TRIBONACCI AND TRIBONACCI-LUCAS TRIGINTADUONIONS

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Abstract. In this paper, we intend to make a new approach to introduce the concept of Tribonacci and Tribonacci-Lucas trigintaduonions and study some properties of these trigintaduonins like norm value, Binet's formula, generating function, summation formula and matrix formulation.

1. Introduction and Preliminaries

In [11] M. Feinberg a fourteen-year-old student introduced the concept of Tribonacci numbers. Tribonacci numbers are like the Fibonacci numbers but instead of starting with two predertermined terms, the sequence starts with three predetermined terms and each term afterward is the sum of the preceding three terms. The Tribonacci sequence $\{U_n\}_{n\geq 0}$ and the Tribonacci-Lucas sequence $\{V_n\}_{n\geq 0}$ are defined by

$$U_n = U_{n-1} + U_{n-2} + U_{n-3}, \quad U_0 = 0,\ U_1 = 1,\ U_2 = 1 \tag{1.1}$$

and

$$V_n = V_{n-1} + V_{n-2} + V_{n-3}, \quad V_0 = 3,\ V_1 = 1,\ V_2 = 3 \tag{1.2}$$

respectively. To know more about Fibonacci and Tribonacci sequences, one may refer to the articles (see [1], [4], [6], [8], [9], [10], [12], [14], [16], [17], [20], [21], [22], [24], [25]). For negative values of n, the sequences $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ can be defined by using the recurrences

$$U_{-n} = -U_{-(n-1)} - U_{-(n-2)} + U_{-(n-3)}$$

and

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$, respectively. Thus, recurrences (1.1) and (1.2) hold for all integers n. Now, the Binet formula for the usual Tribonacci and Tribonacci-Lucas numbers can be expressed as

$$U_n = \frac{\mu^{n+1}}{(\mu-\nu)(\mu-\delta)} + \frac{\nu^{n+1}}{(\nu-\mu)(\nu-\delta)} + \frac{\delta^{n+1}}{(\delta-\mu)(\delta-\nu)} \tag{1.3}$$

and

$$V_n = \mu^n + \nu^n + \delta^n \tag{1.4}$$

respectively, where μ, ν and δ are the roots of the characteristic equation $x^3-x^2-x-1 = 0$. Moreover,

$$\mu = \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3},$$

2010 *Mathematics Subject Classification.* 11B83, 40A05, 40A30.
Key words and phrases. Trigintaduonions, Tribonacci and Tribonacci-Lucas numbers, Binet's formula
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$$\nu = \frac{1 + \varpi \sqrt[3]{19 + 3\sqrt{33}} + \varpi^2 \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\delta = \frac{1 + \varpi^2 \sqrt[3]{19 + 3\sqrt{33}} + \varpi \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where,

$$\varpi = \frac{-1 + i\sqrt{3}}{2}.$$

The generating functions for $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ are defined as

$$\sum_{n=0}^{\infty} U_n x^n = \frac{x}{1 - x - x^2 - x^3} \text{ and } \sum_{n=0}^{\infty} V_n x^n = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}.$$

Now, we discuss some properties of Cayley-Dickson algebras. In the Cayley-Dickson process we extend the field $\mathbb{R}$ of reals to Complex $\mathbb{C}$. It can be carried further to higher dimensions using a doubling process yielding the quaternions, octanions, sedenions and trigintaduonions. The Cayley- Dickson algebras have been studied in several papers (see [3], [19]). Octonions and sedenions are hyper-complex numbers as quateronions have been studied in some of recent papers ([2], [15]). In this paper, we want to extend these ideas and study trigintaduonions which is 32-dimensional Cayley-Dickson algebra. Actually, trigintaduonions is a 32-dimensional hyper-complex number that is a non-associative extension of a sedenion. A trigintaduonions is defined by

$$T = t_0 + \sum_{k=1}^{31} t_k e_k$$

where $t_0, t_1, ..., t_{31}$ are real numbers, e_0 is the identity (or unit) and $e_1, ..., e_{31}$ are called imaginaries. t_0 is called the real part and $\sum_{k=1}^{31} t_k e_k$ is called it's imaginary part. The trigintaduonions product can be presented in the matrix-vector multiplication form as follows [5]:

$$T_1 = a_0 + \sum_{k=1}^{31} a_k e_k,\ T_2 = b_0 + \sum_{k=1}^{31} b_k e_k,\ T_3 = T_1 T_2 = c_0 + \sum_{k=1}^{31} c_k e_k.$$

