Leggett-Williams fixed point theorem type for sums of operators and application in PDEs

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Abstract

In this paper we present an extension of the original version of Leggett-Williams fixed point theorem for a k-set contraction perturbed by an expansive operator. Our approach is applied to prove the existence of non trivial positive solutions for initial value problems (IVPs for short) covering a class two-dimensional nonlinear wave equations.

Key words: fixed point index, cone, sum of operators, Green function, PDEs.
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1 Introduction

The most well-known version of Leggett-Williams fixed point theorem [10] provides conditions which ensure the existence of at least three fixed points in cones of Banach spaces. However this theorem is only an extension of the original version, presented in the same paper by the authors. This version

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presents conditions which guarantee the existence and the localization of at least one fixed point, as well as in krasnosel'skii's cone compression and expansion one.

By Krasnosel'skii's fixed point theorem the fixed point is localized in a conical shell of the form $\{x : a \leq \|x\| \leq b\}$, where $a, b \in (0, +\infty)$, while in the Leggett-Williams it is localized in a conical shell of the form $\{x : a \leq \alpha(x)$ and $\|x\| \leq b\}$, where $a, b \in (0, +\infty)$ and α is a concave positive functional. Although the two approaches are not easily comparable, using the functional α, which, by its definition, cannot coincide with the norm, allows for easier calculations and more versatile results. In this context, it is preferable to use the Leggett-Williams approach.

In [4], Djebali and Mebarki developed a new fixed point index for the sum of an expansive mapping and a set contraction defined in cones of Banach spaces. Then fixed point theorems, including Krasnosel'skii type theorems, have been derived. This results have been applied to obtain existence results for positive solutions of various types of boundary and/or initial value problems (see [2, 3, 7, 8]).

In what follows, let us consider $(E, \|.\|)$ be a real Banach space, $\mathcal{P}$ be a cone in it, and Ω be any subset of $\mathcal{P}$. Let $\alpha : E \to [0, +\infty)$ a continuous concave functional, i.e.,

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y), \quad \text{for all } \; x, y \in E \; \text{and} \; \lambda \in [0, 1].$$

For two numbers $0 < a < c$, we define

$$\mathcal{P}_c = \{x \in \mathcal{P} : \; \|x\| \leq c\},$$

$$S(\alpha, a, c) = \{x \in \mathcal{P} : \; \alpha(x) \geq a \; \text{ and } \; \|x\| \leq c\}$$

such that

$$S(\alpha, a, c) \bigcap \Omega \neq \emptyset.$$

In [10], Leggett and Williams are discussing the existence of at least one solution in $S(\alpha, a, c)$ to the nonlinear operational equation

$$Ax = x \tag{1.1}$$

where A is a given nonlinear map acting in $\mathcal{P}$.

The original Leggett-Williams theorem states the following.

Theorem 1.1. *(Leggett-Williams [10]) Suppose that* $c \geq b > a > 0$, α *is a continuous concave positive functional with* $\alpha(x) \leq \|x\|$ *for all* $x \in \mathcal{P}$ *and* $A : \mathcal{P}_c \to \mathcal{P}$ *is a completely continuous operator such that:*
(i) $\{x \in S(\alpha, a, b) : \ \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b)$,
(ii) $Ax \in \mathcal{P}_c$ *if* $x \in S(\alpha, a, c)$,
(iii) $\alpha(Ax) > a$ *for all* $x \in S(\alpha, a, c)$ *avec* $\|Ax\| > b$. *Then* A *has a fixed point in* $S(\alpha, a, c)$.

Noting that many researchers have been interested in the extension of the above theorem we cite for example [1, 9].

Our main contributions are presented in the next section. We first present a new extension of the original version of Leggett-Williams fixed point theorem for sums of k-set contraction and expansive mapping with constant $h > 1$ when $0 \leq k < h - 1$. The approach used is the generalized fixed point index i_*, developed in [4]. As application of our main results, the existence of non trivial positive solution for IVPs of a class of two-dimensional nonlinear wave equations is considered in the second part. The corresponding Green's function of Problem (2.3) is given in order to transform it into a fixed point problem. Noting that this is a new approach to study the existence of solutions for nonlinear partial differential equations.

2 Main Results

2.1 A Theoretical Result

In this work, instead of equation (1.1), we consider the nonlinear operational equation

$$Tx + Fx = x, \tag{2.1}$$

where $T : \Omega \to E$ is an expansive mapping with constant $h > 1$ and $F : \mathcal{P}_c \to E$ is a k-set contraction with $k < h - 1$. The concept of set contraction is related to that of the Kuratowski measure of noncompactness (see [6]). Using the main properties of the generalized fixed point index i_*, developed by Djebali and Mebarki in [4] for such class of operators, we will establish a generalization of Theorem 1.1 in the case of the sum of two operators.

Recall that the index i_* has the following properties:

Theorem 2.1. *([4, Theorem 2.3]).*

(a) (Normalization). If $U = \mathcal{P}_r,\ 0 \in \Omega$, *and* $Fx = z_0 \in \mathcal{B}(-T0, (h-1)r) \cap \mathcal{P}$, *for all* $x \in \overline{\mathcal{P}_r}$, *then*

$$i_*(T + F, \mathcal{P}_r \cap \Omega, \mathcal{P}) = 1.$$

(b) (Additivity). For any pair of disjoint open subsets U_1, U_2 *in* U *such that* $T + F$ *has no fixed point on* $(\overline{U} \setminus (U_1 \bigcup U_2)) \bigcap \Omega$, *we have*

$$i_*(T + F, U \bigcap \Omega, \mathcal{P}) = i_*(T + F, U_1 \bigcap \Omega, \mathcal{P}) + i_*(T + F, U_2 \bigcap \Omega, \mathcal{P}),$$

where

$$i_*(T + F, U_j \bigcap \Omega, X) := i_*(T + F|_{\overline{U_j}}, U_j \bigcap \Omega, \mathcal{P}), \quad j = 1, 2.$$

(c) (Homotopy Invariance). The fixed point index $i_*(T + H(t, .), U \bigcap \Omega, \mathcal{P})$ *does not depend on the parameter* $t \in [0, 1]$ *whenever*

(i) $H : [0, 1] \times \overline{U} \to E$ *is continuous and* $H(t, x)$ *is uniformly continuous in* t *for each* $x \in \overline{U}$,

(ii) $H([0, 1] \times \overline{U}) \subset (I - T)(\Omega)$,

(iii) $H(t, .) : \overline{U} \to E$ *is a* l*-set contraction with* $0 \le l < h - 1$ *and* k *does not dependent on* $t \in [0, 1]$,

(iv) $Tx + H(t, x) \neq x$ *for all* $t \in [0, 1]$ *and* $x \in \partial U \bigcap \Omega$.

