INITIAL BOUNDARY-VALUE PROBLEM FOR EQUATIONS WITH FRACTIONAL DERIVATIVE WITH DEGENERATION OF THE FIRST KIND

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Abstract. The article considers an initial-boundary value problem for a higher-order equation with a fractional Caputo derivative, in a rectangular domain, degenerating in both variables. The solution to the problem is constructed explicitly, by the method of separation of variables. Uniqueness is shown by the spectral method.

Keywords: Differential equation, high order, degeneration, fractional Caputo derivative, existence, uniqueness, series, uniform convergence.

1. INTRODUCTION

In the domain $D = \{(x,y) : 0 < x < 1, 0 < y < 1\}$, consider the equation

$$(-1)^{k+1} K(y) \, _C D_{0x}^{\alpha} u(x,y) - x^s \frac{\partial^{2k} u}{\partial y^{2k}} = 0, \tag{1}$$

where $0 < \alpha < 1, s \in N \cup \{0\}$, $k \in N, D_{0x}^{\alpha} -$ is the operator of Caputo fractional differentiation, of order α

$$_C D_{0x}^{\alpha} u(x,y) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^x \frac{\frac{\partial u(\tau,y)}{\partial \tau} d\tau}{(x-\tau)^{\alpha}}.$$

The function $K(y)$ satisfies the conditions:
1. $K(y) > 0$ for $y > 0$;
2. $K(0) = 0$;

3. $K(y) \in C[0,1]$.

For equation (1), consider the problem.

Problem A. Find a solution to equation (1) from the class

$$u(x,y) \in C_{x,y}^{0,2k-1}\left(\overline{D}\right) \cap C_y^{2k}(D), {}_C D_{0x}^\alpha \in C(D), \tag{2}$$

satisfying the conditions

$$\frac{\partial^s u(x,0)}{\partial y^s} = \frac{\partial^s u(x,1)}{\partial y^s} = 0, \ 0 \le x \le 1, \ s = 0,1,...,k-1, \tag{3}$$

$$u(0,y) = \varphi(y), \ 0 \le y \le 1, \tag{4}$$

here $\varphi(y) -$ is sufficiently smooth .

Fractional differential equations arise in mathematical modeling of various physical processes and phenomena [1]. Second-order equations of the form (1) with partial derivatives of fractional order $\alpha \in (0,2)$ were studied in [1] - [8] and others. In these papers, the Cauchy problem was considered, the first , the second and mixed boundary value problems, a fundamental solution is found, a general representation of solutions is constructed. Mixed equations and higher-order equations with a fractional derivative were studied in [9] - [12]. Degenerate fractional-order equations were studied in [1], [13]. The research will be carried out by the Fourier method. Previously, by the Fourier method, boundary value problems for equations with a fractional derivative were studied in [6] - [9], [12].

2. Existence of a solution

We are looking for a solution in the form

$$u(x,y) = X(x)Y(y).$$

Then with respect to the variable y, taking into account condition (3), we obtain the following spectral problem:

$$\begin{cases} Y^{(2k)}(y) = (-1)^k \lambda K(y) Y(y), \\ Y^{(s)}(0) = Y^{(s)}(1) = 0, \ s = 0,1,...,k-1. \end{cases} \tag{5}$$

First, we show that $\lambda = 0$ is not an eigenvalue. Indeed, consider problem (5) with $\lambda = 0$

$$\begin{cases} Y^{(2k)}(y) = 0, \\ Y^{(s)}(0) = Y^{(s)}(1) = 0, \ s = 0,1,...,k-1. \end{cases}$$

