma_k\right) \neq 0.$$

We have

$$\det\left(C^{-}_{4m,2m} D^{-}_{4m,2m-1} D^{-}_{4m,1}\right) =$$
$$= \det\left(\cos j\theta_s, \sin j\theta_s, 1, -\cos j\sigma_k, -\sin j\sigma_k, e^{-ij\frac{\pi}{2}}\right)^{j=\overline{0,4m-1}}_{s=\overline{m,2m-1};k=\overline{1,m-1}} =$$
$$= \left(\frac{i}{2}\right)^{2m-1} \det\left(e^{ij\theta_s}, e^{-ij\theta_s}, 1, e^{ij\sigma_k}, e^{-ij\sigma_k}, (-i)^j\right)^{j=\overline{0,4m-1}}_{s=\overline{m,2m-1};k=\overline{1,m-1}} =$$
$$= \left(\frac{i}{2}\right)^{2m-1} M_3\left(-i-1\right)\prod_{k=1}^{m-1}\left(-i-e^{i\sigma_k}\right)\left(-i-e^{-i\sigma_k}\right)\prod_{s=m}^{2m-1}\left(-i-e^{i\theta_s}\right)\left(-i-e^{-i\theta_s}\right) =$$

$$= \left(\tfrac{i}{2}\right)^{2m-1} M_3\,(-i-1)\prod_{k=1}^{m-1} 2i\cos\sigma_k \prod_{s=m}^{2m-1} 2i\sin\theta_s =$$
$$= \left(\tfrac{i}{2}\right)^{2m-1}(2i)^{2m-1} M_3\,(-i-1)\prod_{k=1}^{m-1}\cos\sigma_k \prod_{s=m}^{2m-1}\sin\theta_s =$$
$$= M_3\,(1+i)\prod_{k=1}^{m-1}\cos\sigma_k \prod_{s=m}^{2m-1}\cos\theta_s.$$

We calculate M_3 the following way:

$$M_3 = (-2i)^m \prod_{s=m}^{2m-1}\sin\theta_s$$
$$\cdot\prod_{s=m+1}^{2m-1}\prod_{k=m}^{s-1}\left(2-2\cos\left(\theta_k-\theta_s\right)\right)\left(2-2\cos\left(\theta_k+\theta_s\right)\right)\cdot$$
$$\cdot\prod_{s=m}^{2m-1}\left(2-2\cos\theta_s\right)\prod_{k=1}^{m-1}\left(2-2\cos\sigma_k\right)(-2i)^{m-1}\prod_{k=1}^{m-1}\sin\sigma_k\cdot$$
$$\cdot\prod_{k=2}^{m-1}\prod_{s=1}^{k-1}\left(2-2\cos\left(\sigma_k+\sigma_s\right)\right)\left(2-2\cos\left(\sigma_k-\sigma_s\right)\right)\cdot$$
$$\cdot\prod_{k=1}^{m-1}\prod_{s=m}^{2m-1}\left(2-2\cos\left(\sigma_k+\theta_s\right)\right)\left(2-2\cos\left(\sigma_k-\theta_s\right)\right) = iM_4,$$

where

$$0 \neq M_4 \in R.$$

Consequently,

$$\det\left(C^-_{4m,2m}D^-_{4m,2m-1}D^-_{4m,1}\right) = M_4\,(i-1)\prod_{k=1}^{m-1}\cos\sigma_k \prod_{s=m}^{2m-1}\cos\theta_s = M_5\,(i-1)\,.$$

