

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
<http://semr.math.nsc.ru>

Том 19, №1, стр. 34–48 (2022)

УДК 512.7

DOI 10.33048/semi.2022.19.004

MSC 14C25, 14F25, 14J30

ON ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY OF A COMPACTIFICATION OF THE NÉRON MODEL WITH MULTIPLICATIONS FROM AN IMAGINARY QUADRATIC FIELD

O.V.MAKAROVA

ABSTRACT. It is proved that the Grothendieck standard conjecture of Lefschetz type holds for rational cohomology of degrees 2, 3 of a Künnemann compactification of the Néron minimal model of an absolutely simple principally polarized Abelian variety of non-exceptional dimension divisible by 4 over the field of rational functions of a smooth projective curve provided that the ring of endomorphisms of the generic geometric fibre is an order of an imaginary quadratic field.

Keywords: Abelian variety, Néron minimal model, Künnemann compactification, Grothendieck standard conjecture of Lefschetz type.

INTRODUCTION

Let H be an ample divisor on a smooth complex projective d -dimensional variety X . Then, for any natural number $i \leq d$, the map

$$L^{d-i} : H^i(X, \mathbb{Q}) \xrightarrow{\sim \text{cl}_X(H)^{\sim d-i}} H^{2d-i}(X, \mathbb{Q})$$

is an isomorphism by the strong Lefschetz theorem. The Grothendieck standard conjecture $B(X)$ of Lefschetz type [1] asserts that there exists an algebraic \mathbb{Q} -cycle Z on the Cartesian product $X \times X$ which yields the inverse *algebraic* isomorphism $H^{2d-i}(X, \mathbb{Q}) \xrightarrow{x \mapsto \text{pr}_{2*}(\text{pr}_1^* x \sim \text{cl}_{X \times X}(Z))} H^i(X, \mathbb{Q})$.

MAKAROVA, O.V., ON ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY OF A COMPACTIFICATION OF THE NÉRON MODEL WITH MULTIPLICATIONS FROM AN IMAGINARY QUADRATIC FIELD.

© 2022 MAKAROVA O.V.

Supported by Russian foundation for basic research (grant 18-01-00143).

Received September, 22, 2020, published January, 18, 2022.

A reader can find examples of varieties satisfying the standard conjecture in [1] - [17].

Due to Tankeev [15] and in order to establish notations, we recall some facts concerning Künnemann compactifications of Néron minimal models of Abelian varieties.

Let $\mathcal{M} \rightarrow C$ be the Néron minimal model of the Abelian variety \mathcal{M}_η over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C . Suppose that at any place $s \in C$ the reduction of the Abelian variety \mathcal{M}_η is semi-stable in Grothendieck's sense. In this case the connected component \mathcal{M}_s^0 of the neutral element of the algebraic group \mathcal{M}_s is an extension of an Abelian variety by a linear torus whose dimension r_s is called *the toric (reductive) rank* at the place s [18, Section 2.1.12].

Let R be a Dedekind domain with the fraction field K and let A_η be an Abelian variety over $\eta = \text{Spec } K$ such that all reductions are semi-stable in Grothendieck's sense. As it was shown by Künnemann [19, Section 5.8], in this case there exists a finite extension K' of the field K such that the Abelian variety $A_\eta \otimes_K K'$ has (not necessarily unique) a flat projective regular model P' over the integral closure R' of the ring R in the field K' ; this model P' has strict semi-stable reductions over each localization of the ring R' (in particular, every special fibre P'_s is a union of smooth divisors of multiplicity 1 with normal crossings [20, Section 1.9]), and the scheme P' contains the Néron minimal model \mathcal{A}' of the variety $A_\eta \otimes_K K'$ in the case when all residue fields of the scheme $\text{Spec } R'$ are perfect [20, Section 4.4, Theorem 4.6].

Using the results of Künnemann cited above we may assume after the base change determined by an appropriate ramified covering $\tilde{C} \rightarrow C$ that, for the Néron minimal model $\mathcal{M} \rightarrow C$, there exists a smooth compactification X of the variety \mathcal{M} which is flat and projective over the curve C such that the following conditions hold:

(i) the model X/C has strictly semi-stable reductions (in particular, all fibres of the structure morphism $\pi : X \rightarrow C$ are unions of smooth irreducible components of multiplicity 1 with normal crossings);

(ii) the variety X contains the variety \mathcal{M} as an open dense subscheme;

(iii) the restriction $\pi|_{\mathcal{M}} : \mathcal{M} \rightarrow C$ coincides with the structure morphism of the Néron model;

(iv) the connected component \mathcal{M}_s^0 of the neutral element of any fibre \mathcal{M}_s ($s \in C$) is an extension of an Abelian variety by a linear torus of dimension r_s ;

(v) C -group law $\mathcal{M}^0 \times_C \mathcal{M}^0 \rightarrow \mathcal{M}^0$ can be expanded to a group C -action $\mathcal{M}^0 \times_C X \rightarrow X$.

We call such compactifications of the Néron model by *Künnemann compactifications*.

By definition, the Abelian variety \mathcal{M}_η has a *trivial* trace if, for any finite ramified covering $\tilde{C} \rightarrow C$, the group scheme $\mathcal{M} \times_C \tilde{C} \rightarrow \tilde{C}$ has no non-trivial *constant* Abelian subscheme.

In this article we prove the following main result:

Theorem. *Let $\mathcal{M} \rightarrow C$ be the Néron minimal model of an absolutely simple $(d-1)$ -dimensional principally polarized Abelian variety \mathcal{M}_η over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C .*

Assume that the trace of the Abelian variety \mathcal{M}_η is trivial, $d \geq 5$,

$$\begin{aligned} d-1 \notin \text{Ex}(4) &\stackrel{\text{def}}{=} \left\{ \binom{l+2}{m} \mid (1 < m < (l+2)/2), \right. \\ &\left. \binom{l+2}{m}^{n+1} \mid (1 \leq m < (l+2)/2), \quad l, m, n \in \mathbb{N}_+ = \{1, 2, 3, \dots\} \right\} \\ &= \{9, 10, 15, 16, 21, 25, 27, 28, 35, 36, 45, 49, 55, 56, 64, 66, 78, \\ &\quad 81, 84, 91, 100, \dots\} \end{aligned}$$

and the ring $E_{\mathbb{Q}} \stackrel{\text{def}}{=} \text{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field.

If $d-1$ is divisible by 4,

then there exists a finite ramified covering $\tilde{C} \rightarrow C$ such that, for any K nnemann compactification \tilde{X} of the N ron minimal model of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$, there exist algebraic isomorphisms

$$H^{2d-2}(\tilde{X}, \mathbb{Q}) \cong H^2(\tilde{X}, \mathbb{Q}), \quad H^{2d-3}(\tilde{X}, \mathbb{Q}) \cong H^3(\tilde{X}, \mathbb{Q}).$$

Simple Abelian varieties with multiplications from imaginary quadratic fields are especially interesting because sometimes they have non-trivial *Weil cycles* [21].

The author is grateful to S.G.Tankeev for interesting discussions.

