

ON INT-SOFT QUASI- Γ -IDEALS OF AN ORDERED Γ -SEMIGROUP

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ABSTRACT. In this paper, we introduce the concept of int-soft Γ -semigroup, int-soft quasi- Γ -semigroup and int-soft left (resp., right) Γ -semigroup of ordered Γ -semigroup over an initial universal set U . We investigate some properties of int-soft quasi- Γ -ideals and left (resp., right) Γ -ideals of ordered Γ -semigroup. Moreover, we define critical soft point of ordered Γ -semigroup. By using the notion of critical soft point, we define semiprime int-soft quasi- Γ -ideals of ordered Γ -semigroups. We characterize completely regular ordered Γ -semigroups in terms of their int-soft quasi- Γ -ideals and semiprime int-soft quasi- Γ -ideals. Furthermore, we define semilattice of left and right simple semigroups of ordered Γ -semigroups and characterize semilattice of left and right simple semigroups in terms of their int-soft quasi- Γ -ideal.

1. INTRODUCTION

In 1999, Molodtsov [29] introduced the concept of a soft set as a new mathematical tool to deal with uncertainties that appeared in many fields of research such as social science, environmental science, medical, engineering etc. For example, we have a statement “numbers closer to 20”, the classical set theory which is introduced by Cantor is no longer useful since there is uncertainty whether 19 is closer to 20 or not. These type of variables are non-Boolean. A few more examples of non-boolean variables are young people, tall people. To deal with such uncertainty, Zadeh [42] introduced the concept of fuzzy set in 1965. A fuzzy set of a classical set X is an object of the form $\{(x, f(x)) : x \in X \text{ and } f : X \rightarrow [0, 1]\}$. Many theories like fuzzy set theory have been developed including probability theory [41], intuitionistic fuzzy set theory [5], rough set theory [30]. Molodtsov pointed out that all the theories lack parameterization tool and hence introduced the concept of a soft set. Maji et al. [28] defined operations for soft sets. Later on Ali et al. [3] defined new operations for soft sets. The concept of soft set has proven useful in many different fields such as optimization [25], data analysis [43], simulation [16]. Many researchers had applied the concept of soft set on different algebraic structures such as groups (see [2]), rings (see [6]) and semirings (see [10]). In 2009, Ali and Shabir [40] introduced the notion of soft semigroups. The notion of soft ordered semigroups is defined by Jun et al (see [13]). Soft ideal, soft bi-ideal and soft quasi-ideal of a semigroup are defined by Ali et al. [4].

There are two extensions of soft set called intersectional soft set (briefly int-soft set) and union soft set (briefly uni-soft set). In 2015, Sezer et al. [39] introduced the

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concept of a int-soft semigroup, a int-soft ideal and a int-soft bi-ideal of a semigroup. The notion of int-soft (generalized) bi-ideal of a semigroup is introduced by Jun and Song [14].

In 1981, M.K Sen [34] introduced the notion of a Γ -semigroup as a generalization of semigroup. Later in 1986, Sen with Saha [36] changed the definition that was given by Sen in 1981 and gave more general definition. Ordered Γ -semigroup is defined by Sen and Seth [37] in 1993 as a generalization of ordered semigroup. Many classical notions and results of the theory of semigroups have been extended to Γ -semigroups. Green's relations in Γ -semigroup has been defined in [8, 9, 33, 38]. For other results one can see [35]. Moreover, the concept of fuzzy set (see [18]) has been applied to ordered Γ -semigroup. Changphas and Thongkam ([7]) used the idea of soft set to Γ -semigroups and defined soft Γ -semigroup, soft Γ -subsemigroup, soft l -idealistic (r -idealistic). Also Γ -restricted product is defined in ([7]). Khan [23] applied the concept of fuzzy set to ordered Γ -semigroup and defined the notion of fuzzy interior Γ -ideal of ordered Γ -semigroup. Moreover, Khan [24] also defined generalized fuzzy bi- Γ -ideal of type (θ, λ) of ordered Γ -semigroup.

In this paper, we introduce the concept of int-soft Γ -semigroup, int-soft quasi- Γ -ideal and int-soft left (resp., right) Γ -ideal of ordered Γ -semigroup and investigate their properties. We define critical soft point and by using the concept of critical soft point, we introduce semiprime int-soft quasi- Γ -ideal of ordered Γ -semigroup. Furthermore, characterizations of completely regular by using int-soft quasi- Γ -ideal and semiprime int-soft quasi- Γ -ideal are discussed in detail. Next we define semilattice of left and right simple semigroup of ordered Γ -semigroup and characterize semilattice of left and right simple semigroup in terms of their int-soft quasi- Γ -ideal.