The operations requiring for the matrix-vector multiplication in trigintaduonions are quite alot. Detailed informations about these operations have been presented in the literature (see [5], [7], [28]).
In [13] k-Fibonacci and k-Lucas trigintaduonions have been introduced by Gül as

$$TF_{k,n} = \sum_{i=1}^{31} F_{k,n+i} e_i$$

and

$$TL_{k,n} = \sum_{i=1}^{31} L_{k,n+i} e_i,$$

where $F_{k,n}$ and $L_{k,n}$ are the nth $k-$Fibonacci number and $k-$Lucas number, respectively.

Now, we explain Tribonacci and Tribonacci-Lucas trigintaduonions. The nth Tribonacci trigintaduonions is defined by

$$\widehat{U}_n = \sum_{i=0}^{31} U_{n+i} e_i = U_n + \sum_{i=1}^{31} U_{n+i} e_i \tag{1.5}$$

and nth Tribonacci-Lucas trigintaduonions is

$$\widehat{V}_n = \sum_{i=0}^{31} V_{n+i} e_i = V_n + \sum_{i=1}^{31} V_{n+i} e_i. \tag{1.6}$$

It can be easily shown that the following iterative relation holds:

$$\widehat{U}_n = \widehat{U}_{n-1} + \widehat{U}_{n-2} + \widehat{U}_{n-3} \tag{1.7}$$

and

$$\widehat{V}_n = \widehat{V}_{n-1} + \widehat{V}_{n-2} + \widehat{V}_{n-3}. \tag{1.8}$$

For negative values of n, the sequences $\{\widehat{U}_n\}_{n\geq 0}$ and $\{\widehat{V}_n\}_{n\geq 0}$ can be defined as

$$\widehat{U}_{-n} = -\widehat{U}_{-(n-1)} - \widehat{U}_{-(n-2)} + \widehat{U}_{-(n-3)}$$

and

$$\widehat{V}_{-n} = -\widehat{V}_{-(n-1)} - \widehat{V}_{-(n-2)} + \widehat{V}_{-(n-3)}.$$

The conjugate of $\widehat{U}_n$ and $\widehat{V}_n$ are defined by

$$\overline{\widehat{U}}_n = U_n - \sum_{i=1}^{31} U_{n+i} e_i = U_n - U_{n+1} e_1 - U_{n+2} e_2 - \cdots - U_{n+31} e_{31}$$

and

$$\overline{\widehat{V}}_n = V_n - \sum_{i=1}^{31} V_{n+i} e_i = V_n - V_{n+1} e_1 - V_{n+2} e_2 - \cdots - V_{n+31} e_{31}.$$

The norms of $\widehat{U}_n$ and $\widehat{V}_n$ are

$$\|\widehat{U}_n\|^2 = N^2(\widehat{U}_n) = \widehat{U}_n \overline{\widehat{U}}_n = U_n^2 + U_{n+1}^2 + \cdots + U_{n+31}^2$$

and

$$\|\widehat{V}_n\|^2 = N^2(\widehat{V}_n) = \widehat{V}_n \overline{\widehat{V}}_n = V_n^2 + V_{n+1}^2 + \cdots + V_{n+31}^2.$$

2. Main Results

In this section, we introduce some properties of Tribonacci and Tribonacci-Lucas trigintaduonions like norm value, Binet's formula, generating function and summation formula.