A solution to this problem satisfying the condition $Y^{(s)}(0) = 0, \ s = 0,1,...,k-1$, has the form

$$Y(y) = c_k y^k + c_{k+1} y^{k+1} + ... + c_{2k-1} y^{2k-1},$$

to determine the unknowns $c_j, \ j = k, k+1, ..., 2k-1$, we obtain the system of equations

$$\begin{cases} c_k + c_{k+1} + ... + c_{2k-1} = 0, \\ kc_k + (k+1)c_{k+1} + ... + (2k-1)c_{2k-1} = 0, \\ ... \\ k(k-1)...2 \cdot c_k + (k+1)k...3 \cdot c_{k+1} + ... + (2k-1)(2k-2)...(k+1)c_{2k-1} = 0, \end{cases}$$

the main determinant of this system Δ , has the form

$$\Delta = \begin{vmatrix} 1 & 1 & ... & 1 \\ k & k+1 & ... & 2k-1 \\ . & . & & . \\ k(k-1)...2 & (k+1)k...3 & ... & (2k-1)(2k-2)...(k+1) \end{vmatrix} =$$

$$= \prod_{j=1, j>i}^{k-1} (j - i) \neq 0.$$

Hence $c_j = 0, \ j = k, k + 1, ..., 2k - 1$, from here

$$Y(y) = c_k y^k + c_{k+1} y^{k+1} + ... + c_{2k-1} y^{2k-1} \equiv 0.$$

We reduce problem (6) to the integral equation using the Green function and obtain the necessary estimates for the eigenfunctions. But first, we show that $\lambda > 0$. Indeed, we have

$$\int_0^1 Y(y) \, Y^{(2k)}(y) dy = (-1)^k \lambda \int_0^1 K(y) \, Y^2(y) dy,$$

$$\int_0^1 \left(Y^{(k)} \right)^2 dy = \lambda \int_0^1 K(y) \, Y^2(y) dy,$$

because $\lambda = 0$ is not an eigenvalue, it follows that $\lambda > 0$. It remains to show the existence of eigenvalues and eigenfunctions of problem (5). The integral equation equivalent to problem (5) has the form

$$Y(y) = (-1)^k \lambda \int_0^1 K(y) \, G(y, \xi) \, Y(\xi) \, d\xi, \tag{6}$$

where

$$G(y, \xi) = -\frac{1}{(2k-1)!} \left\{ \begin{array}{ll} G_1(y, \xi), & 0 \leq y \leq \xi, \\ G_2(y, \xi), & \xi \leq y \leq 1, \end{array} \right.$$

- the Green function of problem (5) (see [14]), here

$$G_1(y, \xi) = (1 - \xi)^k y^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i C_{2k-1}^i C_{k-1+j}^j y^{k-i-1} \xi^{j+i},$$

$$G_2(y, \xi) = (1 - y)^k \xi^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i C_{2k-1}^i C_{k-1+j}^j \xi^{k-i-1} y^{j+i},$$

$$C_n^k = \frac{n!}{k! \, (n - k)!}.$$

Rewrite (6) as

$$\sqrt{K(y)} Y(y) = \lambda \int_0^1 \sqrt{K(\xi)} \left[(-1)^k G(y, \xi) \right] \sqrt{K(y)} \left(\sqrt{K(\xi)} Y(\xi) \right) d\xi,$$

we introduce the notation

$$\overline{Y}(y) = \sqrt{K(y)} Y(y),$$

$$\overline{G}(y, \xi) = \sqrt{K(\xi)} \left[(-1)^k G(y, \xi) \right] \sqrt{K(y)},$$

then we have

$$\overline{Y}(y) = \lambda \int_0^1 \overline{G}(y, \xi) \, \overline{Y}(\xi) \, d\xi, \tag{7}$$

(7) - there is an integral equation with a continuous, in both variables, and a symmetric kernel. According to the theory of equations with symmetric kernels, equation (7) has no more than a countable number of eigenvalues and eigenfunctions. So, problem (5) has eigenvalues $\lambda_n > 0$, $n = 1, 2, ...$, and the corresponding eigenfunctions are $Y_n(y)$. Further, we assume that

$$\|Y_n(y)\|^2 = \int_0^1 K(y) Y_n^2(y) dy = 1.$$

then, taking into account (7), we have the Bessel inequality

$$\sum_{n=1}^{\infty} \left(\frac{Y_n(y)}{\lambda_n} \right)^2 \leq \int_0^1 K(y) G^2(y, \xi) d\xi < M < \infty. \qquad (8)$$

Now we find the conditions under which the given function $\varphi(y)$ is expanded in a series according to the eigenfunctions $Y_n(y)$. For this we use the Hilbert-Schmidt theorem.

