AREA OF A TRIANGLE AND ANGLE BISECTORS

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ABSTRACT. Consider a triangle ABC with given lengths l_a, l_b, l_c of its internal angle bisectors. We prove that in general, it is impossible to construct a square of the same area as ABC using a ruler and compass. Moreover, it is impossible to express the area of ABC in radicals of l_a, l_b, l_c.

Keywords: area of a triangle, angle bisectors, ruler and compass construction, Galois group of a polynomial, algebraic equation, solution in radicals.

Consider a triangle ABC on a Euclidean plane. Let $BC = a$, $AC = b$ and $AB = c$. By m_a, m_b, m_c we denote the medians of ABC to the sides a, b and c respectively. Similarly, h_a, h_b, h_c are the altitudes of ABC and l_a, l_b, l_c are the angle bisectors.

Recall that according to Heron's formula, the area S of ABC is equal to

$$S = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = (a+b+c)/2$ is the semiperimeter of the triangle. There are equivalent formulae which express S in terms of the medians

$$S = \frac{1}{3}\sqrt{(m_a+m_b+m_c)(m_a+m_b-m_c)(m_b+m_c-m_a)(m_c+m_a-m_b)},$$

or the altitudes

$$\frac{1}{S} = \sqrt{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)\left(\frac{1}{h_b} + \frac{1}{h_c} - \frac{1}{h_a}\right)\left(\frac{1}{h_c} + \frac{1}{h_a} - \frac{1}{h_b}\right)}.$$

Here we answer the question of *whether it is possible to express the area of a triangle (for example, in radicals) using the lengths of its internal angle bisectors*. This question was posed in [9, problem 12] (published in 1830), but undoubtedly had been known before.

The following facts should also be kept in mind:

(i) If the lengths of corresponding angle bisectors of two triangles are equal then the triangles are congruent.

(ii) For every triplet of positive numbers l_a, l_b, l_c, there exists a triangle with the lengths of the angle bisectors equal to these numbers [8]. This statement answers in the affirmative the question by A. Brocard [2].

(iii) Two previous statements yield that there is a function

$$(l_a, l_b, l_c) \mapsto S,$$

which maps every triplet of positive numbers to the area of a triangle with given angle bisectors.

(iv) In general case, it is impossible to construct a triangle given the lengths of its angle bisectors using a ruler and compass (P. Barbarin, 1896 [1], A. Korselt, 1897 [6]).

In Heron's time, the question of expressing the area of a triangle in terms of the lengths of its angle bisectors could have been posed as follows: *Is it possible to construct a square with the same area as a triangle with given lengths of angle bisectors using a ruler and compass?* The following theorem answers the question negatively.

Theorem 1. *In general case, it is impossible to construct a square of the same area as a triangle with given lengths of angle bisectors using a ruler and compass.*

PROOF. Consider a triangle ABC with angle bisectors $l_a = AL = 1$, $l_b = BL = 1/3$ and $l_c = CL = 1/3$. From (ii) it follows that such a triangle exists. Furthermore, (i) implies that ABC is an isosceles triangle ($AB = AC$). To prove the theorem, it is sufficient to prove that the construction is impossible for this choice of lengths. It is easy to see that the area of ABC is equal to $\operatorname{tg}(A/2)$, so a side of the required square equals to $\sqrt{\operatorname{tg}(A/2)}$. Moreover, given a segment of length 1, the problems of constructing the following objects are equivalent:

- a segment of length $\sqrt{\operatorname{tg}(A/2)}$;
- a segment of length $\operatorname{tg}(A/2)$;
- an angle A;
- a triangle $\triangle ABC$;
- an angle $B/2$;
- a segment of length $\sin(B/2)$.

Therefore, the problem of constructing a square of area equal to the area of ABC using a ruler and compass is equivalent to the problem of constructing an isosceles triangle with angle bisectors of lengths 1, 1/3, 1/3 and is also equivalent to the problem of constructing a half base angle of that triangle.

