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CRUMBLED ICE ON THE SURFACE OF A MULTILAYERED
FLUID

D.O. TSVETKOV

ABSTRACT. We study the problem of the small motions and normal oscillations of a system of two ideal fluids with a free surface partially covered with crumbled ice. By crumbled ice we mean the situation in which heavy particles of some substance float on the free surface and these particles do not interact (or the interaction is small enough to be neglected) when the free surface oscillates. We find sufficient conditions for the existence of a strong solution (with respect to the time variable) to the initial boundary value problem describing the evolution of the specified system. We also study the spectrum of normal oscillations, basic properties of the eigenfunctions, and other questions.

Keywords: initial boundary value problem, differential equation in Hilbert space, Cauchy problem, strong solution, spectral problem

1. INTRODUCTION

1.1. History of the question. During the last half-century, the study of many problems of the hydrodynamics of an ideal fluid has been carried out by methods of functional analysis. Attention also has been paid to the classical problems about the oscillations of a system of two nonmixing ideal fluids filling up an arbitrary basin, which were generalized and summarized in the monograph [1]. In particular, in [1], the problem was considered about small motions of a system of m homogeneous ideal fluids located one above another (like in a sliced pie) so that the fluid of the greatest density ρ_1 occupies the lowest position (with respect to the gravitational acceleration), above which is the fluid next in density, etc.; in other words, the inequalities $\rho_1 > \rho_2 > \dots > \rho_m > 0$ hold. In the study of this problem, a principal role is played by the Weil decomposition of the space of vector-functions $\tilde{L}_2(\Omega)$

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into an orthogonal sum of subspaces. Namely, denoting by Ω_k the domain occupied in a rest state by the fluid of density ρ_k ; the areas of the solid wall of the basin adhering to Ω_k , by S_k , ($k = \overline{1, m}$); the normals n_i to the separation boundaries Γ_i between the i th and $(i + 1)$ th fluids will be directed inside the $(i + 1)$ th fluid ($i = \overline{1, m - 1}$). We will consider the collection $\vec{u} = \{\vec{u}_k\}_{k=1}^m$ of the velocity fields of the fluids as an element of $\vec{L}_2(\Omega)$; in addition, we have the orthogonal decomposition

$$\vec{L}_2(\Omega) = \vec{J}_0(\Omega) \oplus \vec{G}_{h,S}(\Omega) \oplus \vec{G}_{0,\Gamma}(\Omega),$$

where $\vec{J}_0(\Omega) = \{ \vec{w} = \{\vec{w}_k\}_{k=1}^m \in \vec{L}_2(\Omega) \mid \operatorname{div} \vec{w}_k = 0 \text{ (in } \Omega_k), \vec{w}_k \cdot \vec{n}_k = 0 \text{ (on } \partial\Omega_k) \}$,

$$\begin{aligned} \vec{G}_{h,S}(\Omega) = \{ \vec{v} = \{\vec{v}_k\}_{k=1}^m \in \vec{L}_2(\Omega) \mid \vec{v}_k = \nabla \varphi_k, \quad \Delta \varphi_k = 0 \text{ (in } \Omega_k), \\ \partial \varphi_k / \partial n_k = 0 \text{ (on } S_k), \quad \partial \varphi_i / \partial n_i = \partial \varphi_{i+1} / \partial n_i \text{ (on } \Gamma_i, i = \overline{1, m - 1}) \}, \end{aligned}$$

$$\begin{aligned} \vec{G}_{0,\Gamma}(\Omega) = \{ \vec{u} = \{\vec{u}_k\}_{k=1}^m \in \vec{L}_2(\Omega) \mid \vec{u}_k = \nabla p_k, \quad k = \overline{1, m}, \\ \rho_i p_i - \rho_{i+1} p_{i+1} = 0 \text{ (on } \Gamma_i, i = \overline{1, m - 1}) \}. \end{aligned}$$

With this taken into account, the original problem is rewritten as a collection of relations for the initial objects and then, by the method of orthogonal projection onto the chosen subspaces, it is reduced to an equivalent Cauchy problem for a second-order differential equation in a Hilbert space, for which an existence theorem for solutions is proved. However, no transition was carried out from sufficient conditions that require the membership of functions in domains of the matrix operators to sufficient conditions in which constraints are imposed on the initial conditions and the external forces at the corresponding boundaries.

The present article deals with the problem of the small motions and normal oscillations of a multilayered fluid with a free surface consisting of two domains: the surface of the fluid without ice and an area of crumbled ice. By crumbled ice we mean solid particles of some substance floating on the free surface (see, e.g., [2]).

For studying the posed problem, we had to modify the above approach. Namely, we consider the initial objects separately and not in collections, separating trivial components while reasoning. This is due to the fact that, firstly, using this approach, we manage to formulate sufficient conditions for the existence of a solution strong with respect to time in terms of the original problem. And secondly, we plan to use the developed approach also in a problem with the so-called internal flotation (the separation boundary between the layers of the fluid is a solid surface). The interest in such problems is that, in the experimental research of the distribution of the characteristics of water, for example, in the Black Sea, it was discovered that, at the separation boundary between the two main layers (the upper and lower layers), particles of various materials float whose volume density is greater than the density of the upper layer and lower than the density of the lower layer float. These materials include wet wood, seaweed, vegetable residues, "environmental garbage", and the like.

1.2. Contents of the article. The presentation in this paper is carried out by the following scheme: After the introduction, in Section 2, we give the statement of the initial boundary value the problem of small motions of a multilayered fluid with free surface consisting of two domains: the surface of the fluid without ice and

an area of crushed ice. In Subsections 2.2 and 2.3, we construct orthogonal decompositions naturally adapted to the application of the method of orthogonal projection for the original problem. Then the original initial boundary value problem is reduced to an equivalent Cauchy problem for a second-order differential-operator equation in a Hilbert space H . After a detailed study of the properties of the operator coefficients corresponding to the resulting system of equations (Subsection 2.5), in Subsection 2.7, we prove a strong solvability theorem for the obtained Cauchy problem on a finite time interval. Next, we pass from the sufficient conditions (72), in which the membership of functions in the domains of matrix operators is required, to sufficient conditions (Theorem 5) in which constraints are imposed on the initial conditions and the external forces. Finally, for the problem of the normal oscillations of a hydrosystem (in Section 3), the discreteness of the spectrum is proved with two limit points: at the endpoint of the positive axis and at infinity. Asymptotic formulas for two branches of eigenvalues with these limit points are obtained. We prove the orthogonal basis property of the system of eigenfunctions.

2. THE EVOLUTION PROBLEM

2.1. **Mathematical statement of the problem.** Consider a fixed basin partially filled up by a system of two ideal incompressible fluids. Fluids are assumed heavy and, owing to this, the action of the capillar forces is neglected in the problem. Denote by Ω_i ($i = 1, 2$) the domain occupied at a rest state by the fluid of density ρ_i ($\rho_1 > \rho_2$), the corresponding segment of the solid wall will be denoted by S_i ; all parameters concerning the lower fluid will be given index 1, and those concerning the upper fluid, index 2. Represent $\Gamma = \partial\Omega_2 \setminus \overline{S_2} = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are the lower and upper boundaries of Ω_2 respectively; moreover, Γ_2 consists of domains of several types: $\Gamma_2 = \Gamma_{20} \cup \Gamma_{21}$, where Γ_{20} is the area of “pure water,” Γ_{21} is the area of “crumbled ice.” Denote by ρ_0 the surface density of the crumbled ice.

Introduce the coordinate system $Ox_1x_2x_3$ so that the axis Ox_3 be directed opposite to the action of gravity, and the origin lies on the separation surface Γ_1 . Denote by \vec{n}_i the unit vector normal to $\partial\Omega_i$ and directed outside Ω_i . Consider the small motions of the hydrosystem close to a rest state. Let \vec{u}_i be velocity field and let $\zeta_i = \zeta_i(t, \hat{x})$, $\hat{x} \in \Gamma_i$ be the deviation of the freely moving surfaces of the fluids $\Gamma_i(t)$ from Γ_i along the normal \vec{n}_i ; $p_i = p_i(t, x)$, $x \in \Omega_i$ is the deviation of the pressure fields from the equilibrium fields.

The linear statement of the initial boundary value problem looks as follows (see, for example, [1, 2]):

$$(1) \quad \begin{aligned} \frac{\partial \vec{u}_i}{\partial t} &= -\rho_i^{-1} \nabla p_i + \vec{f}_i \quad (\text{in } \Omega_i), & \text{div } \vec{u}_i &= 0 \quad (\text{in } \Omega_i), \\ \vec{u}_i \cdot \vec{n}_i &= 0 \quad (\text{on } S_i), & \partial \zeta_1 / \partial t &= \vec{u}_1 \cdot \vec{n}_1 = \vec{u}_2 \cdot \vec{n}_1 \quad (\text{on } \Gamma_1), \end{aligned}$$

$$(2) \quad \begin{aligned} \frac{\partial \zeta_2}{\partial t} &= \vec{u}_2 \cdot \vec{n}_2 \quad (\text{on } \Gamma_2), & \int_{\Gamma_1} \zeta_1 d\Gamma_1 &= 0, \\ p_1 - p_2 &= \Delta \rho g \zeta_1, & \Delta \rho &:= \rho_1 - \rho_2 > 0 \quad (\text{on } \Gamma_1), \end{aligned}$$

$$(3) \quad p_2 = g\rho_2\zeta_2 \text{ (on } \Gamma_{20}), \quad p_2 = g\rho_2\zeta_2 + \rho_0 \frac{\partial^2 \zeta_2}{\partial t^2} \text{ (on } \Gamma_{21}),$$

$$\int_{\Gamma_2} \zeta d\Gamma = \int_{\Gamma_{20}} \zeta d\Gamma_{20} + \int_{\Gamma_{2,1}} \zeta d\Gamma_{2,1} = 0,$$

$$(4) \quad \vec{u}_i(0, x) = \vec{u}_i^0(x), \quad \zeta_i(0, \hat{x}) = \zeta_i^0(\hat{x}), \quad i = 1, 2.$$

The last two conditions are the initial conditions added to the problem for the completeness of its statement, $\int_{\Gamma_i} \zeta_i d\Gamma_i = 0$ is the condition of the preservation of its volume.

2.2. On one orthogonal decomposition of a Hilbert space. Suppose that we have a domain $\Omega \subset \mathbb{R}^3$. The boundary of the domain $\partial\Omega = \bar{S} \cup \Gamma$, where Γ is a connected set with $\text{mes } \Gamma > 0$. Introduce the space $H_\Gamma^1(\Omega)$ of functions in $H^1(\Omega)$ having zero as the mean value over Γ with the norm

$$(5) \quad \|p\|_{H_\Gamma^1(\Omega)}^2 = \int_{\Omega} |\nabla p|^2 d\Omega < \infty, \quad \int_{\Gamma} p d\Gamma = 0.$$

For $H_\Gamma^1(\Omega)$, we have the orthogonal decomposition (see, for example, [1]):

$$(6) \quad H_\Gamma^1(\Omega) = H_{h,S}^1(\Omega) \oplus H_{0,\Gamma}^1(\Omega),$$

where $H_{0,\Gamma}^1(\Omega) = \{p \in H_\Gamma^1(\Omega) \mid p = 0 \text{ (on } \Gamma)\}$,

$$= \{p \in H_\Gamma^1(\Omega) \mid \Delta p = 0 \text{ (in } \Omega), \frac{\partial p}{\partial n} = 0 \text{ (on } S), \int_{\Gamma} p d\Gamma = 0\},$$

and orthogonality in (6) is understood with respect to the scalar product corresponding to the norm (5).

Suppose now that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, Γ_1 and Γ_2 are connected sets of measure zero located horizontally.