Lemma 2.1. [18] The following formulas are valid:

$$\sum_{i=1}^{n} U_n^2 = \frac{1 + 4U_n U_{n+1} - (U_{n+1} - U_{n-1})^2}{4}, \tag{2.1}$$

$$\sum_{i=1}^{n} V_n^2 = \frac{-V_{n+1}^2 - V_{n-1}^2 + V_{2n+3} + V_{2n-2}}{2} - 2. \tag{2.2}$$

Theorem 2.1. *The norms of nth Tribonacci and Tribonacci-Lucas trigintaduonions are defined as:*

$$\|\widehat{U}_n\|^2 = \frac{4(U_{n+31}U_{n+32} - U_{n-1}U_n) + (U_n - U_{n-2})^2 - (U_{n+32} - U_{n+30})^2}{4},$$

$$\|\widehat{V}_n\|^2 = \frac{-V_{n+32}^2 - V_{n+30}^2 + V_n^2 + V_{n-2}^2 + V_{2n+65} + V_{2n+60} - V_{2n+1} - V_{2n-4}}{2}.$$

Proof. We know that

$$\begin{aligned} \|\widehat{U}_n\|^2 &= \sum_{i=n}^{n+31} U_i^2 = \sum_{i=1}^{n+31} U_i^2 - \sum_{i=1}^{n-1} U_i^2 \\ &= \frac{1 + 4U_{n+31}U_{n+32} - (U_{n+32} - U_{n+30})^2}{4} - \frac{1 + 4U_{n-1}U_n - (U_n - U_{n-2})^2}{4} \\ &= \frac{4(U_{n+31}U_{n+32} - U_{n-1}U_n) + (U_n - U_{n-2})^2 - (U_{n+32} - U_{n+30})^2}{4} \end{aligned}$$

and

$$\begin{aligned} \|\widehat{V}_n\|^2 &= \sum_{i=n}^{n+31} V_i^2 = \sum_{i=1}^{n+31} V_i^2 - \sum_{i=1}^{n-1} V_i^2 \\ &= \left(\frac{-V_{n+32}^2 - V_{n+30}^2 + V_{2n+65} + V_{2n+60}}{2} - 2\right) \\ &- \left(\frac{-V_n^2 - V_{n-2}^2 V_{2n+1} + V_{2n-4}}{2} - 2\right) \\ &= \frac{-V_{n+32}^2 - V_{n+30}^2 + V_n^2 + V_{n-2}^2 + V_{2n+65} + V_{2n+60} - V_{2n+1} - V_{2n-4}}{2}. \end{aligned}$$

□

Theorem 2.2. *Binet's formula for the nth Tribonacci trigintaduonions is*

$$\widehat{U}_n = \frac{\widehat{\mu}\mu^{n+1}}{(\mu-\nu)(\mu-\delta)} + \frac{\widehat{\nu}\nu^{n+1}}{(\nu-\mu)(\nu-\delta)} + \frac{\widehat{\delta}\delta^{n+1}}{(\delta-\mu)(\delta-\nu)}$$

and the nth Tribonacci-Lucas trigintaduonions is

$$\widehat{V}_n = \widehat{\mu}\mu^n + \widehat{\nu}\nu^n + \widehat{\delta}\delta^n.$$

where,
$\widehat{\mu} = e_0 + \mu e_1 + \mu^2 e_2 + \cdots + \mu^{31} e_{31}$,
$\widehat{\nu} = e_0 + \nu e_1 + \nu^2 e_2 + \cdots + \nu^{31} e_{31}$,
$\widehat{\delta} = e_0 + \delta e_1 + \delta^2 e_2 + \cdots + \delta^{31} e_{31}$.

Proof. By using Binet's formula for the Tribonacci numbers (1.3) and definition of (1.5), we have

$$\begin{aligned} \widehat{U}_n = \sum_{i=0}^{31} U_{n+i} e_i &= \sum_{i=0}^{31} \left(\frac{\mu^{n+1+i} e_i}{(\mu-\nu)(\mu-\delta)} + \frac{\nu^{n+1+i} e_i}{(\nu-\mu)(\nu-\delta)} + \frac{\delta^{n+1+i} e_i}{(\delta-\mu)(\delta-\nu)} \right) \\ &= \frac{\widehat{\mu}\mu^{n+1}}{(\mu-\nu)(\mu-\delta)} + \frac{\widehat{\nu}\nu^{n+1}}{(\nu-\mu)(\nu-\delta)} + \frac{\widehat{\delta}\delta^{n+1}}{(\delta-\mu)(\delta-\nu)}. \end{aligned}$$

which is the Binet's formula for the Tribonacci trigintaduonions. Binet's formula for Tribonacci-Lucas trigintaduonions can be proved in the similar way. □