(d) (Solvability). If $i_*(T + F, U \bigcap \Omega, \mathcal{P}) \neq 0$, *then* $T + F$ *has a fixed point in* $U \bigcap \Omega$.

The following result guarantee the existence of at lest one non trivial positive solution of equation (2.1).

Theorem 2.2. *Let* $c \ge b > a > 0$ *three real numbers and* α *be a continuous concave positive functional with* $\alpha(x) \le \|x\|$ *for all* $x \in \mathcal{P}$.
Assume that $T : \Omega \to E$ *is an expansive mapping with constant* $h > 1$, $F : \mathcal{P}_c \to E$ *is a* k*-set contraction with* $0 \le k < h - 1$, *and there exist* $z_0 \in \mathcal{B}(-T0, (h-1)c) \bigcap \mathcal{P}_c$ *such that*

$$tF(\mathcal{P}_c) + (1 - t)z_0 \subset (I - T)(\Omega), \text{ for all } t \in [0, 1]. \tag{2.2}$$

If the following conditions are satisfied:

1. $\{x \in S(\alpha, a, b) :\ \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Tx + Fx) > a$ *if* $x \in S(\alpha, a, b)$;

2. $\alpha(Tx + Fx) > a$ *and* $\alpha(Tx + z_0) \geq a$ *for all* $x \in S(\alpha, a, c) \bigcap \Omega$ *with* $\|Tx + Fx\| > b$;

3. $\|Tx + z_0\| \leq b$ *for all* $x \in S(\alpha, a, c) \bigcap \Omega$, *with* $\alpha(x) = a$,

then $T + F$ *has at lest one fixed point* $x \in S(\alpha, a, c) \bigcap \Omega$, *with* $\alpha(x) > a$.

Proof. Consider the set $U = \{x \in S(\alpha, a, c) : \alpha(x) > a\}$; U is then the interior of $S(\alpha, a, c)$ in $\mathcal{P}_c$.
Suppose that $x \in \partial U \bigcap \Omega$ is a fixe point of $T + F$; then $\alpha(x) = a$ with or $x \in S(\alpha, a, b)$, or $\|x\| > b$.
If $x \in S(\alpha, a, b)$, we get $\alpha(x) = \alpha(Tx + Fx) > a$, which is a contradiction.
If $\|x\| > b$, we get $\|Tx+Fx\| > b$ and $\alpha(x) = \alpha(Tx+Fx) > a$, leading again to a contradiction with (2).
Consequently, the fixed point index $i_* (T + F, U \bigcap \Omega, \mathcal{P})$ is well defined and satisfying the properties (a)-(d) of Theorem 2.1.

Consider the homotopic deformation $H : [0, 1] \times \mathcal{P}_c \to E$ defined by

$$H(t, x) = tFx + (1 - t)z_0.$$

The operator H is continuous and uniformly continuous in t for each x. Moreover, $H(t, .)$ is a k-set contraction for each t and the mapping $T+H(t, .)$ has no fixed point on $\partial U \bigcap \Omega$. Otherwise, there would exist some $x_0 \in \partial U \cap \Omega$ and $t_0 \in [0, 1]$ such that

$$x_0 = Tx_0 + H(t_0, x_0).$$

We have $\alpha(x_0) = a$ and we may distinguish between two cases:
(i) If $\|Tx_0 + Fx_0\| > b$, the concavity of α and the condition (2) lead

$$\begin{aligned} \alpha(x_0) &= \alpha(Tx_0 + t_0 Fx_0 + (1 - t_0)z_0) \\ &= \alpha(t_0(Tx_0 + Fx_0) + (1 - t_0)(Tx_0 + z_0)) \\ &\geq t_0\, \alpha(Tx_0 + Fx_0) + (1 - t_0)\alpha(Tx_0 + z_0) \\ &> a, \end{aligned}$$

which is a contradiction.

(ii) If $\|Tx_0 + Fx_0\| \leq b$, the condition (3) leads

$$\begin{aligned}\|x_0\| &= \|Tx_0 + t_0Fx_0 + (1-t_0)z_0\| \\ &= \|(t_0(Tx_0 + Fx_0) + (1-t_0)(Tx_0 + z_0)\| \\ &\leq t_0\|Tx_0 + Fx_0\| + (1-t_0)\|Tx_0 + z_0\| \\ &\leq b.\end{aligned}$$

Thus, $x_0 \in S(\alpha, a, b)$ and by the condition (1), we get $\alpha(Tx_0 + Fx_0) > a$, which imply that $\alpha(x_0) > a$ and again we come to a contradiction with $\alpha(x_0) = a$.

By properties (a) and (d) of the index i_* in Theorem 2.1, we deduce that

$$i_*\left(T + F, U \bigcap \Omega, \mathcal{P}\right) = i_*\left(T + z_0, U \bigcap \Omega, \mathcal{P}\right) = 1.$$

As a consequence, $T + F$ has a fixed point in $U \bigcap \Omega$.

□

2.2 An Application to IVPs for a Class Two-Dimensional Nonlinear Wave Equations

In this section we will investigate the following IVP

$$u_{tt} - u_{xx} - u_{yy} = f(u), \quad -\infty < x, y < \infty, \quad 0 < t < \infty, \tag{2.3}$$

$$u(x, y, 0) = u_t(x, y, 0) = 0, \quad -\infty < x, y < \infty, \tag{2.4}$$

where

(H1) $f : \mathbb{R} \to (-\infty, 0]$ is a continuous function such that

$$-f(v) \leq \sum_{j=1}^{l} a_j |v|^{l_j}, \quad v \in \mathbb{R},$$

for some nonnegative constants a_j, $j \in \{1, \ldots, l\}$, and for some positive constants l_j, $j \in \{1, \ldots, l\}$, $l \in \mathbb{N}$.

Our main result is as follows.

Theorem 2.3. *Suppose* $(H1)$. *Then the IVP* (2.3), (2.4) *has at least one solution* $u \in \mathcal{C}^2(\mathbb{R}^2 \times [0, \infty))$ *such that* $u \not\equiv 0$ *on* $\mathbb{R}^2 \times [0, \infty)$.