§ 1. SOME REMARKS ON HODGE GROUPS, GLOBAL MONODROMY AND COHOMOLOGY OF LOCAL SYSTEMS

1.1. We may assume that

$$\begin{aligned} \{s \in C \mid \text{the fibre } \mathcal{M}_s \text{ is non-compact}\} &= \Delta \\ &\stackrel{\text{def}}{=} \{\delta \in C \mid \text{Sing}(X_\delta) \neq \emptyset\}. \end{aligned}$$

Set $C' = C \setminus \Delta$, $C' \xrightarrow{j} C$ the canonical embedding,

$$X' = X \setminus \pi^{-1}(\Delta), \quad \pi' = \pi|_{X'} : X' \rightarrow C'.$$

Considering, if necessary, a ramified covering $\tilde{C} \rightarrow C$ and a projective K nnemann model $\tilde{X} \rightarrow \tilde{C}$ of the corresponding N ron model $\tilde{\mathcal{M}} \rightarrow \tilde{C}$ of the generic scheme fibre of the canonical projection $X \times_C \tilde{C} \rightarrow \tilde{C}$, we may assume in virtue of [19, Section 5.8]; [20, Section 4.4, Theorem 4.6] that any singular fibre X_δ is a union of smooth irreducible components of multiplicity 1 with normal crossings. One may also assume that there is a *countable* subset $\Delta_{\text{countable}} \subset C'$ such that, for any point $s \in C' \setminus \Delta_{\text{countable}}$, the closure G of the image of the monodromy representation $\pi_1(C', s) \rightarrow \text{GL}(H^1(X_s, \mathbb{Q}))$ in the Zariski topology of the algebraic group $\text{GL}(H^1(X_s, \mathbb{Q}))$ is a connected semi-simple [22, Corollary 4.2.9] *normal* [23, Theorem 7.3] subgroup of the Hodge group

$$\text{Hg}(X_s) \stackrel{\text{def}}{=} \text{Hg}(H^1(X_s, \mathbb{Q}))$$

of the Abelian variety X_s . We may also assume that local monodromies (Picard - Lefschetz transformations) are *unipotent* and

$$\text{End}_{\overline{\kappa(\eta)}}(X_\eta) = \text{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}).$$

1.2. Consider the normalization $f : Z \rightarrow \pi^{-1}(\Delta)$ of the scheme $\pi^{-1}(\Delta)$. Then Z is a disjoint union of smooth irreducible components of the divisor $\pi^{-1}(\Delta)$. Since

$f : Z \rightarrow \pi^{-1}(\Delta)$ is a resolution of singularities of the subscheme $i_\Delta : \pi^{-1}(\Delta) \hookrightarrow X$, there is a canonical exact sequence of mixed Hodge \mathbb{Q} -structures [24, Corollary (8.2.8)]:

$$H^{n-2}(Z, \mathbb{Q}) \xrightarrow{(i_\Delta f)_*} H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q}),$$

where $(i_\Delta f)_*$ is a morphism of bidegree $(1, 1)$ of pure Hodge structures and φ_n is the restriction map. In particular,

$$(1.1) \quad (i_\Delta f)_* H^{n-2}(Z, \mathbb{Q}) = \text{Ker}[H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q})].$$

1.3. Since $\text{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the semi-simple part $\text{Hg}^{\text{ss}}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ of the reductive Hodge group $\text{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ [25, Proposition B57] is a \mathbb{Q} -simple algebraic group by Borovoi's theorem [26]. Therefore, by the triviality of the trace of the Abelian variety X_η , we have the equality

$$(1.2) \quad G = \text{Hg}^{\text{ss}}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}).$$

By the same reasons there is a canonical isomorphism [22, Proposition 4.4.11]

$$(1.3) \quad \text{End}_{C'}(X') \xrightarrow{\sim} \text{End}_{C'}(R^1 \pi'_* \mathbb{Z}).$$

A choice of a point $s \in C' \setminus \Delta_{\text{countable}}$ determines the canonical embeddings

$$\text{Im}[\pi_1(C', s) \rightarrow \text{GL}(H^1(X_s, \mathbb{Q}))] \subset G \subset \text{Hg}(X_s).$$

Therefore it follows from (1.3) and from the well known equality [25, Lemma B.60]

$$\text{End}_{\text{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that there are canonical maps

$$(1.4) \quad \begin{aligned} & \text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{End}_{\pi_1(C', s)} H^1(X_s, \mathbb{Q}) \\ & \hookrightarrow \text{End}_{\text{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned}$$

The restriction map $\text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, hence it follows from (1.4) that there exists a canonical isomorphism

$$(1.5) \quad \text{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, the Abelian variety X_s is simple and $\text{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an imaginary quadratic field. Hence, by Borovoi's theorem [26], the semi-simple part $\text{Hg}^{\text{ss}}(X_s)$ of the Hodge group $\text{Hg}(X_s)$ is a \mathbb{Q} -simple algebraic group, so that we have the equality

$$(1.6) \quad G = \text{Hg}^{\text{ss}}(X_s) \quad \text{for all } s \in C' \setminus \Delta_{\text{countable}}.$$

Besides, the canonical representation of the Lie algebra

$$\text{Lie } \text{Hg}^{\text{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$$

in the space $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is determined by *minuscule weights* [27], [23, Theorem 0.5.1] in Bourbaki's sense [28]. Since $\dim_{\mathbb{C}}(X_s) = d - 1 \notin \text{Ex}(4)$, it follows that $\text{Lie } \text{Hg}^{\text{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a simple Lie algebra of type A_{d-2} and there exists a canonical decomposition of $\text{Lie } \text{Hg}^{\text{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ -modules [29, Section 2.9]

$$(1.7) \quad H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = E(\omega_1) \oplus E(\omega_1)^\vee = E(\omega_1) \oplus E(\omega_{d-2}).$$

1.4. It follows from the existence of the natural embedding

$$R^2 \pi'_* \mathbb{Q} = \wedge^2 R^1 \pi'_* \mathbb{Q} \hookrightarrow R^1 \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^1 \pi'_* \mathbb{Q}$$

and from (1.3) that $H^0(C', R^2\pi'_*\mathbb{Q})$ is a rational Hodge structure of type (1, 1) because a polarization on X determines an isomorphism of families of Hodge structures [22, Section 4.2.3]

$$[R^1\pi'_*\mathbb{Q}]^\vee \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}(1),$$

the ring $\text{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the component of type (0, 0) of the Hodge \mathbb{Q} -structure $\text{End}_{C'}(R^1\pi'_*\mathbb{Q})$ [22, Section 4.4.6] and there are morphisms of rational Hodge structures

$$\begin{aligned} H^0(C', R^2\pi'_*\mathbb{Q}) &\hookrightarrow H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \\ &\xrightarrow{\sim} H^0(C', [R^1\pi'_*\mathbb{Q}]^\vee \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q})(-1) = \text{End}_{C'}(R^1\pi'_*\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}(-1) \\ &\xrightarrow{\sim} \text{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}(-1). \end{aligned}$$

Taking into account this fact and arguments of [15, § 2], it is easy to show the existence of an algebraic isomorphism

$$H^{2d-2}(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q}).$$

§ 2. SOME CANONICAL DECOMPOSITIONS OF RATIONAL HODGE STRUCTURES OF ODD WEIGHTS

2.1. By the assumption of the theorem, the generic scheme fibre \mathcal{M}_η of the Néron model is a principally polarized Abelian variety; consequently, for any point $s \in C'$, the Abelian variety X_s has a principal polarization determined by certain ample divisor H_s on the variety X_s .