Theorem 1. *Let the function $\varphi(y)$ satisfy the conditions*
1. $\frac{\varphi^{(2k)}(\xi)}{\sqrt{K(\xi)}} \in C[0; 1]$;
2. $\varphi^{(i)}(0) = \varphi^{(i)}(1) = 0$, $i = 0, 1, ..., k - 1$.
Then it can be expanded in a uniformly and absolutely converging series of the form

$$\varphi(y) = \sum_{n=1}^{\infty} \varphi_n Y_n(y),$$

where

$$\varphi_n = \int_0^1 K(y) \varphi(y) Y_n(y) dy,$$

Proof. We show the equality

$$\sqrt{K(y)}\varphi(y) = \int_0^1 \overline{G}(y, \xi) \left((-1)^k \frac{\varphi^{(2k)}(\xi)}{\sqrt{K(\xi)}} \right) d\xi,$$

really

$$\int_0^1 \sqrt{K(\xi)} \left[(-1)^k G(y, \xi) \right] \sqrt{K(y)} \left((-1)^k \frac{\varphi^{(2k)}(\xi)}{\sqrt{K(\xi)}} \right) d\xi =$$

$$= \sqrt{K(y)} \int_0^1 G(y, \xi) \varphi^{(2k)}(\xi) d\xi = \sqrt{K(y)}\varphi(y).$$

Those for the function $\sqrt{K(y)}\varphi(y)$ the conditions of the Hilbert-Schmidt theorem are satisfied and therefore

$$\sqrt{K(y)}\varphi(y) = \sum_{n=1}^{\infty} \sqrt{K(y)}\varphi_n Y_n(y),$$

dividing by $\sqrt{K(y)}$, we have

$$\varphi(y) = \sum_{n=1}^{\infty} \varphi_n Y_n(y).$$

$\square$

We proceed to solve the equation in the variable x. Taking into account condition(4), we obtain the following initial problem:

$$\begin{cases} {}_C D_{0x}^{\alpha} X_n(x) = -\lambda_n x^s X_n(x), \\ X_n(0) = \varphi_n, \end{cases} \tag{9}$$

where

$$\varphi_n = \int_0^1 K(y)\, \varphi(y)\, Y_n(y)\, dy\ ,$$

We will seek solution (9) in the form of a series

$$X_n(x) = \sum_{j=0}^{\infty} c_j x^{\gamma j+\beta},$$

where $c_j, \beta, \gamma-$ are to be defined.
Formally, we have (the validity of the rearrangement of the series and the integral will follow below)

$$
{}_C D_{0x}^{\alpha} X_n(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{\frac{d}{d\tau} \sum_{j=0}^{\infty} c_j \tau^{\gamma j+\beta} d\tau}{(x-\tau)^{\alpha}} =
$$

$$
= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} c_j \int_0^x \frac{(\gamma j+\beta)\, \tau^{\gamma j+\beta-1} d\tau}{(x-\tau)^{\alpha}} =
$$

$$
= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} c_j\, (\gamma j+\beta) \int_0^1 \frac{x^{\gamma j+\beta} z^{\gamma j+\beta-1} dz}{x^{\alpha}(1-z)^{\alpha}} =
$$

$$
= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} c_j\, (\gamma j+\beta)\, x^{\gamma j+\beta-\alpha} \int_0^1 \frac{z^{\gamma j+\beta-1} dz}{(1-z)^{\alpha}} =
$$

$$
= \sum_{j=0}^{\infty} c_j\, (\gamma j+\beta)\, x^{\gamma j+\beta-\alpha} \frac{\Gamma(\gamma j+\beta)}{\Gamma(\gamma j+\beta-\alpha+1)}.
$$

Substituting the last expression in (9), we obtain

$$\sum_{j=0}^{\infty} c_j\, (\gamma j+\beta)\, x^{\gamma j+\beta-\alpha} \frac{\Gamma(\gamma j+\beta)}{\Gamma(\gamma j+\beta-\alpha+1)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\gamma j+\beta+s} \Rightarrow$$