2. PRELIMINARIES

In this section, we give some basic definitions and results, which are necessary for the following sections.

Let S and Γ be any two non-empty sets. By an ordered Γ -semigroup S [37], we mean an algebraic structure (S, Γ, \leq) that satisfies the given conditions:

- (i) (S, Γ) is a Γ -semigroup,
- (ii) (S, \leq) is a poset, and
- (iii) If $p \leq q \implies p\lambda r \leq q\lambda r$ and $r\lambda p \leq r\lambda q$ ($\forall p, q, r \in S$) ($\forall \lambda \in \Gamma$).

We represent the elements of S by English alphabets and the elements of Γ are represented by Greek letters. For $A, B \subseteq S$, we define $A\Gamma B = \{a\gamma b \mid a \in A, \gamma \in \Gamma \text{ and } b \in B\}$. If $A = \{a\}$ and $B = \{b\}$, then we denote $\{a\}\Gamma B$, $A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively, by means of $a\Gamma B$, $A\Gamma b$ and $a\Gamma b$. Let a be any element of S , then we denote $a^2 = a\Gamma a$, $a^3 = a\Gamma a\Gamma a$ or in general $a^n = a\Gamma a\Gamma a \dots \Gamma a$ (n -times), $n \in \mathbb{Z}^+$.

For $\emptyset \neq A \subseteq S$, we define $[A] := \{s \in S \mid s \leq a \text{ for some } a \in A\}$. If $A = \{a\}$, then we write $[a]$ instead of $(\{a\})$ (see [1]). For $x \in S$, we write $A_x = \{(p, q) \in S \mid x \leq p\lambda q \text{ for some } \lambda \in \Gamma\}$ (see [24]).

For other terminologies and definitions relevant to ordered Γ -semigroup, the reader is referred to [21, 31, 32]

Example 2.1. Let $S = \{u, v, w\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be two non-empty sets. Define binary operations on S in the tables given below:

| | | | | | | | | | | | |
|----------|-----|-----|-----|---------|-----|-----|-----|----------|-----|-----|-----|
| α | u | v | w | β | u | v | w | γ | u | v | w |
| u | u | v | w | u | v | w | u | u | w | u | v |
| v | v | w | u | v | w | u | v | v | u | v | w |
| w | w | u | v | w | u | v | w | w | v | w | u |

The order relation " \leq " is define by:

$$\leq := \{(u, u), (v, v), (w, w)\}$$

This is an ordered Γ -semigroup.

A sub- Γ -semigroup H of an ordered Γ -semigroup S is a non-empty subset of S such that $H\Gamma H \subseteq H$.

A left (resp., right) Γ -ideal I of an ordered Γ -semigroup S is a non-empty subset of S which satisfies the following conditions:

- (1) $S\Gamma I \subseteq I$ (resp., $I\Gamma S \subseteq I$),
- (2) $(\forall p \in I \text{ and } \forall s \in S) (s \leq p \implies s \in I)$. In otherwords, $(I) = I$.

I is said to be a two sided Γ -ideal (or simply a Γ -ideal) of S , if I is a left Γ -ideal as well as a right Γ -ideal of S .

A quasi- Γ -ideal Q of an ordered Γ -semigroup S , is a non-empty subset of S which satisfies the given conditions:

- (1) $(S\Gamma Q] \cap (Q\Gamma S] \subseteq Q$,
- (2) $(\forall p \in Q \text{ and } \forall s \in S) (s \leq p \implies s \in Q)$. In otherwords, $(Q) = Q$.

A bi- Γ -ideal B of an ordered Γ -semigroup S , is a non-empty subset of S which satisfies the following conditions:

- (1) B is a sub- Γ -semigroup of S ,
- (2) $B\Gamma S\Gamma B \subseteq B$, and
- (3) $(\forall p \in B \text{ and } \forall s \in S) (s \leq p \implies s \in B)$. In otherwords, $(B) = B$.