It is known that in a triangle with angles α, β, γ and semiperimeter p the length of the bisector l of angle α is equal to

$$l = 2p \frac{\sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2}}.$$

In our case,

$$1 = 2p \frac{\sin^2 \frac{B}{2}}{\cos \frac{A}{2}} = 2p \frac{\sin^2 \frac{B}{2}}{\sin B} = p \operatorname{tg}(B/2)$$

and, since $B = C = \frac{\pi}{2} - \frac{A}{2}$,

$$\frac{1}{3} = 2p \frac{\sin \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{B}{2} \cos \frac{1}{2}(\pi - 2B - B)} = 2p \frac{\cos B \sin \frac{B}{2}}{\cos \frac{B}{2} \sin \frac{3B}{2}} = 2p \frac{\cos B}{\sin \frac{3B}{2}} \operatorname{tg} \frac{B}{2}.$$

Hence $\frac{1}{3} = 2 \frac{\cos B}{\sin \frac{3B}{2}}$ or, in other words, $\sin(3B/2) = 6 \cos B$. Let $\sin(B/2) = x$.

Then $\sin(3B/2) = 3x - 4x^3$, $\cos B = 1 - 2x^2$. We see that $3x - 4x^3 = 6(1 - 2x^2)$ and

$$4x^3 - 12x^2 - 3x + 6 = 0.$$

Recall the following statement from Galois theory (see, for instance, [7, Theorem 2] or [10, page 199]): *Consider segments $l_1, \ldots l_n$ of rational lengths and assume that a segment, with its length l equal to a root of some irreducible polynomial of degree n with rational coefficients, can be constructed from $l_1 \ldots l_n$ using a ruler and compass. Then n is a power of* 2. The polynomial $4x^3 - 12x^2 - 3x + 6$ is irreducible by Eisenstein's criterion [5]. Therefore, a segment of length $\sin(B/2)$ (which is a root of that polynomial) cannot be constructed using a ruler and compass. The theorem is proved.

Note that our proof of theorem 1 contains a proof of the fact that a triangle with bisectors of length 1, 1/3, 1/3 cannot be constructed using a ruler and compass. This proof can also be found in [7, pages 224–225] and [10, pages 201–202] and is listed here for the sake of completeness.

Theorem 1 yields the statement which strengthens the result of Barbarin and Korselt.

Corollary 1. *In general case, it is impossible to construct a triangle of the same area as a triangle with given lengths of angle bisectors using a ruler and compass.*

Corollary 2. *There exists no formula, to express the area of a triangle in terms of the lengths of its angle bisectors in quadratic radicals.*

Concerning the question of whether there exist an explicit expression of a triangle's area in radicals of the lengths of its angle bisectors, a certain confusion persists in mathematical literature. For instance, in [4, page 335] it is stated that von Renthe Fink found such an expression in 1843 [11]. However, his expression includes the

radius of the incircle r in additon to l_a, l_b, l_c. One of the relations he found is as follows [11, page 274, equation 6]:

$$4a_2r^2S^2 - 8a_3r^3S^2 = r^4 + S^2,$$

where $a_2 = l_a^{-2} + l_b^{-2} + l_c^{-2}$ and $a_3 = l_a^{-1}l_b^{-1}l_c^{-1}$. By substituting pr for S, where p is the semiperimeter, we obtain that

$$4a_2r^2p^2 - 8a_3r^3p^2 = r^2 + p^2.$$

Since a general polynomial equation of degree four or lower can be solved in radicals [10, page 126–130], the following statement holds.

Lemma 1. *For every triangle ABC with angle bisectors l_a, l_b, l_c, all of the following objects are either expressible or not expressible by radicals of l_a, l_b, l_c simultaneously:*

- *the radius of the incircle of ABC,*
- *the area of ABC,*
- *the perimeter of ABC.*

In [11], it is also stated that the radius of the incircle of a triangle is a root of a polynomial of degree 16 with some rational functions of the angle bisectors for coefficients.

After almost 100 years since the publication of [11], two papers were published in the same journal. We will see that a negative answer to the problem of expressing the area of a triangle in radicals of the lengths of the angle bisectors follows from these papers.