The solution to problem (9) has the form

$$\sum_{j=0}^{\infty} c_j\, (\gamma j+\beta)\, x^{\gamma j-\alpha} \frac{\Gamma(\gamma j+\beta)}{\Gamma(\gamma j+\beta-\alpha+1)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\gamma j+s} \Rightarrow$$

$$\sum_{j=0}^{\infty} c_j \left(\gamma j + \beta\right) x^{\gamma(j-1)+\gamma-\alpha} \frac{\Gamma\left(\gamma j + \beta\right)}{\Gamma\left(\gamma j + \beta - \alpha + 1\right)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\gamma j + s} \Rightarrow$$

$$\gamma - \alpha = s \Rightarrow \gamma = \alpha + s \Rightarrow$$

$$\sum_{j=0}^{\infty} c_j \left(\left(\alpha + s\right) j + \beta\right) x^{\alpha(j-1)} \frac{\Gamma\left(\left(\alpha + s\right) j + \beta\right)}{\Gamma\left(\left(\alpha + s\right) j + \beta - \alpha + 1\right)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\alpha j} \Rightarrow$$

$$\beta = 0 \Rightarrow$$

$$\sum_{j=0}^{\infty} c_j \left(\left(\alpha + s\right) j\right) x^{\alpha(j-1)} \frac{\Gamma\left(\left(\alpha + s\right) j\right)}{\Gamma\left(\left(\alpha + s\right) j - \alpha + 1\right)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\alpha j} \Rightarrow$$

$$c_j \left(\alpha + s\right) j \frac{\Gamma\left(\left(\alpha + s\right) j\right)}{\Gamma\left(\left(\alpha + s\right) j - \alpha + 1\right)} = -\lambda_n c_{j-1} \Rightarrow$$

$$c_j = \frac{-\lambda_n \Gamma\left(\left(\alpha + s\right) j - \alpha + 1\right)}{\left(\alpha + s\right) j \Gamma\left(\left(\alpha + s\right) j\right)} c_{j-1} = \frac{-\lambda_n \Gamma\left(\left(\alpha + s\right)\left(j - 1\right) + 1 + s\right)}{\Gamma\left(\left(\alpha + s\right) j + 1\right)} c_{j-1} \Rightarrow$$

$$c_j = \frac{\left(-\lambda_n\right) \left(\left(\alpha + s\right)\left(j - 1\right) + s\right) \ldots \left(\left(\alpha + s\right)\left(j - 1\right) + 1\right) \Gamma\left(\left(\alpha + s\right)\left(j - 1\right) + 1\right)}{\Gamma\left(\left(\alpha + s\right) j + 1\right)} c_{j-1} =$$

$$= \frac{\left(-\lambda_n\right) \left(\alpha + s\right)^s \left(\left(j - 1\right) + \frac{1}{\alpha+s}\right) \ldots \left(\left(j - 1\right) + \frac{s}{\alpha+s}\right) \Gamma\left(\left(\alpha + s\right)\left(j - 1\right) + 1\right)}{\Gamma\left(\left(\alpha + s\right) j + 1\right)} c_{j-1} =$$

$$= \frac{\left(-\lambda_n(\alpha + s)^s\right)^j \left(\frac{1}{\alpha+s}\right)_j \ldots \left(\frac{s}{\alpha+s}\right)_j}{\Gamma\left(\left(\alpha + s\right) j + 1\right)} c_0 =$$

$$= \frac{\left(-\lambda_n(\alpha + s)^s\right)^j \Gamma\left(j + \frac{1}{\alpha+s}\right) \ldots \Gamma\left(j + \frac{s}{\alpha+s}\right)}{\Gamma\left(\left(\alpha + s\right) j + 1\right)},$$

$$c_0 = \Gamma\left(\frac{1}{\alpha + s}\right) \ldots \Gamma\left(\frac{s}{\alpha + s}\right).$$