Lemma 2.2. (see [27]) For any non-empty subsets K, L of an ordered Γ -semigroup S , the following holds:

- (1) $K \subseteq (K]$ for any $K \subseteq S$.
- (2) If $K \subseteq L$, then $(K) \subseteq (L)$.
- (3) $(K) = ((K])$.
- (4) $(K)\Gamma(L) \subseteq (K\Gamma L)$.
- (5) $((K)\Gamma(L)) = (K\Gamma L)$.
- (6) $(K \cup L) = (K) \cup (L)$.
- (vii) $(K \cap L) \subseteq (K) \cap (L)$, (if $K \cap L \neq \emptyset$).
- (7) $(p\Gamma S]$, $(S\Gamma p]$ and $(S\Gamma p\Gamma S]$ is a right- Γ -ideal, a left- Γ -ideal and a Γ -ideal of S , for every $p \in S$.

Let p be any element of an ordered Γ -semigroup S . The quasi- Γ -ideal of S generated by an element p is denoted by $Q(p)$, is define as, $Q(p) = (p \cup ((p\Gamma S] \cap (S\Gamma p]))$ (see [27]).

3. BASIC OPERATIONS OF SOFT SETS

Let U be a non-empty set called universal set and E be a non-empty set of all possible parameters with respect to U and $A, B, C \subseteq E$. By $P(U)$ we mean the power set of U and $\xi \subseteq P(U)$.

Definition 3.1. [29]. A pair (f_A, E) is called a soft set over U , where f_A is a mapping given by $f_A : E \rightarrow P(U)$, is defined as:

$$f_A(a) := \begin{cases} \xi, & \text{if } a \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In other words, a soft set is a parameterized family of subsets of the universal set U . For $a \in E$, $f_A(a)$ may be considered as a set of a -approximate elements of the soft set (f_A, E) .

Examples of a soft set can be found in [29]. The set of all soft sets over U is denoted by $S(U)$.

Definition 3.2. (see [4, 7]). Let (f_C, E) and (g_D, E) be any two soft sets over a common universe U , then (f_C, E) is said to be a soft subset of (g_D, E) , denoted by $(f_C, E) \tilde{\subseteq} (g_D, E)$, if $C \subseteq D$ and $f_C(p) \subseteq g_D(p)$, for all $p \in C$.

Definition 3.3. (see [15]). Two soft sets (f_C, E) and (g_D, E) are said to be equal if (f_C, E) is a soft subset of (g_D, E) and (g_D, E) is a soft subset of (f_C, E) .

Definition 3.4. (see [15]). Let (f_C, E) and (g_D, E) be two soft sets over a common universe U . Then, the soft union of f_C and g_D , denoted by $f_C \tilde{\cup} g_D$, is defined to be the soft set $(f_C \tilde{\cup} g_D, E)$ over U in which $f_C \tilde{\cup} g_D$ is defined by $(f_C \tilde{\cup} g_D)(a) = f_C(a) \cup g_D(a)$ for all $a \in E$.

Definition 3.5. (see [15]). Let (f_C, E) and (g_D, E) be two soft sets over a common universe U . Then, the soft intersection of f_C and g_D , denoted by $f_C \tilde{\cap} g_D$, is defined to be the soft set $(f_C \tilde{\cap} g_D, E)$ over U in which $f_C \tilde{\cap} g_D$ is defined by $(f_C \tilde{\cap} g_D)(a) = f_C(a) \cap g_D(a)$ for all $a \in E$.

From now on S represent an ordered Γ -semigroup unless otherwise stated.

Definition 3.6. For any two soft sets (f_S, S) and (g_S, S) over a common universe U , we denote the soft union-intersection product by $f_S \Gamma g_S$, and is define by

$$(f_S \Gamma g_S)(p) = \begin{cases} \bigcup_{(s,t) \in A_p} \{f_S(s) \cap g_S(t)\}, & \text{if } A_p \neq \emptyset, \\ \emptyset, & \text{if } A_p = \emptyset, \end{cases} \quad \text{for all } p \in S.$$

Without any difficulty, the reader can prove that “ Γ ” on $S(U)$ is well defined. Moreover, we have the following lemma.