It is known that the Poincaré bundle \mathcal{P}'_s on the variety $X_s \times \check{X}_s$ is uniquely determined (up to an isomorphism) by the following properties ([30], Ch. 2, § 5):

- (a) $\mathcal{P}'_s|_{X_s \times \{L_s\}} \xrightarrow{\sim} L_s$ for all $L_s \in \check{X}_s = \text{Pic}(X_s)$,
- (b) $\mathcal{P}'_s|_{\{0\} \times \check{X}_s} \xrightarrow{\sim} \mathcal{O}_{\check{X}_s}$.

Since \check{X}_s is an Abelian variety with a principal polarization, then we have the equalities $X_s = \text{Pic}^0(X_s) = \check{X}_s$. It follows easily from the properties (a) and (b) that an element $c_1(\mathcal{P}'_s) \in H^2(X_s \times X_s, \mathbb{Q})$ has Künneth type (1, 1) [30, Ch. 14, Lemma 14.1.9], so that

$$\begin{aligned} c_1(\mathcal{P}'_s) &\in [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})] \cap H^{1,1}(X_s \times X_s, \mathbb{C}) \\ &= [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\text{Hg}(X_s)}. \end{aligned}$$

Finitely, the correspondence $c_1(\mathcal{P}'_s)$ induces the algebraic isomorphism ([2, Section 2A1(ii), Theorem 2A9; [30], Ch. 16, § 16.4, P. 532))

$$H^{2d-3}(X_s, \mathbb{Q}) \xrightarrow{x \mapsto \text{pr}_{2s*}(\text{pr}_{1s}^*(x) \smile c_1(\mathcal{P}'_s))} H^1(\text{Pic}^0(X_s), \mathbb{Q}) = H^1(X_s, \mathbb{Q}).$$

Besides, for any point $s \in C'$ outside some countable subset $\Delta_{\text{countable}}$, the group G (defined in Section 1.1) is a normal subgroup of the Hodge group $\text{Hg}(X_s) = \text{Hg}(H^1(X_s, \mathbb{Q}))$ of a rational Hodge structure $H^1(X_s, \mathbb{Q})$ [23, Theorem 7.3]. We fixe such a point s . It follows from the existence of an inclusion $G \hookrightarrow \text{Hg}(X_s)$ that the correspondence $c_1(\mathcal{P}'_s)$ determines the section

$$\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \xrightarrow{\sim} [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\pi_1(C', s)}$$

of type (1, 1) of a local system of Hodge structures $R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$ inducing the correspondence $c_1(\mathcal{P}'_t)$ for any point $t \in C'$.

By Deligne's theorem, the canonical map $H^2(Y, \mathbb{Q}) \rightarrow H^0(C', R^2\tau'_*\mathbb{Q})$ is a surjective morphism of Hodge \mathbb{Q} -structures ([22], Theorem 4.1.1, a proof of Corollary 4.1.2). Since $\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \subset H^0(C', R^2\tau'_*\mathbb{Q})$ is an element of Hodge type $(1,1)$, it follows from the Lefschetz theorem on divisors that there exists an algebraic \mathbb{Q} -cycle $D^{(1)}$ on the variety Y such that the image of the class $\text{cl}_Y(D^{(1)}) \in H^2(Y, \mathbb{Q}) \cap H^{1,1}(Y, \mathbb{C})$ with respect to the canonical surjective morphism $H^2(Y, \mathbb{Q}) \rightarrow H^0(C', R^2\tau'_*\mathbb{Q})$ coincides with the section $\Lambda'_{1,1}$.

It is known that, for any point $s \in C'$, the algebraic correspondence $c_1(\mathcal{P}'_s)^{\smile 2}$ yields the algebraic isomorphism $H^{2d-4}(X_s, \mathbb{Q}) \xrightarrow{\sim} H^2(X_s, \mathbb{Q})$ ([2], Lemma 2A12, Remark 2A13; [30], Ch. 16, § 16.4, P. 532). Using arguments of [15, Sections 2.1 and 2.3], the canonical commutative diagram

$$\begin{array}{ccc} X \times_C X & \xrightarrow{p_1} & X \\ \downarrow p_2 & \searrow \tau & \downarrow \pi \\ X & \xrightarrow{\pi} & C, \end{array}$$

the canonical embedding $\iota : X \times_C X \hookrightarrow X \times X$, a resolution of singularities $\sigma : Y \rightarrow X \times_C X$ of the fibre product $X \times_C X$ such that σ induces an isomorphism over C' and $(\tau\sigma)^{-1}(\Delta)$ is a union of smooth divisors (of some positive multiplicities) with normal crossings, we see that the element $\Lambda'_{1,1}$ is induced by the class of certain divisor $D^{(1)}$ on the smooth projective compactification Y of the variety $X' \times_{C'} X'$ and the element $\Lambda'_{1,1}{}^{\smile 2}$ yields the isomorphism of local systems $R^{2d-4}\pi'_*\mathbb{Q} \xrightarrow{\sim} R^2\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$(2.1) \quad \begin{array}{ccc} R^{2d-4}\pi'_*\mathbb{Q} & \xrightarrow{(p'_1)^*} & R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q} \xrightarrow{\smile \Lambda'_{1,1}{}^{\smile 2}} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^2\pi'_*\mathbb{Q} \\ & & \xrightarrow{(p'_2)_*} R^2\pi'_*\mathbb{Q}. \end{array}$$

We may also expand (2.1) to a sequence of morphisms of sheaves

$$\begin{array}{ccc} R^{2d-4}\pi_*\mathbb{Q} & \xrightarrow{(p_1\sigma)^*} & R^{2d-4}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile \text{cl}_Y(D^{(1)})^{\smile 2}} R^{2d-2}(\tau\sigma)_*\mathbb{Q} \\ & & \xrightarrow{(p_2\sigma)_*} R^2\pi_*\mathbb{Q}, \end{array}$$

whose composite is an isomorphism outside the finite set Δ . Therefore there exists an isomorphism of bidegree $(3-d, 3-d)$ of mixed Hodge structures

$$(2.2) \quad H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \xrightarrow{[x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)})^{\smile 2})]_1} H^1(C, R^2\pi_*\mathbb{Q}).$$

2.2. Lemma. *For any odd natural number n , there are the equalities*

$$\begin{aligned} H^0(C', R^n\tau'_*\mathbb{Q}) &= H^0(C', R^n\pi'_*\mathbb{Q}) \\ &= H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) = H^2(C, R^n\pi_*\mathbb{Q}) = 0. \end{aligned}$$

Proof. By the condition of the theorem, $d-1$ is divisible by 4. Then, by (1.5) - (1.7), for all $s \in C' \setminus \Delta_{\text{countable}}$, the Lie algebra $\text{Lie } G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \text{Lie } \text{Hg}^{\text{ss}}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is a simple Lie algebra of type A_{d-2} and there is a decomposition of $\text{Lie } G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ -modules

$$H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = E(\omega_1) \oplus E(\omega_1)^\vee = E(\omega_1) \oplus E(\omega_{d-2}).$$

Let $E(\widetilde{\omega}_1)$ be the standard irreducible representation of degree 2 of the Lie algebra $sl_2(\overline{\mathbb{Q}})$ of type A_1 . Since $\dim_{\overline{\mathbb{Q}}} E(\omega_1) = d-1$ is divisible by 4, there exist

(non-canonical) identifications of $sl_2(\overline{\mathbb{Q}})$ -modules

$$E(\tilde{\omega}_1)^{\frac{d-1}{2}} \xrightarrow{\sim} [\overline{\mathbb{Q}}^2]^{\frac{d-1}{2}} = \overline{\mathbb{Q}}^{d-1},$$

so a choice of an appropriate basis of the space $E(\omega_1)$ determine an embedding of Lie algebras $sl_2(\overline{\mathbb{Q}}) \hookrightarrow sl(E(\omega_1))$.