Hence, the solution to problem (9) has the form

$$X_n\left(x\right) = \frac{\varphi_n}{\Gamma\left(\frac{1}{\alpha+s}\right) \ldots \Gamma\left(\frac{s}{\alpha+s}\right)} \sum_{j=0}^{\infty} \frac{\Gamma\left(j + \frac{1}{\alpha+s}\right) \ldots \Gamma\left(j + \frac{s}{\alpha+s}\right) \Gamma\left(j + 1\right)}{\Gamma\left(\left(\alpha + s\right) j + 1\right)} \frac{\left(-\lambda_n(\alpha + s)^s x^{\alpha+s}\right)^j}{j!}.$$

This representation implies the uniqueness of the solution to problem (9).
Because $\alpha + s + 1 > s + 1$, then the last series converges absolutely and uniformly for fixed values of λ_n and for bounded values of x [see 15]. This means that the permutation of the series and the integral in above was legal.
In terms of special functions, the solution to problem (9) can be written in the form

$$X_n\left(x\right) = \frac{\varphi_n}{\Gamma\left(\frac{1}{\alpha+s}\right) \ldots \Gamma\left(\frac{s}{\alpha+s}\right)} {}_{s+1}\Psi_1 \left(\begin{array}{c} \left(1, \frac{1}{\alpha+s}\right), \ldots, \left(1, \frac{s}{\alpha+s}\right), \left(1, 1\right); -\lambda_n\left(\alpha + s\right)^s x^{\alpha+s} \\ \left(\alpha + s, 1\right) \end{array} \right),$$

where

$${}_p\Psi_q \left(\begin{array}{c} \left(\alpha_1, a_1\right), \ldots, \left(\alpha_p, a_p\right), z \\ \left(\beta_1, b_1\right), \ldots, \left(\beta_q, b_p\right) \end{array} \right) = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma\left(\alpha_i m + a_i\right)}{\prod_{i=1}^{q} \Gamma\left(\beta_i m + b_i\right)} \frac{z^m}{m!}$$

- is the generalized Wright function [see 15].

Using the results obtained in [15, Theorem 4.], we obtain an asymptotic expansion of the generalized Wright function for large values λ_n and $x > \delta > 0$:

$$_{s+1}\Psi_1\left(\begin{array}{c}\left(1,\frac{1}{\alpha+s}\right),...,\left(1,\frac{s}{\alpha+s}\right),(1,1)\,;-\lambda_n(\alpha+s)^s x^{\alpha+s}\\(\alpha+s,1)\end{array}\right)\sim$$

$$\sim H_{s+1,1}\left(-\lambda_n(\alpha+s)^s x^{\alpha+s}\right) = H_{s+1,1}(t) =$$

$$= \sum_{m=1}^{s} t^{-\frac{m}{\alpha+s}}S_{s+1,1}(t;m) + t^{-1}S_{s+1,1}(t;s+1),$$

here

$$S_{s+1,1}(t;m) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(k+\frac{m}{\alpha+s}\right)\Gamma\left(1-k-\frac{m}{\alpha+s}\right)\frac{\prod\limits_{r=1,r\neq m}^{s}\Gamma\left(\frac{r}{\alpha+s}-k-\frac{m}{\alpha+s}\right)}{\Gamma\left(1-(\alpha+s)\left(k+\frac{m}{\alpha+s}\right)\right)}t^{-k} =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\pi}{\sin\pi\left(k+\frac{m}{\alpha+s}\right)}\frac{\prod\limits_{r=1,r\neq m}^{s}\frac{-\pi}{\Gamma\left(k+\frac{m}{\alpha+s}-\frac{r}{\alpha+s}+1\right)\sin\pi\left(k+\frac{m}{\alpha+s}-\frac{r}{\alpha+s}\right)}}{\frac{-\pi}{\Gamma((\alpha+s)k+m)\sin\pi((\alpha+s)k+m-1)}}t^{-k} =$$