Similarly,

$$\det\left(B^+_{2m,2m-1},B^-_{2m,1}\right) = M_2 e^{a\left(\frac{\pi k}{l}-2\sum\limits_{j=m+1}^{2m-1}\mu_j\right)} e^{i\frac{\pi k a}{l}} \neq 0,$$

$$\det\left(C^-_{4m,2m}D^-_{4m,2m-1}D^+_{4m,1}\right) = M_3\left(\frac{i}{2}\right)^{2m-1}(i-1)\prod_{k=1}^{m-1}\left(i-e^{i\sigma_k}\right)\left(i-e^{-i\sigma_k}\right)\cdot$$
$$\prod_{s=m}^{2m-1}\left(i-e^{i\theta_s}\right)\left(i-e^{-i\theta_s}\right) = M_3\left(\tfrac{i}{2}\right)^{2m-1}$$
$$\cdot(-2i)^{2m-1}\,(i-1)\prod_{k=1}^{m-1}\cos\sigma_k\prod_{s=m}^{2m-1}\cos\theta_s =$$
$$= M_3\,(i-1)\prod_{k=1}^{m-1}\cos\sigma_k\prod_{s=m}^{2m-1}\cos\theta_s = M_5\,(-1-i)\,.$$

Finally, we have

$$\Delta = iM_6 e^{2b\sum\limits_{s=0}^{m-1}\alpha_s - 2a\sum\limits_{s=m+1}^{2m}\mu_s + \frac{\pi k a}{l}}$$
$$\cdot\left((i-1)\,e^{-i\frac{\pi k a}{l}} + (1+i)\,e^{i\frac{\pi k a}{l}}\right) + \Delta_{1k} =$$

$$= M_7 e^{2b\sum\limits_{s=0}^{m-1}\alpha_s - 2a\sum\limits_{s=m+1}^{2m}\mu_s + \frac{\pi ka}{l}} \sin\left(\frac{\pi ka}{l} + \frac{\pi}{4}\right) + \Delta_{1k} =$$

$$= M_7 e^{\frac{\pi k}{l}A}\left(\sin\left(\frac{\pi ka}{l} + \frac{\pi}{4}\right) + \Delta_{2k}\right),$$

where

$$A = \left(2b\sum_{j=0}^{m-1}\cos\theta_j + 2a\sum_{j=m+1}^{2m-1}(-\cos\sigma_j) + a\right) > 0,$$

$$\Delta_{2k} = B\Delta_{1k}e^{-\frac{\pi k}{l}A},\ M_7, B - const,$$

and, moreover,

$$\lim_{k\to+\infty}\Delta_{2k} = 0.$$

Theorem 1. *If a solution to Problem D exists, then it is unique only if the main determinant Δ of the system (12) is nonzero for all $k \in N$.*

Proof. Suppose that $\Delta \neq 0$ and the boundary conditions (4)–(6) are homogeneous, then the system (11) only has a trivial solution $u_k(y) \equiv 0$, for all $k \in N$. Then, the function $u(x,y) = 0$ almost everywhere, and $u(x,y)$ is continuous in $\overline{\Omega}$, and therefore $u(x,y) \equiv 0$ in $\overline{\Omega}$.

Now, suppose that for some values of a, l, k we have $\Delta = 0$, which is possible if we change the sign of the sine. Then the homogeneous problem (11) has a nonzero solution, which contradicts the uniqueness of solution of Problem D. The theorem is proved. □

3. Solution existence

First, we obtain some estimates for the functions $u_k(y)$. In what follows, in order to decrease the number of notations, all positive constants independent of k will be designated by a single letter T.

Lemma 1. *If the condition (13) is satisfied, then for the function $u_k(y)$ and its derivatives, for sufficiently large numbers k, the following estimates are valid (for $y \neq 0$):*

$$\left|u_k^{(t)}(y)\right| \leq T\frac{k^t\sum\limits_{s=0}^{n-1}\{|\varphi_{ks}| + |\psi_{ks}|\}}{\left|\sin\pi\left(\frac{ka}{l} + \frac{1}{4}\right) + \Delta_{2k}\right|},$$
$t = 0, 1, ..., 2n-1, T > 0$ *is a constant.*