Lemma 3.7. *The set $(S(U), \Gamma, \tilde{\subseteq})$ forms an ordered Γ -semigroup.*

Proof. To show $S(U)$ is an ordered Γ -semigroup, first we show that $(S(U), \Gamma)$ is a Γ -semigroup. Let $f_S, g_S, h_S \in S(U)$. If $A_a = \emptyset$, then we have, $((f_S \Gamma h_S) \Gamma g_S)(a) = \emptyset = (f_S \Gamma (h_S \Gamma g_S))(a)$. Let $A_a \neq \emptyset$, then we have

$$\begin{aligned}
((f_S \Gamma h_S) \Gamma g_S)(a) &= \bigcup_{(s,t) \in A_a} \{(f_S \Gamma h_S)(s) \cap g_S(t)\} \\
&= \bigcup_{(s,t) \in A_a} \left\{ \bigcup_{(k,l) \in A_s} \{f_S(k) \cap h_S(l)\} \cap g_S(t) \right\} \\
&= \bigcup_{(k,l,t) \in A_a} \{(f_S(k) \cap h_S(l)) \cap g_S(t)\} \\
&= \bigcup_{(k,l,\mu t) \in A_a} \{f_S(k) \cap (h_S(l) \cap g_S(t))\} \\
&\subseteq \bigcup_{(k,l,\mu t) \in A_a} \left\{ f_S(k) \cap \left(\bigcup_{(l,t) \in A_{l\mu t}} \{h_S(l) \cap g_S(t)\} \right) \right\} \\
&= \bigcup_{(k,l,\mu t) \in A_a} \{f_S(k) \cap (h_S \Gamma g_S)(l\mu t)\} \\
&= (f_S \Gamma (h_S \Gamma g_S))(a).
\end{aligned}$$

This implies that $((f_S \Gamma h_S) \Gamma g_S)(a) \subseteq (f_S \Gamma (h_S \Gamma g_S))(a)$. Similarly we can show that $(f_S \Gamma (h_S \Gamma g_S))(a) \subseteq ((f_S \Gamma h_S) \Gamma g_S)(a)$. Hence $((f_S \Gamma h_S) \Gamma g_S)(a) = (f_S \Gamma (h_S \Gamma g_S))(a)$. Now we show that the order relation " $\widetilde{\subseteq}$ " on $S(U)$ is compatible with " Γ ". Let $f_S, h_S, g_S \in S(U)$ and $a \in S$ be such that $f_S \widetilde{\subseteq} h_S$. If $A_a = \emptyset$, then $(f_S \Gamma g_S)(a) = \emptyset = (h_S \Gamma g_S)(a)$. Let $A_a \neq \emptyset$, then

$$\begin{aligned}
(f_S \Gamma g_S)(a) &= \bigcup_{(s,t) \in A_a} \{f_S(s) \cap g_S(t)\} \\
&\subseteq \bigcup_{(s,t) \in A_a} \{h_S(s) \cap g_S(t)\} \quad (\text{since } f_S(s) \subseteq h_S(s) \quad \forall s \in S) \\
&= (h_S \Gamma g_S)(a).
\end{aligned}$$

Thus $(f_S \Gamma g_S)(a) \subseteq (h_S \Gamma g_S)(a)$. Similarly we can prove that $(g_S \Gamma f_S)(a) \subseteq (g_S \Gamma h_S)(a)$. Therefore, $(S(U), \Gamma, \widetilde{\subseteq})$ is an ordered Γ -semigroup. \square

Lemma 3.8. For any soft sets (f_S, S) , (h_S, S) and (g_S, S) over U , we have the following:

- (1) $(f_S \Gamma (g_S \widetilde{\cap} h_S), S) = (f_S \Gamma g_S, S) \widetilde{\cap} (f_S \Gamma h_S, S)$,
- (2) $(f_S \Gamma (g_S \widetilde{\cup} h_S), S) = (f_S \Gamma g_S, S) \widetilde{\cup} (f_S \Gamma h_S, S)$,
- (3) $(f_S \widetilde{\cap} (g_S \widetilde{\cup} h_S), S) = (f_S \widetilde{\cap} g_S, S) \widetilde{\cup} (f_S \widetilde{\cap} h_S, S)$,
- (4) $(f_S \widetilde{\cup} (g_S \widetilde{\cap} h_S), S) = (f_S \widetilde{\cup} g_S, S) \widetilde{\cap} (f_S \widetilde{\cup} h_S, S)$.