Theorem 2.3. *The generating functions for the Tribonacci and Tribonacci-Lucas trigintaduonions are:*

$$g(x)=\sum_{n=0}^{\infty}\widehat{U}_n x^n=\frac{\widehat{U}_0+(\widehat{U}_1-\widehat{U}_0)x+\widehat{U}_{-1}x^2}{1-x-x^2-x^3}$$

and

$$g(x)=\sum_{n=0}^{\infty}\widehat{V}_n x^n=\frac{\widehat{V}_0+(\widehat{V}_1-\widehat{V}_0)x+\widehat{V}_{-1}x^2}{1-x-x^2-x^3}.$$

Proof. Now, we define $g(x)=\sum_{n=0}^{\infty}\widehat{U}_n x^n$. Compute that

$g(x)=\widehat{U}_0+\widehat{U}_1x+\widehat{U}_2x^2+\widehat{U}_3x^3+\cdots+\widehat{U}_nx^n+\cdots$

$xg(x)=\widehat{U}_0x+\widehat{U}_1x^2+\widehat{U}_2x^3+\widehat{U}_3x^4+\cdots+\widehat{U}_{n-1}x^n+\cdots$

$x^2g(x)=\widehat{U}_0x^2+\widehat{U}_1x^3+\widehat{U}_2x^4+\widehat{U}_3x^5+\cdots+\widehat{U}_{n-2}x^n+\cdots$

$x^3g(x)=\widehat{U}_0x^3+\widehat{U}_1x^4+\widehat{U}_2x^5+\widehat{U}_3x^6+\cdots+\widehat{U}_{n-3}x^n+\cdots$

By using recurrence $\widehat{U}_n=\widehat{U}_{n-1}+\widehat{U}_{n-2}+\widehat{U}_{n-3}$ and above defined table, we have

$$\begin{aligned}(1-x-x^2-x^3)g(x) &= g(x)-xg(x)-x^2g(x)-x^3g(x)\\ &= \widehat{U}_0+(\widehat{U}_1-\widehat{U}_0)x+(\widehat{U}_2-\widehat{U}_1-\widehat{U}_0)x^2+(\widehat{U}_3-\widehat{U}_2-\\ &\quad \widehat{U}_1-\widehat{U}_0)x^3+(\widehat{U}_4-\widehat{U}_3-\widehat{U}_2-\widehat{U}_1)x^4+\cdots+\\ &\quad (\widehat{U}_n-\widehat{U}_{n-1}-\widehat{U}_{n-2}-\widehat{U}_{n-3})x^n+\cdots\\ &= \widehat{U}_0+(\widehat{U}_1-\widehat{U}_0)x+(\widehat{U}_2-\widehat{U}_1-\widehat{U}_0)x^2.\end{aligned}$$

Since $\widehat{U}_2-\widehat{U}_1-\widehat{U}_0=\widehat{U}_{-1}$, then the generating function for Tribonacci trigintaduonions is

$$g(x)=\frac{\widehat{U}_0+(\widehat{U}_1-\widehat{U}_0)x+\widehat{U}_{-1}x^2}{1-x-x^2-x^3}.$$

Similarly, we can find generating function for Tribonacci-Lucas trigintaduonions. □

In the next theorem by using generating functions, we discuss another forms of Binet's formula for the Tribonacci and Tribonacci-Lucas trigintaduonions .

Theorem 2.4. *For any integer n, the nth Tribonacci trigintaduonions is*

$$\widehat{U}_n=\frac{((\mu^2-\mu)\widehat{U}_0+\mu\widehat{U}_1+\widehat{U}_{-1})\mu^n}{(\mu-\delta)(\mu-\nu)}+\frac{((\nu^2-\nu)\widehat{U}_0+\nu\widehat{U}_1+\widehat{U}_{-1})\nu^n}{(\nu-\delta)(\nu-\mu)}$$

$$+\frac{((\delta^2-\delta)\widehat{U}_0+\delta\widehat{U}_1+\widehat{U}_{-1})\delta^n}{(\delta-\mu)(\delta-\nu)}.$$