2.2.1 Some Preliminary Results

Firstly, we will note that in [5] is shown that the function

$$G(x,y,t,\xi,\eta,\tau) = -\frac{1}{2\pi}\frac{H\left(t-\tau-\sqrt{(x-\xi)^2+(y-\eta)^2}\right)}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}},$$

$-\infty < x,y,\xi,\eta < \infty$, $0 < t,\tau < \infty$, where $H(\cdot)$ denotes the Heaviside function, is the Green function for the considered IVP (2.3), (2.4), i.e., we have

$$\tfrac{\partial^2}{\partial t^2}G(x,y,t,\xi,\eta,\tau) - \tfrac{\partial^2}{\partial x^2}G(x,y,t,\xi,\eta,\tau) - \tfrac{\partial^2}{\partial y^2}G(x,y,t,\xi,\eta,\tau)$$

$$= \delta(x-\xi)\delta(y-\eta)\delta(t-\tau),$$

$-\infty < x,y,\xi,\eta < \infty$, $0 < t,\tau < \infty$,

$$G(x,y,0,\xi,\eta,\tau) = \frac{\partial}{\partial t}G(x,y,0,\xi,\eta,\tau) = 0,$$

$-\infty < x,y,\xi,\eta < \infty$. Here $\delta(\cdot)$ is the Dirac delta function. Thus, if

$$u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x,y,t,\xi,\eta,\tau)f(u(\xi,\eta,\tau))d\tau d\eta d\xi, \qquad (2.5)$$

$-\infty < x,y < \infty$, $0 < t < \infty$, we have

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}u(x,y,t) - \frac{\partial^2}{\partial x^2}u(x,y,t) - \frac{\partial^2}{\partial y^2}u(x,y,t)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Bigg(\frac{\partial^2}{\partial t^2}G(x,y,t,\xi,\eta,\tau) - \frac{\partial^2}{\partial x^2}G(x,y,t,\xi,\eta,\tau)\\
&\quad -\frac{\partial^2}{\partial y^2}G(x,y,t,\xi,\eta,\tau)\Bigg)f(u(\xi,\eta,\tau))d\tau d\eta d\xi\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\delta(x-\xi)\delta(y-\eta)\delta(t-\tau)f(u(\xi,\eta,\tau))d\tau d\eta d\xi\\
&= f(u(x,y,t)), \quad -\infty < x,y < \infty, \quad 0 < t < \infty,
\end{aligned}$$

and

$$u(x,y,0) = \frac{\partial}{\partial t}u(x,y,0) = 0, \quad -\infty < x,y < \infty,$$

i.e., u, defined by (2.5), is a solution of the IVP (2.3), (2.4). Note that

$$G(x,y,t,\xi,\eta,\tau) \le 0, \quad -\infty < x,y,\xi,\eta < \infty, \quad 0 < t,\tau < \infty.$$

Below we suppose

(H2) a, b, c, a_1, b_1, A and ϵ are constants such that

$$c \ge b > a > 0, \quad A > 0, \quad \epsilon \in (0,1),$$

$$\left(1+\tfrac{\epsilon}{2}\right)\tfrac{c}{4} \le b,$$

$$0 < a_1 < b_1,$$

(H3) $g \in \mathcal{C}^2(\mathbb{R}^2 \times [0,\infty))$ be a positive function such that

$$\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1 \le A,$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2 \le A,$$

$$\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$dt_3dt_2dt_1dy_3dy_2dy_1dx_3 \le A,$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \le A,$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$dt_3dt_2dt_1dy_3dx_3dx_2dx_1 \le A,$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_2} g(x_3,y_3,t_3)$$

$$dt_3dt_2dy_3dy_2dy_1dx_3dx_2dx_1 \le A,$$

$$\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t g(x_3,y_3,t_3)$$

$$dt_3dy_3dy_2dy_1dx_3dx_2dx_1 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dt_2dt_1dy_3dx_3dx_2dx_1 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dt_2dy_3dy_2dy_1dx_3dx_2dx_1 \le A,$$

$$-\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)d\tau d\eta d\xi dt_3dy_3dy_2dy_1dx_3dx_2dx_1 \le A,$$

$$-\infty < x,y < \infty,\ 0 < t < \infty.$$

Let $E = \mathcal{C}^2(\mathbb{R}^2 \times [0,\infty))$ be endowed with the norm

$$\|u\| \;=\; \max\Bigg\{ \sup_{\mathbb{R}^2\times[0,\infty)} |u(x,y,t)|, \quad \sup_{\mathbb{R}^2\times[0,\infty)} |u_t(x,y,t)|$$

$$\sup_{\mathbb{R}^2\times[0,\infty)} |u_{tt}(x,y,t)|, \quad \sup_{\mathbb{R}^2\times[0,\infty)} |u_x(x,y,t)|,$$

$$\sup_{\mathbb{R}^2\times[0,\infty)} |u_{xx}(x,y,t)|, \quad \sup_{\mathbb{R}^2\times[0,\infty)} |u_y(x,y,t)|,$$
$$\left.\sup_{\mathbb{R}^2\times[0,\infty)} |u_{yy}(x,y,t)|\right\},$$

provided it exists,

$$\widetilde{\mathcal{P}} = \{u \in E : u \geq 0 \quad \text{on} \quad \mathbb{R}^2 \times [0,\infty)\},$$

$\widetilde{\widetilde{\mathcal{P}}}$ be the set of all equi-continuous families in $\widetilde{\mathcal{P}}$ (an example for an equi-continuous family is the family $\{(3+\sin(t+n))(3+\sin(x+n))(3+\sin(y+n)) : t \in [0,\infty), \quad x, y \in \mathbb{R}\}_{n\in\mathbb{N}}$),

$$\begin{aligned}
\alpha(u) &= \min_{(x,y,t)\in[a_1,b_1]^3} |u(x,y,t)|, \quad u \in E,\\
\mathcal{P} &= \Big\{u \in \widetilde{\widetilde{\mathcal{P}}} \quad \text{on} \quad \mathbb{R}^2 \times [0,\infty),\\
&\qquad \|u\| \leq \frac{c}{4} \quad \text{if} \quad \alpha(u) \leq a,\\
&\qquad \alpha(u) > a \quad \text{if} \quad c \geq \|u\| > b\Big\},\\
\mathcal{P}_c &= \{u \in \mathcal{P} : \|u\| \leq c\},\\
S(\alpha, a, c) &= \{u \in \mathcal{P} : \alpha(u) \geq a, \quad \|u\| \leq c\},\\
\Omega &= \left\{u \in \mathcal{P} : \|u\| \leq c + \frac{3A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right)}{2+\epsilon}\right\}.
\end{aligned}$$

Let $R = 2A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right)$. For $u \in E$, define the operators

$$\begin{aligned}
Q_1u(x,y,t) &= -\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3)\\
&\quad dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1\\
&\quad +\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)\\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)f(u(\xi,\eta,\tau))\\
&\quad d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1,\\
Qu(x,y,t) &= R + Q_1u(x,y,t),
\end{aligned}$$

$-\infty < x, y < \infty$, $0 < t < \infty$.