Consider the set

$$(7) \quad H_{h,S}^1(\Omega) := \{p \in H_\Gamma^1(\Omega) \mid \Delta p = 0 \text{ (in } \Omega), \frac{\partial p}{\partial n} = 0 \text{ (on } S), \\ \int_{\Gamma_1} \frac{\partial p}{\partial n} d\Gamma_1 = 0, \int_{\Gamma_2} \frac{\partial p}{\partial n} d\Gamma_2 = 0, \int_{\Gamma} p d\Gamma = 0\}.$$

In [3], we proved

Lemma 1. *The orthogonal decomposition*

$$(8) \quad H_{h,S}^1(\Omega) = \widehat{H}_{h,S}^1(\Omega) \oplus \{\alpha \varphi_0\},$$

where $\{\alpha \varphi_0\}$ is a one-dimensional subspace and the function φ_0 is the solution to the boundary value problem

$$(9) \quad \Delta \varphi_0 = 0 \text{ (in } \Omega), \quad \frac{\partial \varphi_0}{\partial n} = 0 \text{ (in } S), \\ \varphi_0 = \text{mes } \Gamma_1 \text{ (on } \Gamma_2), \quad \varphi_0 = -\text{mes } \Gamma_2 \text{ (on } \Gamma_1).$$

Decomposition (8) generates a decomposition of the subspace of potential fields $\vec{G}_{h,S}(\Omega)$ into an orthogonal sum:

$$(10) \quad \vec{G}_{h,S}(\Omega) = \vec{\widehat{G}}_{h,S}(\Omega) \oplus \{\alpha \nabla \varphi_0\}.$$

2.3. The method of orthogonal projection. For the domain Ω_1 , introduce a decomposition of the space of vector fields $\vec{L}_2(\Omega_1)$ into an orthogonal sum (see [1]):

$$(11) \quad \vec{L}_2(\Omega_1) = \vec{J}_0(\Omega_1) \oplus \vec{G}_{h,S_1}(\Omega_1) \oplus \vec{G}_{0,\Gamma_1}(\Omega_1).$$

$$\vec{J}_0(\Omega_1) := \{ \vec{u}_1 \mid \operatorname{div} \vec{u}_1 = 0 \text{ (in } \Omega_1), \vec{u}_1 \cdot \vec{n}_1 = 0 \text{ (on } \partial\Omega_1) \},$$

$$\vec{G}_{h,S_1}(\Omega_1) := \{ \vec{v}_1 \mid \vec{v}_1 = \nabla p_1, \vec{v}_1 \cdot \vec{n}_1 = 0 \text{ (on } S_1), \nabla \cdot \vec{v}_1 = 0 \text{ (in } \Omega_1), \int_{\Gamma_1} p_1 d\Gamma_1 = 0 \},$$

$$\vec{G}_{0,\Gamma_1}(\Omega_1) := \{ \vec{w}_1 \mid \vec{w}_1 = \nabla \varphi_1, \varphi_1 = 0 \text{ (on } \Gamma_1) \}.$$

We will assume $\vec{u}_1(t, x)$ and $\nabla p_1(t, x)$ to be functions of the variable t with values in $\vec{L}_2(\Omega_1)$; then, by the equations and the boundary conditions (1)–(2), the orthogonal expansion (11), we have

$$\vec{u}_1(t, x) \in \vec{J}_0(\Omega_1) \oplus \vec{G}_{h,S_1}(\Omega_1), \quad \nabla p_1(t, x) \in \vec{G}_{0,\Gamma_1}(\Omega_1) \oplus \vec{G}_{h,S_1}(\Omega_1).$$

Therefore, for each t , we will search for them in the form

$$(12) \quad \vec{u}_1(t, x) = \vec{v}_1(t, x) + \nabla \Phi_1(t, x), \quad \vec{v}_1(t, x) \in \vec{J}_0(\Omega_1), \quad \nabla \Phi_1(t, x) \in \vec{G}_{h,S_1}(\Omega_1),$$

$$\nabla p_1(t, x) = \nabla p_{1,1}(t, x) + \nabla p_{1,2}(t, x), \quad \nabla p_{1,1}(x, t) \in \vec{G}_{h,S_1}(\Omega_1), \quad \nabla p_{1,2}(t, x) \in \vec{G}_{0,\Gamma_1}(\Omega_1).$$

Denote by $P_{0,1}$, P_{h,S_1} , and P_{0,Γ_1} the orthoprojections onto the spaces $\vec{J}_0(\Omega_1)$, $\vec{G}_{h,S_1}(\Omega_1)$, $\vec{G}_{0,\Gamma_1}(\Omega_1)$ respectively. Then, inserting (12) in the first equation of (1) and applying the orthoprojections, we infer

$$(13) \quad \frac{\partial \vec{v}_1}{\partial t} = P_{0,1} \vec{f}_1 \quad (\text{in } \Omega_1),$$

$$(14) \quad \rho_1 \frac{\partial}{\partial t} \nabla \Phi_1 = -\nabla p_{1,1} + \rho_1 P_{h,S_1} \vec{f}_1 \quad (\text{in } \Omega_1),$$

$$(15) \quad \vec{0} = -\nabla p_{1,2} + \rho_1 P_{0,\Gamma_1} \vec{f}_1 \quad (\text{in } \Omega_1).$$

From (13), reckoning with the initial conditions (4), we immediately obtain

$$\vec{v}_1(t, x) = \int_0^t P_{0,1} \vec{f}_1(\tau, x) d\tau + P_{0,1} \vec{u}_1^0.$$

Therefore, it suffices to confine ourselves to considering relation (14) and also the boundary conditions and initial data with the corresponding change $p_1 \rightarrow p_{1,1}$ since $p_1 = p_{1,1} + p_{1,2}$, $p_{1,2} = 0$ (on Γ_1).

For the domain Ω_2 , introduce the decomposition of the space of vector fields $\vec{L}_2(\Omega_2)$ into an orthogonal sum:

$$(16) \quad \vec{L}_2(\Omega_2) = \vec{J}_0(\Omega_2) \oplus \vec{G}_{h,S_2}(\Omega_2) \oplus \vec{G}_{0,\Gamma}(\Omega_2).$$

The subspace $\vec{G}_{h,S_2}(\Omega_2)$ of (16) consists of potential harmonic fields with zero normal component on the solid wall S_2 for which the condition of the preservation of the volume along the whole boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. In the problem under study, due to the incompressibility of the fluids, the condition of the preservation of the volume must be fulfilled on each of the boundaries Γ_1 and Γ_2 separately. It follows that the subspace $\vec{G}_{h,S_2}(\Omega_2)$ is wider than required. In this connection, we make use of the decomposition of this subspace into the orthogonal sum of two subspaces (see (10)) naturally adapted to the problem.

Reckoning with (10) and (16), introduce the orthogonal decomposition

$$(17) \quad \vec{L}_2(\Omega_2) = \vec{J}_0(\Omega_2) \oplus \vec{G}_{h,S_2}(\Omega_2) \oplus \{ \alpha \nabla \varphi_0 \} \oplus \vec{G}_{0,\Gamma}(\Omega_2),$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$. Introduce also the orthoprojections onto the corresponding subspaces: $P_{0,2}, P_{\widehat{h,S_2}}, P_\varphi, P_{0,\Gamma}$.

As above, by the solenoidality condition and the nonpenetration condition on the solid wall S_2 , we assume that $\vec{u}_2 \in \vec{J}_0(\Omega_2) \oplus \vec{G}_{\widehat{h,S_2}}(\Omega_2)$.

The field ∇p_2 is potential; therefore,

$$\nabla p_2 \in \vec{G}_{\widehat{h,S_2}}(\Omega_2) \oplus \{\alpha \nabla \varphi_0\} \oplus \vec{G}_{0,\Gamma}(\Omega_2).$$

Represent the fields \vec{u}_2 and ∇p_2 in the form:

$$\begin{aligned} \vec{u}_2 &= \vec{v}_2 + \nabla \Phi_2, & \vec{v}_2 &\in \vec{J}_0(\Omega_2), & \nabla \Phi_2 &\in \vec{G}_{\widehat{h,S_2}}(\Omega_2), \\ (18) \quad \nabla p_2 &= \nabla p_{2,1} + \nabla p_{2,2} + \alpha(t) \nabla \varphi_0, & \nabla p_{2,1} &\in \vec{G}_{\widehat{h,S_2}}(\Omega_2), & \nabla p_{2,2} &\in \vec{G}_{0,\Gamma}(\Omega_2). \end{aligned}$$

Insert these representations in the motion equation for an ideal fluid from Ω_2 and apply to it the orthoprojections corresponding to decomposition (17). We infer

$$(19) \quad \frac{\partial \vec{v}_2}{\partial t} = P_{0,2} \vec{f}_2 \quad (\text{in } \Omega_2),$$

$$(20) \quad \rho_2 \frac{\partial}{\partial t} \nabla \Phi_2 = -\nabla p_{2,1} + \rho_2 P_{\widehat{h,S_2}} \vec{f}_2 \quad (\text{in } \Omega_2),$$

$$(21) \quad \alpha(t) \nabla \varphi_0 = \rho_2 P_\varphi \vec{f}_2 \quad (\text{in } \Omega_2),$$

$$(22) \quad \nabla p_{2,2} = \rho_2 P_{0,\Gamma} \vec{f}_2 \quad (\text{in } \Omega_2).$$

From (19), with account taken of the initial conditions (4), we obtain

$$\vec{v}_2(t, x) = \int_0^t P_{0,2} \vec{f}_2(\tau, x) d\tau + P_{0,2} \vec{u}_2^0.$$

It follows from (22) that the component $\nabla p_{2,2}$ of the pressure field ∇p_2 is determined directly from the field of external forces \vec{f}_2 . Moreover, the potential of this field vanishes on Γ and hence is not involved in the boundary conditions. Note also that the elements of the subspace $\{\alpha \nabla \varphi_0\}$ satisfy conditions (26); therefore, from (21) we find all the coefficients α and hence the component $\alpha \nabla \varphi_0$.

Agree to refer to solutions to equations (13), (19) and also the components of the gradients of the pressures (15), (21), (22) as trivial solutions. Thus, the main equations we will consider are equations (14) and (20).

2.4. Statement of the problem after the separation of trivial relations.

Introduce the displacement fields of fluid particles by setting

$$(23) \quad \nabla \Phi_i = \frac{\partial}{\partial t} \nabla \Psi_i, \quad \nabla \Psi_1 \in \vec{G}_{h,S_1}(\Omega_1), \quad \nabla \Psi_2 \in \vec{G}_{\widehat{h,S_2}}(\Omega_2).$$

Here the deviations ζ_i of the moving boundaries Γ_i in the process of motion are obviously equal to

$$(24) \quad \zeta_1 = \frac{\partial \Psi_1}{\partial n_1} = \frac{\partial \Psi_2}{\partial n_1} \quad (\text{on } \Gamma_1), \quad \zeta_2 = \frac{\partial \Psi_2}{\partial n_2} \quad (\text{on } \Gamma_2).$$

Furthermore, if $P_{h,S_1} \vec{f}_1 = \nabla F_1 \in \vec{G}_{h,S_1}(\Omega_1)$, $P_{\widehat{h,S_2}} \vec{f}_2 = \nabla F_2 \in \vec{G}_{\widehat{h,S_2}}(\Omega_2)$, then (14) and (20) imply

$$(25) \quad p_{k,1} = \rho_k \left(F_k - \frac{\partial^2}{\partial t^2} \Psi_k \right) + c_k(t) \quad (\text{in } \Omega_k, \quad k = 1, 2),$$

with arbitrary functions $c_k(t)$ depending only on the time variable (the so-called Cauchy–Lagrange integrals).

With account taken of what was said and also after the separation of trivial relations, the initial boundary value problem (1), (4) is formulated as follows:

$$(26) \quad \Delta \Psi_i = 0 \text{ (in } \Omega_i), \quad \frac{\partial \Psi_i}{\partial n_i} = 0 \text{ (on } S_i), \quad \int_{\Gamma_i} \Psi_i d\Gamma_i = 0,$$

$$\zeta_1 = \frac{\partial \Psi_1}{\partial n_1} = \frac{\partial \Psi_2}{\partial n_1} \text{ (on } \Gamma_1), \quad \zeta_2 = \frac{\partial \Psi_2}{\partial n_2} \text{ (on } \Gamma_2),$$

$$(27) \quad \frac{\partial^2}{\partial t^2} (-\rho_1 \Psi_1 + \rho_2 \Psi_2) - g \Delta \rho \frac{\partial \Psi_1}{\partial n_1} = \rho_2 F_2 - \rho_1 F_1 + c_2(t) - c_1(t) \text{ (on } \Gamma_1),$$

$$(28) \quad \rho_2 \frac{\partial^2}{\partial t^2} \Psi_2 + g \rho_2 \frac{\partial \Psi_2}{\partial n_2} = \rho_2 F_2 + c_2(t) \text{ (on } \Gamma_{20}),$$

$$(29) \quad \rho_2 \frac{\partial^2}{\partial t^2} \Psi_2 + \rho_0 \frac{\partial^2}{\partial t^2} \frac{\partial \Psi_2}{\partial n_2} + g \rho_2 \frac{\partial \Psi_2}{\partial n_2} = \rho_2 F_2 + c_2(t) \text{ (on } \Gamma_{21}),$$

$$\int_{\Gamma_1} \frac{\partial \Psi_1}{\partial n_1} d\Gamma_1 = 0, \quad \int_{\Gamma_2} \frac{\partial \Psi_2}{\partial n_2} d\Gamma_2 = \int_{\Gamma_{20}} \frac{\partial \Psi_2}{\partial n_2} d\Gamma_{20} + \int_{\Gamma_{21}} \frac{\partial \Psi_2}{\partial n_2} d\Gamma_{21} = 0,$$

$$(30) \quad \frac{\partial}{\partial t} \nabla \Psi_1(0, x) = P_{h, S_1} \vec{u}_1^0(x), \quad \frac{\partial}{\partial t} \nabla \Psi_2(0, x) = P_{\widehat{h}, S_2} \vec{u}_2^0(x),$$

$$\zeta_i(0, \hat{x}) = \zeta_i^0(\hat{x}) = \left(\frac{\partial \Psi_i}{\partial n_i} \right)_{\Gamma_i} (0, \hat{x}), \quad i = 1, 2.$$

2.5. Reduction of the system to a differential-operator equation. For passing to an operator statement of the problem under study, consider a number of auxiliary boundary value problems.