and the nth Tribonacci-Lucas trigintaduonions is

$$\widehat{V}_n=\frac{((\mu^2-\mu)\widehat{V}_0+\mu\widehat{V}_1+\widehat{V}_{-1})\mu^n}{(\mu-\delta)(\mu-\nu)}+\frac{((\nu^2-\nu)\widehat{V}_0+\nu\widehat{V}_1+\widehat{V}_{-1})\nu^n}{(\nu-\delta)(\nu-\mu)}$$

$$+\frac{((\delta^2-\delta)\widehat{V}_0+\delta\widehat{V}_1+\widehat{V}_{-1})\delta^n}{(\delta-\mu)(\delta-\nu)}.$$

Proof. To prove this theorem, we use generating function. Since

$$1-x-x^2-x^3=(1-\mu x)(1-\nu x)(1-\delta x)$$

Now, the generating function of $\widehat{U}_n$ is given by

$$\begin{aligned} g(x) &= \frac{\widehat{U}_0 + (\widehat{U}_1 - \widehat{U}_0)x + \widehat{U}_{-1}x^2}{1 - x - x^2 - x^3} = \frac{\widehat{U}_0 + (\widehat{U}_1 - \widehat{U}_0)x + \widehat{U}_{-1}x^2}{(1-\mu x)(1-\nu x)(1-\delta x)} \\ &= \frac{P}{(1-\mu x)} + \frac{Q}{(1-\nu x)} + \frac{R}{(1-\delta x)} \\ &= \frac{P(1-\nu x)(1-\delta x) + Q(1-\mu x)(1-\delta x) + R(1-\mu x)(1-\nu x)}{(1-\mu x)(1-\nu x)(1-\delta x)} \\ &= \frac{(P+Q+R) + (-P\nu - P\delta - Q\mu - Q\delta - R\mu - R\nu)x + (P\nu\delta + Q\mu\delta + R\mu\nu)x^2}{(1-\mu x)(1-\nu x)(1-\delta x)}. \end{aligned}$$

We need to determine the value of P, Q and R, then we have the following system of equations:

$$P + Q + R = \widehat{U}_0$$

$$-P\nu - P\delta - Q\mu - Q\delta - R\mu - R\nu = \widehat{U}_1 - \widehat{U}_0$$

$$P\nu\delta + Q\mu\delta + R\mu\nu = \widehat{U}_{-1}$$

Then, we get

$$P = \frac{\widehat{U}_{-1} + \mu\widehat{U}_1 + \mu^2\widehat{U}_0 - \mu\widehat{U}_0}{\mu^2 - \mu\nu - \mu\delta + \nu\delta} = \frac{(\mu^2 - \mu)\widehat{U}_0 + \mu\widehat{U}_1 + \widehat{U}_{-1}}{(\mu - \delta)(\mu - \nu)}$$

$$Q = \frac{\widehat{U}_{-1} + \nu\widehat{U}_1 + \nu^2\widehat{U}_0 - \nu\widehat{U}_0}{\nu^2 - \mu\nu + \mu\delta - \nu\delta} = \frac{(\nu^2 - \nu)\widehat{U}_0 + \nu\widehat{U}_1 + \widehat{U}_{-1}}{(\nu - \delta)(\nu - \mu)}$$

$$R = \frac{\widehat{U}_{-1} + \delta\widehat{U}_1 + \delta^2\widehat{U}_0 - \delta\widehat{U}_0}{\delta^2 + \mu\nu - \mu\delta - \nu\delta} = \frac{(\delta^2 - \delta)\widehat{U}_0 + \delta\widehat{U}_1 + \widehat{U}_{-1}}{(\delta - \mu)(\delta - \nu)}$$