Lemma 2.4. *Suppose $(H1)-(H3)$. For $u \in E$ and $\|u\| \le c$, we have*

$$\|Q_1u\| \le A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right) \quad \textit{and} \quad \|Qu\| \le 3A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right).$$

For $u \in \mathcal{P}_c$, we have $Qu \ge 0$ on $\mathbb{R}^2 \times [0,\infty)$.

Proof. Let $u \in E$ and $\|u\| \leq c$. Then

$$\begin{aligned}
|Q_1 u(x,y,t)| &= \bigg| -\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^{t}\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3) \\
&\quad dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1 \\
&\quad + \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^{t}\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3) \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)f(u(\xi,\eta,\tau)) \\
&\quad d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1 \bigg| \\
&\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^{t}\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)| \\
&\quad dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1 \\
&\quad + \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^{t}\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3) \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)f(u(\xi,\eta,\tau)) \\
&\quad d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1 \\
&\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^{t}\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)| \\
&\quad dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2dx_1
\end{aligned}$$

$$-\sum_{j=1}^{l} a_j \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)|u(\xi,\eta,\tau)|^{l_j}$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$\leq \quad A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$, and

$$\left|\frac{\partial}{\partial t}Q_1 u(x,y,t)\right| \;=\; \Big| -\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3)$$

$$dt_3 dt_2 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dt_2 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1 \Big|$$

$$\leq \quad \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3 dt_2 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dt_2 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3dt_2dy_3dy_2dy_1dx_3dx_2dx_1$$

$$-\sum_{j=1}^{l} a_j \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)|u(\xi,\eta,\tau)|^{l_j}$$

$$d\tau d\eta d\xi dt_3dt_2dy_3dy_2dy_1dx_3dx_2dx_1$$

$$\leq A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$, and

$$\left|\frac{\partial^2}{\partial t^2} Q_1 u(x,y,t)\right| = \Bigg| -\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t} g(x_3,y_3,t_3)u(x_3,y_3,t_3)$$

$$dt_3dy_3dy_2dy_1dx_3dx_2dx_1$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3dy_3dy_2dy_1dx_3dx_2dx_1\Bigg|$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3dy_3dy_2dy_1dx_3dx_2dx_1$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$-\sum_{j=1}^{l} a_j \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)|u(\xi,\eta,\tau)|^{l_j}$$

$$d\tau d\eta d\xi dt_3 dy_3 dy_2 dy_1 dx_3 dx_2 dx_1$$

$$\leq A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$, and

$$\begin{aligned}
\left|\frac{\partial}{\partial x} Q_1 u(x,y,t)\right| &= \Big| -\int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3) \\
&\quad dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3 dx_2 \\
&\quad + \int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3) \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau)) \\
&\quad d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3 dx_2 \Big| \\
&\le \int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|
\end{aligned}$$

$$dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2$$

$$-\sum_{j=1}^{l} a_j\int_{-\infty}^{x}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)|u(\xi,\eta,\tau)|^{l_j}$$

$$d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dy_1dx_3dx_2$$

$$\leq A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$, and

$$\left|\frac{\partial^2}{\partial x^2}Q_1u(x,y,t)\right| = \Big| -\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3)$$

$$dt_3dt_2dt_1dy_3dy_2dy_1dx_3$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_0^t\int_0^{t_1}\int_0^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3 \Bigg|$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3$$

$$-\sum_{j=1}^{l} a_j \int_{-\infty}^{x}\int_{-\infty}^{y}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) |u(\xi,\eta,\tau)|^{l_j}$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dy_2 dy_1 dx_3$$

$$\leq A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$, and

$$\begin{aligned}
\left|\frac{\partial}{\partial y} Q_1 u(x,y,t)\right| &= \Bigg| -\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3) \\
&\quad dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \\
&\quad + \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3) \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau)) \\
&\quad d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \Bigg| \\
&\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)| \\
&\quad dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \\
&\quad + \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3) \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau)) \\
&\quad d\tau d\eta d\xi dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \\
&\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)| \\
&\quad dt_3dt_2dt_1dy_3dy_2dx_3dx_2dx_1 \\
&\quad - \sum_{j=1}^{l} a_j \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{-\infty}^{y_2}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3) \\
&\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) |u(\xi,\eta,\tau)|^{l_j}
\end{aligned}$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dy_2 dx_3 dx_2 dx_1$$

$$\leq \quad A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$, and

$$\left|\frac{\partial^2}{\partial y^2} Q_1 u(x,y,t)\right| = \Big| -\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)u(x_3,y_3,t_3)$$

$$dt_3 dt_2 dt_1 dy_3 dx_3 dx_2 dx_1$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dx_3 dx_2 dx_1 \Big|$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3 dt_2 dt_1 dy_3 dx_3 dx_2 dx_1$$

$$+\int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau) f(u(\xi,\eta,\tau))$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dx_3 dx_2 dx_1$$

$$\leq \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)|u(x_3,y_3,t_3)|$$

$$dt_3 dt_2 dt_1 dy_3 dx_3 dx_2 dx_1$$

$$-\sum_{j=1}^{l} a_j \int_{-\infty}^{x}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{y}\int_{0}^{t}\int_{0}^{t_1}\int_{0}^{t_2} g(x_3,y_3,t_3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x_3,y_3,t_3,\xi,\eta,\tau)|u(\xi,\eta,\tau)|^{l_j}$$

$$d\tau d\eta d\xi dt_3 dt_2 dt_1 dy_3 dx_3 dx_2 dx_1$$

$$\leq \quad A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right),$$

$-\infty < x, y < \infty$, $0 < t < \infty$. Let $u \in \mathcal{P}_c$. Then

$$\begin{aligned} Qu(x,y,t) &= R + Q_1 u(x,y,t) \\ &\geq R - |Q_1 u(x,y,t)| \\ &\geq A\left(c+\sum_{j=1}^{l} a_j c^{l_j}\right) > 0, \end{aligned}$$

$-\infty < x, y < \infty$, $0 < t < \infty$. This completes the proof. □

For $u \in E$, define the operators

$$\begin{aligned} Tu &= -(1+\epsilon)u, \\ Fu &= (2+\epsilon)u + Qu. \end{aligned}$$

Lemma 2.5. *Suppose* $(H1)-(H3)$. *If* $u \in E$ *is a fixed point of the operator* $T+F$, *then it is a solution of the IVP* (2.3), (2.4).