$$= \frac{(-1)^{m+s+1}\pi^{s-1}}{\sin\frac{m\pi}{\alpha+s}}\sum_{k=0}^{\infty}\frac{(-1)^{k(s-1)}}{k!}\frac{\Gamma((\alpha+s)k+m)\sin\pi k(\alpha+s)}{\prod\limits_{r=1,r\neq m}^{s}\Gamma\left(k+\frac{m}{\alpha+s}-\frac{r}{\alpha+s}+1\right)\sin\pi\left(\frac{m}{\alpha+s}-\frac{r}{\alpha+s}\right)}t^{-k},$$

$$S_{s+1,1}(t;s+1) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(k+1)\frac{\prod\limits_{r=1}^{s}\Gamma\left(\frac{r}{\alpha+s}-k-1\right)}{\Gamma(1-(\alpha+s)(k+1))}t^{-k} =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{1}\frac{\prod\limits_{r=1}^{s}\frac{-\pi}{\Gamma\left(k+2-\frac{r}{\alpha+s}\right)\sin\pi\left(-\frac{r}{\alpha+s}+k+1\right)}}{\frac{\pi}{\Gamma((\alpha+s)(k+1)+2)\sin\pi(\alpha+s)(k+1)}}t^{-k} =$$

$$= \pi^{s-1}(-1)^s\sum_{k=0}^{\infty}(-1)^{k(1+s)}\frac{\Gamma((\alpha+s)(k+1)+2)\sin\pi(\alpha+s)(k+1)}{\prod\limits_{r=1}^{s}\Gamma\left(k+2-\frac{r}{\alpha+s}\right)\sin\frac{\pi r}{\alpha+s}}t^{-k}.$$

Taking into account the latter, we have that there exists a number T such that for all $n > T$ and for $x > \delta > 0$ the estimate

$$|X_n(x)| \leq \frac{M}{\lambda_n}|\varphi_n|x^{\alpha-1},\ 0 < M - const.$$

So, the formal solution of the posed problem A has the form

$$u(x,y) = \sum_{n=0}^{\infty}X_n(x)Y_n(y). \tag{10}$$

Let us show that (10) is a classical solution to Eq. (1); indeed (for $x > \delta > 0$)

$$|_C D_{0x}^{\alpha}u(x,y)| \leq \sum_{n=0}^{\infty}|_C D_{0x}^{\alpha}X_n(x)|\,|Y_n(y)| = \sum_{n=0}^{\infty}|\lambda_n x^s X_n(x)|\,|Y_n(y)| \leq$$

$$\leq M_1\sum_{n=0}^{\infty}|\varphi_n|\,|Y_n(y)|,\ 0 < M_1 - const.$$

Next, we apply the Cauchy-Bunyakovsky inequality

$$\sum_{n=0}^{\infty} |\varphi_n| \, |Y_n(y)| = \sum_{n=0}^{\infty} |\lambda_n \varphi_n| \left| \frac{Y_n(y)}{\lambda_n} \right| \leq \sqrt{\sum_{n=1}^{\infty} |\lambda_n^2 \varphi_n^2|} \, \sqrt{\sum_{n=0}^{\infty} \left| \frac{Y_n^2(y)}{\lambda_n^2} \right|},$$

we have

$$\varphi_n = \int_0^1 K(y)\, \varphi(y)\, Y_n(y)\, dy =$$

$$= \frac{(-1)^k}{\lambda_n} \int_0^1 \varphi(y)\, Y_n^{(2k)}(y)\, dy =$$

$$= \frac{(-1)^k}{\lambda_n} \int_0^1 \varphi^{(2k)}(y)\, Y_n(y)\, dy =$$

$$= \frac{(-1)^k}{\lambda_n} \int_0^1 K(y)\, \frac{\varphi^{(2k)}(y)}{K(y)} Y_n(y)\, dy.$$

We apply Bessel's inequality

$$\sum_{n=0}^{\infty} \lambda_n^2 |\varphi_n(y)|^2 \leq \int_0^1 K(y) \left(\frac{\varphi^{(2k)}(y)}{K(y)} \right)^2 dy = \int_0^1 \left(\frac{\varphi^{(2k)}(y)}{\sqrt{K(y)}} \right)^2 dy < \infty. \qquad (11)$$

Taking into account (8) and (11), we obtain that the series

$$_C D_{0x}^{\alpha} u(x, y) = \sum_{n=0}^{\infty} {}_C D_{0x}^{\alpha} X_n(x)\, Y_n(y) -$$

converges uniformly. Now, in order for the calculations made above to be legal, we impose the following restrictions on the $\varphi(y)$ function. So, the theorem is valid.