and

$$\begin{aligned} g(x) &= \frac{(\mu^2 - \mu)\widehat{U}_0 + \mu\widehat{U}_1 + \widehat{U}_{-1}}{(\mu - \delta)(\mu - \nu)} \sum_{n=0}^{\infty} \mu^n x^n + \frac{(\nu^2 - \nu)\widehat{U}_0 + \nu\widehat{U}_1 + \widehat{U}_{-1}}{(\nu - \delta)(\nu - \mu)} \sum_{n=0}^{\infty} \nu^n x^n \\ &+ \frac{(\delta^2 - \delta)\widehat{U}_0 + \delta\widehat{U}_1 + \widehat{U}_{-1}}{(\delta - \mu)(\delta - \nu)} \sum_{n=0}^{\infty} \delta^n x^n \\ &= \sum_{n=0}^{\infty} \Bigg(\frac{((\mu^2 - \mu)\widehat{U}_0 + \mu\widehat{U}_1 + \widehat{U}_{-1})\mu^n}{(\mu - \delta)(\mu - \nu)} + \frac{((\nu^2 - \nu)\widehat{U}_0 + \nu\widehat{U}_1 + \widehat{U}_{-1})\nu^n}{(\nu - \delta)(\nu - \mu)} \\ &\qquad + \frac{((\delta^2 - \delta)\widehat{U}_0 + \delta\widehat{U}_1 + \widehat{U}_{-1})\delta^n}{(\delta - \mu)(\delta - \nu)} \Bigg) x^n. \end{aligned}$$

Then, we get the Binet formula for the Tribonacci trigintaduonins is

$$\begin{aligned} \widehat{U}_n = &\frac{((\mu^2 - \mu)\widehat{U}_0 + \mu\widehat{U}_1 + \widehat{U}_{-1})\mu^n}{(\mu - \delta)(\mu - \nu)} + \frac{((\nu^2 - \nu)\widehat{U}_0 + \nu\widehat{U}_1 + \widehat{U}_{-1})\nu^n}{(\nu - \delta)(\nu - \mu)} \\ &+ \frac{((\delta^2 - \delta)\widehat{U}_0 + \delta\widehat{U}_1 + \widehat{U}_{-1})\delta^n}{(\delta - \mu)(\delta - \nu)}. \end{aligned}$$

Similarly, one can proof Binet's formula of the Tribonacci-Lucas trigintaduonins. □

Now, we discuss the formula which gives the summation of the first n Tribonacci and Tribonacci-Lucas numbers.

Lemma 2.2. ([9], [12]) For every integer $n \geq 0$, we have

$$\sum_{s=0}^{n} U_s = U_0 + \frac{1}{2}(U_{n+2} + U_n - 1) = \frac{1}{2}(U_{n+2} + U_n - 1) \tag{2.3}$$

and

$$\sum_{s=0}^{n} V_s = \frac{V_{n+2} + V_n}{2}. \tag{2.4}$$

Theorem 2.5. *The summation formula for the Tribonacci and Tribonacci-Lucas trigintaduonions are*

$$\sum_{s=0}^{n} \widehat{U}_s = \frac{1}{2}(\widehat{U}_{n+2} + \widehat{U}_n + c_1)$$

and

$$\sum_{s=0}^{n} \widehat{V}_s = \frac{1}{2}(\widehat{V}_{n+2} + \widehat{V}_n + c_2)$$

respectively, where
$c_1 = -1 + e_1(-1 - 2U_0) + e_2(-1 - 2(U_0 + U_1)) + e_3(-1 - 2(U_0 + U_1 + U_2)) + \cdots + e_{31}(-1 - 2(U_0 + U_1 + \cdots + U_{30}))$
and
$c_2 = e_1(-2V_0) + e_2(-2(V_0 + V_1)) + e_3(-2(V_0 + V_1 + V_2)) + \cdots + e_{31}(-2(V_0 + V_1 + \cdots + V_{30}))$.

Proof. By using eq. (1.5) and eq. (2.3), we have

$$\begin{aligned}
\sum_{s=0}^{n} \widehat{U}_s &= \sum_{s=0}^{n} U_s + e_1 \sum_{s=0}^{n} U_{s+1} + e_2 \sum_{s=0}^{n} U_{s+2} + \cdots + e_{31} \sum_{s=0}^{n} U_{s+31} \\
&= (U_0 + \cdots + U_n) + e_1(U_1 + \cdots + U_{n+1}) + e_2(U_2 + \cdots + U_{n+2}) \\
&\quad + \cdots + e_{31}(U_{31} + \cdots + U_{n+31})
\end{aligned}$$