Proof. We have

$$\begin{aligned} u &= Tu + Fu \\ &= -(1+\epsilon)u + (2+\epsilon)u + Qu \\ &= u + Qu, \end{aligned}$$

whereupon $Qu = 0$. We differentiate trice in x, trice in y and trice in t, the last equation and we get

$$\begin{aligned}
0 &= -g(x,y,t)u(x,y,t) \\
&\quad + g(x,y,t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x,y,t,\xi,\eta,\tau)f(u(\xi,\eta,\tau))d\tau d\eta d\xi,
\end{aligned}$$

$-\infty < x, y < \infty$, $0 < t < \infty$. Hence, we arrive at (2.5) and therefore u is a solution of the IVP (2.3), (2.4). This completes the proof. □

2.2.2 Proof of Theorem 2.3

1. For $u \in \mathcal{P}$, we have
$$\|Tu\| = (1+\epsilon)\|u\|,$$
i.e., $T : \mathcal{P} \to E$ is an expansive operator with $h = 1 + \epsilon$.

2. For $u \in \mathcal{P}_c$, we get
$$\begin{aligned}
\|Fu\| &\leq (2+\epsilon)\|u\| + \|Qu\| \\
&\leq (2+\epsilon)c + 3A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right).
\end{aligned}$$
Therefore $F : \mathcal{P}_c \to E$ is uniformly bounded. Since $F : \mathcal{P}_c \to E$ is continuous, we obtain that $F(\mathcal{P}_c)$ is equi-continuous and then $F : \mathcal{P}_c \to E$ is relatively compact. Consequently $F : \mathcal{P}_c \to E$ is 0-set contraction.

3. Let $\lambda \in [0,1]$, $z_0 \in B(0,\epsilon c) \bigcap \mathcal{P}_c$ and $u \in \mathcal{P}_c$ be arbitrarily chosen. Take
$$z = \frac{\lambda(2+\epsilon)u + \lambda Qu + (1-\lambda)z_0}{2+\epsilon}.$$
We have
$$z = \frac{\lambda Fu + (1-\lambda)z_0}{2+\epsilon}$$
or
$$\lambda Fu + (1-\lambda)z_0 = (I-T)z.$$

Next, $z \in \mathcal{P}$ and

$$\begin{aligned}
\|z\| &\leq \frac{\lambda(2+\epsilon)\|u\| + \lambda\|Qu\| + (1-\lambda)\|z_0\|}{2+\epsilon} \\
&\leq \frac{\lambda(2+\epsilon)c + 3A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right) + (1-\lambda)\epsilon c}{2+\epsilon} \\
&\leq \frac{\lambda(2+\epsilon)c + 3A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right) + (1-\lambda)(2+\epsilon)c}{2+\epsilon} \\
&= \frac{(2+\epsilon)c + 3A\left(c + \sum_{j=1}^{l} a_j c^{l_j}\right)}{2+\epsilon}.
\end{aligned}$$

Thus, $z \in \Omega$ and

$$\lambda F(\mathcal{P}_c) + (1-\lambda)z_0 \subset (I-T)(\Omega), \quad \lambda \in [0,1].$$

4. Note that

$$b \in \{u \in S(\alpha, a, b) : \alpha(u) > a\}, \quad \text{i.e.} \quad \{u \in S(\alpha, a, b) : \alpha(u) > a\} \neq \emptyset.$$

Moreover, if $u \in S(\alpha, a, b)$, then $\alpha(u) \geq a$ and

$$\begin{aligned}
\alpha(Tu + Fu) &= \alpha(u + Qu) \\
&> \alpha(u) \\
&\geq a.
\end{aligned}$$

5. Let $u \in S(\alpha, a, c)$ and $\|Tu + Fu\| > b$. Then, by the definition of $\mathcal{P}$, we have $\alpha(Tu + Fu) > a$. Let $z_0 = \frac{\epsilon}{2}u$. Then

$$\begin{aligned}
\|z_0\| &= \frac{\epsilon}{2}\|u\| \\
&\leq \frac{\epsilon}{2}c
\end{aligned}$$

and $z_0 \in B(0, \epsilon c) \bigcap \mathcal{P}_c$. Hence,

$$\begin{aligned}
\alpha(Tu + z_0) &= \alpha\left(-(1+\epsilon)u + \frac{\epsilon}{2}u\right) \\
&= \alpha\left(-\left(1+\frac{\epsilon}{2}\right)u\right) \\
&= \left(1+\frac{\epsilon}{2}\right)\alpha(u) \\
&\geq \left(1+\frac{\epsilon}{2}\right)a \\
&\geq a.
\end{aligned}$$

6. Let $u \in S(\alpha, a, c)$ and $\alpha(u) = a$. Then $\|u\| \leq \frac{c}{4}$. Take $z_0 = \frac{\epsilon}{2}u$. We have $z_0 \in B(0, \epsilon c) \bigcap \mathcal{P}_c$ and

$$\begin{aligned}
\|Tu + z_0\| &= \left\|-(1+\epsilon)u + \frac{\epsilon}{2}u\right\| \\
&= \left(1+\frac{\epsilon}{2}\right)\|u\| \\
&\leq \left(1+\frac{\epsilon}{2}\right)\frac{c}{4} \\
&\leq b.
\end{aligned}$$

By 1, 2, 3, 4, 5, 6 and Theorem 2.2, we conclude that the operator $T + F$ has a fixed point $u \in S(\alpha, a, c) \bigcap \Omega$ with $\alpha(u) > a$. This completes the proof.