Proof. It is easy to show that

$$\left|u_k^{(t)}(y)\right| \leq Tk^t\left|u_k(y)\right|.$$

Therefore, it suffices to prove the estimate for $t = 0$. We will prove the lemma, considering that the order of the equation is $4m$ and $y < 0$ (other cases are treated similarly). Then we have

$$|u_k(y)| \leq |d_0|\, e^{\frac{\pi k}{l}y} + \sum_{s=1}^{2m-1} e^{\mu_s y}\left(\left|d_s^1\right| + \left|d_s^2\right|\right) + |d_{2n}|\, e^{-\frac{\pi k}{l}y}.$$

Solving the system (12) using the Cramer method, we obtain the following estimates:

$$\left|d_0 e^{\frac{\pi k}{l}y}\right| = \left|\frac{\Delta(d_0)}{\Delta}\right| e^{\frac{\pi k}{l}y} \le T\frac{e^{\frac{\pi k}{l}y}\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}\cdot\left|O\left(e^{\frac{\pi k}{l}A}\right)\right|}{\left|e^{\frac{\pi k}{l}A}\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|} \le$$
$$\le T\frac{\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}}{\left|\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|},$$

$$\left|d_{2m} e^{-\frac{\pi k}{l}y}\right| = \left|\frac{\Delta(d_{2m})}{\Delta}\right| e^{-\frac{\pi k}{l}y} \le$$
$$\le T\frac{e^{-\frac{\pi k}{l}y}\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}\cdot\left|O\left(e^{\frac{\pi k}{l}(A-a)}\right)\right|}{\left|e^{\frac{\pi k}{l}A}\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|} \le$$
$$\le T\frac{\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}}{\left|\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|}.$$

If $\mu_s > 0$, then

$$\left|d_s^1 e^{\mu_s y}\right| = \left|\frac{\Delta\left(d_s^1\right)}{\Delta}\right| e^{\mu_s y} \le T\frac{e^{\mu_s y}\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}\cdot\left|O\left(e^{\frac{\pi k}{l}A}\right)\right|}{\left|e^{\frac{\pi k}{l}A}\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|} \le$$
$$\le T\frac{\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}}{\left|\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|}.$$

If $\mu_s < 0$, then

$$\left|d_s^1 e^{\mu_s y}\right| = \left|\frac{\Delta\left(d_s^1\right)}{\Delta}\right| e^{\mu_s y} \le$$
$$\le T\frac{e^{\mu_s y}\sum\limits_{s=0}^{2m-1}\{|\varphi_{sk}|+|\psi_{sk}|\}\left|O\left(e^{\frac{\pi k}{l}A+\mu_s a}\right)\right|}{\left|e^{\frac{\pi k}{l}A}\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|} \le$$
$$\le T\frac{\sum\limits_{s=0}^{2m-1}\{|\varphi_{ks}|+|\psi_{ks}|\}}{\left|\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|}.$$

Here $\Delta(d_0), \Delta(d_{2m}), \Delta\left(d_s^1\right)$ are the identifiers obtained from the determinant Δ by replacing the corresponding columns with the right-hand side of the system (12). The lemma is proved. □

Next, we need to find the conditions under which the expression

$$\left|\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right|$$

is separated from zero, i.e. starting from some number $k \ge k_1 \ge 1$ the following estimate holds:

$$\left|\left(\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)+\Delta_{2k}\right)\right| > \delta > 0.$$

Because $\lim\limits_{k\to\infty}\Delta_{2k} = 0$, it is enough to find the conditions under which

$$\left|\sin\left(\frac{\pi ka}{l}+\frac{\pi}{4}\right)\right| > \delta_1 > 0$$

for any values of k. The following lemma is true.