Auxiliary Problem I.

$$\Delta \Psi_1 = 0 \text{ (in } \Omega_1), \quad \frac{\partial \Psi_1}{\partial n_1} = 0 \text{ (on } S_1), \quad \int_{\Gamma_1} \Psi_1 d\Gamma_1 = 0,$$

$$\frac{\partial \Psi_1}{\partial n_1} = \eta_0 \text{ (on } \Gamma_1), \quad \int_{\Gamma_1} \eta_0 d\Gamma_1 = 0.$$

Endow the space $H_1 = L_{2, \Gamma_1} := L_2(\Gamma_1) \ominus \{1_{\Gamma_1}\}$ with a rigging in the form $H_1^+ \subset H_1 \subset H_1^-$, where $H_1^+ = H_1^{1/2}(\Gamma_1) \cap H_1 =: H_{\Gamma_1}^{1/2}$, $H_1^- = (H_1^+)^* =: \widetilde{H}_{\Gamma_1}^{-1/2}$. The symbol \sim designates the class of functions in $H_{\Gamma_1}^{-1/2}$ extendible by zero to the whole boundary $\partial\Omega_1$ in the class $H^{-1/2}(\partial\Omega_1)$ (see [4]). If $\eta_0 \in \widetilde{H}_{\Gamma_1}^{-\frac{1}{2}}$ then the problem has a unique solution $\Psi_1 \in H_{\Gamma_1}^1(\Omega_1)$. Introduce the following operator from the solution to Problem I: $P_{\Gamma_1} \Psi_1|_{\Gamma_1} =: S_0 \eta_0$, the operator S_0 is selfadjoint, positive, and compact in $L_{2, \Gamma_1} = H_1$ (for details, see, for example, [5]).

Auxiliary Problem II.

$$\Delta \Psi_{2,1} = 0 \text{ (in } \Omega_2), \quad \frac{\partial \Psi_{2,1}}{\partial n_2} = 0 \text{ (on } S_2), \quad \int_{\Gamma = \Gamma_1 \cup \Gamma_2} \Psi_{2,1} d\Gamma = 0,$$

$$\frac{\partial \Psi_{2,1}}{\partial n_2} = \eta_1 \text{ (on } \Gamma_1), \quad \frac{\partial \Psi_{2,1}}{\partial n_2} = 0 \text{ (on } \Gamma_2).$$

By analogy with the previous arguments, if $\eta_1 \in \widetilde{H}_{\Gamma_1}^{-\frac{1}{2}}$ then Problem II has a unique solution $\Psi_1 \in H_{\Gamma}^1(\Omega_2)$; moreover, $P_{\Gamma_1}\Psi_{2,1} =: S_1\eta_1$, where the operator S_1 is self-adjoint, positive, and compact in $L_{2,\Gamma_1} = H_1$.

Auxiliary Problem III.

$$\begin{aligned} \Delta\Psi_{2,2} &= 0 \text{ (in } \Omega_2), & \frac{\partial\Psi_{2,2}}{\partial n_2} &= 0 \text{ (on } S_2), & \int_{\Gamma=\Gamma_1\cup\Gamma_2} \Psi_{2,2} d\Gamma &= 0, \\ \frac{\partial\Psi_{2,2}}{\partial n_2} &= 0 \text{ (on } \Gamma_1), & \frac{\partial\Psi_{2,2}}{\partial n_2} &= \eta_2 \text{ (on } \Gamma_2 = \Gamma_{20} \cup \Gamma_{21}). \end{aligned}$$

We will regard the function η_2 as an element of the space $H_2 = L_{2,\Gamma_2} := L_2(\Gamma_2) \ominus \{1_{\Gamma_2}\}$ and search for it in the form of a pair of functions $\eta_2 = (\eta_{2,0}; \eta_{2,1})$, where $\eta_{2,0} = \eta_2|_{\Gamma_{20}}$ and $\eta_{2,1} = \eta_2|_{\Gamma_{21}}$, i.e., of functions defined on the corresponding domains Γ_{20} and Γ_{21} .

Consider the following subspaces of H_2 :

$$(31) \quad H_{20} := \{ (\eta_{2,0}; \eta_{2,1}) \mid \eta_{2,0} \in L_2(\Gamma_{20}) \ominus \{1_{\Gamma_{20}}\}, \eta_{2,1} \equiv 0 \},$$

$$(32) \quad H_{21} := \{ (\eta_{2,0}; \eta_{2,1}) \mid \eta_{2,1} \in L_2(\Gamma_{21}) \ominus \{1_{\Gamma_{21}}\}, \eta_{2,0} \equiv 0 \}.$$

The space H_2 can be decomposed into the orthogonal sum of three spaces (see [2] for details):

$$(33) \quad H_2 = H_{20} \oplus H_{21} \oplus \widehat{H},$$

where \widehat{H} is the one-dimensional subspace in H_2 spanned by the vector $\widehat{\varphi}$: $\widehat{H} = \{ \widehat{v} \mid \widehat{v} = \alpha\widehat{\varphi}, \forall \alpha \in \mathbb{C}, \widehat{\varphi} = (\text{mes } \Gamma_{21}; -\text{mes } \Gamma_{20}) \}$. On the space H_2 , introduce the orthoprojections P_{20} , P_{21} , and \widehat{P} onto the subspaces H_{20} , H_{21} , and \widehat{H} respectively. They will act by the following rules:

$$\begin{aligned} P_{20}w &= (w_{2,0} - \widetilde{w}_{2,0}; 0), & \widetilde{w}_{2,0} &= (\text{mes } \Gamma_{20})^{-1} \int_{\Gamma_{20}} w_{2,0} d\Gamma_{20}, \\ (34) \quad P_{21}w &= (0; w_{2,1} - \widetilde{w}_{2,1}), & \widetilde{w}_{2,1} &= (\text{mes } \Gamma_{21})^{-1} \int_{\Gamma_{21}} w_{2,1} d\Gamma_{21}, \\ \widehat{P}w &= (I - P_{20} - P_{21})w = (\widetilde{w}_{2,0}; \widetilde{w}_{2,1}). \end{aligned}$$

Pass to constructing the potential $\Psi_{2,2}$ in the domain Ω_2 by expressing it in terms of η_2 . For obtaining the general form of the function $\Psi_{2,2}$ taking into account the representation of η_2 in the form

$$(35) \quad \eta_2 = (\eta_{2,0} - \widetilde{\eta}_{2,0}; 0) + (0; \eta_{2,1} - \widetilde{\eta}_{2,1}) + (\widetilde{\eta}_{2,0}; \widetilde{\eta}_{2,1}) =: P_{20}\eta_2 + P_{21}\eta_2 + \widehat{P}\eta_2,$$

consider three auxiliary problems.

Auxiliary Problem III.1.

$$\begin{aligned} \Delta\Psi_{2,2}^0 &= 0 \text{ (in } \Omega_2), & \frac{\partial\Psi_{2,2}^0}{\partial n_2} &= 0 \text{ (on } S_2), & \int_{\Gamma=\Gamma_1\cup\Gamma_2} \Psi_{2,2}^0 d\Gamma &= 0, \\ \frac{\partial\Psi_{2,2}^0}{\partial n_2} &= 0 \text{ (on } \Gamma_1), & \frac{\partial\Psi_{2,2}^0}{\partial n_2} &= \eta_{2,0} - \widetilde{\eta}_{2,0} \text{ (on } \Gamma_{20}), & \frac{\partial\Psi_{2,2}^0}{\partial n_2} &= 0 \text{ (on } \Gamma_{21}). \end{aligned}$$

Since $H_{20} \subset H_2$, the necessary condition for the solvability of Problem III.1 is fulfilled, and hence this problem has a unique solution $\Psi_{2,2}^0 = \Psi_{2,2}^0(x)$ in $H_{\Gamma}^1(\Omega_2)$.

Introduce the operator T_0 , which assigns to a function $P_{20}\eta_2$ the solution to Problem III.1:

$$\Psi_{2,2}^0 = \Psi_{2,2}^0|_{\Omega_2} =: T_0 P_{20}\eta_2 = T_0(\eta_{2,0} - \tilde{\eta}_{2,0}; 0) =: T_0 w_0, \quad w_0 := P_{20}\eta_2 \in H_{20}.$$

Now, consider the values of the function Ψ_{20}^0 on the boundary of Γ_2 . Introduce the trace operator of the boundary Γ_2 : $\gamma_2(\Psi_{2,2}^0|_{\Omega_2}) := \Psi_{2,2}^0|_{\Gamma_2}$ and represent the function $\Psi_{2,2}^0|_{\Gamma_2}$ as the sum of its projections onto the subspaces H_{20} , H_{21} , and \hat{H} :

$$(36) \quad \Psi_{2,2}^0|_{\Gamma_2} = P_{20}\gamma T_0 P_{20}\eta_2 + P_{21}\gamma T_0 P_{20}\eta_2 + \hat{P}\gamma T_0 P_{20}\eta_2 \\ =: C_{00}w_0 + C_{10}w_0 + C_{20}w_0.$$

Auxiliary Problem III.2.

$$\Delta \Psi_{2,2}^1 = 0 \text{ (in } \Omega_2), \quad \frac{\partial \Psi_{2,2}^1}{\partial n_2} = 0 \text{ (on } S_2), \quad \int_{\Gamma=\Gamma_1 \cup \Gamma_2} \Psi_{2,2}^1 d\Gamma = 0, \\ \frac{\partial \Psi_{2,2}^1}{\partial n_2} = 0 \text{ (on } \Gamma_1), \quad \frac{\partial \Psi_{2,2}^1}{\partial n_2} = 0 \text{ (on } \Gamma_{20}), \quad \frac{\partial \Psi_{2,2}^1}{\partial n_2} = \eta_{2,1} - \tilde{\eta}_{2,1} \text{ (on } \Gamma_{21}).$$

Auxiliary Problem III.2 has a unique solution $\Psi_{2,2}^1 = \Psi_{2,2}^1(x) \in H_{\Gamma}^1(\Omega_2)$. Introduce the operator T_1 assigning to a function $P_{21}\eta_2$ the solution to Problem III.2:

$$\Psi_{2,2}^1 = \Psi_{2,2}^1|_{\Omega_2} =: T_1 P_{21}\eta_2 = T_1(0; \eta_{2,1} - \tilde{\eta}_{2,1}) =: T_1 w_1, \quad w_1 = P_{21}\eta_2 \in H_{21}.$$

Consider the values of the values of the function $\Psi_{2,2}^1$ on the boundary Γ_2 and represent the function $\Psi_{2,2}^1|_{\Gamma_2}$ as the sum of the projections to the subspaces H_{20} , H_{21} , and \hat{H} :

$$(37) \quad \Psi_{2,2}^1|_{\Gamma_2} = P_{20}\gamma T_1 P_{21}\eta_2 + P_{21}\gamma T_1 P_{21}\eta_2 + \hat{P}\gamma T_1 P_{21}\eta_2 \\ =: C_{01}w_1 + C_{11}w_1 + C_{21}w_1.$$

Auxiliary Problem III.3.

$$\Delta \Psi_{2,2}^2 = 0 \text{ (in } \Omega_2), \quad \frac{\partial \Psi_{2,2}^2}{\partial n_2} = 0 \text{ (on } S_2), \quad \int_{\Gamma=\Gamma_1 \cup \Gamma_2} \Psi_{2,2}^2 d\Gamma = 0, \\ \frac{\partial \Psi_{2,2}^2}{\partial n_2} = 0 \text{ (on } \Gamma_1), \quad \frac{\partial \Psi_{2,2}^2}{\partial n_2} = \hat{P}\eta_2 \text{ (on } \Gamma_2).$$

Since \hat{H} is a one-dimensional subspace, it suffices to consider Auxiliary Problem III.3 with function $\hat{\varphi}$ instead of $\hat{P}\eta_2$, i.e., with the boundary conditions on Γ_{20} and Γ_{21} of the form

$$\frac{\partial \Psi_{2,2}^2}{\partial n_2} = \text{mes } \Gamma_{21} \text{ (on } \Gamma_{20}), \quad \frac{\partial \Psi_{2,2}^2}{\partial n_2} = -\text{mes } \Gamma_{20} \text{ (on } \Gamma_{21}).$$

Problem III.3 has a unique solution $\Psi_{2,2}^2 = \alpha \hat{\Psi} \in H_{\Gamma}^1(\Omega_2)$, where $\hat{\Psi}$ is the solution to Problem III.3 with the boundary conditions rewritten above. Introduce the operator T_2 assigning to a function $\hat{P}\eta_2$ the solution to Problem III.3:

$$\Psi_{2,2}^2 =: T_2 \hat{P}\eta_2 =: T_2 \hat{w}, \quad \hat{w} = \hat{P}\eta_2 \in \hat{H}.$$

Represent the function $\Psi_{2,2}|_{\Gamma_2}$ as the sum of its projections to the subspaces H_{20} , H_{21} , and \widehat{H} :

$$(38) \quad \Psi_{2,2}|_{\Gamma_2} = P_{20}\gamma T_2 \widehat{P}\eta_2 + P_{21}\gamma T_2 \widehat{P}\eta_2 + \widehat{P}\gamma T_2 \widehat{P}\eta_2 =: C_{02}\widehat{w} + C_{12}\widehat{w} + C_{22}\widehat{w}.$$

In this case, the operators C_{02} , C_{12} , and C_{22} are one-dimensional.

