

SIBERIAN ELECTRONIC
MATHEMATICAL REPORTS

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Volume 17, pages 1–14 (2020)

UDC

512.7

DOI [10.26907/2541-7713.2020.17.14](https://doi.org/10.26907/2541-7713.2020.17.14)

MSC 14C25,

14F25, 14J30

**ON ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY
OF A COMPACTIFICATION OF THE NÉRON MODEL
WITH MULTIPLICATIONS FROM AN IMAGINARY
QUADRATIC FIELD**

O.V.MAKAROVA

ABSTRACT. It is proved that the Grothendieck standard conjecture of Lefschetz type holds for rational cohomology of degrees 2, 3 of a Künnemann compactification of the Néron minimal model of an absolutely simple principally polarized Abelian variety of non-exceptional dimension over the field of rational functions of a smooth projective curve provided that the Hodge group of the generic geometric fibre of the Néron minimal model is semi-simple and the ring of endomorphisms of the generic geometric fibre is an order of an imaginary quadratic field.

O.V.MAKAROVA, ON ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY OF A COMPACTIFICATION OF THE NÉRON MODEL WITH MULTIPLICATIONS FROM AN IMAGINARY QUADRATIC FIELD.

© 2020 MAKAROVA O.V.

Supported by Russian foundation for basic research (grant 18-01-00143).

Received September, 22, 2020, published ??????, ??, 2020.

Keywords: Abelian variety, Néron minimal model, Künnemann compactification, Grothendieck standard conjecture of Lefschetz type.

INTRODUCTION

Let H be an ample divisor on a smooth complex projective d -dimensional variety X . Then, for any natural number $i \leq d$, the map

$$L^{d-i} : H^i(X, \mathbb{Q}) \xrightarrow{\sim \text{cl}_X(H)^{\sim d-i}} H^{2d-i}(X, \mathbb{Q})$$

is an isomorphism by the strong Lefschetz theorem. The Grothendieck standard conjecture $B(X)$ of Lefschetz type [1] asserts that there exists an algebraic \mathbb{Q} -cycle Z on the Cartesian product $X \times X$ which determines the inverse *algebraic* isomorphism $H^{2d-i}(X, \mathbb{Q}) \xrightarrow{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \sim \text{cl}_{X \times X}(Z))} H^i(X, \mathbb{Q})$.

A reader can find examples of varieties satisfying the standard conjecture in [1] - [17].

Let $\mathcal{M} \rightarrow C$ be the Néron minimal model of the Abelian variety \mathcal{M}_η over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C . Suppose that at any place $s \in C$ the reduction of the Abelian variety \mathcal{M}_η is semi-stable in Grothendieck's sense. In this case the connected component \mathcal{M}_s^0 of the neutral element of the algebraic group \mathcal{M}_s is an extension of an Abelian variety by a linear torus whose dimension r_s is called *the toric (reductive) rank* at the place s [18, Section 2.1.12].

Let R be a Dedekind domain with the fraction field K and let A_η be an Abelian variety over $\eta = \text{Spec } K$ such that all reductions are semi-stable in Grothendieck's sense. As it was shown by Künnemann [19, Section 5.8], in this case there exists a finite extension K' of the field K such that the Abelian variety $A_\eta \otimes_K K'$ has (not necessarily unique) a flat projective regular model P' over the integral closure R' of the ring R in the field K' ; this model P' has strict semi-stable reductions over each localization of the ring R' (in particular, every special fibre P'_s is a union of smooth divisors of multiplicity 1 with normal crossings [20, Section 1.9]), and the scheme P' contains the Néron minimal model \mathcal{A}' of the variety $A_\eta \otimes_K K'$ in the case when all residue fields of the scheme $\text{Spec } R'$ are perfect [20, Section 4.4, Theorem 4.6].

After the base change determined by an appropriate ramified covering $\tilde{C} \rightarrow C$, we may assume by the results of Künnemann cited above that, for the Néron minimal model $\mathcal{M} \rightarrow C$, there exists a smooth

compactification X of the variety \mathcal{M} which is flat and projective over the curve C such that the following conditions hold:

(i) the model X/C has strictly semi-stable reductions (in particular, all fibres of the structure morphism $\pi : X \rightarrow C$ are unions of smooth irreducible components of multiplicity 1 with normal crossings);

(ii) the variety X contains the variety \mathcal{M} as an open dense subscheme;

(iii) the restriction $\pi|_{\mathcal{M}} : \mathcal{M} \rightarrow C$ coincides with the structure morphism of the Néron model;

(iv) the connected component \mathcal{M}_s^0 of the neutral element of any fibre \mathcal{M}_s ($s \in C$) is an extension of an Abelian variety by a linear torus of dimension r_s ;

(v) C -group law $\mathcal{M}^0 \times_C \mathcal{M}^0 \rightarrow \mathcal{M}^0$ can be expanded to a group C -action $\mathcal{M}^0 \times_C X \rightarrow X$.

We call such compactifications of the Néron model by *Künnemann compactifications*.

By definition, the Abelian variety \mathcal{M}_η has a *trivial* trace if, for any finite ramified covering $\tilde{C} \rightarrow C$, the group scheme $\mathcal{M} \times_C \tilde{C} \rightarrow \tilde{C}$ has no non-trivial *constant* Abelian subscheme.

In this article we prove the following main result:

Theorem. *Let $\mathcal{M} \rightarrow C$ be the Néron minimal model of an absolutely simple $(d - 1)$ -dimensional principally polarized Abelian variety \mathcal{M}_η over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C .*

Assume that the trace of the Abelian variety \mathcal{M}_η is trivial, $d \geq 5$,

$$d - 1 \notin \text{Ex}(4) \stackrel{\text{def}}{=} \left\{ \binom{l+2}{m} \mid (1 < m < (l+2)/2) \right\},$$

$$\begin{aligned} & \left\{ \binom{l+2}{m} \right\}^{n+1} \mid (1 \leq m < (l+2)/2), \quad l, m, n \in \mathbb{N}_+ = \{1, 2, 3, \dots\} \\ & = \{9, 10, 15, 16, 21, 25, 27, 28, 35, 36, 45, 49, 55, 56, 64, 66, 78, 81, 84, 91, 100, \dots\} \end{aligned}$$

and the ring $E_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field.

If, for some embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the Hodge group

$$\text{Hg}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C})$$

is a semi-simple algebraic \mathbb{Q} -group (this condition automatically holds, if the Abelian variety \mathcal{M}_η has a totally degenerated reduction of multiplicative type at some place $s \in C$),

or $d - 1$ is divisible by 4,

then there exists a finite ramified covering $\tilde{C} \rightarrow C$ such that, for any Künnemann compactification \tilde{X} of the Néron minimal model of

the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \kappa(\tilde{\eta})$, there exist algebraic isomorphisms

$$H^{2d-2}(\tilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^2(\tilde{X}, \mathbb{Q}), \quad H^{2d-3}(\tilde{X}, \mathbb{Q}) \xrightarrow{\sim} H^3(\tilde{X}, \mathbb{Q}).$$

Simple Abelian varieties with multiplications from an imaginary quadratic field are especially interesting because sometimes they have non-trivial *Weil cycles* [21].