Lemma 2. *Assume that one of the following conditions is satisfied: either* $\frac{a}{l} \in N$, *or* $\frac{a}{l} = \frac{s}{t}$ $\left(\frac{a}{l} \notin N\right)$, $s, t \in N$, $(s, t) = 1$, $(t, 4) = 1$. *Then for every number* k, *the following inequality holds:*

$$\left|\sin\left(\frac{\pi ka}{l} + \frac{\pi}{4}\right)\right| > \delta_1 > 0. \tag{14}$$

Proof. Suppose that $\frac{a}{l} \in N$, then $\left|\sin\left(\frac{\pi ka}{l} + \frac{\pi}{4}\right)\right| = \frac{\sqrt{2}}{2}$, and everything is proved. Now assume that $\frac{a}{l} = \frac{s}{t}$ $\left(\frac{a}{l} \notin N\right)$, $s, t \in N$, $(s, t) = 1$, $(t, 4) = 1$, then $\frac{ka}{l} = \frac{ks}{t} = k_1 + \frac{k_2}{t}$, where $k_1, k_2 \in N$, $1 \le k_2 \le t - 1$. We have $\left|\sin\left(\frac{\pi ka}{l} + \frac{\pi}{4}\right)\right| = \left|\sin\left(\frac{\pi k_2}{t} + \frac{\pi}{4}\right)\right| \ne 0$ (because $(t, 4) = 1$), and for every number k the following inequality is true:

$$\left|\sin\left(\frac{\pi ka}{l} + \frac{\pi}{4}\right)\right| \ge \delta_1 = \min_{1 \le k_2 \le t-1} \left|\sin\left(\frac{\pi k_2}{t} + \frac{\pi}{4}\right)\right| > 0.$$

This implies the estimate (14). The lemma is proved. □

Note that if $\tau = \frac{a}{l}$ is an irrational number, it is not always possible to separate the denominator of the expression Δ from zero. But in some cases one can find the dependence of the "smallness" of the denominator on the number k. The following lemma is true.

Lemma 3. *If* $\tau = \frac{a}{l} > 0$ *is an irrational algebraic number of a degree* $p \ge 2$, *then there exists a number* $T > 0$, *such that for all* $p \in N$, $0 < \varepsilon < 1$ *the following estimate holds:*

$$\left|\sin\left(\pi k\tau + \frac{\pi}{4}\right)\right| \ge \frac{T}{k^{1+\varepsilon}}. \tag{15}$$

Proof. Following the reasoning in [8], we get

$$\left|\sin\left(\tau\pi k + \frac{\pi}{4}\right)\right| = \left|\sin\left(\pi k\left(\tau - \frac{m - \frac{1}{4}}{k}\right)\right)\right|,$$

where $m \in N$ is arbitrary. Now we choose m, so that the inequality

$$\left|\tau - \frac{m - \frac{1}{4}}{k}\right| < \frac{1}{2k} \tag{16}$$

is true. Set

$$m = \begin{cases} [\tau k], & \{\tau k\} < \frac{1}{4}, \\ [\tau k] + 1, & \{\tau k\} \ge \frac{1}{4}, \end{cases}$$

where $[\tau k]$, $\{\tau k\}$ are, respectively, the integer and fractional parts of the irrational number τk. Using the convexity of the function $y = \sin x$ on the interval $\left(0, \frac{\pi}{2}\right)$, we get the inequality

$$|\sin x| > \frac{2|x|}{\pi}, \; 0 < |x| < \frac{\pi}{2}. \tag{17}$$

Now let $m \in N$ be such that (16) holds, then using (17), we get

$$\left|\sin\left(\pi k\left(\tau - \frac{m - \frac{1}{4}}{k}\right)\right)\right| \ge 2k\left|\tau - \frac{m - \frac{1}{4}}{k}\right| = 2k\left|\tau - \frac{4m - 1}{4k}\right|. \tag{18}$$

It is known from [8] that for any algebraic number τ of degree $p \ge 2$ and an arbitrary $0 < \varepsilon < 1$ there exists $\delta_2 > 0$, such that for all integers s, q, $q > 0$, we have the inequality

$$\left|\tau - \frac{s}{q}\right| \ge \frac{\delta_2}{q^{2+\varepsilon}}. \tag{19}$$