Proposition 1. *Due to the membership of $\nabla\Psi_2$ in $\vec{G}_{\widehat{h,S_2}}(\Omega_2)$ and the definition of $\vec{G}_{\widehat{h,S_2}}(\Omega_2)$, the potential Ψ_2 is representable with the use of the solutions to the auxiliary problems in the form:*

$$(39) \quad \Psi_2 = \Psi_{2,1} + \Psi_{2,2};$$

we assume that

$$(40) \quad \eta_1 = \frac{\partial\Psi_{2,1}}{\partial n_2} = -\frac{\partial\Psi_{2,1}}{\partial n_1} = -\frac{\partial\Psi_1}{\partial n_1} = -\eta_0 = -\zeta_1 \quad (\text{on } \Gamma_1),$$

$$(41) \quad \eta_2 = \frac{\partial\Psi_{2,2}}{\partial n_2} = \zeta_2 \quad (\text{on } \Gamma_2).$$

That said, decompose the space $\vec{G}_{\widehat{h,S_2}}(\Omega_2)$ as the following direct sum:

$$(42) \quad \vec{G}_{\widehat{h,S_2}}(\Omega_2) = \vec{G}_1(\Omega_2) \dot{+} \vec{G}_2(\Omega_2),$$

where

$$\vec{G}_1(\Omega_2) := \left\{ \nabla p \mid \Delta p = 0 \text{ (in } \Omega_2), \frac{\partial p}{\partial n_2} = 0 \text{ (on } S_2), \frac{\partial p}{\partial n_2} = 0 \text{ (on } \Gamma_2), \int_{\Gamma} p \, d\Gamma = 0 \right\},$$

$$\vec{G}_2(\Omega_2) := \left\{ \nabla p \mid \Delta p = 0 \text{ (in } \Omega_2), \frac{\partial p}{\partial n_2} = 0 \text{ (on } S_2), \frac{\partial p}{\partial n_2} = 0 \text{ (on } \Gamma_1), \int_{\Gamma} p \, d\Gamma = 0 \right\}.$$

Reckoning with (39), rewrite relations (27)–(29) in the form

$$(43) \quad \frac{d^2}{dt^2} (-\rho_1\Psi_1 + \rho_2\Psi_{2,1} + \rho_2\Psi_{2,2})_{\Gamma_1} - g\Delta\rho\eta_0 = \rho_2 F_2|_{\Gamma_1} - \rho_1 F_1|_{\Gamma_1} + c_2(t) - c_1(t),$$

$$\rho_2 \frac{d^2}{dt^2} (\Psi_{2,1}|_{\Gamma_{20}}; \Psi_{2,1}|_{\Gamma_{21}}) + \rho_2 \frac{d^2}{dt^2} (\Psi_{2,2}|_{\Gamma_{20}}; \Psi_{2,2}|_{\Gamma_{21}}) + \rho_0 \frac{d^2}{dt^2} (0; \eta_{2,1}) + \rho_2 g (\eta_{2,0}; \eta_{2,1}) = \rho_2 (F_2|_{\Gamma_{20}}; F_2|_{\Gamma_{21}}) + (c(t); c(t)).$$

Henceforth, we assume all functions depending on t to be functions of t with values in the corresponding Hilbert space; in this connection, we replace $\partial/\partial t$ with d/dt in the equations of the problem.

In correspondence with decomposition (35), represent the solution to Problem III as the sum of the solutions to the three auxiliary problems:

$$(44) \quad \Psi_{2,2} = \Psi_{2,2}^0 + \Psi_{2,2}^1 + \Psi_{2,2}^2.$$

First of all, by (36), (37), (38), and (44), we have

$$(45) \quad (\Psi_{2,2}|_{\Gamma_{20}}; \Psi_{2,2}|_{\Gamma_{21}}) = \Psi_{2,2}|_{\Gamma_2} = \Psi_{2,2}^0|_{\Gamma_2} + \Psi_{2,2}^1|_{\Gamma_2} + \Psi_{2,2}^2|_{\Gamma_2} = C_{00}w_0 + C_{10}w_0 + C_{20}w_0 + C_{01}w_1 + C_{11}w_1 + C_{21}w_1 + C_{02}\widehat{w} + C_{12}\widehat{w} + C_{22}\widehat{w},$$

where the elements C_{ik} are defined by (36), (37), and (38). Therefore, by these definitions, we respectively have

$$C_{00}w_0 + C_{01}w_1 + C_{02}\widehat{w} \in H_{20}, \quad C_{10}w_0 + C_{11}w_1 + C_{12}\widehat{w} \in H_{21},$$

$$C_{20}w_0 + C_{21}w_1 + C_{22}\widehat{w} \in \widehat{H}.$$

Further, we obviously have the relation

$$(46) \quad (\eta_{2,0}; \eta_{2,1}) = (\eta_{2,0} - \tilde{\eta}_{2,0}; 0) + (0; \eta_{2,1} - \tilde{\eta}_{2,1}) + (\tilde{\eta}_{2,0}; \tilde{\eta}_{2,1}) = w_0 + w_1 + \hat{w}.$$

Let P_{Γ_2} be the orthoprojection onto $H_2 = L_2(\Gamma_2) \ominus \{1_{\Gamma_2}\}$. Then easy calculations show that

$$P_{\Gamma_2}(0; \eta_{2,1}) = (0; \eta_{2,1} - \tilde{\eta}_{2,1}) + P_{\Gamma_2}(0; \tilde{\eta}_{2,1}) = (0; \eta_{2,1} - \tilde{\eta}_{2,1}) + \alpha(\tilde{\eta}_{2,0}; \tilde{\eta}_{2,1}) = w_1 + \alpha \hat{w},$$

$$0 < \alpha := \frac{\text{mes } \Gamma_{20}}{\text{mes } \Gamma_{20} + \text{mes } \Gamma_{21}} < 1.$$

Project both sides of the first equation in (43) to the space $H_1 = L_2(\Gamma_1) \ominus \{1_{\Gamma_1}\}$ and project the second to the subspaces H_{20} , H_{21} , and \hat{H} respectively. Then it is convenient to write system (43) in the orthogonal sum of Hilbert spaces

$$H := H_1 \oplus H_2, \quad H_2 = H_{20} \oplus H_{21} \oplus \hat{H}$$

in the form

$$(47) \quad \frac{d^2}{dt^2} M \zeta + I_0 \zeta = \hat{F}, \quad \zeta = (\eta_0; w)^t, \quad w = (w_0; w_1; \hat{w})^t,$$

$$(48) \quad M = \begin{pmatrix} \rho_1 S_0 + \rho_2 S_1 & -\rho_2 S_3 \\ -\rho_2 S_2 & \rho_2 C + \rho_0 I_\alpha \end{pmatrix}, \quad I_0 = \begin{pmatrix} g \Delta \rho I_1 & 0 \\ 0 & g \rho_2 I_2 \end{pmatrix},$$

$$(49) \quad C = \begin{pmatrix} C_{00} & C_{01} & C_{02} \\ C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{pmatrix}, \quad I_i \text{ are the identity operators in } H_i,$$

$I_\alpha = \text{diag}(0; 1; \alpha)$, $\hat{F} = (\hat{F}_1; \hat{F}_2)^t := (\rho_1 P_{\Gamma_1} F_1 - \rho_2 P_{\Gamma_1} F_2; P_{\Gamma_2}(F_2|_{\Gamma_{20}}; F_2|_{\Gamma_{21}}))^t$;
 $P_{\Gamma_2}(F_2|_{\Gamma_{20}}; F_2|_{\Gamma_{21}}) = P_{20}(F_2|_{\Gamma_{20}}; F_2|_{\Gamma_{21}}) + P_{21}(F_2|_{\Gamma_{20}}; F_2|_{\Gamma_{21}}) + \hat{P}(F_2|_{\Gamma_{20}}; F_2|_{\Gamma_{21}})$,
 $S_2 \eta_0 := -P_{\Gamma_2}(\Psi_{2,1}|_{\Gamma_{20}}; \Psi_{2,1}|_{\Gamma_{21}})$, $S_3 w := \Psi_{2,2}|_{\Gamma_1}$, S_i ($i = 0, 1$) are the operators of Auxiliary Problems I and II.

The initial conditions (30) for equation (47) can also be written down:

$$(50) \quad \zeta^0 = (\zeta_1^0(\hat{x}), w^0(\hat{x}))^t, \quad \zeta^1 = (\zeta_1^1(\hat{x}), w^1(\hat{x}))^t,$$

where $\zeta_1^1(\hat{x}) = [(P_{h,S_1} \vec{u}_1^0(x)) \cdot \vec{n}_1]_{\Gamma_1}$,

$$w^0 = (w_0^0; w_1^0; \hat{w}^0)^t, \quad w_i^0 = P_{2i} \zeta_2^0 \quad (i = 1, 2), \quad \hat{w}^0 = \hat{P} \zeta_2^0,$$

$$w^1(\hat{x}) = (w_0^1; w_1^1; \hat{w}^1)^t, \quad w_i^1 = P_{2i} \left[(P_{\widehat{h,S_2}} \vec{u}_2^0(x)) \cdot \vec{n}_2 \right], \quad \hat{w}^1 = \hat{P} \left[(P_{\widehat{h,S_2}} \vec{u}_2^0(x)) \cdot \vec{n}_2 \right];$$

moreover, the initial data must satisfy the following kinematic condition by decomposition (42):

$$(51) \quad \gamma_1 P_{h,S_1} \vec{u}_1^0(x) = -\gamma_2 \Pi_1 P_{\widehat{h,S_2}} \vec{u}_2^0(x) \quad (\text{on } \Gamma_1).$$

Here Π_1 stands for the projection onto the space $\vec{G}_1(\Omega_2)$ and γ_i is the operation of taking the normal trace on Γ_1 for fields defined in Ω_i ($i = 1, 2$).

Theorem 1. *The initial boundary value problem (26)–(30) is equivalent to the Cauchy problem (47)–(51) for a second-order differential equation in the Hilbert space H .*

Observe the properties of the operator blocks in (47).

Lemma 2. *The operator C (see (49)) is a selfadjoint, compact, and positive operator acting in the space $H_2 = H_{20} \oplus H_{21} \oplus \hat{H}$.*

The proof repeats verbatim the proof of Lemma 2 in [2].

Lemma 3. *The operator M (see (47)) is bounded, selfadjoint, and positive.*

Proof. Boundedness follows from the fact that all the operator coefficients of the matrix M are bounded.

Find the quadratic form of the operator M ; for every $\zeta \in H$, we have

$$\begin{aligned} (M\zeta, \zeta) &= \left(\begin{pmatrix} (\rho_1 S_0 + \rho_2 S_1)\eta_0 - \rho_2 S_3 w \\ -\rho_2 S_2 \eta_0 + (\rho_2 C + \rho_0 I_\alpha)w \end{pmatrix}, \begin{pmatrix} \eta_0 \\ w \end{pmatrix} \right) \\ &= ((\rho_1 S_0 + \rho_2 S_1)\eta_0, \eta_0) - \rho_2 (S_3 w, \eta_0) - \rho_2 (S_2 \eta_0, w) + ((\rho_2 C + \rho_0 I_\alpha)w, w) \\ &= \rho_1 (S_0 \eta_0, \eta_0) + (\rho_2 S_1 \eta_0 - \rho_2 S_3 w, \eta_0) + (-\rho_2 S_2 \eta_0 + \rho_2 C w, w) + \rho_0 (I_\alpha w, w) \\ &= \rho_1 (S_0 \eta_0, \eta_0) + \rho_2 (-P_{\Gamma_1} \Psi_2, \eta_0) + \rho_2 (P_{\Gamma_2} \Psi_2, w) + \rho_0 (I_\alpha w, w). \end{aligned}$$

Moreover,

$$\begin{aligned} (S_0 \eta_0, \eta_0) &= \left(P_{\Gamma_1} \Psi_1, \frac{\partial \Psi_1}{\partial n_1} \right) = \int_{\Gamma_1} P_{\Gamma_1} \Psi_1 \frac{\partial \Psi_1}{\partial n_1} d\Gamma_1 = \int_{\partial \Omega_1} P_{\Gamma_1} \Psi_1 \frac{\partial \Psi_1}{\partial n_1} dS_1 \\ &= \int_{\partial \Omega_1} \Psi_1 \cdot (\nabla \Psi_1 \cdot \vec{n}_1) dS_1 = \int_{\Omega_1} \nabla \Psi_1 \cdot \nabla \Psi_1 d\Omega_1 = \int_{\Omega_1} |\nabla \Psi_1|^2 d\Omega_1; \\ - (P_{\Gamma_1} \Psi_2, \eta_0) + (P_{\Gamma_2} \Psi_2, w) &= \int_{\Gamma_1} P_{\Gamma_1} \Psi_2 \frac{\partial \Psi_{2,1}}{\partial n_2} d\Gamma_1 + \int_{\Gamma_2} P_{\Gamma_2} \Psi_2 \frac{\partial \Psi_{2,2}}{\partial n_2} d\Gamma_2 \\ &= \int_{\partial \Omega_2} \Psi_2 \left(\frac{\partial \Psi_{2,1}}{\partial n_2} + \frac{\partial \Psi_{2,2}}{\partial n_2} \right) dS_2 = \int_{\partial \Omega_2} \Psi_2 \frac{\partial \Psi_2}{\partial n_2} dS_2 = \int_{\partial \Omega_2} \Psi_2 \cdot (\nabla \Psi_2 \cdot \vec{n}_2) dS_2 \\ &= \int_{\Omega_2} \nabla \Psi_2 \cdot \nabla \Psi_2 d\Omega_2 = \int_{\Omega_2} |\nabla \Psi_2|^2 d\Omega_2. \end{aligned}$$

This, we obtain

$$(52) \quad (M\zeta, \zeta) = \rho_1 \int_{\Omega_1} |\nabla \Psi_1|^2 d\Omega_1 + \rho_2 \int_{\Omega_2} |\nabla \Psi_2|^2 d\Omega_2 + \rho_0 \|w_1\|_{H_{21}}^2 + \rho_0 \|\widehat{w}\|_{\widehat{H}}^2.$$

Reckoning with the boundedness of the operator M in (52), we can show that it is selfadjoint and positive. \square

2.6. Auxiliary assertions. Consider the Cauchy problem for a second-order linear differential equation of the following form:

$$(53) \quad A^{1/2} \frac{d^2}{dt^2} (A^{1/2} u) + Bu = f(t), \quad u(0) = u^0, \quad u'(0) = u^1,$$

where the operator coefficients satisfy the conditions

$$(54) \quad 0 < A = A^* \in \mathcal{L}(H), \quad B = B^* \gg 0.$$

Here $\mathcal{L}(H)$ is the space of bounded linear operators acting in the space H .