The author is grateful to S.G.Tankeev for interesting discussions.

§ 1. SOME REMARKS ON HODGE GROUPS, GLOBAL MONODROMY AND COHOMOLOGY OF LOCAL SYSTEMS

1.1. First of all note that, if the Abelian variety \mathcal{M}_η has a totally degenerated reduction of multiplicative type at some place $\delta \in C$, then the Hodge group $\mathrm{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ is a semi-simple algebraic \mathbb{Q} -group [22, Theorem 4.1].

We may assume that

$$\{s \in C \mid \text{the fibre } \mathcal{M}_s \text{ is non-compact}\} = \Delta \stackrel{\text{def}}{=} \{\delta \in C \mid \mathrm{Sing}(X_\delta) \neq \emptyset\}.$$

Set $C' = C \setminus \Delta$, $C' \xrightarrow{j} C$ the canonical embedding, $X' = X \setminus \pi^{-1}(\Delta)$, $\pi' = \pi|_{X'} : X' \rightarrow C'$.

Considering, if necessary, a ramified covering $\tilde{C} \rightarrow C$ and a projective Künnemann model $\tilde{X} \rightarrow \tilde{C}$ of the corresponding Néron model $\tilde{\mathcal{M}} \rightarrow \tilde{C}$ of the generic scheme fibre of the canonical projection $X \times_C \tilde{C} \rightarrow \tilde{C}$, we may assume in virtue of [19, Section 5.8]; [20, Section 4.4, Theorem 4.6] that any singular fibre X_δ is a union of smooth irreducible components of multiplicity 1 with normal crossings. One may also assume that there is a *countable* subset $\Delta_{\text{countable}} \subset C'$ such that, for any point $s \in C' \setminus \Delta_{\text{countable}}$, the closure G of the image of the monodromy representation $\pi_1(C', s) \rightarrow \mathrm{GL}(H^1(X_s, \mathbb{Q}))$ in the Zariski topology of the algebraic group $\mathrm{GL}(H^1(X_s, \mathbb{Q}))$ is a connected semi-simple [23, Corollary 4.2.9] *normal* [24, Theorem 7.3] subgroup of the Hodge group

$$\mathrm{Hg}(X_s) \stackrel{\text{def}}{=} \mathrm{Hg}(H^1(X_s, \mathbb{Q}))$$

of the Abelian variety X_s . We may also assume that local monodromies (Picard - Lefschetz transformations) *are unipotent* and

$$\mathrm{End}_{\kappa(\eta)}(X_\eta) = \mathrm{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}).$$

1.2. Consider the normalization $f : Z \rightarrow \pi^{-1}(\Delta)$ of the scheme $\pi^{-1}(\Delta)$. Then Z is a disjoint union of smooth irreducible components of the divisor $\pi^{-1}(\Delta)$. Since $f : Z \rightarrow \pi^{-1}(\Delta)$ is a resolution of singularities

of the subscheme $i_\Delta : \pi^{-1}(\Delta) \hookrightarrow X$, there is a canonical exact sequence of mixed Hodge \mathbb{Q} -structures [25, Corollary (8.2.8)]:

$$H^{n-2}(Z, \mathbb{Q}) \xrightarrow{(i_\Delta f)_*} H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q}),$$

where $(i_\Delta f)_*$ is a morphism of bidegree $(1, 1)$ of pure Hodge structures and φ_n is the restriction map. In particular,

$$(1.1) \quad (i_\Delta f)_* H^{n-2}(Z, \mathbb{Q}) = \text{Ker}[H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q})].$$

1.3. Since $\text{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the semi-simple part $\text{Hg}^{\text{ss}}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ of the reductive Hodge group $\text{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ [26, Proposition B57] is a \mathbb{Q} -simple algebraic group by Borovoi's theorem [27]. Therefore, by the triviality of the trace of the Abelian variety X_η , we have the equality

$$(1.2) \quad G = \text{Hg}^{\text{ss}}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}).$$

By the same reasons there is a canonical isomorphism [23, Proposition 4.4.11]

$$(1.3) \quad \text{End}_{C'}(X') \xrightarrow{\sim} \text{End}_{C'}(R^1 \pi'_* Z).$$

A choice of a point $s \in C' \setminus \Delta_{\text{countable}}$ determines the canonical embeddings

$$\text{Im}[\pi_1(C', s) \rightarrow \text{GL}(H^1(X_s, \mathbb{Q}))] \subset G \subset \text{Hg}(X_s).$$

Therefore it follows from (1.3) and from the well known equality [26, Lemma B.60]

$$\text{End}_{\text{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that there are canonical maps

$$(1.4) \quad \begin{aligned} & \text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{End}_{\pi_1(C', s)} H^1(X_s, \mathbb{Q}) \\ & \leftrightarrow \text{End}_{\text{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

The restriction map $\text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, hence it follows from (1.4) that there exists a canonical isomorphism

$$(1.5) \quad \text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, the Abelian variety X_s is simple and $\text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field. Hence, by Borovoi's theorem [27], the semi-simple part $\text{Hg}^{\text{ss}}(X_s)$ of the Hodge group $\text{Hg}(X_s)$ is a \mathbb{Q} -simple algebraic group, so that we have the equality

$$(1.6) \quad G = \text{Hg}^{\text{ss}}(X_s) \quad \text{for all } s \in C' \setminus \Delta_{\text{countable}}.$$

Besides, the canonical representation of the Lie algebra $\mathrm{Lie\,Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ in the space $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is determined by *minuscule weights* [28], [24, Theorem 0.5.1] in Bourbaki's sense [29]. Since $\dim_{\mathbb{C}}(X_s) = d - 1 \notin \mathrm{Ex}(4)$, it follows that $\mathrm{Lie\,Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a simple Lie algebra of type A_{d-2} and there exists a canonical decomposition of $\mathrm{Lie\,Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ -modules [30, Section 2.9]

$$(1.7) \quad H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = E(\omega_1) \oplus E(\omega_1)^\vee = E(\omega_1) \oplus E(\omega_{d-2}).$$

1.4. It follows from the existence of the natural embedding

$$R^2\pi'_*\mathbb{Q} = \wedge^2 R^1\pi'_*\mathbb{Q} \hookrightarrow R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$$

and from (1.3) that $H^0(C', R^2\pi'_*\mathbb{Q})$ is a rational Hodge structure of type $(1, 1)$ because a polarization on X determines an isomorphism of families of Hodge structures [23, Section 4.2.3]

$$[R^1\pi'_*\mathbb{Q}]^\vee \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}(1),$$

the ring $\mathrm{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the component of type $(0, 0)$ of the Hodge \mathbb{Q} -structure $\mathrm{End}_{C'}(R^1\pi'_*\mathbb{Q})$ [23, Section 4.4.6] and there are morphisms of rational Hodge structures

$$\begin{aligned} H^0(C', R^2\pi'_*\mathbb{Q}) &\hookrightarrow H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \\ &\xrightarrow{\sim} H^0(C', [R^1\pi'_*\mathbb{Q}]^\vee \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q})(-1) = \mathrm{End}_{C'}(R^1\pi'_*\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}(-1) \\ &\xrightarrow{\sim} \mathrm{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}(-1). \end{aligned}$$

Taking into account this fact and arguments of [15, § 2], it is easy to show the existence of an algebraic isomorphism $H^{2d-2}(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$.