Proof. (1) Let $a \in S$. If $A_a = \emptyset$, then $(f_S \Gamma (g_S \widetilde{\cap} h_S))(a) = \emptyset = ((f_S \Gamma g_S) \widetilde{\cap} (f_S \Gamma h_S))(a)$. Let $A_a \neq \emptyset$, then

$$\begin{aligned}
(f_S \Gamma (g_S \widetilde{\cap} h_S))(a) &= \bigcup_{(s,t) \in A_a} \{f_S(s) \cap (g_S \widetilde{\cap} h_S)(t)\} \\
&= \bigcup_{(s,t) \in A_a} \{f_S(s) \cap (g_S(t) \cap h_S(t))\} \\
&= \bigcup_{(s,t) \in A_a} \{(f_S(s) \cap g_S(t)) \cap (f_S(s) \cap h_S(t))\} \\
&= \left(\bigcup_{(s,t) \in A_a} \{f_S(s) \cap g_S(t)\} \right) \cap \left(\bigcup_{(s,t) \in A_a} \{f_S(s) \cap h_S(t)\} \right) \\
&= (f_S \Gamma g_S)(a) \cap (f_S \Gamma h_S)(a) \\
&= ((f_S \Gamma g_S) \widetilde{\cap} (f_S \Gamma h_S))(a),
\end{aligned}$$

hence, $(f_S \Gamma (g_S \tilde{\cap} h_S), S) = (f_S \Gamma g_S, S) \tilde{\cap} (f_S \Gamma h_S, S)$.

(2) Let $a \in S$. If $A_a = \emptyset$, then $(f_S \Gamma (g_S \tilde{\cup} h_S))(a) = \emptyset = ((f_S \Gamma g_S) \tilde{\cup} (f_S \Gamma h_S))(a)$. Let $A_a \neq \emptyset$, then,

$$\begin{aligned} (f_S \Gamma (g_S \tilde{\cup} h_S))(a) &= \bigcup_{(s,t) \in A_a} \{f_S(s) \cap (g_S \tilde{\cup} h_S)(t)\} \\ &= \bigcup_{(s,t) \in A_a} \{f_S(s) \cap (g_S(t) \cup h_S(t))\} \\ &= \bigcup_{(s,t) \in A_a} \{(f_S(s) \cap g_S(t)) \cup (f_S(s) \cap h_S(t))\} \\ &= \left(\bigcup_{(s,t) \in A_a} \{f_S(s) \cap g_S(t)\} \right) \cup \left(\bigcup_{(s,t) \in A_a} \{f_S(s) \cap h_S(t)\} \right) \\ &= (f_S \Gamma g_S)(a) \cup (f_S \Gamma h_S)(a) \\ &= ((f_S \Gamma g_S) \tilde{\cup} (f_S \Gamma h_S))(a). \end{aligned}$$

Hence, $(f_S \Gamma (g_S \tilde{\cup} h_S), S) = (f_S \Gamma g_S, S) \tilde{\cup} (f_S \Gamma h_S, S)$.

The proof of (3) and (4) are straightforward. \square

Definition 3.9. In an ordered Γ -semigroup (S, Γ, \leq) , the soft set “ \emptyset_S ” of S over U is defined as:

$$\emptyset_S : S \longrightarrow P(U), \quad s \longmapsto \emptyset_S(s) = \emptyset, \quad \forall s \in S,$$

is the “least element” of $S(U)$, is said to be a null soft set over U .

The soft set “ T_S ” of S over U is defined as:

$$T_S : S \longrightarrow P(U), \quad s \longmapsto T_S(s) = U, \quad \forall s \in S,$$

is the “greatest element” of $S(U)$, is said to be a whole soft set over U .

Definition 3.10. Let $\emptyset \neq C \subseteq S$, we denote, the characteristic soft set over U by “ χ_C ”, and is defined by:

$$\chi_C(p) = \begin{cases} U, & \text{if } p \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 3.11. Let C, D be any two non-empty subsets of an ordered Γ -semigroup S , then the following are true:

- (1) $C \subseteq D$ iff $\chi_C \tilde{\subseteq} \chi_D$,
- (2) $\chi_C \tilde{\cap} \chi_D = \chi_{C \cap D}$,
- (3) $\chi_C \tilde{\cup} \chi_D = \chi_{C \cup D}$,
- (4) $\chi_C \Gamma \chi_D = \chi_{(C \Gamma D)}$.

Proof. The proof of (i), (ii) and (iii) is straightforward.

(iv) Let $m \in S$. Then $\chi_C \Gamma \chi_D = \chi_{(C \Gamma D)}$. Indeed: let $m \in (C \Gamma D)$, then $\chi_{(C \Gamma D)}(m) = U$.

Since $m \leq c \lambda d$ for some $c \in C$, $d \in D$ and $\lambda \in \Gamma$, so $(c, d) \in A_m$. Thus we have

$$(\chi_C \Gamma \chi_D)(m) = \bigcup_{(s,t) \in A_m} \chi_C(s) \cap \chi_D(t) \supseteq \chi_C(c) \cap \chi_D(d) = U \cap U = U.$$

Thus $(\chi_C \Gamma \chi_D)(m) \supseteq U$. But $\chi_C \Gamma \chi_D$ is a soft subset of S , so $(\chi_C \Gamma \chi_D)(m) \subseteq U$. Therefore, $(\chi_C \Gamma \chi_D)(m) = U = \chi_{(C \Gamma D)}(m)$.