In particular, the operators A^{-1} and B can be unbounded selfadjoint operators defined on domains $\mathcal{D}(A^{-1})$ and $\mathcal{D}(B)$ dense in H .

The results proved below are justifications of the corresponding assertions in [1, p. 57–62], see also [6, p. 38–76, 158–170].

Definition 1. We say that a function $u(t)$, $0 \leq t \leq T$, is a strong solution to the Cauchy problem (53) with values in $H_{A^{-1}} = \mathcal{D}(A^{-1/2}) \subset H$ if the following conditions are fulfilled: $u(t) \in \mathcal{D}(B)$ and $Bu(t) \in C([0, T]; \mathcal{D}(A^{-1/2}))$, the function $A^{1/2}u(t)$ is twice continuously differentiable, $A^{1/2}d^2/dt^2(A^{1/2}u(t)) \in C([0, T]; \mathcal{D}(A^{-1/2}))$, and equation (53) holds for any $t \in [0, T]$.

Theorem 2. If the following conditions are fulfilled:

$$(55) \quad u^0 \in \mathcal{D}(A^{-1/2}B), \quad u^1 \in \mathcal{D}(B^{1/2}), \quad f(t) \in C^1([0, T]; \mathcal{D}(A^{-1/2})),$$

then the Cauchy problem (53) has a unique strong solution (with values in $\mathcal{D}(A^{-1/2})$) on $[0, T]$.

Proof. Consider the Cauchy problem for the equation

$$(56) \quad \frac{d^2v}{dt^2} + A^{-1/2}BA^{-1/2}v = A^{-1/2}f(t), \quad v(0) = A^{1/2}u^0, \quad v'(0) = A^{1/2}u^1$$

that is obtained from (53) after the change

$$(57) \quad A^{1/2}u(t) = v(t)$$

and applying the operator $A^{-1/2}$ from the left.

Here the operator $A^{-1/2}BA^{-1/2}$ is defined on the domain

$$(58) \quad \mathcal{D}(A^{-1/2}BA^{-1/2}) = \mathcal{R}(A^{1/2}B^{-1}A^{1/2}), \quad \mathcal{D}(A^{1/2}B^{-1}A^{1/2}) = H.$$

Since the operators B and A^{-1} are positive definite, we can verify that so is the operator $A^{-1/2}BA^{-1/2}$.

Perform a change of variable in (56) by introducing a new sought function by the rule

$$(59) \quad \frac{dw}{dt} := -iB^{1/2}A^{-1/2}v(t), \quad w(0) = 0.$$

If dw/dt is a continuously differentiable function (this assumption will be justified below) then (since the operator $B^{1/2}A^{-1/2}$ is closed)

$$(60) \quad \frac{d^2w}{dt^2} = -i \frac{d}{dt} (B^{1/2}A^{-1/2}v) = -iB^{1/2}A^{-1/2} \frac{dv}{dt},$$

$$(61) \quad w'(0) = -iB^{1/2}A^{-1/2}v(0).$$

Write down (56), (60)–(61) with account taken of change (59) in the form of the system of equations

$$(62) \quad \begin{aligned} \frac{d^2}{dt^2} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} + i \begin{pmatrix} 0 & A^{-1/2}B^{1/2} \\ B^{1/2}A^{-1/2} & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} &= \begin{pmatrix} A^{-1/2}f(t) \\ 0 \end{pmatrix}, \\ v'(0) = A^{1/2}u^1, \quad w'(0) &= -iB^{1/2}u^0. \end{aligned}$$

Introducing the new sought function $y(t) := (dv/dt; dw/dt)^t$, where, as above, the index $(\cdot; \cdot)^t$ stands for transposition (in this case, of a row vector) and also the new given function $\hat{f}(t) := (A^{-1/2}f(t); 0)^t$ and the operator matrix

$$(63) \quad \mathcal{A} := \begin{pmatrix} 0 & A^{-1/2}B^{1/2} \\ B^{1/2}A^{-1/2} & 0 \end{pmatrix},$$

defined on the domain

$$(64) \quad \mathcal{D}(\mathcal{A}) := \mathcal{D}(B^{1/2}A^{-1/2}) \oplus \mathcal{D}(A^{-1/2}B^{1/2}),$$

, which is dense in the space $H^2 = H \oplus H$, rewrite problem (62) as the first order differential equation

$$(65) \quad \frac{dy}{dt} + i\mathcal{A}y = \widehat{f}(t), \quad y(0) = (A^{1/2}u^1; -iB^{1/2}u^0)^t$$

in the Hilbert space H^2 .

Here the operator matrix \mathcal{A} is (in general) unbounded and selfadjoint operator: its symmetry is obvious, and its range $\mathcal{R}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^{-1}) = H^2$ because

$$(66) \quad \mathcal{A}^{-1} = (\mathcal{A}^{-1})^* = \begin{pmatrix} 0 & A^{1/2}B^{-1/2} \\ B^{-1/2}A^{1/2} & 0 \end{pmatrix}$$

is a bounded selfadjoint operator defined on the whole space H^2 . This implies that the operator $-i\mathcal{A}$ is a generator of a (strongly continuous) C_0 -group of unitary operators $U(t) := \exp(-it\mathcal{A})$ (see [7]), and the solution to problem (65) is expressed by the formula

$$(67) \quad y(t) = U(t)y(0) + \int_0^t U(t-s)\widehat{f}(s) ds.$$

If

$$(68) \quad y(0) \in \mathcal{D}(\mathcal{A}), \quad \widehat{f}(t) \in C^1([0, T]; H^2),$$

then problem (65) has a unique strong solution on $[0, T]$ (see [6, p. 164]). It is easy to check that conditions (68) are fulfilled if conditions (55) hold. Then problem (65), i.e., problem (62), has a unique strong solution for $t \in [0, T]$. Moreover, all functions in (62) are continuous functions of t with values in H .

Using the initial condition $w'(0) = -iB^{1/2}u^0$ and integrating equation (60) over t from 0 to t , we get equation (59). Since, for a strong solution (see the first equality in (62)), we have

$$(69) \quad \frac{dw}{dt} \in \mathcal{D}(A^{-1/2}B^{1/2}), \quad A^{-1/2}B^{1/2}\frac{dw}{dt} \in C([0, T]; H),$$

inserting (59) in the first equation of (62), we obtain equation (56), where now all the summands are continuous functions of t with values in H , $t \in [0, T]$.

Performing in (56) the inverse change (57) and applying the bounded operator $A^{1/2}$ to both sides, we conclude that equation (53) holds, where all summands are continuous functions of t with values in $\mathcal{D}(A^{1/2}) = H_{A^{-1}} \subset H$. The theorem is proved. \square

Proposition 2. *Since the operator $0 < A = A^* \in \mathcal{L}(H)$, the operator $A^{1/2}$ can be inserted into the derivative sign d^2/dt^2 , and hence the conditions of Theorem 2 are also sufficient for the existence of a strong solution for the equation*

$$(70) \quad \frac{d^2}{dt^2}(Au) + Bu = f(t).$$

Proposition 3. *If the operator B is bounded (in this case, $\mathcal{D}(B) = H$), conditions (55) for the strong solvability of the Cauchy problem (53) can be rewritten as follows:*

$$(71) \quad u^0 \in \mathcal{D}(A^{-1/2}), \quad u^1 \in H, \quad f(t) \in C^1([0, T]; \mathcal{D}(A^{-1/2})).$$

2.7. An existence theorem for a strong solution. As a consequence of Theorem 2 and Proposition 3, we obtain the following theorem:

Theorem 3. *If the following conditions hold:*

$$(72) \quad \zeta^0 \in \mathcal{D}(M^{-1/2}), \quad \zeta^1 \in H, \quad \widehat{F}(t) \in C^1\left([0, T]; \mathcal{D}(M^{-1/2})\right),$$

then there exists a unique strong solution to problem (47), (50).

From the sufficient conditions (72), where the membership of functions in the domains of the matrix operators is required, we can pass to sufficient conditions where constraints are imposed on the initial conditions and external forces.

For this we prove the following theorem:

Theorem 4. *Suppose we have two Hilbert spaces E_1 and E_2 with Hilbert riggings $E_1^+ \subset E_1 \subset E_1^-$ and $E_2^+ \subset E_2 \subset E_2^-$. In the Hilbert space $E = E_1 \times E_2$, consider the operator matrix*

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where the operators possess the following properties: The operator $J : E_2 \rightarrow E_2$ is bounded and positive definite. A_{ij} acts boundedly from E_j^- into E_i^+ ($i, j = 1, 2$). Moreover, the operator $A_{ij} : E_j^- \rightarrow E_i$ is compact. The operator $A_{ii}^{-1} : E_i^+ \rightarrow E_i^-$ is also bounded. The matrix operator $C = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is selfadjoint, positive, and compact in E . Then the operator \mathcal{A}^{-1} is bounded as an operator from $E^+ = E_1^+ \times E_2$ into $E^- = E_1^- \times E_2$ and $\mathcal{D}(\mathcal{A}^{-1/2}) = E^+$.

Proof. Consider the equation

$$(73) \quad \mathcal{A}y = v.$$

For arbitrary $v \in E^+$, with account taken of the operator \mathcal{A} , we obtain the system of equations

$$(74) \quad A_{11}y_1 + A_{12}y_2 = v_1, \quad A_{21}y_1 + (J + A_{22})y_2 = v_2.$$

Since $v \in E^+$, we have $v_1 \in E_1^+$, and, for $y_2 \in E_2$, we obtain $A_{12}y_2 \in E_1^+$. Then, from the first equation (74) we infer

$$(75) \quad y_1 = A_{11}^{-1}(v_1 - A_{12}y_2) \in E_1^-.$$

Hence, $A_{21}y_1 \in E_2^+ \subset E_2$, and, after inserting (75) in the second equation of (74), we have

$$(76) \quad (J + \widehat{A})y_2 = v_2 - A_{21}A_{11}^{-1}v_1, \quad \widehat{A} := A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

The properties of the operators A_{ij} implies that the operator \widehat{A} is compact as an operator acting in E_2 . Furthermore, since C is selfadjoint and positive, \widehat{A} is also selfadjoint and positive. Indeed, for $\forall y_2 \in E_2$ and $y_1 := -A_{11}^{-1}A_{12}y_2$, we obtain

$$\begin{aligned} 0 < (Cy, y) &= (A_{11}y_1, y_1) + (A_{12}y_2, y_1) + (A_{21}y_1, y_2) + (A_{22}y_2, y_2) \\ &= (-A_{12}y_2, y_1) + (A_{12}y_2, y_1) + (-A_{21}A_{11}^{-1}A_{12}y_2, y_2) + (A_{22}y_2, y_2) \\ &= (\widehat{A}y_2, y_2), \quad y_2 \neq 0, \end{aligned}$$

i.e., \widehat{A} is a positive operator.

Then, in problem (76), $(J + \widehat{A}) \gg 0$, and hence it has a bounded inverse acting in E_1 . Therefore,

$$(77) \quad y_2 = (J + \widehat{A})^{-1} (v_2 - A_{21}A_{11}^{-1}v_1) \in E_2.$$

Inserting (77) in (75), we infer

$$(78) \quad y_1 = \left[A_{11}^{-1} + A_{11}^{-1}A_{12}(J + \widehat{A})^{-1}A_{21}A_{11}^{-1} \right] v_1 - A_{11}^{-1}A_{12}(J + \widehat{A})^{-1}v_2.$$

Thus, from (77), (78) we finally obtain the relation

$$(79) \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}QA_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}Q \\ -QA_{21}A_{11}^{-1} & Q \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where $Q := (J + \widehat{A})^{-1}$, which gives the solution to the problem under consideration, i.e.,

$$(80) \quad y = \mathcal{A}^{-1}v.$$

It follows from (79) that if $v_1 \in E_1^+$, $v_2 \in E_2$, then

$$(81) \quad y_1 \in E_1^-, \quad y_2 \in E_2.$$

Indeed, if $v_1 \in E_1^+$ then

$$\begin{aligned} A_{11}^{-1}v_1 \in E_1^- &\implies A_{21}A_{11}^{-1}v_1 \in E_2^+ \subset E_2 \implies QA_{21}A_{11}^{-1}v_1 \in E_2 \subset E_2^- \implies \\ &\implies A_{12}QA_{21}A_{11}^{-1}v_1 \in E_1^+ \implies A_{11}^{-1}A_{12}QA_{21}A_{11}^{-1}v_1 \in E_1^- \end{aligned}$$

and hence $(A_{11}^{-1} + A_{11}^{-1}A_{12}QA_{21}A_{11}^{-1})v_1 \in E_1^-$. we similarly conclude that

$$-A_{11}^{-1}A_{12}Qv_2 \in E_1^-, \quad -QA_{21}A_{11}^{-1}v_1 \in E_2, \quad Qv_2 \in E_2,$$

which gives properties (81).