Using (18) and (19), we get

$$\left|\sin\left(\pi k\tau+\frac{\pi}{4}\right)\right|\geq 2k\left|\tau-\frac{4m-1}{4k}\right|\geq 2k\frac{\delta_2}{(4k)^{2+\varepsilon}}=\frac{T}{k^{1+\varepsilon}},$$

which is the estimate (15). □

It should be noted that the resulting estimate (14) coincides with the estimate obtained in [11], where the inverse problem for the equation (1) was considered in the cases $n=1$, while the estimate (15) coincides with the estimate obtained in [8], where the Dirichlet problem for the equation (1) with smooth coefficients was considered.

Given the above, we obtain the conditions under which the series

$$u(x,y)=\sum_{k=1}^{\infty}u_k(y)\,X_k(x) \tag{20}$$

is a classic solution to the problem. Formally, we have

$$\left|D_x^{2n}u(x,y)\right|=\left(\tfrac{\pi}{l}\right)^{2n}\left|\sum_{k=1}^{\infty}(-1)^n k^{2n}u_k(y)X_k(x)\right|\leq$$
$$\leq T\sum_{k=1}^{\infty}k^{2n}\frac{\sum_{s=0}^{n-1}\{|\varphi_{ks}|+|\psi_{ks}|\}}{\left|\sin\pi\left(\frac{ka}{l}+\frac{1}{4}\right)+\Delta_{2k}\right|}.$$

Now, if $\Delta\neq 0$, $\frac{a}{l}$ satisfies the conditions of Lemma 2 and (13) holds, therefore we have the estimate

$$\left|D_x^{2n}u(x,y)\right|\leq T\sum_{k=1}^{\infty}k^{2n}\sum_{s=0}^{n-1}\{|\varphi_{ks}|+|\psi_{ks}|\}. \tag{21}$$

If $\Delta\neq 0$, $\frac{a}{l}$ satisfies the conditions of Lemma 3 and (13) holds, therefore

$$\left|D_x^{2n}u(x,y)\right|\leq T\sum_{k=1}^{\infty}k^{2n+1+\varepsilon}\sum_{s=0}^{n-1}\{|\varphi_{ks}|+|\psi_{ks}|\},\ 0<\varepsilon<1.$$

It remains to impose the conditions on the boundary functions.

Theorem 2. *Suppose the following:*
1. *the condition (13) is satisfied;*
2. *the conditions of Lemma 2 are satisfied;*
3. $\varphi_s(x),\psi_s(x)\in C^{2n+1}[0;l]$, $\varphi_s^{2m}(0)=\varphi_s^{2m}(l)=\psi_s^{2m}(0)=\psi_s^{2m}(l)=0,\ s=0,1,...n-1,\ \ m=0,1,...n-1.$

Then the series (20) is a classical solution to the problem D.

Proof. Consider the following two cases:
Case 1. Let the determinant Δ of the system (12) be nonzero. Then the solution of the problem D is unique. We seek a solution in the form of a series (20). Taking into consideration the above reasoning, we show the convergence of the series (21). We study the convergence for $s=0$, and the convergence of the remaining terms is shown similarly. We have

$$\sum_{k=k_0}^{\infty}k^{2n}|\varphi_{k0}|=\sum_{k=1}^{\infty}\frac{1}{k}k^{2n+1}|\varphi_{k0}|\leq\sqrt{\sum_{k=1}^{\infty}\frac{1}{k^2}}\sqrt{\sum_{k=1}^{\infty}\left(k^{2n+1}\varphi_{k0}\right)^2}\leq$$