For natural numbers p, q , the Klebsh - Gordon formula [28, Ch. VIII, § 9, n^o 4] yields the decomposition

$$\begin{aligned} & E(p\tilde{\omega}_1) \otimes_{\mathbb{Q}} E(q\tilde{\omega}_1) \\ &= E((p+q)\tilde{\omega}_1) \oplus E((p+q-2)\tilde{\omega}_1) \oplus E((p+q-4)\tilde{\omega}_1) \oplus \dots \end{aligned}$$

Therefore, if p is an odd natural number, then the representation of $sl_2(\overline{\mathbb{Q}})$ in $\wedge^p(E(\omega_1))$ is a sum of representations of type $E(\tilde{\omega}_1), E(3\tilde{\omega}_1), \dots, E(p\tilde{\omega}_1)$. On the other hand, if q is an even natural number, then the representation of $sl_2(\overline{\mathbb{Q}})$ in $\wedge^q(E(\omega_1))$ is a sum of representations of type $E(0), E(2\tilde{\omega}_1), E(4\tilde{\omega}_1), \dots, E(q\tilde{\omega}_1)$. Consequently, by the Schur lemma, one has for odd $p+q$:

$$\begin{aligned} & \text{Hom}_{G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}}(\wedge^p E(\omega_1), \wedge^q(E(\omega_1))) \\ &= \text{Hom}_{G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}}(\wedge^p E(\omega_1), \wedge^q(E(\omega_1))^\vee) = 0. \end{aligned}$$

Finally, in virtue of the theorem on local invariant cycles, the canonical maps

$$R^n(\tau\sigma)_*\mathbb{Q} \rightarrow j_*R^n(\tau'\sigma')_*\mathbb{Q}, \quad R^n\pi_*\mathbb{Q} \rightarrow j_*R^n\pi'_*\mathbb{Q}$$

are surjective with the kernels concentrated on the finite set Δ ([31], Proposition (15.12); [32], Section (3.7)). Therefore it remains to note that by [31, Proposition (10.5)], one has

$$\begin{aligned} & H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) = H^2(C, j_*R^n(\tau'\sigma')_*\mathbb{Q}) \\ & \xrightarrow{\sim} H^0(C', R^n(\tau'\sigma')_*\mathbb{Q})^\vee = H^0(C', R^n\tau'_*\mathbb{Q})^\vee; \\ & H^2(C, R^n\pi_*\mathbb{Q}) = H^2(C, j_*R^n\pi'_*\mathbb{Q}) \xrightarrow{\sim} H^0(C', R^n\pi'_*\mathbb{Q})^\vee. \end{aligned}$$

The lemma is proved.

2.3. Let

$$\begin{aligned} K_{nX} &= \text{Ker}[H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n\pi_*\mathbb{Q})], \\ K_{nY} &= \text{Ker}[H^n(Y, \mathbb{Q}) \rightarrow H^0(C, R^n(\tau\sigma)_*\mathbb{Q})]. \end{aligned}$$

The Leray spectral sequences $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ and $E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$ are degenerated: $E_2^{p,q} = E_\infty^{p,q}$ [31, Corollary (15.15)]. Hence, for any natural number n , there are exact sequences of Hodge \mathbb{Q} -structures ([13], Formula (2.4))

$$(2.3) \quad 0 \rightarrow H^2(C, R^{n-2}\pi_*\mathbb{Q}) \rightarrow K_{nX} \xrightarrow{\alpha_{nX}} H^1(C, R^{n-1}\pi_*\mathbb{Q}) \rightarrow 0,$$

$$(2.4) \quad 0 \rightarrow H^2(C, R^{n-2}(\tau\sigma)_*\mathbb{Q}) \rightarrow K_{nY} \xrightarrow{\alpha_{nY}} H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \rightarrow 0.$$

It follows from Lemma 2.2, (2.3), (2.4), the theorem on local invariant cycles and the strong Lefschetz theorem for fibres of a smooth morphism π' that

$$(2.5) \quad K_{3X} = H^1(C, R^2\pi_*\mathbb{Q});$$

$$(2.6) \quad K_{(2d-3)X} = H^1(C, R^{2d-4}\pi_*\mathbb{Q});$$

$$(2.7) \quad \begin{aligned} \text{cl}_X(H)^{\smile d-3} \smile K_{3X} &= \text{cl}_X(H)^{\smile d-3} \smile H^1(C, R^2\pi_*\mathbb{Q}) \\ &= \text{cl}_X(H)^{\smile d-3} \smile H^1(C, j_*R^2\pi'_*\mathbb{Q}) = K_{(2d-3)X}. \end{aligned}$$

Moreover, taking into account the non-degeneracy of the canonical pairing [31, Proposition (10.5)]

$$\begin{aligned} &H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q}) \times H^1(C, j_*R^2\pi'_*\mathbb{Q}) \\ &\xrightarrow{x \times x' \mapsto x \smile x'} H^2(C, j_*R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \end{aligned}$$

we see, that the canonical pairing

$$K_{(2d-3)X} \times K_{3X} \xrightarrow{x \times x' \mapsto x \smile x'} H^{2d}(X, \mathbb{Q})$$

is non-degenerate. Therefore the restriction of the *non-degenerate* [2, Section 1.2.A] bilinear form

$$\begin{aligned} \Phi : H^3(X, \mathbb{Q}) \times H^3(X, \mathbb{Q}) &\xrightarrow{x \times y \mapsto x \smile y \smile \text{cl}_X(H)^{\smile d-3}} \\ &H^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow{(2\pi i)^d} \mathbb{Q}. \end{aligned}$$

to the subspace $K_{3X} \subset H^3(X, \mathbb{Q})$ is non-degenerate. Hence, as it was noticed in Section 1.2 of [15], there exists the decomposition of rational Hodge structures

$$(2.8) \quad H^3(X, \mathbb{Q}) = K_{3X} \oplus K_{3X}^\perp,$$

where K_{3X}^\perp is the orthogonal complement of the subspace $K_{3X} \hookrightarrow H^3(X, \mathbb{Q})$ with respect to the form Φ . In virtue of (2.7) we have:

$$\begin{aligned} K_{3X}^\perp &= \{x \in H^3(X, \mathbb{Q}) \mid x \smile \text{cl}_X(H)^{\smile d-3} \smile K_{3X} = 0\} \\ &= \{x \in H^3(X, \mathbb{Q}) \mid x \smile K_{(2d-3)X} = 0\}. \end{aligned}$$

Consequently, the decomposition (2.8) is *canonical* and it does not depend on the choice of an ample divisor H ; moreover, by (2.7), (2.8) and by the strong Lefschetz theorem, there is a *canonical* decomposition of rational Hodge structures

$$(2.9) \quad H^{2d-3}(X, \mathbb{Q}) = K_{(2d-3)X} \oplus K_{(2d-3)X}^\perp,$$

where

$$\begin{aligned} K_{(2d-3)X}^\perp &\stackrel{\text{def}}{=} \text{cl}_X(H)^{\smile d-3} \smile K_{3X}^\perp \\ &= \{x \in H^{2d-3}(X, \mathbb{Q}) \mid K_{3X} \smile x = 0\}. \end{aligned}$$