Thus, we have proved that the operator \mathcal{A}^{-1} acts boundedly from $E^+ = E_1^+ \times E_2$ into $E^- = E_1^- \times E_2$. It is also obvious that \mathcal{A} acts boundedly from E^- into E^+ and the restriction of the operator \mathcal{A} to $E = E_1 \times E_2$ is a bounded positive selfadjoint operator.

Since E^+ , E , and E^- constitute the rigging

$$(82) \quad E^+ \subset E \subset E^-,$$

generated by the operator \mathcal{A} (or \mathcal{A}^{-1}), it follows that, by the constructions from Yu. M. Berezanskiĭ's book (see [8, p. 45–53]), we have

$$(83) \quad E^+ = \mathcal{D}(\mathcal{A}^{-1/2}) = E_1^+ \times E_2. \quad \square$$

For applying the proven theorem, rewrite equation (47) as follows:

$$(84) \quad \frac{d^2}{dt^2} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} + J \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \rho_g I_1 & 0 \\ 0 & g\rho_2 I_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

where $y_1 := (\eta_0; w_0)^t \in H_1 \oplus H_{20}$, $y_2 := (w_1; \widehat{w})^t \in H_{21} \oplus \widehat{H}$; moreover,

$$\begin{aligned} M_{11} &= \begin{pmatrix} \rho_1 S_0 + \rho_2 S_1 & -\rho_2 S_3^0 \\ -\rho_2 S_2^0 & \rho_2 C_{00} \end{pmatrix}, & M_{12} &= \begin{pmatrix} -\rho_2 S_3^1 & -\rho_2 \widehat{S}_3 \\ \rho_2 C_{01} & \rho_2 C_{02} \end{pmatrix}, \\ M_{21} &= \begin{pmatrix} -\rho_2 S_2^1 & \rho_2 C_{10} \\ -\rho_2 \widehat{S}_2 & \rho_2 C_{20} \end{pmatrix}, & M_{22} &= \begin{pmatrix} \rho_2 C_{11} & \rho_2 C_{12} \\ \rho_2 C_{21} & \rho_2 C_{22} \end{pmatrix}, \end{aligned}$$

$$S_j^i = P_{2i}S_j, \widehat{S}_j = \widehat{P}S_j \quad (j = 2, 3, i = 0, 1), \quad J = \text{diag}(1, \alpha), \quad g_1 = (\widehat{F}_1; P_{\Gamma_{20}}\widehat{F}_2)^t, \\ g_2 = (P_{\Gamma_{21}}\widehat{F}_2; \widehat{P}\widehat{F}_2)^t, \quad \rho_g = (g\Delta\rho; g\rho_2)^t.$$

Put $E_1 := H_1 \oplus H_{20}$ and $E_2 := H_{21} \oplus \widehat{H}$.

The rigging of E_1 has the form $E_1^+ \subset E_1 \subset E_1^-$, where $E_1^+ = H_1^+ \times H_{20}^+$, $(E_1^+)^* = E_1^-$, $H_1^+ := H_{\Gamma_1}^{1/2} = H^{1/2}(\Gamma_1) \cap H_1$, $H_1 = L_2(\Gamma_1) \ominus \{1_{\Gamma_1}\}$

$$(85) \quad H_{20}^+ := H_{\Gamma_{20}}^{1/2} = \left\{ u \in H^{1/2}(\Gamma_2) : u \equiv 0 \text{ (on } \Gamma_{21}), \int_{\Gamma_{20}} u \, d\Gamma_{20} = 0 \right\},$$

For E_2 we have: $E_2^+ \subset E_2 \subset E_2^-$, where $E_2^+ = H_{21}^+ \times H_{22}^+$, $(E_2^+)^* = E_2^-$,

$$H_{21}^+ := H_{\Gamma_{21}}^{1/2} = \left\{ u \in H^{1/2}(\Gamma_2) : u \equiv 0 \text{ (on } \Gamma_{20}), \int_{\Gamma_{21}} u \, d\Gamma_{21} = 0 \right\},$$

$$H_{22}^+ = H^{1/2}(\Gamma_2) \cap \widehat{H}.$$

Proposition 4. *The construction of the spaces H_{2i}^+ and the study of their properties are carried out by analogy with the case when the boundary of the domain consists of a solid wall and a moving surface of one type (see, for example, [1], [5]). As a consequence, we have the following properties of the operators C_{ij} .*

1. *The operator C_{ij} is a bounded operator acting from H_{2j}^- into H_{2i}^+ ; moreover, it is compact as an operator acting from H_{2j}^- into H_{2i} .*

2. *The operator C_{ii}^{-1} is bounded as an operator acting from H_{2i}^+ into H_{2i}^- ; moreover, $C_{ii}^{-1/2}$ acts boundedly from H_{2i}^+ into H_{2i} and from H_{2i} into H_{2i}^- .*

Note that similar assertions also hold for the operators S_k ($k = 0, 1$), S_j^i , \widehat{S}_j ($j = 2, 3, i = 0, 1$).

It is not hard to show that all the requirements of Theorem 4 are fulfilled for the operator

$$M = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

of (84). Then, basing on Theorems 3, we prove the following

Lemma 4. *If the conditions*

$$(86) \quad \zeta^0 \in E^+ = (H_1 \oplus H_{20})^+ \times (H_{21} \oplus \widehat{H}),$$

$$(87) \quad \zeta^1 \in H_1 \oplus H_{20} \oplus H_{21} \oplus \widehat{H} = H, \quad \widehat{F}(t) \in C^1([0, T]; E^+)$$

are fulfilled then there exists a unique strong solution to problem (47), (50).

Starting from the statements of problems (1) – (4), (26) – (30), (47) – (51), we now give the definitions (coordinated with each other) of the so-called solutions to these problems strong with respect to the variable t .

Definition 2. *Refer as a strong solution (with respect to the variable t) to problem (1)–(4) on $[0, T]$ to a set of functions $\vec{u}_i(t, x)$, $p_i(t, x)$, and $\zeta_i(t, \hat{x})$ ($i = 1, 2$), satisfying the following conditions:*

$$1. \quad \vec{u}_i(t) \in C^1([0, T]; \vec{J}_{0, S_i}(\Omega_i)), \quad \vec{J}_{0, S_i}(\Omega_i) := \vec{J}_0(\Omega_i) \oplus \vec{G}_{h, S_i}(\Omega_i),$$

$$\nabla p_i(t, x) \in C([0, T]; \vec{G}(\Omega_i)), \quad \vec{G}(\Omega_i) := \vec{G}_{h, S_i}(\Omega_i) \oplus \vec{G}_{0, \Gamma_i}(\Omega_i) \quad (i = 1, 2),$$

and the first equation in (1) holds for every $t \in [0, T]$;

2. $\zeta_1 \in C([0, T]; H_{\Gamma_1}^{1/2})$, $\zeta_2 \in C([0, T]; H_2)$; moreover, $P_{20}\zeta_2 \in H_{\Gamma_{20}}^{1/2}$;
3. The following boundary condition is fulfilled on Γ_1 and Γ_2 :

$$p_1 - p_2 = g\Delta\rho_0\zeta_1 \in C([0, T]; H_{\Gamma_1}^{1/2}),$$

$$p_2 = g\rho_2\zeta_2 \in C([0, T]; H_{\Gamma_{20}}^{1/2}), \quad p_2 = g\rho_2\zeta_2 + \rho_0 \frac{\partial^2 \zeta_2}{\partial t^2} \in C([0, T]; L_2(\Gamma_{21})),$$

where all summands are functions continuous in t with values in the corresponding spaces;

4. The initial conditions (4) are fulfilled.

Definition 3. Refer as a strong solution (with respect to the variable t) to problem (26)–(30) on $[0, T]$ to a collection of functions $\Psi_i(t, x)$, $\zeta_i(t, \hat{x})$ ($i = 1, 2$) satisfying the following conditions:

1. $\nabla\Psi_1 \in C^2([0, T]; \vec{G}_{h,S_1}(\Omega_1))$; $\nabla\Psi_2 \in C^2([0, T]; \vec{G}_{h,S_2}(\Omega_2))$,

where $\vec{G}_{h,S_2}(\Omega_2) = \vec{G}_1(\Omega_2) \dot{+} \vec{G}_2(\Omega_2)$ (see (42) for details);

2. The following kinematic conditions hold for $\Psi_i(t, x)$ and $\zeta_i(t, \hat{x})$:

$$\frac{\partial\Psi_1}{\partial n_1} = \frac{\partial\Psi_2}{\partial n_2} = \zeta_1 \in C^1([0, T]; H_{\Gamma_1}^{1/2}), \quad \frac{\partial\Psi_2}{\partial n_2} = \zeta_2 \in C^1([0, T]; H_2), \quad P_{20}\zeta_2 \in H_{\Gamma_{20}}^{1/2};$$

3. The boundary conditions (27)–(29) are fulfilled, where all the summands are functions continuous in t with values in the corresponding spaces;

4. The initial conditions (30) are fulfilled.

Definition 4. Refer to a function $\zeta(t)$ defined on $[0, T]$ as a strong solution to problem (47)–(51) with values in $H = H_1 \oplus H_2$ if the following conditions are fulfilled:

1. $\zeta(t) \in C([0, T]; H)$, $M\zeta(t) \in C^2([0, T]; H)$;
2. The equation and the conditions of (47)–(51) hold.

Theorem 5. Suppose the fulfillment of the conditions:

1. $\vec{u}_i^0 \in \vec{J}_{0,S_i}(\Omega_i)$ ($i = 1, 2$) (see (11), (17) for details),
moreover, $\gamma_1 P_{h,S_1} \vec{u}_1^0(x) = -\gamma_2 \Pi_1 P_{h,S_2} \vec{u}_2^0(x)$ (on Γ_1) (see (51) for details);
2. $\zeta_1^0 \in H_1^+ = H_{\Gamma_1}^{1/2}$, $\zeta_2^0 \in H_2$, $P_{20}\zeta_2^0 \in H_{\Gamma_{20}}^+$ (see (85) for details),
3. $\zeta_1^1 = \left[\left(P_{h,S_1} \vec{u}_1^0(x) \cdot \vec{n}_1 \right)_{\Gamma_1} \right] \in H_1$, $\zeta_2^1 = \left[\left(P_{h,S_2} \vec{u}_2^0(x) \right) \cdot \vec{n}_2 \right]_{\Gamma_2} \in H_2$,
4. $\vec{f}_i(t) \in C^1([0, T]; \vec{L}_2(\Omega_i))$.

Then each of the problems (1)–(4), (26)–(30), (47)–(51) has a unique solution strong with respect to t .

Proof. Carry out the proof in stages, passing consecutively from problem (47)–(51) to (26)–(30) and then to the initial problem (1)–(4).

Passage from problem (47)–(51) to (26)–(30). Suppose the fulfillment of the hypotheses of Theorem 5, then $(\zeta_1^0; P_{20}\zeta_2^0)^t = (\eta_2^0; w_0^0)^t = y_1^0 \in H_1^+ \times H_{20}^+$, $(P_{21}\zeta_2^0; \widehat{P}\zeta_2^0)^t = (w_1^0; \widehat{w}^0)^t = y_2^0 \in H_{21} \oplus \widehat{H}$, and hence, $\zeta^0 = (y_1^0; y_2^0)^t \in (H_1 \oplus H_{20})^+ \times (H_{21} \oplus \widehat{H})$. Since $y_1^1 = (\zeta_1^1; w_0^1)^t = y_1^1 \in H_1 \oplus H_{20}$, $(w_1^1; \widehat{w}^1)^t = y_2^1 \in H_{21} \oplus \widehat{H}$, then $\zeta^1 = (y_1^1; y_2^1)^t \in H$. Further, by the assertions (see [5] for details)

$$\begin{aligned} \left\{ \vec{f}_1(t) \in C^1([0, T]; \vec{L}_2(\Omega_1)) \right\} &\Leftrightarrow \left\{ P_{h,S_1} \vec{f}_1 = \nabla F_1 \in C^1([0, T]; \vec{G}_{h,S_1}(\Omega_1)) \right\} \\ &\Leftrightarrow \left\{ F_1|_{\Gamma_1} \in C^1([0, T]; H_1^+ = H_{\Gamma_1}^{1/2}) \right\}, \end{aligned}$$

$$\begin{aligned} \left\{ \vec{f}_2(t) \in C^1 \left([0, T]; \vec{L}_2(\Omega_2) \right) \right\} &\Leftrightarrow \left\{ P_{\widehat{h}, S_1} \vec{f}_2 = \nabla F_2 \in C^1 \left([0, T]; \vec{G}_{\widehat{h}, S_1}(\Omega_2) \right) \right\} \\ &\Leftrightarrow \left\{ F_2|_{\Gamma} \in C^1 \left([0, T]; H_2^+ = H_{\Gamma}^{1/2} \right) \right\}, \end{aligned}$$

we infer that $\widehat{F} = (g_1; g_2)^t$, where $g_1 = (\widehat{F}_1; P_{\Gamma_{20}} \widehat{F}_2)^t$, $g_2 = (P_{\Gamma_{20}} \widehat{F}_2; \widehat{P} \widehat{F}_2)^t$, belongs to the space $(H_1 \oplus H_{20})^+ \times (H_{21} \oplus \widehat{H})$.