$$\leq \frac{\pi}{\sqrt{6}} \left\| \varphi_0^{(2n+1)}(x) \right\|_{L_2}.$$

Case 2. Now let $\Delta = 0$. Because Δ_{2k}, taking into account that the estimate in Lemma 1 tends to zero (for $k \to +\infty$) exponentially and the estimate (14) holds, we conclude that Δ can vanish only for some finite values of $k = k_1, k_2, ..., k_p < k_0$, where $k_1 < k_2 < ... < k_p$, $i = \overline{1,p}$, k_0 and p are given natural numbers. Then, for the solvability of problem (1)–(6), we only need to satisfy the conditions $\varphi_{sk} = \int\limits_0^l \varphi_s(x) X_k dx = 0$, $\psi_{sk} = \int\limits_0^l \psi_s(x) X_k(x) dx$, $s = \overline{0,(n-1)}$, $k = k_1, k_2, ..., k_p$. Then, the solution itself (not necessarily unique) has the form

$$u(x,y) = \sum_{k=1}^{\infty} (k \neq k_1, k_2, ..., k_p) u_k(y) X_k(x) + \sum_m \widetilde{u_m}(y) X_k(x),$$

where in the last sum m takes values $k_1, k_2, ..., k_p$, and the function $\widetilde{u_m}(y)$ is a nonzero solution (not necessarily unique) of the homogeneous system (12). □

Theorem 3. *Suppose the following:*
1. *the condition (13) is satisfied;*
2. *the conditions of Lemma 3 are satisfied;*
3. $\varphi_s(x), \psi_s(x) \in C^{2n+3}[0;l]$, $\varphi_s^{2m}(0) = \varphi_s^{2m}(l) = \psi_s^{2m}(0) = \psi_s^{2m}(l) = 0$, $s = 0, 1, ...n-1$, $m = 0, 1, ...n+1$.

Then the series (20) is a classical solution to the problem D.

Proof. The proof is similar to that of Theorem 2. □

4. Stability of solutions

Suppose that

$$\|u(x,y)\|_{L_2} = \left(\int\limits_0^l |u(x,y)|^2 dx \right)^{1/2},$$

$$\|u(x,y)\|_{C(\overline{D})} = \max_{\overline{D}} |u(x,y)|,$$

$$\|f(x)\|_{W_2^n} = \left(\int\limits_0^l \sum_{k=0}^{n} \left(f^{(k)}(x) \right)^2 dx \right)^{1/2}, \ n \in N \cup \{0\}.$$

Theorem 4. *Suppose that the conditions of Theorem 2 are fulfilled. Then, for a unique solution to the problem D, we have the following estimates*

$$\|u(x,y)\|_{L_2} \leq T \sum_{s=0}^{n-1} \left(\|\varphi_s\|_{L_2} + \|\psi_s\|_{L_2} \right),$$

$$\|u(x,y)\|_{C(\overline{D})} \leq T \sum_{s=0}^{n-1} \left(\|\varphi_s\|_{W_2^1} + \|\psi_s\|_{W_2^1} \right),$$

where the constant $T > 0$ is independent of φ_s, ψ_s, $s = \overline{0, n-1}$.

Proof. All constants will be designated by one letter T as has been mentioned above. Because the function system $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi k}{l}x\right\}$ is orthonormal and full in the space $L_2\left[0,l\right]$, the equality

$$\left\|u\left(x,y\right)\right\|_{L_2}^2=\sum_{k=1}^{\infty}u_k^2\left(y\right)$$

is true. From Lemma 1 we have

$$\left\|u\left(x,y\right)\right\|_{L_2}^2\leq T\sum_{k=1}^{\infty}\left(\sum_{s=0}^{n-1}\left\{\left|\varphi_{ks}\right|+\left|\psi_{ks}\right|\right\}\right)^2\leq T\sum_{s=0}^{n-1}\left\{\left\|\varphi_s\left(x\right)\right\|_{L_2}^2+\left\|\psi_s\left(x\right)\right\|_{L_2}^2\right\}.$$

Consequently,

$$\left\|u(x,y)\right\|_{L_2}\leq T\sum_{s=0}^{n-1}\left(\left\|\varphi_s\right\|_{L_2}+\left\|\psi_s\right\|_{L_2}\right).$$