§ 2. SOME CANONICAL DECOMPOSITIONS OF RATIONAL HODGE STRUCTURES OF ODD WEIGHTS

2.1. By the assumptions of the theorem, the generic scheme fibre \mathcal{M}_η of the Néron model is an Abelian variety with a principal polarization; consequently, for any point $s \in C'$, the Abelian variety X_s has a principal polarization determined by some ample divisor H_s on the variety X_s . Taking into account arguments of ([3], § 4), we may assume that there exists a rigid Poincaré bundle \mathcal{P}' on the Abelian scheme $X' \times_{C'} X'$, which induces the Poincaré bundle \mathcal{P}'_s on the Cartesian product $X_s \times X_s$, for every point $s \in C'$.

One may assume that

$$c_1(\mathcal{P}'_s) = \wp(H^1(X_s, \mathbb{Q})) \in H^2(X_s \times X_s, \mathbb{Q}) \cap H^{1,1}(X_s \times X_s, \mathbb{C}),$$

where $\wp(H^1(X_s, \mathbb{Q}))$ is the Poincaré class in the sense of [15, Section 1.2]; it is *algebraic* by the Lefschetz theorem on divisors and it is

determined uniquely (up to a non-zero scalar multiple) by the corresponding bilinear form

$$\Phi_s : H^1(X_s, \mathbb{Q}) \times H^1(X_s, \mathbb{Q}) \xrightarrow{x \times y \mapsto \langle x \smile \text{cl}_{X_s}(H_s) \smile^{d-2} \smile y \rangle} \mathbb{Q}$$

as a generator of the space $[H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\text{Sp}(H^1(X_s, \mathbb{Q}), \Phi_s)}$ of invariants of the diagonal action $x \otimes y \mapsto \mu(x) \otimes \mu(y)$ of the group $\text{Sp}(H^1(X_s, \mathbb{Q}), \Phi_s)$ on the tensor square of the \mathbb{Q} -space $H^1(X_s, \mathbb{Q})$.

By [2, Lemma 2A12, Remark 2A13] the correspondence $c_1(\mathcal{P}'_s)^{-2}$ yields an algebraic isomorphism $H^{2d-4}(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_s, \mathbb{Q})$.

Using arguments of [15, Sections 2.1 and 2.3], we see that the element $\Lambda'_{1,1} \smile^{-2}$ yields the isomorphism of local systems $R^{2d-4}\pi'_*\mathbb{Q} \xrightarrow{\sim} R^2\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$(2.1) \quad \begin{array}{ccc} R^{2d-4}\pi'_*\mathbb{Q} & \xrightarrow{(p'_1)^*} & R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q} \xrightarrow{\smile \Lambda'_{1,1} \smile^{-2}} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^2\pi'_*\mathbb{Q} \\ & & \xrightarrow{(p'_2)_*} R^2\pi'_*\mathbb{Q}. \end{array}$$

We may also expand (2.1) to a sequence of morphisms of sheaves

$$R^{2d-4}\pi_*\mathbb{Q} \xrightarrow{(p_1\sigma)^*} R^{2d-4}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile \text{cl}_Y(D^{(1)}) \smile^{-2}} R^{2d-2}(\tau\sigma)_*\mathbb{Q} \xrightarrow{(p_2\sigma)_*} R^2\pi_*\mathbb{Q},$$

whose composite is an isomorphism outside the finite set Δ . Therefore there exists an isomorphism of bidegree $(3-d, 3-d)$ of mixed Hodge structures

$$(2.2) \quad H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \xrightarrow{[x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)}) \smile^{-2})]_1} H^1(C, R^2\pi_*\mathbb{Q}).$$

2.2. Lemma. *For any odd natural number n , there are the equalities*

$$H^0(C', R^n\tau'_*\mathbb{Q}) = H^0(C', R^n\pi'_*\mathbb{Q}) = H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) = H^2(C, R^n\pi_*\mathbb{Q}) = 0.$$

Proof. Assume first that the Hodge group $\text{Hg}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ is semi-simple. In this case one has by (1.2)

$$\begin{aligned} H^0(C', R^n\tau'_*\mathbb{Q}) &\xrightarrow{\sim} H^n([X_\eta \times_{\kappa(\eta)} X_\eta] \otimes_{\kappa(\eta)} \mathbb{C}, \mathbb{Q})^G \\ &= H^n([X_\eta \times_{\kappa(\eta)} X_\eta] \otimes_{\kappa(\eta)} \mathbb{C}, \mathbb{Q})^{\text{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})} = 0 \end{aligned}$$

because, for any element $\theta \in \mathbb{R}$, the element $e^{i\theta} \in U^1 \stackrel{\text{def}}{=} \{z \in \mathbb{C}^\times \mid |z| = 1\}$ acts on the Hodge component $H^{p,n-p}([X_\eta \times_{\kappa(\eta)} X_\eta] \otimes_{\kappa(\eta)} \mathbb{C}, \mathbb{C})$ as the multiplication by an element $e^{ip\theta} e^{-i(n-p)\theta} = e^{i(2p-n)\theta}$, the Hodge group $\text{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ is defined as the smallest algebraic \mathbb{Q} -subgroup of $\text{GL}(H^1(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}, \mathbb{Q}))$ whose group of \mathbb{R} -points contains

the image of the group U^1 in $\mathrm{GL}(H^1(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}, \mathbb{R}))$ [26, Definition B51]; consequently, the group

$$\mathrm{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C} = G \otimes_{\mathbb{Q}} \mathbb{C}$$

acts on the space $H^n([X_\eta \times_{\kappa(\eta)} X_\eta] \otimes_{\kappa(\eta)} \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ without non-zero fixed points. Since $H^0(C', R^n \pi'_* \mathbb{Q}) \subset H^0(C', R^n \tau'_* \mathbb{Q})$, we get the equality $H^0(C', R^n \pi'_* \mathbb{Q}) = 0$.