The first of these papers was published in 1937 by H. Wolff [13]. It contains a proof, based on geometric reasons, that $\dfrac{1}{2r}$ is a root of the following polynomial:

$$
\begin{aligned}
W(t) = t^{10} &- \frac{5}{2}a_2t^8 + \frac{7}{2}a_3t^7 + \frac{33}{16}a_2^2t^6 - \frac{47}{8}a_2a_3t^5 + \\
&+ \left(\frac{1}{4}a_2a_4 - \frac{5}{8}a_2^3 + \frac{61}{16}a_3^2\right)t^4 + \left(\frac{5}{2}a_2^2a_3 - \frac{1}{4}a_4a_3\right)t^3 + \\
&+ \left(\frac{1}{16}a_2^4 - \frac{1}{4}a_2^2a_4 - \frac{25}{8}a_2a_3^2\right)t^2 + \left(\frac{1}{2}a_2a_3a_4 - \frac{1}{8}a_2^3a_3 + \frac{5}{4}a_3^3\right)t + \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \left(\frac{1}{16}a_2^2a_3^2 - \frac{1}{4}a_4a_3^2\right),
\end{aligned}
$$

where

$$a_2 = l_a^{-2} + l_b^{-2} + l_c^{-2}, \quad a_3 = l_a^{-1}l_b^{-1}l_c^{-1}, \quad \text{and} \quad a_4 = l_a^{-2}l_b^{-2} + l_b^{-2}l_c^{-2} + l_c^{-2}l_a^{-2}.$$

It is also proved that the polynomial $W(t)$ is irreducible over the field $\mathbb{Q}(a_2, a_3, a_4)$. A year later B.L. van der Waerden [12] showed that the Galois group of the Wolff polynomial $W(t)$ over $\mathbb{Q}(a_2, a_3, a_4)$ is isomorphic to S_{10}, the symmetric group of degree 10 (which is non-solvable). So the roots of a polynomial $W(t)$ cannot be expressed in radicals of a_2, a_3, a_4 (see, for instance, [10, pages 89–90]). These papers provides no explicit conclusion on solvability of the equation $W(t) = 0$ over $\mathbb{Q}(l_a, l_b, l_c)$ in radicals. However, it is quite easy to prove the following:

Lemma 2. *Roots of the polynomial $W(t)$ cannot be expressed in radicals of l_a, l_b, l_c.*

PROOF. Assume that a root of the polynomial $W(t)$ can be expressed in radicals of l_a, l_b, l_c. Let us show that then it can also be expressed in radicals of a_2, a_3, a_4, which contradicts the results by van der Waerden and Wolff. To do that, it suffices to show that l_a, l_b, l_c themselves can be expressed in radicals of a_2, a_3, a_4.

Consider the following polynomials:

$$U(t) = t^3 - a_2 t^2 + a_4 t - a_3^2 \quad \text{and} \quad V(t) = U(t^2).$$

By Vieta's formulas, the roots of the polynomial $U(t)$ are equal to $l_a^{-2}, l_b^{-2}, l_c^{-2}$, and the roots of $V(t)$ are

$$\pm \frac{1}{l_a}, \quad \pm \frac{1}{l_b}, \quad \pm \frac{1}{l_c}.$$

Cardano's formula [10, page 130] implies that the roots of the cubic equation $U(t) = 0$ and the bicubic equation $V(t) = 0$ can be expressed in radicals of the coefficients a_2, a_3^2, a_4. Hence l_a, l_b, l_c can be expressed in radicals of a_2, a_3, a_4. This contradiction proves the lemma.

Now the following statement is a direct corollary of Lemma 2.

Theorem 2. *The radius of the incircle, as well as the area and the perimeter of a triangle cannot be expressed in radicals of the lengths of this triangle's angle bisectors.*

Using a direct computer calculation in [3], one can show that if $l_a = 1$, $l_b = 2$ and $l_c = 3$ then the Galois group of the polynomial $W(t)$ over $\mathbb{Q}$ is isomorphic to S_{10}. So the following statement holds.

Theorem 3. *The radius of the incircle, the area and the perimeter of a triangle with the length of the angle bisectors 1, 2 and 3 cannot be expressed in radicals of rational numbers.*

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