Suppose $m \notin (C \Gamma D)$, if $A_m = \emptyset$, then $\chi_{(C \Gamma D)}(m) = \emptyset = (\chi_C \Gamma \chi_D)(m)$. Let $A_m \neq \emptyset$, since $m \notin (C \Gamma D)$, this implies that there does not exist any $c \in C$ or $d \in D$ such that $m \leq c \lambda d$ for some $\lambda \in \Gamma$.

Let $m \leq a\mu b$ for some $a \in C^c$ or $b \in D^c$ and $\mu \in \Gamma$. If $a \in C^c$, then $\chi_C(a) = \emptyset$. Since $(a, b) \in A_m$, thus we have

$$(\chi_C \Gamma \chi_D)(m) = \bigcup_{(a,b) \in A_m} \chi_C(a) \cap \chi_D(b) = \emptyset = \chi_{(C \Gamma D]}(m)$$

If $b \in D^c$, then $\chi_D(b) = \emptyset$. Thus,

$$(\chi_C \Gamma \chi_D)(m) = \bigcup_{(a,b) \in A_m} \chi_C(a) \cap \chi_D(b) = \emptyset = \chi_{(C \Gamma D]}(m).$$

Hence in any case, $\chi_C \Gamma \chi_D = \chi_{(C \Gamma D]}$. \square

Definition 3.12. A soft set (f_S, S) of an ordered Γ -semigroup S over U is called int-soft Γ -semigroup over U if:

$$(f_S(p\gamma q) \supseteq f_S(p) \cap f_S(q)) \quad (\forall p, q \in S \text{ and } \gamma \in \Gamma).$$

Example 3.13. Let $S = \{p, q, r\}$ and $\Gamma = \{\lambda, \mu\}$ be two non-empty sets. Let $U = \{e, x, x^2, y, xy, x^2y\}$ is a universal set. Then S is an ordered Γ -semigroup with respect to the binary operations and order relation define below:

$$\begin{array}{c|c|c|c} \lambda & p & q & r \\ \hline p & p & q & r \\ \hline q & q & p & r \\ \hline r & r & r & r \end{array} \quad \begin{array}{c|c|c|c} \mu & p & q & r \\ \hline p & q & p & r \\ \hline q & p & q & r \\ \hline r & r & r & r \end{array}$$

The order relation is defined by:

$$\leq := \{(p, p), (q, q), (r, r), (p, r), (q, r)\}$$

Let f_S be a soft set define by:

$f_S(p) = \{e, x, x^2\}$, $f_S(q) = \{y, xy\}$, $f_S(r) = \{e, x^2y\}$. Then clearly f_S is an int-soft Γ -semigroup of an ordered Γ -semigroup S .

4. INT-SOFT QUASI- Γ -IDEALS

Definition 4.1. A soft set (f_S, S) of an ordered Γ -semigroup S over U is called int-soft left (resp., right) Γ -ideal if:

- (1) $(\forall p, q \in S) (\forall \lambda \in \Gamma) (f_S(p\lambda q) \supseteq f_S(q))$ (resp., $(f_S(q\lambda p) \supseteq f_S(q))$),
- (2) $(\forall p, q \in S) (\text{If } p \leq q \implies f_S(p) \supseteq f_S(q))$.

Example 4.2. Let $S = [0, 1]$ and $\Gamma = \{1/n : n \in \mathbb{N}\}$. Then S is an ordered Γ -semigroup under usual multiplication and usual partial order. Let $U = \mathbb{N}$ be a universal set. We define a soft set f_S over U given below:

$$f_S(p) = \begin{cases} \mathbb{N} \setminus \{p\}, & \text{if } p \in [0, 1/2], \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly it is int-soft left Γ -ideal as well as int-soft right Γ -ideal of S over U .

Proposition 4.3. Let f_S be any soft set of an ordered Γ -semigroup S over U , then the following statements are equivalent:

- (1) A soft set f_S is an int-soft right (resp., int-soft left) Γ -ideal of S .
- (2) (i) $f_S \Gamma \chi_S \tilde{\subseteq} f_S$ (resp., $\chi_S \Gamma f_S \tilde{\subseteq} f_S$),
(ii) $(\forall p, q \in S) (\text{If } p \leq q \implies f_S(p) \supseteq f_S(q))$.