Now assume that $d - 1$ is divisible by 4. Then, by (1.5) - (1.7), for all $s \in C' \setminus \Delta_{\text{countable}}$, the Lie algebra $\mathrm{Lie} G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \mathrm{Lie} \mathrm{Hg}^{\mathrm{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a simple Lie algebra of type A_{d-2} and there is a decomposition of $\mathrm{Lie} G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ -modules

$$H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = E(\omega_1) \oplus E(\omega_1)^\vee = E(\omega_1) \oplus E(\omega_{d-2}).$$

Let $E(\tilde{\omega}_1)$ be the standard irreducible representation of degree 2 of the Lie algebra $sl_2(\overline{\mathbb{Q}})$ of type A_1 . Since $\dim_{\overline{\mathbb{Q}}} E(\omega_1) = d - 1$ is divisible by 4, there exist (non-canonical) identifications of $sl_2(\overline{\mathbb{Q}})$ -modules

$$E(\tilde{\omega}_1)^{\frac{d-1}{2}} \xrightarrow{\sim} [\overline{\mathbb{Q}}^2]^{\frac{d-1}{2}} = \overline{\mathbb{Q}}^{d-1},$$

so a choice of an appropriate basis of the space $E(\omega_1)$ determine an embedding of Lie algebras $sl_2(\overline{\mathbb{Q}}) \hookrightarrow sl(E(\omega_1))$.

For natural numbers p, q , the Klebsh - Gordon formula [29, Ch. VIII, § 9, n^o 4] yields the decomposition

$$E(p\tilde{\omega}_1) \otimes_{\mathbb{Q}} E(q\tilde{\omega}_1) = E((p+q)\tilde{\omega}_1) \oplus E((p+q-2)\tilde{\omega}_1) \oplus E((p+q-4)\tilde{\omega}_1) \oplus \dots$$

Therefore, if p is an odd natural number, then the representation of $sl_2(\overline{\mathbb{Q}})$ in $\wedge^p(E(\omega_1))$ is a sum of representations of type $E(\tilde{\omega}_1)$, $E(3\tilde{\omega}_1)$, \dots , $E(p\tilde{\omega}_1)$. On the other hand, if q is an even natural number, then the representation of $sl_2(\overline{\mathbb{Q}})$ in $\wedge^q(E(\omega_1))$ is a sum of representations of type $E(0)$, $E(2\tilde{\omega}_1)$, $E(4\tilde{\omega}_1)$, \dots , $E(q\tilde{\omega}_1)$. Consequently, by the Schur lemma, one has for odd $p + q$:

$$\mathrm{Hom}_{G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}}(\wedge^p E(\omega_1), \wedge^q(E(\omega_1))) = \mathrm{Hom}_{G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}}(\wedge^p E(\omega_1), \wedge^q(E(\omega_1))^\vee) = 0.$$

Finally, in virtue of the theorem on local invariant cycles, the canonical maps

$$R^n(\tau\sigma)_* \mathbb{Q} \rightarrow j_* R^n(\tau'\sigma')_* \mathbb{Q}, \quad R^n \pi_* \mathbb{Q} \rightarrow j_* R^n \pi'_* \mathbb{Q}$$

are surjective with the kernels concentrated on the finite set Δ ([31], Proposition (15.12); [32], Section (3.7)). Therefore it remains to note

that by [31, Proposition (10.5)], one has

$$\begin{aligned} H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) &= H^2(C, j_*R^n(\tau'\sigma')_*\mathbb{Q}) \\ &\xrightarrow{\sim} H^0(C', R^n(\tau'\sigma')_*\mathbb{Q})^\vee = H^0(C', R^n\tau'_*\mathbb{Q})^\vee; \\ H^2(C, R^n\pi_*\mathbb{Q}) &= H^2(C, j_*R^n\pi'_*\mathbb{Q}) \xrightarrow{\sim} H^0(C', R^n\pi'_*\mathbb{Q})^\vee. \end{aligned}$$

The lemma is proved.

2.3. Let $K_{nX} = \text{Ker}[H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n\pi_*\mathbb{Q})]$, $K_{nY} = \text{Ker}[H^n(Y, \mathbb{Q}) \rightarrow H^0(C, R^n(\tau\sigma)_*\mathbb{Q})]$. The Leray spectral sequences $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ and $E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$ are degenerated: $E_2^{p,q} = E_\infty^{p,q}$ [31, Corollary (15.15)]. Hence, for any natural number n , there are exact sequences of Hodge \mathbb{Q} -structures ([13], Formula (2.4))

$$(2.3) \quad 0 \rightarrow H^2(C, R^{n-2}\pi_*\mathbb{Q}) \rightarrow K_{nX} \xrightarrow{\alpha_{nX}} H^1(C, R^{n-1}\pi_*\mathbb{Q}) \rightarrow 0,$$

$$(2.4)$$

$$0 \rightarrow H^2(C, R^{n-2}(\tau\sigma)_*\mathbb{Q}) \rightarrow K_{nY} \xrightarrow{\alpha_{nY}} H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \rightarrow 0.$$

It follows from Lemma 2.2, (2.3), (2.4), the theorem on local invariant cycles and the strong Lefschetz theorem for fibres of the smooth morphism π' that

$$(2.5) \quad K_{3X} = H^1(C, R^2\pi_*\mathbb{Q});$$

$$(2.6) \quad K_{(2d-3)X} = H^1(C, R^{2d-4}\pi_*\mathbb{Q});$$

$$\text{cl}_X(H)^{\smile d-3} \smile K_{3X} = \text{cl}_X(H)^{\smile d-3} \smile H^1(C, R^2\pi_*\mathbb{Q})$$

$$(2.7) \quad = \text{cl}_X(H)^{\smile d-3} \smile H^1(C, j_*R^2\pi'_*\mathbb{Q}) = K_{(2d-3)X}.$$

Moreover, taking into account the non-degeneracy of the canonical pairing [31, Proposition (10.5)]

$$\begin{aligned} &H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q}) \times H^1(C, j_*R^2\pi'_*\mathbb{Q}) \\ &\xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_*R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \end{aligned}$$

we see, that the canonical pairing $K_{(2d-3)X} \times K_{3X} \xrightarrow{x \times x' \mapsto x \smile x'} H^{2d}(X, \mathbb{Q})$ is non-degenerate. Therefore the restriction of the *non-degenerate* [2, Section 1.2.A] bilinear form

$$\Phi : H^3(X, \mathbb{Q}) \times H^3(X, \mathbb{Q}) \xrightarrow{x \times y \mapsto x \smile y \smile \text{cl}_X(H)^{\smile d-3}} H^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow{\sim} \mathbb{Q}.$$

to the subspace $K_{3X} \subset H^3(X, \mathbb{Q})$ is non-degenerate. Hence, as it was noticed in Section 1.2 of [15], there exists the decomposition of rational Hodge structures

$$(2.8) \quad H^3(X, \mathbb{Q}) = K_{3X} \oplus K_{3X}^\perp,$$

where K_{3X}^\perp is the orthogonal complement of the subspace $K_{3X} \hookrightarrow H^3(X, \mathbb{Q})$ with respect to the form Φ . In virtue of (2.7) we have:

$$K_{3X}^\perp = \{x \in H^3(X, \mathbb{Q}) \mid x \smile \text{cl}_X(H)^{\smile d-3} \smile K_{3X} = 0\} \\ \{x \in H^3(X, \mathbb{Q}) \mid x \smile K_{(2d-3)X} = 0\}.$$