and

$$\begin{aligned}
2\sum_{s=0}^{n} \widehat{U}_s &= (U_{n+2} + U_n - 1) + e_1(U_{n+3} + U_{n+1} - 1 - 2U_0) \\
&\quad + e_2(U_{n+4} + U_{n+3} - 1 - 2(U_0 + U_1)) \\
&\quad \vdots \\
&\quad + e_{31}(U_{n+33} + U_{n+31} - 1 - 2(U_0 + U_1 + \cdots + U_{30})) \\
&= \widehat{U}_{n+2} + \widehat{U}_n + c_1
\end{aligned}$$

where $c_1 = -1 + e_1(-1 - 2U_0) + e_2(-1 - 2(U_0 + U_1)) + e_3(-1 - 2(U_0 + U_1 + U_2)) + \cdots + e_{31}(-1 - 2(U_0 + U_1 + \cdots + U_{30}))$. Thus, we have

$$\sum_{s=0}^{n} \widehat{U}_s = \frac{1}{2}(\widehat{U}_{n+2} + \widehat{U}_n + c_1).$$

Similarly, one can prove summation formula for the Tribonacci-Lucas trigintaduonions. □

3. Matrix representation of Tribonacci and Tribonacci-Lucas trigintaduonions

Suppose the sequence $\{X_n\}$ which is defined by the third-order recurrence relation

$$X_n = X_{n-1} + X_{n-2} + X_{n-3}, \qquad X_0 = X_1 = 0,\ X_2 = 1.$$

Some mathematicians call $\{X_n\}$ as a Tribonacci sequence instead of $\{U_n\}$. By using Binet's formula, the numbers X_n can be defined as

$$X_n = \frac{\mu^n}{(\mu-\nu)(\mu-\delta)} + \frac{\nu^n}{(\nu-\mu)(\nu-\delta)} + \frac{\delta^n}{(\delta-\mu)(\delta-\nu)}$$

and for negative numbers X_{-n}, it is expressed by

$$X_{-n} = \begin{vmatrix} X_{n+1} & X_{n+2} \\ X_n & X_{n+1} \end{vmatrix} = X_{n+1}^2 - X_{n+2}X_n.$$

Let us consider a square matrix D of order 3 as:

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det D = 1$ and

$$D^n = \begin{pmatrix} X_{n+2} & X_{n+1}+X_n & X_{n+1} \\ X_{n+1} & X_n + X_{n-1} & X_n \\ X_n & X_{n-1}+X_{n-2} & X_{n-1} \end{pmatrix}. \tag{3.1}$$

To know more about (3.1) matrix (see [1]). Matrix formulation of U_n and V_n can be given as

$$\begin{pmatrix} U_{n+2} \\ U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} U_2 \\ U_1 \\ U_0 \end{pmatrix} \tag{3.2}$$

and

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}. \tag{3.3}$$

The matrix D was defined and used in [23]. To know more above matrix see ([26], [27]). Now, we define the matrix D_U and D_V as

$$D_U = \begin{pmatrix} \widehat{U}_4 & \widehat{U}_3 + \widehat{U}_2 & \widehat{U}_3 \\ \widehat{U}_3 & \widehat{U}_2 + \widehat{U}_1 & \widehat{U}_2 \\ \widehat{U}_2 & \widehat{U}_1 + \widehat{U}_0 & \widehat{U}_1 \end{pmatrix}$$

and

$$D_V = \begin{pmatrix} \widehat{V}_4 & \widehat{V}_3 + \widehat{V}_2 & \widehat{V}_3 \\ \widehat{V}_3 & \widehat{V}_2 + \widehat{V}_1 & \widehat{V}_2 \\ \widehat{V}_2 & \widehat{V}_1 + \widehat{V}_0 & \widehat{V}_1 \end{pmatrix}.$$

These matrix D_U and D_V can be called the Tribonacci trigintaduonions matrix and Tribonacci-Lucas trigintaduonions matrix, respectively.