2.4. For any point $s \in C$, we denote by $\iota_{X_s/X} : X_s \hookrightarrow X$ the canonical embedding. The morphism π is proper, therefore the fibre of the sheaf $R^n\pi_*\mathbb{Q}$ over a point $s \in C$ coincides with the space $H^n(X_s, \mathbb{Q})$ ([33], Ch. II, § 4, Remark 4.17.1; [34], Ch. VI, § 2, Corollary 2.5). Consequently, the restriction map $\iota_{X_s/X}^*$ coincides with the composite ([35], Vol. II, Ch. 4, Section 4.3.1)

$$H^n(X, \mathbb{Q}) \rightarrow E_\infty^{0,n}(\pi) \rightarrow E_2^{0,n}(\pi) = H^0(C, R^n\pi_*\mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q}).$$

Thus the map $\iota_{X_s/X}^*$ is the composite of canonical maps

$$H^n(X, \mathbb{Q}) \rightarrow H^0(C, R^n \pi_* \mathbb{Q}) \hookrightarrow \prod_{s \in C} H^n(X_s, \mathbb{Q}) \rightarrow H^n(X_s, \mathbb{Q}),$$

where the \mathbb{Q} -space $\prod_{s \in C} H^n(X_s, \mathbb{Q})$ is identified with the \mathbb{Q} -space of *discontinuous* global sections of the sheaf $R^n \pi_* \mathbb{Q}$ ([33], Ch. II, § 4, Section 4.4.4). Evidently

$$(2.10) \quad \omega \in K_{nX} \Leftrightarrow (\forall s \in C) \quad \iota_{X_s/X}^*(\omega) = 0.$$

§ 3. CONSTRUCTIONS OF ALGEBRAIC ISOMORPHISMS OF COHOMOLOGY OF ODD WEIGHTS

3.1. Lemma. *There is the equality $(i_{\Delta} f)_* H^1(Z, \mathbb{Q}) = K_{3X}^{\perp}$.*

Proof. In virtue of Lemma 2.2 and the surjectivity [31, Corollary (15.14)] of the canonical edge map $H^3(X, \mathbb{Q}) \rightarrow H^0(C, R^3 \pi_* \mathbb{Q})$ of the Leray spectral sequence $E_2^{p,q}(\pi)$ one may assume that $\Delta \neq \emptyset$ because (2.8) induces a canonical isomorphism $K_{3X}^{\perp} \xrightarrow{\sim} H^0(C, R^3 \pi_* \mathbb{Q})$.

For any irreducible smooth projective variety W , we denote by

$$\langle \rangle: H^{2 \dim W}(W, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$$

the orientation isomorphism of Weil cohomology ([2], Section 1.2.A) determined by a choice of an element $\sqrt{-1} \in \mathbb{C}$.

Irreducible components of a smooth divisor Z are naturally identified with irreducible components $X_{\delta i_{\delta}}$ ($i_{\delta} \in \{1, \dots, m_{\delta}\}$) of the divisor $\pi^{-1}(\Delta) = \sum_{\delta \in \Delta} X_{\delta}$. Denote by $\iota_{X_{\delta i_{\delta}}/X}: X_{\delta i_{\delta}} \hookrightarrow X$, $\iota_{X_{\delta i_{\delta}}/X_{\delta}}: X_{\delta i_{\delta}} \hookrightarrow X_{\delta}$, $\iota_{X_{\delta i_{\delta}}/Z}: X_{\delta i_{\delta}} \hookrightarrow Z$ the canonical embeddings. In accordance with the functoriality of cohomology with coefficients in the field \mathbb{Q} there is the equality $\iota_{X_{\delta i_{\delta}}/X}^* = \iota_{X_{\delta i_{\delta}}/X_{\delta}}^* \iota_{X_{\delta}/X}^*$. On the other hand, $\iota_{X_{\delta}/X}^* K_{(2d-3)X} = 0$ by (2.10). Hence $\iota_{X_{\delta i_{\delta}}/X}^* K_{(2d-3)X} = 0$. By definition, the operator $\iota_{X_{\delta i_{\delta}}/X}^*$ is conjugate to the operator $\iota_{X_{\delta i_{\delta}}/X}^*$ ([2], Section 1.2.A; [34], Ch. VI, § 11, Remark 11.6), therefore

$$\begin{aligned} & \langle \iota_{X_{\delta i_{\delta}}/X}^* H^1(X_{\delta i_{\delta}}, \mathbb{Q}) \smile K_{(2d-3)X} \rangle \\ &= \langle H^1(X_{\delta i_{\delta}}, \mathbb{Q}) \smile \iota_{X_{\delta i_{\delta}}/X}^* K_{(2d-3)X} \rangle. \end{aligned}$$

Consequently, $\iota_{X_{\delta i_{\delta}}/X}^* H^1(X_{\delta i_{\delta}}, \mathbb{Q}) \smile K_{(2d-3)X} = 0$. From the commutativity of the diagram

$$\begin{array}{ccc} X_{\delta i_{\delta}} & \xrightarrow{\iota_{X_{\delta i_{\delta}}/X}} & X \\ \parallel & & \uparrow i_{\Delta} f \\ X_{\delta i_{\delta}} & \xrightarrow{\iota_{X_{\delta i_{\delta}}/Z}} & Z \end{array}$$

of canonical morphisms and ([34], Ch. VI, § 11, Remark 11.6 (c)) we obtain the equality

$$(3.1) \quad (i_{\Delta} f)_*(\iota_{X_{\delta i_{\delta}}/Z})_* |_{H^1(X_{\delta i_{\delta}}, \mathbb{Q})} = \iota_{X_{\delta i_{\delta}}/X}^* |_{H^1(X_{\delta i_{\delta}}, \mathbb{Q})}.$$

Thus $(i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \smile K_{(2d-3)X} = 0$, so that from (2.7) we obtain the equality

$$(i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \smile \text{cl}_X(H)^{-d-3} \smile K_{3X} = 0$$

and the existence of the inclusion

$$(3.2) \quad (i_{\Delta} f)_* H^1(Z, \mathbb{Q}) \subset K_{3X}^{\perp}.$$

In accordance with Lemma 2.2 we have the equalities

$$H^2(C, R^1\pi_*\mathbb{Q}) = H^0(C', R^3\pi'_*\mathbb{Q}) = 0.$$

On the other hand, the theorem on local invariant cycles and the Leray spectral sequence for the embedding $j : C' \subset C$ yield the embedding of mixed Hodge \mathbb{Q} -structures $H^1(C, R^2\pi_*\mathbb{Q}) \hookrightarrow H^1(C', R^2\pi'_*\mathbb{Q})$ ([31], P. 457, Corollary (13.10), Remark (14.5)). Besides, the canonical map

$$H^3(X, \mathbb{Q}) \rightarrow H^0(C, R^3\pi_*\mathbb{Q})$$

is surjective ([31], Corollary (15.14)). Consequently, taking into account (1.1), (2.3), the diagram (15.1) in [31], the degeneracy of the Leray spectral sequence $E_2^{p,q}(\pi') = H^p(C', R^q\pi'_*\mathbb{Q})$ ([22], Theorem 4.1.1), the equality $H^2(C', R^1\pi'_*\mathbb{Q}) = 0$ (because the cohomological dimension of the affine curve C' equals 1 ([34], Ch. VI, § 7, Theorem 7.2)) and the commutativity of the diagram of morphisms

$$\begin{array}{ccc} X' & \subset & X \\ \downarrow \pi' & & \downarrow \pi \\ C' & \subset & C, \end{array}$$

we obtain the commutative diagram of mixed Hodge \mathbb{Q} -structures with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & H^1(C, R^2\pi_*\mathbb{Q}) & \rightarrow & H^3(X, \mathbb{Q}) & \rightarrow & H^0(C, R^3\pi_*\mathbb{Q}) \rightarrow 0 \\ & & \cap & & \downarrow \varphi_3 & & \downarrow \\ 0 & \rightarrow & H^1(C', R^2\pi'_*\mathbb{Q}) & \rightarrow & H^3(X', \mathbb{Q}) & \rightarrow & 0. \end{array}$$