Consequently, the decomposition (2.8) is *canonical* and it does not depend on the choice of an ample divisor H ; moreover, by (2.7), (2.8) and by the strong Lefschetz theorem, there is a *canonical* decomposition of rational Hodge structures

$$(2.9) \quad H^{2d-3}(X, \mathbb{Q}) = K_{(2d-3)X} \oplus K_{(2d-3)X}^\perp,$$

where

$$K_{(2d-3)X}^\perp \stackrel{\text{def}}{=} \text{cl}_X(H)^{\smile d-3} \smile K_{3X}^\perp = \{x \in H^{2d-3}(X, \mathbb{Q}) \mid K_{3X} \smile x = 0\}.$$

2.4. For any point $s \in C$, we denote by $\iota_{X_s/X} : X_s \hookrightarrow X$ the canonical embedding. The morphism π is proper, therefore the fibre of the sheaf $R^n \pi_* \mathbb{Q}$ over a point $s \in C$ coincides with the space $H^n(X_s, \mathbb{Q})$ ([33], Ch. II, § 4, Remark 4.17.1; [34], Ch. VI, § 2, Corollary 2.5). Consequently, the restriction map $\iota_{X_s/X}^*$ coincides with the composite ([35], Vol. II, Ch. 4, Section 4.3.1)

$$H^n(X, \mathbb{Q}) \rightarrow E_\infty^{0,n}(\pi) \rightarrow E_2^{0,n}(\pi) = H^0(C, R^n \pi_* \mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q}).$$

Thus the map $\iota_{X_s/X}^*$ is the composite of canonical maps

$$H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n \pi_* \mathbb{Q}) \hookrightarrow \prod_{s \in C} H^n(X_s, \mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q}),$$

where the \mathbb{Q} -space $\prod_{s \in C} H^n(X_s, \mathbb{Q})$ is identified with the \mathbb{Q} -space of *discontinuous* global sections of the sheaf $R^n \pi_* \mathbb{Q}$ ([33], Ch. II, § 4, Section 4.4.4). Evidently

$$(2.10) \quad \omega \in K_{nX} \Leftrightarrow (\forall s \in C) \quad \iota_{X_s/X}^*(\omega) = 0.$$

§ 3. CONSTRUCTIONS OF ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY OF ODD WEIGHTS

3.1. Lemma. *There is the equality $(i_\Delta f)_* H^1(Z, \mathbb{Q}) = K_{3X}^\perp$.*

Proof. In virtue of Lemma 2.2 and the surjectivity [31, Corollary (15.14)] of the canonical edge map $H^3(X, \mathbb{Q}) \rightarrow H^0(C, R^3 \pi_* \mathbb{Q})$ of the Leray spectral sequence $E_2^{p,q}(\pi)$ one may assume that $\Delta \neq \emptyset$ because (2.8) induces a canonical isomorphism $K_{3X}^\perp \xrightarrow{\sim} H^0(C, R^3 \pi_* \mathbb{Q})$.

Irreducible components of a smooth divisor Z are naturally identified with irreducible components X_{δ_i} of the divisor $\pi^{-1}(\Delta) = \sum_{\delta \in \Delta} X_\delta$. Denote by $\iota_{X_{\delta_i}/X} : X_{\delta_i} \hookrightarrow X$, $\iota_{X_{\delta_i}/X_\delta} : X_{\delta_i} \hookrightarrow X_\delta$, $\iota_{X_{\delta_i}/Z} : X_{\delta_i} \hookrightarrow Z$ the canonical embeddings. In accordance with the functoriality

of cohomology with coefficients in the field \mathbb{Q} there is the equality $\iota_{X_{\delta i}/X}^* = \iota_{X_{\delta i}/X_{\delta}}^* \iota_{X_{\delta}/X}^*$. On the other hand, $\iota_{X_{\delta}/X}^* K_{(2d-3)X} = 0$ by (2.10). Hence $\iota_{X_{\delta i}/X}^* K_{(2d-3)X} = 0$. By definition, the operator $\iota_{X_{\delta i}/X^*}$ is conjugate to the operator $\iota_{X_{\delta i}/X}^*$ ([2], Section 1.2.A; [34], Ch. VI, § 11, Remark 11.6), therefore

$$\langle \iota_{X_{\delta i}/X^*} H^1(X_{\delta i}, \mathbb{Q}) \smile K_{(2d-3)X} \rangle = \langle H^1(X_{\delta i}, \mathbb{Q}) \smile \iota_{X_{\delta i}/X}^* K_{(2d-3)X} \rangle.$$

Consequently, $\iota_{X_{\delta i}/X^*} H^1(X_{\delta i}, \mathbb{Q}) \smile K_{(2d-3)X} = 0$. From the commutativity of the diagram

$$\begin{array}{ccc} X_{\delta i} & \xrightarrow{\iota_{X_{\delta i}/X}} & X \\ \parallel & & \uparrow i_{\Delta} f \\ X_{\delta i} & \xrightarrow{\iota_{X_{\delta i}/Z}} & Z \end{array}$$

of canonical morphisms and ([34], Ch. VI, § 11, Remark 11.6 (c)) we obtain the equality

$$(3.1) \quad (i_{\Delta} f)_* (\iota_{X_{\delta i}/Z})_* |_{H^1(X_{\delta i}, \mathbb{Q})} = \iota_{X_{\delta i}/X^*} |_{H^1(X_{\delta i}, \mathbb{Q})}.$$

Thus $(i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \smile K_{(2d-3)X} = 0$, so that from (2.7) we obtain the equality

$$(i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-3} \smile K_{3X} = 0$$

and the existence of the inclusion

$$(3.2) \quad (i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \subset K_{3X}^{\perp}.$$