Thus, if the conditions of Theorem 5 are fulfilled then the functions ζ^0 , ζ^1 , and \widehat{F} satisfy the hypotheses of Lemma 4. Consequently,

$$(88) \quad \frac{d^2}{dt^2} (M_{11}y_1 + M_{12}y_2) \in C([0, T]; (H_1 \oplus H_{20})^+),$$

$$(89) \quad \frac{d^2}{dt^2} (M_{21}y_1 + (M_{22} + J)y_2) \in C([0, T]; H_{21} \oplus \widehat{H}).$$

Since $(M_{22} + J)$ is bounded and positive definite, from the condition $d^2/dt^2(M_{22} + J)y_2 \in C([0, T]; H_{21} \oplus \widehat{H})$, acting by the bounded operator $(M_{22} + J)^{-1}$, we obtain that $d^2y_2/dt^2 \in C([0, T]; H_{21} \oplus \widehat{H})$. By the properties of the operators M_{ij} , we conclude that

$$\frac{d^2}{dt^2} (M_{12}y_2) \in C([0, T]; (H_1 \oplus H_{20})^+), \quad \frac{d^2}{dt^2} (M_{22}y_2) \in C([0, T]; (H_{21} \oplus \widehat{H})^+).$$

Then (88) implies that $d^2/dt^2(M_{11}y_1) \in C([0, T]; (H_1 \oplus H_{20})^+)$, and hence

$$M_{11}^{-1/2} \frac{d^2}{dt^2} (M_{11}y_1) = \frac{d^2}{dt^2} (M_{11}^{1/2}y_1) \in C([0, T]; (H_1 \oplus H_{20})).$$

Therefore,

$$M_{21}M_{11}^{-1/2} \frac{d^2}{dt^2} (M_{11}^{1/2}y_1) = \frac{d^2}{dt^2} (M_{21}y_1) \in C([0, T]; (H_{21} \oplus \widehat{H})^+).$$

Recalling the representations of the operators M_{ij} and the relationship with the functions Ψ_i and also by the embeddings $H_{20}^+ \subset H_{\Gamma}^{1/2}$ and $(H_{21} \oplus \widehat{H})^+ \subset H_{\Gamma}^{1/2}$, we obtain

$$\Psi_1|_{\Gamma_1} \in C^2([0, T]; H_1^+ = H_{\Gamma_1}^{1/2}) \implies \nabla \Psi_1(t, x) \in C^2([0, T]; \vec{G}_{\widehat{h}, S_1}(\Omega_1)).$$

$$\Psi_2|_{\Gamma_2} \in C^2([0, T]; H_2^+ = H_{\Gamma}^{1/2}) \implies \nabla \Psi_2(t, x) \in C^2([0, T]; \vec{G}_{\widehat{h}, S_1}(\Omega_2)).$$

Moreover, the conditions

$$\frac{\partial \Psi_1}{\partial n_1} = \frac{\partial \Psi_2}{\partial n_2} = \zeta_1 \in C^1([0, T]; H_{\Gamma_1}^{1/2}), \quad \frac{\partial \Psi_2}{\partial n_2} = \zeta_2 \in C^1([0, T]; H_2), \quad P_{20}\zeta_2 \in H_{\Gamma_{20}}^{1/2},$$

valid for the solution to problem (47)–(51), are fulfilled.

Hence, by definition 3, the collection of functions $\Psi_i(t, x)$, $\zeta_i(t, \hat{x})$ ($i = 1, 2$) is a solution to problem (26)–(30) strong with respect to t .

Passage from problem (26)–(30) to problem (1)–(4).

Relying upon the above-proven facts, reckoning with representations (12) and (18), connections (24), (25), relations (27)–(27), it is easy to check that problem (1)–(4) has a strong solution (with respect to t) in the sense of definition 2. \square

3. EIGENOSCILLATIONS

3.1. The spectrum of the problem and the basis property of eigenelements. In the absence of external forces (except for the gravitational field), i.e., for $\widehat{F}(t, x) \equiv 0$, consider the eigenoscillations — the solutions to problem (47) depending on time by the law $\exp(i\omega t)$: $\zeta(t, x) = e^{i\omega t}\zeta(x)$. For the amplitude elements $\zeta(x)$, we obtain the spectral problem

$$(90) \quad \lambda M\zeta = I_0\zeta, \quad \lambda := \omega^2.$$

For $\lambda = 0$, we obtain $\zeta = 0$. Hence, $\lambda = 0$ is not an eigenvalue for the spectral problem (90). Divide both sides by λ and change the notation:

$$(91) \quad M\zeta = \mu I_0\zeta, \quad \mu := 1/\lambda.$$

By definition, the operator I_0 is invertible, and the inverse operator I_0^{-1} is bounded and positive. Hence, there exists $I_0^{-1/2}$. Introduce the notation

$$(92) \quad I_0^{1/2}\zeta =: \eta.$$

Insert (92) in equation (91), act at both sides of the equation by the bounded operator $I_0^{-1/2}$ and obtain

$$(93) \quad I_0^{-1/2}MI_0^{-1/2}\eta = \mu\eta.$$

We obtain the eigenvalue problem for the operator

$$J := I_0^{-1/2}MI_0^{-1/2} = I_0^{-1/2}(S_C + \rho_0\widehat{I})I_0^{-1/2} = I_0^{-1/2}S_C I_0^{-1/2} + \rho_0 I_0^{-1/2}\widehat{I}I_0^{-1/2},$$

where

$$\begin{aligned} M &= \begin{pmatrix} \rho_1 S_0 + \rho_2 S_1 & -\rho_2 S_3 \\ -\rho_2 S_2 & \rho_2 C + \rho_0 I_\alpha \end{pmatrix} \\ &= \begin{pmatrix} \rho_1 S_0 + \rho_2 S_1 & -\rho_2 S_3 \\ -\rho_2 S_2 & \rho_2 C \end{pmatrix} + \rho_0 \begin{pmatrix} 0 & 0 \\ 0 & I_\alpha \end{pmatrix} =: S_C + \rho_0 \widehat{I}; \end{aligned}$$

$$(94) \quad \rho_0 I_0^{-1/2}\widehat{I}I_0^{-1/2} = \frac{\rho_0}{g\rho_2} \cdot \begin{pmatrix} 0 & 0 \\ 0 & I_\alpha \end{pmatrix}, \quad I_\alpha = \text{diag}(0; 1; \alpha).$$

Obviously, J is a selfadjoint bounded positive definite operator, and hence its spectrum is real and positive. Moreover, the operator J is the sum of two bounded selfadjoint operators, and the operator $I_0^{-1/2}S_C I_0^{-1/2}$ is compact. Hence, we can use Weil's theorem, and hence the limit spectrum σ_e of J coincides with the limit spectrum of the operator $\rho_0 I_0^{-1/2}\widehat{I}I_0^{-1/2}$. In accordance with representation (94), its limit spectrum consists of the points 0 and $\rho_0/g\rho_2$ (here the space \widehat{H} is one-dimensional, and hence the point α does not belong to the limit spectrum). Recall that the limit spectrum of an operator is the set of the points of the continuous spectrum, the limit points of the point spectrum, and the eigenvalues of infinite multiplicity.

Check that the point $\rho_0/g\rho_2$ is a limit of a branch of eigenvalues and not an eigenvalue of infinite multiplicity.

Rewrite (91) componentwise for the case $\mu = \rho_0/g\rho_2$; we have

$$(95) \quad \begin{cases} (\rho_1 S_0 + \rho_2 S_1) \eta_0 - \rho_2 S_3 w = (\rho_0/g\rho_2) \cdot g\Delta\rho\eta_0, \\ -\rho_2 S_2^0 \eta_0 + \rho_2 C_{00} w_0 + \rho_2 C_{01} w_1 + \rho_2 C_{02} \widehat{w} = \rho_0 w_0, \\ -\rho_2 S_2^1 \eta_0 + \rho_2 C_{10} w_0 + \rho_2 C_{11} w_1 + \rho_2 C_{12} \widehat{w} + \rho_0 w_1 = \rho_0 w_1, \\ -\rho_2 \widehat{S}_2 \eta_0 + \rho_2 C_{20} w_0 + \rho_2 C_{21} w_1 + \rho_2 C_{22} \widehat{w} + \rho_0 \alpha \widehat{w} = \rho_0 \widehat{w}, \end{cases}$$

where $S_2^i \eta_0 := -P_{2i}(\Psi_{2,1}|_{\Gamma_{20}}; \Psi_{2,1}|_{\Gamma_{21}})$, $\widehat{S}_2 \eta_0 := -\widehat{P}(\Psi_{2,1}|_{\Gamma_{20}}; \Psi_{2,1}|_{\Gamma_{21}})$. After reducing, the third equation takes the form:

$$(96) \quad -\rho_2 S_2^1 \eta_0 + \rho_2 C_{10} w_0 + \rho_2 C_{11} w_1 + \rho_2 C_{12} \widehat{w} = 0$$

Express w_1 from (96) w_1 and insert it in the remaining equations of (95); this will give the system that can be written down in block-matrix form as

$$(97) \quad \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \eta_0 \\ w_0 \\ \widehat{w} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_0 \alpha \end{pmatrix} \begin{pmatrix} \eta_0 \\ w_0 \\ \widehat{w} \end{pmatrix} = \begin{pmatrix} (\rho_0/\rho_2)\Delta\rho\eta_0 \\ \rho_0 w_0 \\ \rho_0 \widehat{w} \end{pmatrix},$$

$$A_{11} := \rho_1 S_0 + \rho_2(S_1 - S_3^1 C_{11}^{-1} S_2^1), \quad A_{12} := \rho_2(-S_3^0 + S_3^1 C_{11}^{-1} C_{10}),$$

$$A_{13} := \rho_2(-\widehat{S}_3 + S_3^1 C_{11}^{-1} C_{12}), \quad A_{21} := \rho_2(-S_2^0 + C_{01} C_{11}^{-1} S_2^1),$$

$$A_{22} := \rho_2(C_{00} - C_{01} C_{11}^{-1} C_{10}), \quad A_{23} := \rho_2(C_{12} - C_{01} C_{11}^{-1} C_{12}),$$

$$A_{31} := \rho_2(-\widehat{S}_2 + C_{21} C_{11}^{-1} S_2^1), \quad A_{32} := \rho_2(C_{20} - C_{21} C_{11}^{-1} C_{10}),$$

$$A_{33} := \rho_2(C_{22} - C_{21} C_{11}^{-1} C_{12}), \quad S_j^i := P_{2i} S_j, \quad \widehat{S}_j := \widehat{P} S_j, \quad j = 2, 3, \quad i = 0, 1.$$

Consider, for example, the operator $C_{01} C_{11}^{-1} C_{10}$. The operator C_{10} acts boundedly from H_{20}^- into H_{21}^+ . The operator C_{11}^{-1} acts boundedly from H_{21}^+ into H_{21}^- . The operator C_{01} is compact as an operator acting from H_{21}^- into H_{20} . Then, by the embedding $H_{20}^+ \subset H_{20} \subset H_{20}^-$, we conclude that $C_{01} C_{11}^{-1} C_{10}$ is a compact operator in H_{20} . The compactness of the remaining operators in (97), where the operators S_i are compact by construction (see the auxiliary problems), is proved similar. Consequently, the operator corresponding to the first summand in (97) is compact. By Weil's theorem, the limit spectrum in (97) coincides with the limit spectrum of the second summand. Since the space \widehat{H} is one dimensional, for this operator, $\rho_0 \neq 0$ can only be a multiple eigenvalue. Consequently, also for the operator J of (93), the point $\rho_0/g\rho_2$ can be only an eigenvalue of finite multiplicity.

Thus, the limit spectrum of the operator J of (93) consists of two points: 0 and $\rho_0/g\rho_2$, and to each of the points, there converges a branch of eigenvalues of J . Then, for the squares of the eigenfrequencies of the oscillations ω_k , we obtain the limit spectrum $+\infty$ and the points $g\rho_2/\rho_0$; moreover, there exist corresponding branches of eigenvalues converging to these eigenvalues. Since all eigenvalues of the selfadjoint operator J have finite multiplicity, the union of the eigenelements $\{\eta_k\}_{k=1}^{+\infty}$ of J constitutes an orthogonal basis of H (see [9, p. 409, 411]). What was exposed enables us to finally formulate the spectral theorem for problem (90).

Theorem 6. *Problem (90) has discrete positive spectrum $\{\lambda_k\}_{k=1}^{+\infty}$ with limit points at infinity and $\lambda_0 = g\rho_2/\rho_0 > 0$. The set of all eigenelements $\{\zeta_k\}_{k=1}^{+\infty}$ of problem (90) constitutes an orthogonal basis in H .*

3.2. Asymptotics of the branches of eigenvalues. The presence of two branches of eigenvalues (see Theorem 6) in the spectral problem (90) generates the problem of finding the asymptotical behavior of these branches.