Therefore (2.5), (2.8) yield the commutative diagram

$$(3.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & K_{3X} & \rightarrow & K_{3X} \oplus K_{3X}^\perp & \rightarrow & K_{3X}^\perp \rightarrow 0 \\ & & \cap & & \downarrow \varphi_3 & & \downarrow \\ 0 & \rightarrow & H^1(C', R^2\pi'_*\mathbb{Q}) & \rightarrow & H^3(X', \mathbb{Q}) & \rightarrow & 0. \end{array}$$

Evidently $\text{Im}(\varphi_3) = K_{3X}$, therefore (3.3) yields the commutative diagram of canonical morphisms of Hodge \mathbb{Q} -structures

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{3X} & \rightarrow & K_{3X} \oplus K_{3X}^\perp & \rightarrow & K_{3X}^\perp \rightarrow 0 \\ & & \parallel & & \downarrow \varphi_3 & & \downarrow \\ 0 & \rightarrow & K_{3X} & \rightarrow & K_{3X} & \rightarrow & 0. \end{array}$$

The corresponding exact sequence of Hodge \mathbb{Q} -structures ([13], Section 2.6) of the snake-like diagram ([36], § 1, Proposition 2) and (1.1) yield the canonical identifications

$$(i_{\Delta f})_* H^1(Z, \mathbb{Q}) = \text{Ker}(\varphi_3) = K_{3X}^\perp.$$

The lemma is proved.

3.2. Lemma. *There is a canonical embedding $(p_2\sigma)_*(K_{(2d+1)Y}) \subset K_{3X}$.*

Proof. Since

$$K_{3X} = \{x \in H^3(X, \mathbb{Q}) \mid x \smile K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3} = 0\},$$

it suffices to check the equality

$$(p_2\sigma)_*(K_{(2d+1)Y}) \smile K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3} = 0,$$

which is equivalent ([2], Section 1.2.A; [34], Ch. VI, § 11, Remark 11.6) to the equalities

$$(3.4) \quad \begin{aligned} & \langle (p_2\sigma)_*(K_{(2d+1)Y}) \smile K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3} \rangle = \\ & \langle K_{(2d+1)Y} \smile (p_2\sigma)^*(K_{3X}^\perp \smile \text{cl}_X(H)^{\smile d-3}) \rangle = 0. \end{aligned}$$

If $\Delta = \emptyset$, then the formula (3.4) is true in virtue of Lemma 3.1. From now on we assume that $\Delta \neq \emptyset$.

By (2.4) and Lemma 2.2 we have the canonical identification

$$(3.5) \quad K_{(2d+1)Y} = H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}).$$

It is known that the Gysin map $\iota_{X_{\delta i_\delta}/X*} : H^k(X_{\delta i_\delta}, \mathbb{Q}) \rightarrow H^{k+2}(X, \mathbb{Q})$ is given by $\alpha \mapsto \alpha \smile \text{cl}_X(X_{\delta i_\delta})$ ([15], Formula (4.20)). On the other hand, the strong Lefschetz theorem for the variety $X_{\delta i_\delta}$ yields the existence of the embedding

$$H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile \iota_{X_{\delta i_\delta}/X*}^* \text{cl}_X(H)^{\smile d-3} \subset H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}).$$

Therefore the projection formula ([2], Section 1.2.A) yields the inclusion

$$\begin{aligned} & \iota_{X_{\delta i_\delta}/X*} H^1(X_{\delta i_\delta}, \mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-3} \\ & \subset \iota_{X_{\delta i_\delta}/X*} H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}) = H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}) \smile \text{cl}_X(X_{\delta i_\delta}). \end{aligned}$$

Finitely, it follows from (3.1) that there is the embedding

$$(3.6) \quad \begin{aligned} & (i_{\Delta}f)_* H^1(Z, \mathbb{Q}) \smile \text{cl}_X(H)^{\smile d-3} \\ & \subset \sum_{\delta \in \Delta, i_\delta \in \{1, \dots, m_\delta\}} H^{2d-5}(X_{\delta i_\delta}, \mathbb{Q}) \smile \text{cl}_X(X_{\delta i_\delta}). \end{aligned}$$

By definition ([35], Vol. II, Ch. 4, Section 4.2.1), for any point $s \in C'$, the \smile -product by the class $\text{cl}_X(X_{\delta i_\delta}) \in H^2(X, \mathbb{Q})$ acts on the fibre $H^q(X_s, \mathbb{Q}) = [j_* R^q \pi'_* \mathbb{Q}]_s$ of the sheaf $j_* R^q \pi'_* \mathbb{Q}$ as the \smile -product by the class $\iota_{X_s/X}^*(\text{cl}_X(X_{\delta i_\delta}))$. It follows from the evident equality

$$\iota_{X_s/X}^*(\text{cl}_X(X_{\delta i_\delta})) = 0$$

that

$$(3.7) \quad j_* R^q \pi'_* \mathbb{Q} \smile \text{cl}_X(X_{\delta i_\delta}) = 0.$$

In virtue of the theorem on local invariant cycles and by Künneth's formula on fibres of a smooth morphism $\tau' : Y' = X' \times_{C'} X' \rightarrow C'$ there is a canonical decomposition

$$(3.8) \quad H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}) = \oplus_{p+q=2d} H^1(C, j_*(R^p \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^q \pi'_* \mathbb{Q})).$$

It follows from (3.7) that

$$H^1(C, j_*(R^p \pi'_* \mathbb{Q} \otimes_{\mathbb{Q}} R^q \pi'_* \mathbb{Q})) \smile [1 \otimes_{\mathbb{Q}} \text{cl}_X(X_{\delta i_\delta})] = 0,$$

so that (3.4) follows from Lemma 3.1, (3.5), (3.6), (3.8). The lemma is proved.