In accordance with Lemma 2.2 we have the equalities

$$H^2(C, R^1 \pi_* \mathbb{Q}) = H^0(C', R^3 \pi'_* \mathbb{Q}) = 0.$$

On the other hand, the theorem on local invariant cycles and the Leray spectral sequence for the embedding $j : C' \subset C$ yield the embedding of mixed Hodge \mathbb{Q} -structures $H^1(C, R^2 \pi_* \mathbb{Q}) \hookrightarrow H^1(C', R^2 \pi'_* \mathbb{Q})$ ([31], P. 457, Corollary (13.10), Remark (14.5)). Besides, the canonical map

$$H^3(X, \mathbb{Q}) \rightarrow H^0(C, R^3 \pi_* \mathbb{Q})$$

is surjective ([31], Corollary (15.14)). Consequently, taking into account (1.1), (2.3), the diagram (15.1) in [31], the degeneracy of the Leray spectral sequence $E_2^{p,q}(\pi') = H^p(C', R^q \pi'_* \mathbb{Q})$ ([23], Theorem 4.1.1), the equality $H^2(C', R^1 \pi'_* \mathbb{Q}) = 0$ (because the cohomological dimension of the affine curve C' equals 1 ([34], Ch. VI, § 7, Theorem 7.2)) and the commutativity of the diagram of morphisms

$$\begin{array}{ccc} X' & \subset & X \\ \downarrow \pi' & & \downarrow \pi \\ C' & \subset & C, \end{array}$$

we obtain the commutative diagram of mixed Hodge \mathbb{Q} -structures with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(C, R^2\pi_*\mathbb{Q}) & \rightarrow & H^3(X, \mathbb{Q}) & \rightarrow & H^0(C, R^3\pi_*\mathbb{Q}) \rightarrow 0 \\ & & \cap & & \downarrow \varphi_3 & & \downarrow \\ 0 & \rightarrow & H^1(C', R^2\pi'_*\mathbb{Q}) & \rightarrow & H^3(X', \mathbb{Q}) & \rightarrow & 0. \end{array}$$

Therefore (2.5), (2.8) yield the commutative diagram

$$(3.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & K_{3X} & \rightarrow & K_{3X} \oplus K_{3X}^\perp & \rightarrow & K_{3X}^\perp \rightarrow 0 \\ & & \cap & & \downarrow \varphi_3 & & \downarrow \\ 0 & \rightarrow & H^1(C', R^2\pi'_*\mathbb{Q}) & \rightarrow & H^3(X', \mathbb{Q}) & \rightarrow & 0. \end{array}$$

Evidently $\text{Im}(\varphi_3) = K_{3X}$, therefore (3.3) yields the commutative diagram of canonical morphisms of Hodge \mathbb{Q} -structures

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{3X} & \rightarrow & K_{3X} \oplus K_{3X}^\perp & \rightarrow & K_{3X}^\perp \rightarrow 0 \\ & & \parallel & & \downarrow \varphi_3 & & \downarrow \\ 0 & \rightarrow & K_{3X} & \rightarrow & K_{3X} & \rightarrow & 0. \end{array}$$

The corresponding exact sequence of Hodge \mathbb{Q} -structures ([13], Section 2.6) of the snake-like diagram ([36], § 1, Proposition 2) and (1.1) yield the canonical identifications

$$(i_{\Delta}f)_* H^1(Z, \mathbb{Q}) = \text{Ker}(\varphi_3) = K_{3X}^\perp.$$

The lemma is proved.

3.2. Lemma. *There is a canonical embedding $(p_2\sigma)_*(K_{(2d+1)Y}) \subset K_{3X}$.*

Proof. Since $K_{3X} = \{x \in H^3(X, \mathbb{Q}) \mid x \smile K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3} = 0\}$, it suffices to check the equality

$$(p_2\sigma)_*(K_{(2d+1)Y}) \smile K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3} = 0,$$

which is equivalent ([2], Section 1.2.A; [34], Ch. VI, § 11, Remark 11.6) to the equalities

$$(3.4) \quad \begin{aligned} & \langle (p_2\sigma)_*(K_{(2d+1)Y}) \smile K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3} \rangle = \\ & \langle K_{(2d+1)Y} \smile (p_2\sigma)^*(K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3}) \rangle = 0. \end{aligned}$$

If $\Delta = \emptyset$, then the formula (3.4) is true in virtue of Lemma 3.1. From now on we assume that $\Delta \neq \emptyset$.

By (2.4) and Lemma 2.2 we have the canonical identification

$$(3.5) \quad K_{(2d+1)Y} = H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}).$$

It is known that the Gysin map $\iota_{X_{\delta i}/X^*} : H^k(X_{\delta i}, \mathbb{Q}) \rightarrow H^{k+2}(X, \mathbb{Q})$ is given by $\alpha \mapsto \alpha \smile \text{cl}_X(X_{\delta i})$ ([15], Formula (4.20)). On the other hand, the strong Lefschetz theorem for the variety $X_{\delta i}$ yields the existence of the embedding

$$H^1(X_{\delta i}, \mathbb{Q}) \smile \iota_{X_{\delta i}/X}^* \text{cl}_X(H)^{\smile d-3} \subset H^{2d-5}(X_{\delta i}, \mathbb{Q}).$$

Therefore the projection formula ([2], Section 1.2.A) yields the inclusion

$$\begin{aligned} & \iota_{X_{\delta i}/X*} H^1(X_{\delta i}, \mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-3} \\ & \subset \iota_{X_{\delta i}/X*} H^{2d-5}(X_{\delta i}, \mathbb{Q}) = H^{2d-5}(X_{\delta i}, \mathbb{Q}) \smile \text{cl}_X(X_{\delta i}). \end{aligned}$$

Finitely, it follows from (3.1) that there is the embedding

$$(3.6) \quad (i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-3} \subset \sum_{\delta, i} H^{2d-5}(X_{\delta i}, \mathbb{Q}) \smile \text{cl}_X(X_{\delta i}).$$

By definition ([35], Vol. II, Ch. 4, Section 4.2.1), for any point $s \in C'$, the \smile -product by the class $\text{cl}_X(X_{\delta i}) \in H^2(X, \mathbb{Q})$ acts on the fibre $H^q(X_s, \mathbb{Q}) = [j_* R^q \pi'_* \mathbb{Q}]_s$ of the sheaf $j_* R^q \pi'_* \mathbb{Q}$ as the \smile -product by the class $\iota_{X_s/X}^*(\text{cl}_X(X_{\delta i}))$. It follows from the evident equality

$$\iota_{X_s/X}^*(\text{cl}_X(X_{\delta i})) = 0$$

that

$$(3.7) \quad j_* R^q \pi'_* \mathbb{Q} \smile \text{cl}_X(X_{\delta i}) = 0.$$

In virtue of the theorem on local invariant cycles and by Künneth's formula on fibres of a smooth morphism $\tau' : Y' = X' \times_{C'} X' \rightarrow C'$ there is a canonical decomposition

$$(3.8) \quad H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q}) = \bigoplus_{p+q=2d} H^1(C, j_*(R^p \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^q \pi'_* \mathbb{Q})).$$

It follows from (3.7) that

$$H^1(C, j_*(R^p \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^q \pi'_* \mathbb{Q})) \smile [1 \otimes_{\mathbb{Q}} \text{cl}_X(X_{\delta i})] = 0,$$

so that (3.4) follows from Lemma 3.1, (3.5), (3.6), (3.8). The lemma is proved.