Proof. (1) \implies (2). Let $p \in S$ and f_S be an int-soft right Γ -ideal of S . If $A_p = \emptyset$, then we have, $(f_S \Gamma \chi_S)(p) = \emptyset \subseteq f_S(p)$. Let $A_p \neq \emptyset$, then we have, $(f_S \Gamma \chi_S)(p) = \bigcup_{(s,t) \in A_p} f_S(s) \cap \chi_S(t) = \bigcup_{(s,t) \in A_p} f_S(s)$

On the other hand, $f_S(s) \subseteq f_S(p)$ for every $(s, t) \in A_p$. Indeed: Since $(s, t) \in A_p$, so $p \leq s\lambda t$ for some $\lambda \in \Gamma$. Since f_S is an int-soft right Γ -ideal over U , so we have $f_S(p) \supseteq f_S(s\lambda t) \supseteq f_S(s)$ implies $f_S(p) \supseteq f_S(s)$ or $f_S(s) \subseteq f_S(p)$. Hence $(f_S \Gamma \chi_S)(p) \subseteq f_S(p)$. Since f_S is an int-soft right Γ -ideal of S , so property (ii) holds.

(2) \implies (1). By hypothesis for every $s, t \in S$ and $\lambda \in \Gamma$, we have

$$f_S(s\lambda t) \supseteq (f_S \Gamma \chi_S)(s\lambda t) = \bigcup_{(u,v) \in A_{s\lambda t}} f_S(u) \cap \chi_S(v) \supseteq f_S(s) \cap \chi_S(t) = f_S(s).$$

Therefore, $f_S(s\lambda t) \supseteq f_S(s)$ and so f_S is an int-soft right Γ -ideal of S over U . \square

Proposition 4.4. Let (S, Γ, \leq) be an ordered Γ -semigroup and $\emptyset \neq C \subseteq S$, then the following statements are equivalent:

- (1) C is a right Γ -ideal (resp., left Γ -ideal) of S .
- (2) The characteristic soft set (χ_C, S) is an int-soft right (resp., left) Γ -ideal over U .

Proof. (1) \implies (2). Let C is a right Γ -ideal of S , so $C\Gamma S \subseteq C$. Then we have, $\chi_C \Gamma \chi_S = \chi_{(C\Gamma S)} \subseteq \chi_C$. Thus $\chi_C \Gamma \chi_S \subseteq \chi_C$. Let $p, q \in S$ be such that $p \leq q$. If $q \in C$, then $\chi_C(q) = U$. Since $p \leq q$ and C is a quasi- Γ -ideal of S , we have $p \in C$. Hence $\chi_C(p) = U = \chi_C(q)$. If $q \notin C$, then $\chi_C(q) = \emptyset \subseteq \chi_C(p)$ or $\chi_C(p) \supseteq \chi_C(q)$. So by proposition 4.3, (χ_C, S) is an int-soft right Γ -ideal over U .

(2) \implies (1). Let $c \in C\Gamma S$ imply $c \in (C\Gamma S)$. Thus $\chi_{(C\Gamma S)} = U$, but $\chi_{(C\Gamma S)}(c) = \chi_C \Gamma \chi_S(c)$. Hence $U = \chi_C \Gamma \chi_S(c) \subseteq \chi_C(c)$ (by proposition 4.3). Hence $U \subseteq \chi_C(c)$, but $\chi_C(c) \subseteq U$. So $\chi_C(c) = U$ imply $c \in C$. Thus $C\Gamma S \subseteq C$. Let $p, q \in S$ be such that $p \in C$ and $q \leq p$, then $q \in C$. Indeed: Since $q \leq p$, then $\chi_C(q) \supseteq \chi_C(p) = U$ (by hypothesis) but $\chi_C(q) \subseteq U$, so $\chi_C(q) = U$. Thus $q \in C$. Therefore, C is a right Γ -ideal of S . \square

Definition 4.5. An element s of an ordered Γ -semigroup (S, Γ, \leq) is said to be left (resp., right) regular if there exist some $r \in S$ and $\lambda, \mu \in \Gamma$ such that $s \leq r\lambda s\mu$ (resp., $s \leq s\lambda s\mu r$).