We have

$$\begin{aligned}\left\|u(x,y)\right\|_{C(\overline{D})}&\leq T\sum_{k=1}^{\infty}\sum_{s=0}^{n-1}\left\{\left|\varphi_{ks}\right|+\left|\psi_{ks}\right|\right\}\leq\\ &\leq T\sqrt{\sum_{k=1}^{\infty}\frac{1}{k^2}}\left(\sum_{s=0}^{n-1}\left(\sqrt{\sum_{k=1}^{\infty}\left(k\varphi_{ks}\right)^2}+\sqrt{\sum_{k=1}^{\infty}\left(k\psi_{ks}\right)^2}\right)\right)\leq\\ &\leq T\left(\sum_{s=0}^{n-1}\sqrt{\int_0^l\left(\varphi'_s\left(x\right)\right)^2dx}+\sqrt{\int_0^l\left(\psi'_s\left(x\right)\right)^2dx}\right)\leq\\ \leq T\left(\sum_{s=0}^{n-1}\sqrt{\int_0^l\left(\varphi_s\left(x\right)\right)^2+\left(\varphi'_s\left(x\right)\right)^2dx}\right.&\left.+\sqrt{\int_0^l\left(\psi_s\left(x\right)\right)^2+\left(\psi'_s\left(x\right)\right)^2dx}\right)=\\ &=T\sum_{s=0}^{n-1}\left(\left\|\varphi_s\right\|_{W_2^1}+\left\|\psi_s\right\|_{W_2^1}\right).\end{aligned}$$

Theorem 4 is proved. □

Similar results can be obtained if the conditions of Theorem 3 are satisfied.

References

[1] A.V. Bitsadze, *Incorrectness of Dirichlet's problem for the mixed type of equations in mixed regions*, Dokl. Akad. Nauk SSSR, **122**:2 (1958), 167–170. Zbl 0145.35403

[2] J.R. Cannon, *A Dirichlet problem for an equation of mixed type with a discontinuous coefficient* , Ann. Mat. Pura Appl., IV. Ser. **61** (1963), 371–377. Zbl 0116.29903

[3] A.M. Nakhushev, *A uniqueness condition of the Dirichlet problem for an equation of mixed type in a cylindrical domain*, Differ. Uravn., **6**:1 (1970), 190–191. Zbl 0214.10401

[4] B.I. Ptashnik, *Ill-posed boundary-value problems for partial differential equations*, Naukova Dumka, Kiev, 1984. MR0772024

[5] A.V. Bitsadze, *Equations of Mixed Type*, Academy of Sciences of the USSR, Moscow, 1959. Zbl 0087.09403

[6] K.B. Sabitov, *A remark on the theory of initial-boundary value problems for the equation of rods and beams*, Differ. Equ., **53**:1 (2017), 86–98. Zbl 1368.35070

[7] K.B. Sabitov, *Cauchy problem for the beam vibration equation*, Differ. Equ., **53**:5 (2017), 658–664. Zbl 1372.35182

[8] K.B. Sabitov, *The Dirichlet problem for higher-order partial differential equations*, Math. Notes, **97**:2 (2015), 255–267. Zbl 1320.35159

[9] B.Yu. Irgashev, *Spectral problem for an equation of high even order*, Russ. Math., **60**:7 (2016), 37–46. Zbl 1348.35155

[10] B.Yu. Irgashev, *On one boundary-value problem for an equation of higher even order*, Russ. Math., **61**:9 (2017), 10–26. Zbl 1383.35053

[11] K.B. Sabitov, I.A. Khadzhi, *The boundary-value problem for the Lavrent'ev-Bitsadze equation with unknown right-hand side*, Russ. Math., **55**:5 (2011), 35–42. Zbl 1236.35096

Bahrom Yusuphanovich Irgashev
Namangan Engineering Construction Institute, Uzbekistan
12, I.Karimov ave.,
Namangan, 160103, Uzbekistan
E-mail address: `bahromirgasev@gmail.com`