3.3. There is a non-degenerate canonical pairing [31, Proposition (10.5)]

$$\begin{aligned} H^1(C, j_* R^{2d-4} \pi'_* \mathbb{Q}) \times H^1(C, j_* R^2 \pi'_* \mathbb{Q}) & \xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_* R^{2d-2} \pi'_* \mathbb{Q}) \\ & = H^2(C, R^{2d-2} \pi_* \mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}, \end{aligned}$$

identifying (in accordance with the theorem on local invariant cycles) the space $H^1(C, R^{2d-4} \pi'_* \mathbb{Q})^{\vee}$ with $H^1(C, R^2 \pi'_* \mathbb{Q})$. By the similar reason we have the identification $H^1(C, R^{2d-4}(\tau\sigma)_* \mathbb{Q})^{\vee} = H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q})$. Therefore by (2.5), (3.5) and Lemma 3.2 we obtain a commutative diagram

$$(3.9) \quad \begin{array}{ccc} K_{3X} & \xleftarrow{(p_2\sigma)_* K_{(2d+1)Y}} & K_{(2d+1)Y} \\ \parallel & & \parallel \\ H^1(C, R^2 \pi_* \mathbb{Q}) & \xleftarrow{[(p_2\sigma)_*]_1} & H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q}). \end{array}$$

Finally, in accordance with ([37], Ch. 2, § 8, Formula (5)) and by (2.10) we obtain the equality

$$\iota_{X_s/X}^*(K_{(2d-3)Y} \smile \text{cl}_Y(D^{(1)})^{\smile 2}) = \iota_{X_s/X}^*(K_{(2d-3)Y}) \smile \iota_{X_s/X}^*(\text{cl}_Y(D^{(1)})^{\smile 2}) = 0,$$

so in virtue of (2.6), (2.10) there is a commutative diagram

$$(3.10) \quad \begin{array}{ccc} K_{(2d-3)Y} & \xrightarrow{\smile \text{cl}_Y(D^{(1)})^{\smile 2}} & K_{(2d+1)Y} \\ \parallel & & \parallel \\ H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[(\smile \text{cl}_Y(D^{(1)})^{\smile 2})]_1} & H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}). \end{array}$$

On the other hand, the commutative diagram of morphisms

$$\begin{array}{ccc} X & \xleftarrow{p_1\sigma} & Y \\ \uparrow \iota_{X_s/X} & & \uparrow \iota_{Y_s/Y} \\ X_s & \xleftarrow{p_1s\sigma_s} & Y_s \end{array}$$

yields the commutative diagram of canonical maps

$$\begin{array}{ccc} H^{2d-3}(X, \mathbb{Q}) & \xrightarrow{(p_1\sigma)^*} & H^{2d-3}(Y, \mathbb{Q}) \\ \downarrow \iota_{X_s/X}^* & & \downarrow \iota_{Y_s/Y}^* \\ H^{2d-3}(X_s, \mathbb{Q}) & \xrightarrow{(p_1s\sigma_s)^*} & H^{2d-3}(Y_s, \mathbb{Q}), \end{array}$$

so that, for any point $s \in C$, it follows from (2.10) that $\iota_{X_s/X}^*(K_{(2d-3)X}) = 0$ and

$$\iota_{Y_s/Y}^*(p_1\sigma)^*(K_{(2d-3)X}) = 0.$$

The morphism $\tau\sigma$ is proper, therefore by arguments of Section 2.4 the canonical restriction map

$$\iota_{Y_s/Y}^* : H^{2d-3}(Y, \mathbb{Q}) \rightarrow H^{2d-3}(Y_s, \mathbb{Q})$$

is the composite of canonical maps

$$H^{2d-3}(Y, \mathbb{Q}) \rightarrow H^0(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q}) \hookrightarrow \prod_{s \in C} H^{2d-3}(Y_s, \mathbb{Q}) \rightarrow H^{2d-3}(Y_s, \mathbb{Q}).$$

Thus, it follows from the equalities $\iota_{Y_s/Y}^*(p_1\sigma)^*(K_{(2d-3)X}) = 0$ ($s \in C$) and from (2.10) that $(p_1\sigma)^*K_{(2d-3)X} \hookrightarrow K_{(2d-3)Y}$. By (2.4), (2.6) and Lemma 2.2 we have a commutative diagram

$$(3.11) \quad \begin{array}{ccc} K_{(2d-3)X} & \xrightarrow{(p_1\sigma)^*} & K_{(2d-3)Y} \\ \parallel & & \parallel \\ H^1(C, R^{2d-4}\pi_*\mathbb{Q}) & \xrightarrow{[(p_1\sigma)^*]_1} & H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q}). \end{array}$$

Gluing together diagrams (3.9) - (3.11), we see that (2.2) takes the form

$$(3.12) \quad K_{(2d-3)X} \xrightarrow{x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)})^{\smile 2})} K_{3X}.$$

For $x \in H^{2d-3}(X, \mathbb{Q})$, the projection formula [2, Section 1.2.A] yields the equalities

$$\begin{aligned} (p_2\sigma)_* \left((p_1\sigma)^* x \smile [\text{cl}_Y(D^{(1)})]^{-2} \right) &= [\text{pr}_2 \iota\sigma]_* \left([\text{pr}_1 \iota\sigma]^* x \smile [\text{cl}_Y(D^{(1)})]^{-2} \right) \\ &= \text{pr}_{2*}(\iota\sigma)_* \left((\iota\sigma)^* \text{pr}_1^* x \smile [\text{cl}_Y(D^{(1)})]^{-2} \right) \\ &= \text{pr}_{2*} \left(\text{pr}_1^* x \smile (\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}] \right). \end{aligned}$$

Therefore an algebraic class $(\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}] \in H^6(X \times X, \mathbb{Q})$ determines the isomorphism (3.12), which takes the form

$$(3.13) \quad K_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \smile (\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}])} K_{3X}.$$

3.4. By arguments of [15, Section 3.5], we have the isomorphism

$$(3.14) \quad K_{(2d-3)X}^\perp \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^*(x) \smile \wp(K_{3X}^\perp))} K_{3X}^\perp,$$

where $\wp(K_{3X}^\perp)$ is an *algebraic* (by Lemma 3.1 and [15, Lemma 3.8]) Poincaré class.

On the other hand, taking into account decompositions (2.8), (2.9), the existence of *algebraic* isomorphisms (3.13), (3.14) and arguments of [15, Sections 4.1 - 4.4], it is easy to show the existence of an *algebraic* isomorphism

$$H^{2d-3}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \smile (u_{3,3} + u_{3,3^\perp} + h_{10} + \wp(K_{3X}^\perp)))} H^3(X, \mathbb{Q}),$$

where $u_{3,3} \in K_{3X} \otimes_{\mathbb{Q}} K_{3X}$, $u_{3,3^\perp} \in K_{3X} \otimes_{\mathbb{Q}} K_{3X}^\perp$ are components of an algebraic class $u \stackrel{\text{def}}{=} (\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}] \in H^6(X \times X, \mathbb{Q})$ and h_{10} is an appropriate element of the \mathbb{Q} -space $\bigoplus_{p+q=6, p \neq 3} H^p(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(X, \mathbb{Q})$. The theorem is proved.