3.3. There is a non-degenerate canonical pairing [31, Proposition (10.5)]

$$\begin{aligned} & H^1(C, j_* R^{2d-4} \pi'_* \mathbb{Q}) \times H^1(C, j_* R^2 \pi'_* \mathbb{Q}) \xrightarrow{x \times x' \mapsto x \smile x'} \\ & H^2(C, j_* R^{2d-2} \pi'_* \mathbb{Q}) = H^2(C, R^{2d-2} \pi_* \mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}, \end{aligned}$$

identifying (in accordance with the theorem on local invariant cycles) the space $H^1(C, R^{2d-4} \pi'_* \mathbb{Q})^\vee$ with $H^1(C, R^2 \pi'_* \mathbb{Q})$. By the similar reason we have the identification $H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q})^\vee = H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q})$. Therefore by the theorem on

local invariant cycles, (2.5), (3.5) and Lemma 3.2 we obtain a commutative diagram

$$(3.9) \quad \begin{array}{ccc} K_{3X} & \xleftarrow{(p_2\sigma)^*|_{K_{(2d+1)Y}}} & K_{(2d+1)Y} \\ \parallel & & \parallel \\ H^1(C, j_* R^2 \pi'_* \mathbb{Q}) & \xleftarrow{[(p_2\sigma)^*]_1 = [p'_2]_*]_1} & H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q}) = H^1(C, j_* R^{2d} \tau'_* \mathbb{Q}). \end{array}$$

Finally, in accordance with ([37], Ch. 2, § 8, Formula (5)) and by (2.10) we obtain the equalities

$$\begin{aligned} & \iota_{X_s/X}^*(K_{(2d-3)Y} \smile \text{cl}_Y(D^{(1)})^{\smile 2}) \\ &= \iota_{X_s/X}^*(K_{(2d-3)Y}) \smile \iota_{X_s/X}^*(\text{cl}_Y(D^{(1)})^{\smile 2}) = 0, \end{aligned}$$

so in virtue of (2.6), (2.10) there is a commutative diagram

$$(3.10) \quad \begin{array}{ccc} K_{(2d-3)Y} & \xrightarrow{\smile \text{cl}_Y(D^{(1)})^{\smile 2}} & K_{(2d+1)Y} \\ \parallel & & \parallel \\ H^1(C, R^{2d-4}(\tau\sigma)_* \mathbb{Q}) & \xrightarrow{[\smile \text{cl}_Y(D^{(1)})^{\smile 2}]_1} & H^1(C, R^{2d}(\tau\sigma)_* \mathbb{Q}). \end{array}$$

On the other hand, the commutative diagram of morphisms

$$\begin{array}{ccc} X & \xleftarrow{p_1\sigma} & Y \\ \uparrow \iota_{X_s/X} & & \uparrow \iota_{Y_s/Y} \\ X_s & \xleftarrow{p_{1s}\sigma_s} & Y_s \end{array}$$

yields the commutative diagram of canonical maps

$$\begin{array}{ccc} H^{2d-3}(X, \mathbb{Q}) & \xrightarrow{(p_1\sigma)^*} & H^{2d-3}(Y, \mathbb{Q}) \\ \downarrow \iota_{X_s/X}^* & & \downarrow \iota_{Y_s/Y}^* \\ H^{2d-3}(X_s, \mathbb{Q}) & \xrightarrow{(p_{1s}\sigma_s)^*} & H^{2d-3}(Y_s, \mathbb{Q}), \end{array}$$

so that, for any point $s \in C$, it follows from (2.10) that

$$\iota_{X_s/X}^*(K_{(2d-3)X}) = 0$$

and

$$\iota_{Y_s/Y}^*(p_1\sigma)^*(K_{(2d-3)X}) = 0.$$

The morphism $\tau\sigma$ is proper, therefore by arguments of Section 2.4 the canonical restriction map

$$\iota_{Y_s/Y}^* : H^{2d-3}(Y, \mathbb{Q}) \rightarrow H^{2d-3}(Y_s, \mathbb{Q})$$

is the composite of canonical maps

$$\begin{aligned} H^{2d-3}(Y, \mathbb{Q}) &\rightarrow H^0(C, R^{2d-3}(\tau\sigma)_* \mathbb{Q}) \hookrightarrow \prod_{s \in C} H^{2d-3}(Y_s, \mathbb{Q}) \\ &\rightarrow H^{2d-3}(Y_s, \mathbb{Q}). \end{aligned}$$

Thus, it follows from the equalities $\iota_{Y_s/Y}^*(p_1\sigma)^*(K_{(2d-3)X}) = 0$ ($s \in C$) and from (2.10) that $(p_1\sigma)^*K_{(2d-3)X} \hookrightarrow K_{(2d-3)Y}$. By (2.4), (2.6) and Lemma 2.2 we have a commutative diagram

$$(3.11) \quad \begin{array}{ccc} K_{(2d-3)X} & \xrightarrow{(p_1\sigma)^*} & K_{(2d-3)Y} \\ \parallel & & \parallel \\ H^1(C, R^{2d-4}\pi_* \mathbb{Q}) & \xrightarrow{[(p_1\sigma)^*]_1} & H^1(C, R^{2d-4}(\tau\sigma)_* \mathbb{Q}). \end{array}$$

Gluing together diagrams (3.9) - (3.11), we see that (2.2) takes the form

$$(3.12) \quad K_{(2d-3)X} \xrightarrow{x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \text{cl}_Y(D^{(1)})^{-2})} K_{3X}.$$

For $x \in H^{2d-3}(X, \mathbb{Q})$, the projection formula [2, Section 1.2.A] yields the equalities

$$\begin{aligned} & (p_2\sigma)_* \left((p_1\sigma)^*x \smile [\text{cl}_Y(D^{(1)})]^{-2} \right) \\ &= [\text{pr}_2 \iota\sigma]_* \left([\text{pr}_1 \iota\sigma]^*x \smile [\text{cl}_Y(D^{(1)})]^{-2} \right) \\ &= \text{pr}_{2*}(\iota\sigma)_* \left((\iota\sigma)^* \text{pr}_1^*x \smile [\text{cl}_Y(D^{(1)})]^{-2} \right) \\ &= \text{pr}_{2*} \left(\text{pr}_1^*x \smile (\iota\sigma)_* [\text{cl}_Y(D^{(1)})]^{-2} \right). \end{aligned}$$

Therefore an algebraic class $(\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}] \in H^6(X \times X, \mathbb{Q})$ determines the isomorphism (3.12), which takes the form

$$(3.13) \quad K_{(2d-3)X} \xrightarrow{x \mapsto \text{pr}_{2*}(\text{pr}_1^*x \smile (\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}])} K_{3X}.$$

3.4. By arguments of [15, Section 3.5], we have the isomorphism

$$(3.14) \quad K_{(2d-3)X}^\perp \xrightarrow{x \mapsto \text{pr}_{2*}(\text{pr}_1^*(x) \smile \wp(K_{3X}^\perp))} K_{3X}^\perp,$$

where $\wp(K_{3X}^\perp)$ is an *algebraic* (by Lemma 3.1 and [15, Lemma 3.8]) Poincaré class.

On the other hand, taking into account decompositions (2.8), (2.9), the existence of *algebraic* isomorphisms (3.13), (3.14) and arguments of [15, Sections 4.1 - 4.4], it is easy to show the existence of an *algebraic* isomorphism

$$H^{2d-3}(X, \mathbb{Q}) \xrightarrow{x \mapsto \text{pr}_{2*}(\text{pr}_1^*x \smile (u_{3,3} + u_{3,3^\perp} + h_{10} + \wp(K_{3X}^\perp)))} H^3(X, \mathbb{Q}),$$

where $u_{3,3} \in K_{3X} \otimes_{\mathbb{Q}} K_{3X}$, $u_{3,3^\perp} \in K_{3X} \otimes_{\mathbb{Q}} K_{3X}^\perp$ are components of an algebraic class $u \stackrel{\text{def}}{=} (\iota\sigma)_* [[\text{cl}_Y(D^{(1)})]^{-2}] \in H^6(X \times X, \mathbb{Q})$ and h_{10} is an appropriate element of the \mathbb{Q} -space $\bigoplus_{p+q=6, p \neq 3} H^p(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(X, \mathbb{Q})$. The theorem is proved.