Theorem 3.1. *For $n \geq 0$, the following holds:*
(i)

$$D_U \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{U}_{n+4} & \widehat{U}_{n+3} + \widehat{U}_{n+2} & \widehat{U}_{n+3} \\ \widehat{U}_{n+3} & \widehat{U}_{n+2} + \widehat{U}_{n+1} & \widehat{U}_{n+2} \\ \widehat{U}_{n+2} & \widehat{U}_{n+1} + \widehat{U}_{n} & \widehat{U}_{n+1} \end{pmatrix}, \tag{3.4}$$

(ii)

$$D_V \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{V}_{n+4} & \widehat{V}_{n+3} + \widehat{V}_{n+2} & \widehat{V}_{n+3} \\ \widehat{V}_{n+3} & \widehat{V}_{n+2} + \widehat{V}_{n+1} & \widehat{V}_{n+2} \\ \widehat{V}_{n+2} & \widehat{V}_{n+1} + \widehat{V}_{n} & \widehat{V}_{n+1} \end{pmatrix}. \tag{3.5}$$

Proof. To prove (i) we use mathematical induction on n. If $n = 0$, then the result is obvious. We suppose that it is true for $n = \upsilon$, then we have

$$D_U D^{\upsilon} = \begin{pmatrix} \widehat{U}_{\upsilon+4} & \widehat{U}_{\upsilon+3} + \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+3} \\ \widehat{U}_{\upsilon+3} & \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} & \widehat{U}_{\upsilon+2} \\ \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+1} + \widehat{U}_{\upsilon} & \widehat{U}_{\upsilon+1} \end{pmatrix}.$$

By using eq. (1.7), then for $\upsilon \geq 0$, we have $\widehat{U}_{\upsilon+3} = \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} + \widehat{U}_{\upsilon}$. Now, by induction hypothesis we have

$$\begin{aligned} D_U D^{\upsilon+1} &= (D_U D^{\upsilon}) D \\ &= \begin{pmatrix} \widehat{U}_{\upsilon+4} & \widehat{U}_{\upsilon+3} + \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+3} \\ \widehat{U}_{\upsilon+3} & \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} & \widehat{U}_{\upsilon+2} \\ \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+1} + \widehat{U}_{\upsilon} & \widehat{U}_{\upsilon+1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \widehat{U}_{\upsilon+4} + \widehat{U}_{\upsilon+3} + \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+4} + \widehat{U}_{\upsilon+3} & \widehat{U}_{\upsilon+4} \\ \widehat{U}_{\upsilon+3} + \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} & \widehat{U}_{\upsilon+3} + \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+3} \\ \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} + \widehat{U}_{\upsilon} & \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} & \widehat{U}_{\upsilon+2} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{U}_{\upsilon+5} & \widehat{U}_{\upsilon+4} + \widehat{U}_{\upsilon+3} & \widehat{U}_{\upsilon+4} \\ \widehat{U}_{\upsilon+4} & \widehat{U}_{\upsilon+3} + \widehat{U}_{\upsilon+2} & \widehat{U}_{\upsilon+3} \\ \widehat{U}_{\upsilon+3} & \widehat{U}_{\upsilon+2} + \widehat{U}_{\upsilon+1} & \widehat{U}_{\upsilon+2} \end{pmatrix}. \end{aligned}$$

Hence, (3.4) holds for all non-negative integers n. (ii) can be proved in the similar way. □

Corollary 3.2. *For $n \geq 0$, the following holds:*
(i) $\widehat{U}_{n+2} = \widehat{U}_2 X_{n+2} + (\widehat{U}_1 + \widehat{U}_0) X_{n+1} + \widehat{U}_1 X_n$,
(ii) $\widehat{V}_{n+2} = \widehat{V}_2 X_{n+2} + (\widehat{V}_1 + \widehat{V}_0) X_{n+1} + \widehat{V}_1 X_n$.

Proof. The proof of (i) can be easily seen by the coefficient (3.2) of the matrix D_U and (3.1). Similarly, the proof of (ii) can be seen by the coefficient (3.3) of the matrix D_V and (3.1).
If the matrix D replaces with the matrices E and F, then we obtain the same results as the matrix D, matrices E and F are defined by

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

□

Conflict of interest: The authors declare that there are no conflicts of interest.

Acknowledgement: The corresponding author thanks the Council of Scientific and Industrial Research (CSIR), India for partial support under Grant No. 25(0288)/18/EMR-II, dated 24/05/2018.

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