REFERENCES

- [1] A.Grothendieck, *Standard conjectures on algebraic cycles*, Algebraic Geometry, Internat. Colloq. (Bombay, 1968), Oxford Univ. Press, London, 1969, 193–199.
- [2] S.L.Kleiman, *Algebraic cycles and the Weil conjectures*, Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam; Masson, Paris, 1968, 359–386.
- [3] S.G. Tankeev, *On the standard conjecture for complex Abelian schemes over smooth projective curves*, *Izv. Math.*, **67**:3 (2003), 597–635.
- [4] S. G. Tankeev, *On the numerical equivalence of algebraic cycles on potentially simple Abelian schemes of prime relative dimension*, *Izv. Math.*, **69**:1 (2005), 143–162.
- [5] S.G.Tankeev, *Monoidal transformations and conjectures on algebraic cycles*, *Izv. Math.*, **71**:3 (2007), 629–655.
- [6] D.Lieberman, *Numerical and homological equivalence of algebraic cycles on Hodge manifolds*, *Amer. J. Math.*, **90**:2 (1968), 366–374.
- [7] S.G.Tankeev, *On the standard conjecture of Lefschetz type for complex projective threefolds. II*, *Izv. Math.*, **75**:5 (2011), 1047–1062.
- [8] D.Arapura, *Motivation for Hodge cycles*, *Adv. Math.*, **207**:2 (2006), 762–781.
- [9] F.Charles, E.Markman, *The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces*, *Compositio Mathematica*, **149**:3 (2013), 481–494.
- [10] O.V.Nikol'skaya, *On algebraic cycles on a fibre product of families of K3 surfaces*, *Izv. Math.*, **77**:1 (2013), 143–162.
- [11] O.V.Nikol'skaya, *On algebraic cycles on fibre products of non-isotrivial families of regular surfaces with geometric genus 1*, *Modeling and analysis of information systems*, **23**:4 (2016), 440–465
- [12] S.G.Tankeev, *On the standard conjecture and the existence of the Chow - Lefschetz decomposition for complex projective varieties*, *Izv. Math.*, **79**:1 (2015), 177–207.
- [13] S.G.Tankeev, *On the inductive approach to the standard conjecture for a fibred complex variety with strong semi-stable degeneracies*, *Izv. Math.*, **81**:6 (2017), 1253–1285.
- [14] F.Charles, *Remarks on the Lefschetz standard conjecture and hyperkähler varieties*, *Comment. Math. Helv.*, **88** (2013), 449–468.
- [15] S.G.Tankeev, "On algebraic isomorphisms of rational cohomology of a Künneman compactification of the Néron minimal model", *Siberian Electronic Mathematical Reports*, **17** (2020), 89-125.
- [16] S.G.Tankeev, "On the standard conjecture for a fibre product of three elliptic surfaces with pairwise-disjoint discriminant loci", *Izv. Math.*, **83**:3 (2019), 613–653.
- [17] S.G.Tankeev, "On the standard conjecture for fibred by curves 3-dimensional variety with non-injective Kodaira - Spencer map", *Izv. Math.*, **84**:5 (2020), 211-232.
- [18] A.Grothendieck, *Modèles de Néron et monodromie*, Groupes de monodromie en géométrie algébrique, Lecture Notes in Mathematics, **288**, SGA 7 I, Exposé IX, Springer-Verlag, Berlin - Heidelberg - New York 1972, 313–523.
- [19] K.Künnemann, *Height pairings for algebraic cycles on abelian varieties*, *Ann. Scient. Éc. Norm. Sup.*, 4^e série, **34** (2001), 503–523.

- [20] K.Künnemann, *Projective regular models for Abelian varieties, semistable reduction, and the height pairing*, Duke Math. J., **95**:1 (1998), 161–212.
- [21] B.Moonen, Yu.Zarhin, "Weil classes on Abelian varieties", *J. reine angew. Math.*, **496** (1998), 83-92.
- [22] A. Silverberg, Yu.G. Zarhin, "Hodge groups of abelian varieties with purely multiplicative reduction", *Izv. Ross. Akad. Nauk. Ser. Mat.*, **60**:2 (1996), 149-158.
- [23] P.Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math., **40**:1 (1971), 5–57.
- [24] Yu.G.Zarhin, *Weights of simple Lie algebras in cohomology of algebraic varieties*, Math. USSR-Izv., **24**:2 (1985), 245–281.
- [25] P.Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math., **44** (1974), 5–77.
- [26] B.B.Gordon, *A survey of the Hodge conjecture for Abelian varieties*, in: J.D.Lewis, *A survey of the Hodge conjecture*, second edition, CRM Monograph Series, Centre de Recherches Mathématiques Université de Montréal, vol. **10** (1999), 297–356.
- [27] M.V.Borovoi, "The Hodge group and the algebra of endomorphisms of an Abelian variety", Questions of the group theory and homological algebra, Yaroslavl, 1981, pp. 124-128.
- [28] P. Deligne, *Variétés de Shimura: interprétation modulaires et techniques de construction de modèles canoniques*, Proc. Symp. Pure Math., 1979, v. **33**, pt. 2, 247–290.
- [29] N. Bourbaki, *Groupes et algèbres de Lie, Chaps. 1-8*, Actualités Sci. Indust., nos. 1285, 1349, 1337, 1364, Hermann, Paris, 1971, 1972, 1968, 1975.
- [30] S.G.Tankeev, *Cycles on Abelian varieties and exceptional numbers*, Izv. Math., **60**:2 (1996), 391–424.
- [31] S.Zucker, *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, Ann. Math., **109** (1979), 415–476.
- [32] C.H. Clemens, *Degeneration of Kähler manifolds*, Duke Math. J., **44**:2 (1977), 215–290.
- [33] R.Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958.
- [34] J.S. Milne, *Etale cohomology*, Princeton Univ. Press, Princeton, New Jersey, 1980.
- [35] C.Voisin, *Hodge theory and complex algebraic geometry I, II*, Cambridge University Press, Cambridge, 2002.
- [36] N.Bourbaki, *Éléments de mathématique*, Algèbre. Ch. X: Algèbre homologique, Masson, Paris, 1980.
- [37] G. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967.

OL'GA VLADIMIROVNA MAKAROVA
VLADIMIR STATE UNIVERSITY,
GORKIJ STREET, 87,
600000, VLADIMIR, RUSSIA
E-mail address: papichonok@yandex.ru