2.3 Example

Consider the IVP

$$u_{tt} - u_{xx} - u_{yy} = -u^2 - 3\frac{u^4}{1+u^8} - 4u^{10}, \quad -\infty < x, y < \infty, \quad 0 < t < \infty, \tag{2.6}$$

$$u(x, y, 0) = u_t(x, y, 0) = 0, \quad -\infty < x, y < \infty. \tag{2.7}$$

Here

$$f(u) = -u^2 - 3\frac{u^4}{1+u^8} - 4u^{10}, \quad u \in \mathbb{R}.$$

Then

$$-f(u) \leq u^2 + 3u^4 + 4u^{10}, \quad u \in \mathbb{R}.$$

For $\epsilon = \frac{1}{2}$, the conditions $(H2)$ take the form

$$c \geq b > a > 0, \quad A > 0,$$

$$\frac{5}{2}c + 3A\left(c + c^2 + 3c^4 + 4c^{10}\right) < \frac{1}{8},$$

$$\frac{5c}{16} \leq b, \quad 0 < a_1 < b_1.$$

Then, we take

$$a_1 = \frac{1}{4}, \quad b_1 = \frac{1}{2}, \quad a = \frac{1}{128}, \quad b = \frac{1}{64}, \quad c = \frac{1}{32}, \quad A = \frac{1}{10000000000}$$

and we get that $(H2)$ holds. Observe that

$$\begin{aligned}
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} G(x,y,t,\xi,\eta,\tau)n d\tau d\eta d\xi \\
&= \frac{1}{2\pi}\int_0^{\infty}\int_{\sqrt{(x-\xi)^2+(y-\eta)^2}\leq t-\tau} \frac{1}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}} d\xi d\eta d\tau \\
&= \int_0^t \int_0^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho d\tau \\
&= \int_0^t (t-\tau) d\tau \\
&= \frac{t^2}{2}.
\end{aligned}$$

Now we will construct a function g so that $(H3)$ holds. Let

$$h(t) = \log\frac{1+t^4\sqrt{2}+t^8}{1-t^4\sqrt{2}+t^8}, \quad l(t) = \arctan\frac{t^4\sqrt{2}}{1-t^8}, \quad t \geq 0.$$

We have

$$\begin{aligned}
h'(t) &= \frac{1}{(1+t^4\sqrt{2}+t^8)(1-t^4\sqrt{2}+t^8)}\Big((4\sqrt{2}t^3+8t^7)(1-t^4\sqrt{2}+t^8) \\
&\quad -(1+t^4\sqrt{2}+t^8)(-4\sqrt{2}t^3+8t^7)\Big) \\
&= \frac{1}{(1+t^4\sqrt{2}+t^8)(1-t^4\sqrt{2}+t^8)}\Big(4\sqrt{2}t^3-8t^7+4\sqrt{2}t^{11}+8t^7 \\
&\quad -8\sqrt{2}t^{11}+8t^{15}+4\sqrt{2}t^3-8t^7+8t^7-8\sqrt{2}t^{11}+4\sqrt{2}t^{11}-8t^{15}\Big) \\
&= -\frac{8\sqrt{2}t^3(t^8-1)}{(1+t^4\sqrt{2}+t^8)(1-t^4\sqrt{2}+t^8)}, \quad t\geq 0.
\end{aligned}$$

Thus,

$$\sup_{t\geq 0} h(t) = h(1) = \log\frac{2+\sqrt[4]{2}}{2-\sqrt[4]{2}},$$

h is an increasing function on $[0,1]$ and it is a decreasing function on $[1,\infty)$. Next,

$$\begin{aligned}
l'(t) &= \frac{1}{1+\frac{2t^8}{(1-t^8)^2}}\frac{4\sqrt{2}t^3(1-t^8)+8t^7t^4\sqrt{2}}{(1-t^8)^2} \\
&= \frac{4\sqrt{2}t^3-4\sqrt{2}t^{11}+8\sqrt{2}t^{11}}{1+t^{16}} \\
&= \frac{4\sqrt{2}t^3(1+t^8)}{1+t^{16}}, \quad t\geq 0.
\end{aligned}$$

Therefore l is an increasing function on $[0,\infty)$. Note that

$$\begin{aligned}\lim_{t\to\infty} th(t) &= \lim_{t\to\infty}\frac{h(t)}{\frac{1}{t}}\\ &= \lim_{t\to\infty}\frac{h'(t)}{-\frac{1}{t^2}}\\ &= \lim_{t\to\infty}\frac{8\sqrt{2}t^5(t^8-1)}{(t^8+\sqrt{2}t^4+1)(t^8-\sqrt{2}t^4+1)}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} t^2h(t) &= \lim_{t\to\infty}\frac{h(t)}{\frac{1}{t^2}}\\ &= \lim_{t\to\infty}\frac{h'(t)}{-\frac{2}{t^3}}\\ &= \lim_{t\to\infty}\frac{4\sqrt{2}t^6(t^8-1)}{(t^8+\sqrt{2}t^4+1)(t^8-\sqrt{2}t^4+1)}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty} tl(t) &= \lim_{t\to\infty}\frac{l(t)}{\frac{1}{t}}\\ &= \lim_{t\to\infty}\frac{l'(t)}{-\frac{1}{t^2}}\\ &= -\lim_{t\to\infty}\frac{4\sqrt{2}t^5(1+t^8)}{1+t^{16}}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to\infty} t^2 l(t) &= \lim_{t\to\infty} \frac{l(t)}{\frac{1}{t^2}} \\
&= \lim_{t\to\infty} \frac{l'(t)}{-\frac{2}{t^3}} \\
&= -2\sqrt{2}\lim_{t\to\infty} \frac{t^6(t^8+1)}{1+t^{16}} \\
&= 0.
\end{aligned}$$

Consequently, there exists a constant $B > 1$ such that

$$\frac{1}{16\sqrt{2}}\log\frac{1+t^4\sqrt{2}+t^8}{1-t^4\sqrt{2}+t^8}+\frac{1}{8\sqrt{2}}\arctan\frac{t^4\sqrt{2}}{1-t^8}\le B,$$

$$t\left(\frac{1}{16\sqrt{2}}\log\frac{1+t^4\sqrt{2}+t^8}{1-t^4\sqrt{2}+t^8}+\frac{1}{8\sqrt{2}}\arctan\frac{t^4\sqrt{2}}{1-t^8}\right)\le B,$$

$$t^2\left(\frac{1}{16\sqrt{2}}\log\frac{1+t^4\sqrt{2}+t^8}{1-t^4\sqrt{2}+t^8}+\frac{1}{8\sqrt{2}}\arctan\frac{t^4\sqrt{2}}{1-t^8}\right)\le B, \quad t\ge 0.$$

Note that, by [11](pp. 707, Integral 79), we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2},$$

and, by [11](pp. 704, Integral 30), we have

$$\int\frac{z^2}{(1+z^2)^3}dz=-\frac{z}{4(1+z^2)^2}+\frac{z}{8(1+z^2)}+\frac{1}{8}\arctan z.$$

Let

$$f_1(t)=\frac{t^3}{B(1+t^{16})}, \quad t\ge 0,$$

$$f_2(x)=\frac{|x|^3}{2B(3+\pi)(1+x^2)^4(1+x^{16})(1+x^6)}, \quad -\infty<x<\infty.$$