Let us first consider the branch of eigenvalues with limit point at zero. As was proved, $\mu = 0$ is not an eigenvalue; therefore, henceforth, we assume that $\mu \neq 0$. Write problem (84) as a system:

$$(98) \quad M_{11}y_1 + M_{12}y_2 = \rho_g \mu y_1,$$

$$(99) \quad M_{21}y_1 + (M_{22} + J)y_2 = \rho_{2g} \mu y_2.$$

We can express y_2 from (99):

$$(100) \quad y_2 = -(M_{22} + J - \mu \rho_{2g} I)^{-1} M_{21} y_1.$$

Indeed, $Q := (M_{22} + J) \gg 0$, and hence Q is boundedly invertible. Consequently, for small μ , the operator $(M_{22} + J - \mu \rho_{2g} I)$ is also boundedly invertible.

Inserting the obtained expression for y_2 in (98) and taking into account that $\mu = 1/\lambda$, we obtain the spectral problem

$$(101) \quad \left[I - \frac{\lambda}{\rho_g} M_{11} + \frac{\lambda}{\rho_g} M_{12} \left(M_{22} + J - \frac{\rho_{2g}}{\lambda} I \right)^{-1} M_{21} \right] y_1 = 0.$$

Perform the following transformations for the operator $(M_{22} + J - \rho_{2g}/\lambda)^{-1}$:

$$(102) \quad \begin{aligned} & \left(M_{22} + J - \frac{\rho_{2g}}{\lambda} I \right)^{-1} = \left(Q - \frac{\rho_{2g}}{\lambda} I \right)^{-1} = \left(Q^{1/2} \left(I - \frac{\rho_{2g}}{\lambda} Q^{-1} \right) Q^{1/2} \right)^{-1} \\ & = Q^{-1/2} \left(\sum_{k=0}^{\infty} \frac{(\rho_{2g})^k Q^{-k}}{\lambda^k} \right) Q^{-1/2} = Q^{-1} + \frac{\rho_{2g}}{\lambda} Q^{-2} + O\left(\frac{1}{\lambda^2}\right) \quad (\lambda \rightarrow \infty). \end{aligned}$$

Insert (102) in (101); we obtain

$$(103) \quad \left[I - \frac{\lambda}{\rho_g} M_{11} + \frac{\lambda}{\rho_g} M_{12} \left(Q^{-1} + \frac{\rho_{2g}}{\lambda} Q^{-2} + O\left(\frac{1}{\lambda^2}\right) \right) M_{21} \right] y_1 = 0.$$

This yields

$$(104) \quad \left[I - \frac{\lambda}{\rho_g} (M_{11} - M_{12} Q^{-1} M_{21}) + \frac{\rho_{2g}}{\rho_g} M_{12} Q^{-2} M_{21} + G(\lambda) \right] y_1 = 0,$$

where $G(\lambda)$ is an analytic operator-function for $\lambda \rightarrow \infty$, and also $G(\lambda) \rightarrow 0$. The operator $M_{12} Q^{-2} M_{21}$ is compact, being the product of two compact operators and a bounded operator. Then the bunch of problem (104) satisfies the Markus–Matsaev theorem (see, for example, [1, p. 71]), and the asymptotics of the problem is determined by the asymptotics of the reduced bunch

$$(105) \quad \left[I - \frac{\lambda}{\rho_g} (M_{11} - M_{12} Q^{-1} M_{21}) \right] y_1 = 0$$

if the operator $(M_{11} - M_{12} Q^{-1} M_{21})$ has power asymptotics. Demonstrate that this is true. Consider the eigenvalue problem for this operator:

$$(106) \quad (M_{11} - M_{12} Q^{-1} M_{21}) y_1 = \rho_g \nu y_1.$$

Introduce one more variable by the law

$$(107) \quad y_0 := -Q^{-1} M_{21} y_1.$$

From this, reckoning with the form of the operator Q , from the eigenvalue problem (106), pass to the equivalent problem

$$(108) \quad M_{11}y_1 + M_{12}y_0 = \rho_g \nu y_1,$$

$$(109) \quad M_{21}y_1 + (M_{22} + J)y_0 = 0.$$

To this problem there corresponds a problem for the displacement potentials of the form

$$(110) \quad \begin{aligned} \Delta \Psi_k &= 0 \quad (\text{in } \Omega_k), \quad \frac{\partial \Psi_k}{\partial n_k} = 0 \quad (\text{on } S_k), \quad \int_{\Gamma_k} \Psi_k \, d\Gamma_k = 0, \\ (\rho_1 \Psi_1 - \rho_2 \Psi_2)_{\Gamma_1} &= \nu g \Delta \rho \cdot \left(\frac{\partial \Psi_1}{\partial n_1} \right)_{\Gamma_1}, \quad \frac{\partial \Psi_1}{\partial n_1} = \frac{\partial \Psi_2}{\partial n_1} = \zeta_1 \quad (\text{on } \Gamma_1), \\ \rho_2 \Psi_2|_{\Gamma_{20}} &= \nu \rho_2 g \cdot \left(\frac{\partial \Psi_2}{\partial n_2} \right)_{\Gamma_{20}}, \quad (\text{on } \Gamma_{20}), \\ \rho_0 J \left(\frac{\partial \Psi_2}{\partial n_2} \right)_{\Gamma_{21}} &+ \rho_2 \Psi_2|_{\Gamma_{21}} = 0 \quad (\text{on } \Gamma_{21}). \end{aligned}$$

Problem (110) can be studied by variational methods. But this problem is close to the analogous problem appearing under the oscillations of a system of $m = 3$ homogeneous ideal fluids located one above another considered in [1]. By analogy with the arguments carried out in [1], one can show that problem (110) indeed has power asymptotics. We will give only some explanations about it. The relations

$$\begin{aligned} 0 &= -\rho_2 \int_{\Omega_2} \Delta \Psi_2 \cdot \Psi_2 \, d\Omega_2 = \rho_2 \int_{\Omega_2} |\nabla \Psi_2|^2 \, d\Omega_2 - \int_{\partial\Omega_2} \frac{\partial \Psi_2}{\partial n_2} \cdot (\rho_2 \Psi_2) \, dS \\ &= \rho_2 \int_{\Omega_2} |\nabla \Psi_2|^2 \, d\Omega_2 - \int_{\Gamma_{20}} \frac{\partial \Psi_2}{\partial n_2} \cdot (\rho_2 \Psi_2) \, d\Gamma_{20} - \int_{\Gamma_{21}} \frac{\partial \Psi_2}{\partial n_2} \cdot (\rho_2 \Psi_2) \, d\Gamma_{21} \\ &\quad - \int_{\Gamma_1} \frac{\partial \Psi_2}{\partial n_2} \cdot (\rho_2 \Psi_2) \, d\Gamma_1 = \rho_2 \int_{\Omega_2} |\nabla \Psi_2|^2 \, d\Omega_2 + \frac{\rho_2^2}{\rho_0} \int_{\Gamma_{21}} J^{-1} \Psi_2 \cdot \Psi_2 \, d\Gamma_{21} \\ &\quad - \rho_2 g \nu \int_{\Gamma_{20}} \left| \frac{\partial \Psi_2}{\partial n_2} \right|^2 \, d\Gamma_{20} - \int_{\Gamma_1} \frac{\partial \Psi_2}{\partial n_2} \cdot (\rho_2 \Psi_2) \, d\Gamma_1; \\ 0 &= -\rho_1 \int_{\Omega_1} \Delta \Psi_1 \cdot \Psi_1 \, d\Omega_1 = \rho_1 \int_{\Omega_1} |\nabla \Psi_1|^2 \, d\Omega_1 - \int_{\partial\Omega_1} \frac{\partial \Psi_1}{\partial n_1} \cdot (\rho_1 \Psi_1) \, dS \\ &= \rho_1 \int_{\Omega_1} |\nabla \Psi_1|^2 \, d\Omega_1 - \int_{\Gamma_1} \frac{\partial \Psi_1}{\partial n_1} \cdot (\rho_1 \Psi_1) \, d\Gamma_1. \end{aligned}$$

imply that the eigenvalues of problem (106) can be found as consecutive minima of the variational relation

$$\nu_k = \min_{M_{k-1}} \max_{\Phi \in M_{k-1}} \frac{\sum_{k=1}^2 \rho_k \int_{\Omega_k} |\nabla \Psi_k|^2 \, d\Omega_k + \frac{\rho_2^2}{\rho_0} \int_{\Gamma_{21}} J^{-1} \Psi_2 \cdot \Psi_2 \, d\Gamma_{21}}{\rho_2 g \int_{\Gamma_{20}} \left| \frac{\partial \Psi_2}{\partial n_2} \right|^2 \, d\Gamma_{20} + \Delta \rho g \int_{\Gamma_1} \left| \frac{\partial \Psi_1}{\partial n_1} \right|^2 \, d\Gamma_1},$$

where the minimum is taken over the subspaces M_{k-1} of dimension $k - 1$. But the second summand in the numerator is quite subordinated to first summand and

does not influence the main term of the asymptotics (see [10]). Indeed, for the second summand, we infer

$$\int_{\Gamma_{21}} J^{-1}\Psi_2 \cdot \Psi_2 \, d\Gamma_{21} = (J^{-1}\gamma_2\Psi_2, \gamma_2\Psi_2)_{H_{21}} = (\gamma_2^*J^{-1}\gamma_2\Psi_2, \Psi_2)_{H_{21}},$$

and since the operator $\gamma_2 : H^1(\Omega_2) \rightarrow L_2(\Gamma_{21})$ is compact and J^{-1} is bounded, we conclude that this summand is quite subordinated to the first summand.

On the other hand, to the variational relation

$$\nu_k = \min_{M_{k-1}} \max_{\Phi \in M_{k-1}} \frac{\sum_{k=1}^2 \rho_k \int_{\Omega_k} |\nabla \Psi_k|^2 \, d\Omega_k}{\rho_2 g \int_{\Gamma_{20}} \left| \frac{\partial \Psi_2}{\partial n_2} \right|^2 \, d\Gamma_{20} + \Delta \rho g \int_{\Gamma_1} \left| \frac{\partial \Psi_1}{\partial n_1} \right|^2 \, d\Gamma_1},$$

there corresponds the previously considered problem (see [1, c.195]) with the known power asymptotics

$$\begin{aligned} \nu_k &= C^{1/2} k^{-1/2} [1 + o(1)] \quad (k \rightarrow \infty), \\ (111) \quad C &:= \frac{1}{4\pi g^2} \left(\frac{\rho_1 + \rho_2}{\Delta \rho} \right)^2 \text{mes } \Gamma_1 + \frac{1}{4\pi g^2} \text{mes } \Gamma_2. \end{aligned}$$

Therefore, problem (106) has the same power asymptotics.

Lemma 5. *In problem (90), to the branches of eigenvalues with limit point at $+\infty$ there correspond waves typical for the system of fluids in a basin with free boundary and asymptotic behavior*

$$\omega_{1,k}^2 = C^{-1/2} k^{1/2} \quad (k \rightarrow \infty),$$

where C is defined by (111).

Now, for problem (90), consider the branch of eigenvalues with limit point $\mu_0 = \rho_0/\rho_2 g$. Perform the change

$$(112) \quad \nu := \mu - \rho_0/(\rho_2 g).$$

By analogy with the previous arguments, we can show that the main term of the asymptotics is defined in this case only by the operator C_{11} (see (49)) corresponds to the problem

$$\begin{aligned} \Delta \Psi_2 &= 0 \quad (\text{in } \Omega_2), \quad \frac{\partial \Psi_2}{\partial n_2} = 0 \quad (\text{on } S_2), \quad \int_{\Gamma=\Gamma_1 \cup \Gamma_2} \Psi_2 \, d\Gamma = 0, \\ \frac{\partial \Psi_2}{\partial n_2} &= 0 \quad (\text{on } \Gamma_1), \quad \frac{\partial \Psi_2}{\partial n_2} = 0 \quad (\text{on } \Gamma_{20}), \quad \frac{\partial \Psi_2}{\partial n_2} = \nu^{-1} g^{-1} \Psi_2 \quad (\text{on } \Gamma_{21}), \end{aligned}$$

which has power asymptotics (see, for example, [1],[11])

$$\nu_k = g^{-1} \left(\frac{\text{mes } \Gamma_{21}}{4\pi} \right)^{1/2} k^{-1/2} [1 + o(1)] \quad (k \rightarrow +\infty).$$

Then, returning to $\mu_k = 1/\lambda_k$, we obtain

$$\lambda_k = \frac{\rho_2 g}{\rho_0} \cdot \frac{1}{1 + (\rho_2 g/\rho_0)\nu_k} = \frac{\rho_2 g}{\rho_0} \cdot \left(1 - \frac{\rho_2 g}{\rho_0} \nu_k + o(\nu_k) \right) \quad (k \rightarrow +\infty).$$

Thus, we have the following lemma:

Lemma 6. *In problem (90), to the branch of eigenvalues with limit point $\lambda_0 = \rho_2 g / \rho_0$, there corresponds the type of waves conditioned by the presence of crumbled ice on the surface. The frequencies of the eigenoscillations of these waves have asymptotic behavior*

$$\omega_{2,k}^2 = \frac{\rho_2 g}{\rho_0} \left(1 - \frac{\rho_2}{\rho_0} \left(\frac{\text{mes} \Gamma_{21}}{4\pi} \right)^{1/2} k^{-1/2} [1 + o(1)] \right) \quad (k \rightarrow \infty).$$

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DENIS OLEGOVICH TSVETKOV
 CRIMEAN FEDERAL UNIVERSITY, TAURIDA ACADEMY,
 4, VERNADSKOGO AVE.,
 SIMFEROPOL, 295007, RUSSIA
E-mail address: tsvetdo@gmail.com