Theorem 2. *Let the function $\varphi(y)$, satisfy the following conditions:*

$$\frac{\varphi^{(2k)}(\xi)}{\sqrt{K(\xi)}} \in C[0;1], \; \varphi^{(s)}(0) = \varphi^{(s)}(1) = 0,$$

$$s = 0, 1, ..., k - 1,$$

then a solution to Problem A exists.

The uniform convergence of the series

$$\frac{\partial^{2k} u(x, y)}{\partial y^{2k}} = \sum_{n=0}^{\infty} X_n(x) \frac{\partial^{2k} Y_n(y)}{\partial y^{2k}} = (-1)^k K(y) \sum_{n=0}^{\infty} \lambda_n X_n(x)\, Y_n(y).$$

3.Uniqueness

Let the function $u(x, y)$ be a solution to Problem A with zero initial and boundary conditions. Consider its Fourier coefficients with respect to the system of eigenfunctions of problem (5)

$$u_n(x) = \int_0^1 K(y)\, u(x, y)\, Y_n(y)\, dy$$

it is easy to show that $u_n(x)$ is a solution to the problem

$$\begin{cases} {}_CD^{\alpha}_{0x}u_n(x) = -\lambda_n x^s u_n(x), \\ u_n(0) = 0. \end{cases}$$

This problem has only a zero solution, i.e.

$$\int_0^1 K(y) u(x,y) Y_n(y) \, dy = 0, \ \forall n.$$

Because

$$\overline{G}(y,\xi) - \text{symmetric, continuous,}$$

$$\int_0^1 \overline{G}^2(y,\xi) \, d\xi < \infty, \ \int_0^1 \overline{G}^2(y,\xi) \, dy < \infty, \ \int_0^1 \int_0^1 \overline{G}^2(y,\xi) \, dy d\xi < \infty, \ \lambda_n > 0, \ \forall n,$$

then the conditions of Mercer's theorem are fulfilled and hence

$$\overline{G}(y,\xi) = \sum_{n=0}^{\infty} \frac{\overline{Y_n}(y) \, \overline{Y_n}(\xi)}{\lambda_n}.$$

Hence we have

$$\sqrt{K(y)}u(x,y) = \int_0^1 \overline{G}(y,\xi) \left((-1)^k \frac{1}{\sqrt{K(\xi)}} \frac{\partial^{2k} u(x,\xi)}{\partial \xi^{2k}} \right) d\xi =$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{\overline{Y}_n(y) \overline{Y}_n(\xi)}{\lambda_n} \left((-1)^k \frac{1}{\sqrt{K(\xi)}} \frac{\partial^{2k} u(x,\xi)}{\partial \xi^{2k}} \right) d\xi =$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{K(y)}Y_n(y)}{\lambda_n} \int_0^1 \left((-1)^k \frac{\sqrt{K(\xi)}Y_n(\xi)}{\sqrt{K(\xi)}} \frac{\partial^{2k} u(x,\xi)}{\partial \xi^{2k}} \right) d\xi =$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{K(y)}Y_n(y)}{\lambda_n} \int_0^1 \left((-1)^k Y_n(\xi) \frac{\partial^{2k} u(x,\xi)}{\partial \xi^{2k}} \right) d\xi =$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{K(y)}Y_n(y)}{\lambda_n} \int_0^1 \left((-1)^k Y^{(2k)}{}_n(\xi) u(x,\xi) \right) d\xi =$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{K(y)}Y_n(y)}{\lambda_n} \int_0^1 (\lambda_n K(\xi) Y_n(\xi) u(x,\xi)) \, d\xi = 0.$$

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