Then, we can take

$$g(x,y,t)=\frac{1}{100000000000000000000}\frac{1}{t^2+1}f_1(t)f_2(x)f_2(y),$$

$-\infty < x, y < \infty, \quad 0 < t < \infty$. To check that $(H3)$ holds, we have a need to estimate the following integrals

$$
\begin{aligned}
I_1(t) &= \int_0^t f_1(s)ds, \\
I_2(t) &= \int_0^t \int_0^{t_1} f_1(s)dsdt_1 = \int_0^t (t-s)f_1(s)ds, \\
I_3(t) &= \int_0^t \int_0^{t_1} \int_0^{t_2} f_1(s)dsdt_2dt_1 = \frac{1}{2}\int_0^t (t-s)^2 f_1(s)ds, \quad t \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
J_1(x) &= \int_{-\infty}^x f_2(s)ds, \\
J_2(x) &= \int_{-\infty}^x \int_{-\infty}^{x_1} f_2(s)dsdx_1 = \int_{-\infty}^x (x-s)f_2(s)ds, \\
J_3(x) &= \int_{-\infty}^x \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_2(s)dsdx_2dx_1 = \frac{1}{2}\int_{-\infty}^x (x-s)^2 f_2(s)ds, \quad -\infty < x < \infty.
\end{aligned}
$$

1. Firstly, we will estimate $I_1(t)$, $I_2(t)$, $I_3(t)$ for $t \geq 0$. We have

$$
\begin{aligned}
I_1(t) &= \frac{1}{4B}\int_0^t \frac{ds^4}{1+s^{16}} \\
&= \frac{1}{4B}\int_0^{t^4} \frac{dz}{1+z^4} \\
&= \frac{1}{4B}\left(\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}\right)\Bigg|_{z=0}^{z=t^4} \\
&= \frac{1}{4B}\left(\frac{1}{4\sqrt{2}}\log\frac{1+t\sqrt{2}+t^2}{1-t\sqrt{2}+t^2} + \frac{1}{2\sqrt{2}}\arctan\frac{t\sqrt{2}}{1-t^2}\right) \\
&\leq 1, \quad t \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(t) &= \int_0^t (t-s) f_1(s) ds \\
&\leq t I_1(t) \\
&= \frac{t}{4B}\left(\frac{1}{4\sqrt{2}}\log\frac{1+t\sqrt{2}+t^2}{1-t\sqrt{2}+t^2}+\frac{1}{2\sqrt{2}}\arctan\frac{t\sqrt{2}}{1-t^2}\right) \\
&\leq 1, \quad t \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(t) &= \frac{1}{2}\int_0^t (t-s)^2 f_1(s) ds \\
&\leq \frac{t^2}{2} I_1(t) \\
&= \frac{t^2}{8B}\left(\frac{1}{4\sqrt{2}}\log\frac{1+t\sqrt{2}+t^2}{1-t\sqrt{2}+t^2}+\frac{1}{2\sqrt{2}}\arctan\frac{t\sqrt{2}}{1-t^2}\right) \\
&\leq 1, \quad t \geq 0.
\end{aligned}
$$

2. Now, we will estimate $J_1(x)$, $J_2(x)$ and $J_3(x)$ for $-\infty < x < \infty$.

   (a) Let $x \leq 0$. Note that, for $s \in (-\infty, x]$, we have

$$x^2 \leq s^2, \quad x \leq -s, \quad -sx \leq s^2,$$

   and

$$(x-s)^2 = x^2 - 2xs + s^2 \leq 4s^2, \quad x - s \leq -2s.$$

Then

$$
\begin{aligned}
J_1(x) &= \frac{1}{2B(3+\pi)} \int_{-\infty}^{x} \frac{|s|^3}{(1+s^2)^4(1+s^{16})(1+s^6)} ds \\
&\le \frac{1}{2B(3+\pi)} \int_{-\infty}^{x} \frac{ds}{1+s^2} \\
&= \frac{1}{2B(3+\pi)} \arctan s \Big|_{s=-\infty}^{s=x} \\
&\le \frac{\pi}{2B(3+\pi)} \\
&\le 1, \quad -\infty < x < \infty,
\end{aligned}
$$

$$
\begin{aligned}
J_2(x) &= \frac{1}{2B(3+\pi)} \int_{-\infty}^{x} \frac{(x-s)|s|^3}{(1+s^2)^4(1+s^{16})(1+s^6)} ds \\
&\le \frac{1}{2B(3+\pi)} \int_{-\infty}^{x} \frac{x-s}{(1+s^2)^3} ds \\
&\le -\frac{1}{B(3+\pi)} \int_{-\infty}^{x} \frac{s}{(1+s^2)^3} ds \\
&= -\frac{1}{2B(3+\pi)} \int_{-\infty}^{x} \frac{d(1+s^2)}{(1+s^2)^3} \\
&= \frac{1}{4B(3+\pi)(1+s^2)^2} \Big|_{s=-\infty}^{s=x} \\
&= \frac{1}{4B(3+\pi)(1+x^2)^2} \\
&\le 1, \quad -\infty < x < \infty,
\end{aligned}
$$

$$
\begin{aligned}
J_3(x) &= \frac{1}{4B(3+\pi)}\int_{-\infty}^{x}\frac{(x-s)^2|s|^3}{(1+s^2)^4(1+s^{16})(1+s^6)}ds\\
&\leq \frac{1}{B(3+\pi)}\int_{-\infty}^{x}\frac{s^2}{(1+s^2)^3}ds\\
&= \frac{1}{B(3+\pi)}\left(-\frac{s}{4(1+s^2)^2}+\frac{s}{8(1+s^2)}+\frac{1}{8}\arctan s\right)\Bigg|_{s=-\infty}^{s=x}\\
&= \frac{1}{B(3+\pi)}\left(-\frac{x}{4(1+x^2)^2}+\frac{x}{8(1+x^2)}+\frac{1}{8}\arctan x+\frac{\pi}{16}\right)\\
&\leq \frac{1}{B(3+\pi)}\left(2+\frac{\pi}{8}\right)\\
&\leq \frac{1}{2B}\\
&\leq 1, \quad -\infty < x < \infty.
\end{aligned}
$$