REFERENCES

- [1] A. Grothendieck, *Standard conjectures on algebraic cycles*, Algebr. Geom., Bombay Colloq. 1968, 193–199. Zbl 0201.23301
- [2] S.L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix Exposés Cohomologie Schémas, Adv. Stud. Pure Math., **3**, North-Holland, Amsterdam; Masson, Paris, 1968, 359–386. Zbl 0198.25902
- [3] S.G. Tankeev, *On the standard conjecture for complex abelian schemes over smooth projective curves*, Izv. Math., **67**:3 (2003), 597–635. Zbl 1072.14011
- [4] S.G. Tankeev, *On the numerical equivalence of algebraic cycles on potentially simple abelian schemes of prime relative dimension*, Izv. Math., **69**:1 (2005), 143–162. Zbl 1075.14004
- [5] S.G. Tankeev, *Monoidal transformations and conjectures on algebraic cycles*, Izv. Math., **71**:3 (2007), 629–655. Zbl 1139.14011
- [6] D. Lieberman, *Numerical and homological equivalence of algebraic cycles on Hodge manifolds*, Am. J. Math., **90**:2 (1968), 366–374. Zbl 0159.50501
- [7] S.G. Tankeev, *On the standard conjecture of Lefschetz type for complex projective threefolds. II*, Izv. Math., **75**:5 (2011), 1047–1062. Zbl 1234.14009
- [8] D. Arapura, *Motivation for Hodge cycles*, Adv. Math., **207**:2 (2006), 762–781. Zbl 1109.14012

- [9] F. Charles, E. Markman, *The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces*, Compos. Math., **149**:3 (2013), 481–494. Zbl 1312.14012
- [10] O.V. Nikol'skaya, *On algebraic cycles on a fibre product of families of K3-surfaces*, Izv. Math., **77**:1 (2013), 143–162. Zbl 1408.14029
- [11] O.V. Nikol'skaya, *On algebraic cycles on fibre products of non-isotrivial families of regular surfaces with geometric genus 1*, Modeling and analysis of information systems, **23**:4 (2016), 440–465.
- [12] S.G. Tankeev, *On the standard conjecture and the existence of the Chow-Lefschetz decomposition for complex projective varieties*, Izv. Math., **79**:1 (2015), 177–207. Zbl 1318.14012
- [13] S.G. Tankeev, *On an inductive approach to the standard conjecture for a fibred complex variety with strong semistable degeneracies*, Izv. Math., **81**:6 (2017), 1253–1285. Zbl 1400.14067
- [14] F. Charles, *Remarks on the Lefschetz standard conjecture and hyperkähler varieties*, Comment. Math. Helv., **88**:2 (2013), 449–468. Zbl 1276.14011
- [15] S.G. Tankeev, *On algebraic isomorphisms of rational cohomology of a Künnemann compactification of the Néron minimal model*, Sib. Elektron. Mat. Izv., **17** (2020), 89–125. Zbl 1433.14006
- [16] S.G. Tankeev, *On the standard conjecture for a fibre product of three elliptic surfaces with pairwise-disjoint discriminant loci*, Izv. Math., **83**:3 (2019), 613–653. Zbl 1420.14016
- [17] S.G. Tankeev, *On the standard conjecture for a 3-dimensional variety fibred by curves with a non-injective Kodaira-Spencer map*, Izv. Math., **84**:5 (2020), 1016–1035. Zbl 1460.14024
- [18] A. Grothendieck, *Modèles de Néron et monodromie*, in *Sémin. Géom. Algébrique, Bois-Marie 1967–1969, SGA 7 I, Exp. No. 9*, Lect. Notes Math., **288**, Springer-Verlag, Berlin etc., 1972, 313–523. Zbl 0248.14006
- [19] K. Künnemann, *Height pairings for algebraic cycles on abelian varieties*, Ann. Sci. Éc. Norm. Supér., (4), **34**:4 (2001), 503–523. Zbl 1073.14527
- [20] K. Künnemann, *Projective regular models for abelian varieties, semistable reduction, and the height pairing*, Duke Math. J., **95**:1 (1998), 161–212. Zbl 0955.14017
- [21] B. Moonen, Yu. Zarhin, *Weil classes on abelian varieties*, J. Reine Angew. Math., **496** (1998), 83–92. Zbl 0887.14004
- [22] P. Deligne, *Théorie de Hodge. II*, Publ. Math., Inst. Hautes Étud. Sci., **40**:1 (1971), 5–57. Zbl 0219.14007
- [23] Yu.G. Zarhin, *Weights of simple Lie algebras in cohomology of algebraic varieties*, Math. USSR, Izv., **24**:2 (1985), 245–281. Zbl 0579.14019
- [24] P. Deligne, *Théorie de Hodge. III*, Publ. Math., Inst. Hautes Étud. Sci., **44** (1974), 5–77. Zbl 0237.14003
- [25] B.B. Gordon, *A survey of the Hodge conjecture for Abelian varieties*, in: J.D. Lewis, *A survey of the Hodge conjecture. With an appendix by B. Brent Gordon*, CRM Monograph Series, **10**, AMS, Providence, 1999, 297–356. Zbl 0922.14004
- [26] M.V. Borovoi, *The Hodge group and the algebra of endomorphisms of an abelian variety*, in *Vopr. Teor. Grupp Gomologicheskoy Algebr*, Yaroslavl, 1981, 124–128. Zbl 0494.14017
- [27] P. Deligne, *Variétés de Shimura: Interprétation modulaires, et techniques de construction de modeles canoniques*, Proc. Symp. Pure Math., **33**:2 (1979), 247–290. Zbl 0437.14012
- [28] N. Bourbaki, *Groupes et algèbres de Lie, Chaps. 1–8*, Actualités Sci. Indust., **1285**, **1349**, **1337**, **1364**, Hermann, Paris, 1971, 1972, 1968, 1975. Zbl 0213.04103, 0244.22007, 0186.33001, 0329.17002
- [29] S.G. Tankeev, *Cycles on abelian varieties and exceptional numbers*, Izv. Math., **60**:2 (1996), 391–424. Zbl 0899.14021
- [30] C. Birkenhake, H. Lange, *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften, **302**, Springer-Verlag, Berlin, 1992. Zbl 0779.14012
- [31] S. Zucker, *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, Ann. Math. (2), **109** (1979), 415–476. Zbl 0446.14002
- [32] C.H. Clemens, *Degeneration of Kähler manifolds*, Duke Math. J., **44**:2 (1977), 215–290. Zbl 0353.14005
- [33] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958. Zbl 0080.16201
- [34] J.S. Milne, *Étale cohomology*, Princeton Univ. Press, Princeton, 1980. Zbl 0433.14012

- [35] C. Voisin, *Hodge theory and complex algebraic geometry. I,II*, Société Mathématique de France, Paris, 2002, Cambridge University Press, Cambridge, 2003. Zbl 1032.14001, 1032.14002
- [36] N. Bourbaki, *Éléments de mathématique, Algèbre. Ch. X: Algèbre homologique*, Masson, Paris etc., 1980. Zbl 0455.18010
- [37] G. Bredon, *Sheaf theory*, McGraw-Hill, New York etc., 1967. Zbl 0158.20505

OL'GA VLADIMIROVNA MAKAROVA
VLADIMIR STATE UNIVERSITY,
87, GORKIJ STR.,
VLADIMIR, 600000, RUSSIA
Email address: papichonok@yandex.ru