(b) Let $x \geq 0$. Then, using the previous case, f_2 is an even function and the computations for I_1, I_2 and I_3, we find

$$
\begin{aligned}
J_1(x) &= \int_{-\infty}^{x} f_2(s)ds\\
&= \int_{-\infty}^{-x} f_2(s)ds + \int_{-x}^{x} f_2(s)ds\\
&= J_1(-x) + 2\int_{0}^{x} f_2(s)ds\\
&\leq \frac{\pi}{2B(3+\pi)} + \frac{1}{B(3+\pi)}\int_0^x \frac{s^3}{(1+s^2)^3(1+s^{16})(1+s^6)(1+s^2)}ds\\
&\leq \frac{\pi}{2B(3+\pi)} + \frac{1}{B(3+\pi)}\int_0^x \frac{s^3}{1+s^{16}}ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi}{2B(3+\pi)} + \frac{1}{3+\pi} I_1(x) \\
&\leq \frac{\pi}{2B(3+\pi)} + \frac{1}{3+\pi} \\
&\leq \frac{1}{2} + \frac{1}{2} \\
&= 1, \quad -\infty < x < \infty.
\end{aligned}
$$

Now, using that $sf_2(s)$ is an odd function on $(-\infty, \infty)$, the computations for J_2 in the previous case and using that $x - s \leq -2s$ for $s \in (-\infty, -x)$, we find

$$
\begin{aligned}
J_2(x) &= \int_{-\infty}^{x} (x-s) f_2(s) ds \\
&= \int_{-\infty}^{-x} (x-s) f_2(s) ds + \int_{-x}^{x} (x-s) f_2(s) ds \\
&= \int_{-\infty}^{-x} (x-s) f_2(s) ds + x \int_{-x}^{x} f_2(s) ds \\
&= \int_{-\infty}^{-x} (x-s) f_2(s) ds + 2x \int_{0}^{x} f_2(s) ds \\
&= \frac{x}{B(3+\pi)} \int_0^x \frac{s^3}{(1+s^2)^4(1+s^{16})(1+s^6)} ds - 2 \int_{-\infty}^{-x} s f_2(s) ds \\
&\leq \frac{x}{B(3+\pi)} \int_0^x \frac{s^3}{1+s^{16}} ds - \frac{1}{B(3+\pi)} \int_{-\infty}^{-x} \frac{s}{(1+s^2)^3} ds \\
&= \frac{x}{3+\pi} I_1(x) + \frac{1}{2B(3+\pi)(1+x^2)^2} \\
&\leq \frac{1}{3+\pi} + \frac{1}{2} \\
&\leq 1, \quad -\infty < x < \infty,
\end{aligned}
$$

and since $(x-s)^2 \le 4s^2$, $s \in (-\infty, -x)$,

$$
\begin{aligned}
J_3(x) &= \frac{1}{2}\int_{-\infty}^{x}(x-s)^2 f_2(s)ds \\
&= \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + \frac{1}{2}\int_{-x}^{x}(x^2-2xs+s^2)f_2(s)ds \\
&= \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + \frac{1}{2}\int_{-x}^{x}(x^2+s^2)f_2(s)ds \\
&= \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + \int_{0}^{x}(x^2+s^2)f_2(s)ds \\
&= \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + x^2\frac{1}{2B(3+\pi)}\int_0^x \frac{s^3}{(1+s^2)^4(1+s^{16})(1+s^6)}ds \\
&\quad + \frac{1}{2B(3+\pi)}\int_0^x \frac{s^5}{(1+s^2)^4(1+s^{16})(1+s^6)}ds \\
&\le \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + \frac{x^2}{2B(3+\pi)}\int_0^x \frac{s^3}{1+s^{16}}ds \\
&\quad + \frac{1}{2B(3+\pi)}\int_0^x \frac{s^3}{1+s^{16}}ds \\
&= \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + \frac{1}{2(3+\pi)}x^2 I_1(x) + \frac{1}{2(3+\pi)}I_1(x) \\
&\le \frac{1}{2}\int_{-\infty}^{-x}(x-s)^2 f_2(s)ds + \frac{1}{3+\pi} \\
&\le 2\int_{-\infty}^{-x} s^2 f_2(s)ds + \frac{1}{3+\pi} \\
&= \frac{1}{B(3+\pi)}\int_{-\infty}^{-x}\frac{s^2}{(1+s^2)^3}ds + \frac{1}{3+\pi} \\
&\le \frac{1}{2B} + \frac{1}{2} \\
&\le 1, \quad -\infty < x < \infty.
\end{aligned}
$$

Consequently $(H3)$ holds. By Theorem 2.3, it follows that the IVP (2.6), (2.7) has at least one solution $u \in \mathcal{C}^2(\mathbb{R}^2 \times [0, \infty))$ such that $u \not\equiv 0$ on $\mathbb{R}^2 \times [0, \infty)$.

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References

[1] D. R. Anderson, R. I. Avery and J. Henderson, Functional Expansion-Compression Fixed Point Theorem of Leggett-Williams Type, *Electron. J. Differential Equations,* Vol. 2010 (2010), No. 63, 1-9.

[2] L. Benzenati and K. Mebarki, Multiple positive fixed points for the sum of expansive mappings and k-set contractions, *Math. Meth. Appl. Sci.* Vol.42, no.13 (1919), 4412–4426.

[3] L. Benzenati, S.G. Georgiev and K. Mebarki, Existence of positive solutions for some kinds of BVPs in Banach spaces, *Accepted in Studia Universitatis Babeş-Bolyai Mathematica.*

[4] S. Djebali and K. Mebarki, Fixed point index for expansive perturbation of k-set contraction mappings, *Top. Meth. Nonli. Anal.,* Vol 54, No 2 (2019), 613–640. Doi: 10.12775/TMNA.2019.055.

[5] D. Duffy, Green's function with Applications, *CRC Press,* 2001.

[6] D. Guo, Y. I. Cho and J. Zhu, Partial Ordering Methods in Nonlinear Problems, *Shangdon Science and Technology Publishing Press, Shangdon,* 1985.

[7] S.G. Georgiev and K. Mebarki, Existence of positive solutions for a class ODEs, FDEs and PDEs via fixed point index theory for the sum of operators, *Communications on Applied Nonlinear Analysis,* Vol 26, No. 4 (2019), 16–40.

[8] S.G. Georgiev and K. Mebarki, On fixed point index theory for the sum of operators and applications in a class ODEs and PDEs, Submitted.

[9] K. G. Mavridis, Two Modifications of The Leggett-Williams Fixed Point Theorem and Their Applications, *Electron. J. Differential Equations,* Vol. 2010(2010), No.53, 1-11.

[10] R.W. Leggett and L.R. Williams, Multiple positive fixed-points of nonlinear operators on orderd Banach spaces, *Indiana University Mathematics Journal,* Vol. 28, 1979, 673–688.

[11] A. Polyanin and A. Manzhirov, Hoandbook of integral equations, *CRC Press,* 1998.