S is said to be left (resp., right) regular, if all the elements of S are left (resp., right) regular. Equivalent definitions are:

- (1) For every $K \subseteq S$, $K \subseteq (S\Gamma K\Gamma K)$ (resp., $K \subseteq (K\Gamma K\Gamma S)$),
- (2) For every element $s \in S$, $s \in (S\Gamma s\Gamma s)$ (resp., $s \in (s\Gamma s\Gamma S)$).

Definition 4.6. An element s of an ordered Γ -semigroup (S, Γ, \leq) is said to be regular if there exist some element $r \in S$ and $\lambda, \mu \in \Gamma$ such that $s \leq s\lambda r\mu s$.

If every element of an ordered Γ -semigroup S is regular then S is called regular ordered Γ -semigroup. Equivalent definitions are:

- (1) For every $K \subseteq S$, $K \subseteq (K\Gamma S\Gamma K)$,
- (2) For every $s \in S$, $s \in (s\Gamma S\Gamma s)$.

Proposition 4.7. In an ordered Γ -semigroup, we have

$$f_S \tilde{\cap} g_S \tilde{\supseteq} f_S \Gamma g_S.$$

for every int-soft right Γ -ideal f_S and for every int-soft left Γ -ideal g_S over U .

Proof. Let $p \in S$. If $A_p = \emptyset$, then $(f_S \Gamma g_S)(p) = \emptyset \subseteq (f_S \tilde{\cap} g_S)(p)$. Let $A_p \neq \emptyset$, then we have

$$(f_S \Gamma g_S)(p) = \bigcup_{(s,t) \in A_p} \{f_S(s) \cap g_S(t)\}.$$

Since $(s, t) \in A_p$, so $p \leq s\lambda t$ for some $\lambda \in \Gamma$. Since f_S is an int-soft right Γ -ideal, so we have $f_S(p) \supseteq f_S(s\lambda t) \supseteq f_S(s)$ implies $f_S(p) \supseteq f_S(s)$. Also g_S is an int-soft left Γ -ideal, so we have $g_S(p) \supseteq g_S(s\lambda t) \supseteq g_S(t)$ implies $g_S(p) \supseteq g_S(t)$. Thus we have,

$$(f_S \Gamma g_S)(p) = \bigcup_{(s,t) \in A_p} \{f_S(s) \cap g_S(t)\} \subseteq f_S(p) \cap g_S(p) = (f_S \tilde{\cap} g_S)(p).$$

Therefore, $f_S \Gamma g_S \tilde{\subseteq} f_S \tilde{\cap} g_S$ or $f_S \tilde{\cap} g_S \tilde{\supseteq} f_S \Gamma g_S$. \square

Proposition 4.8. Let (S, Γ, \leq) be a regular ordered Γ -semigroup. Then for each int-soft right (resp., left) Γ -ideal f_S of S and for each soft subset g_S of S , we have

$$f_S \tilde{\cap} g_S \tilde{\subseteq} f_S \Gamma g_S \text{ (resp., } g_S \tilde{\cap} f_S \tilde{\subseteq} g_S \Gamma f_S).$$

Proof. Let f_S be an int-soft right Γ -ideal and g_S be any soft set over U . Let $p \in S$. As S is regular, then $p \leq p\lambda q\mu p$ for some $q \in S$ and $\lambda, \mu \in \Gamma$. Thus $(p\lambda q, p) \in A_p$. Since $A_p \neq \emptyset$, we have

$$(f_S \Gamma g_S)(p) = \bigcup_{(s,t) \in A_p} \{f_S(s) \cap g_S(t)\} \supseteq f_S(p\lambda q) \cap g_S(p)$$

Since f_S is an int-soft right Γ -ideal of S , so $f_S(p\lambda q) \supseteq f_S(p)$. Then we have, $(f_S \Gamma g_S)(p) \supseteq f_S(p\lambda q) \cap g_S(p) \supseteq f_S(p) \cap g_S(p)$. Therefore, $f_S \Gamma g_S \tilde{\supseteq} f_S \tilde{\cap} g_S$ or $f_S \tilde{\cap} g_S \tilde{\subseteq} f_S \Gamma g_S$. \square

Corollary 4.9. In a regular ordered Γ -semigroup S , we have

$$f_S \tilde{\cap} g_S = f_S \Gamma g_S.$$

for every int-soft right Γ -ideal f_S and for every int-soft left Γ -ideal g_S over U .

Definition 4.10. Let (S, Γ, \leq) be an ordered Γ -semigroup. Then a soft set (f_S, S) of S over U is said to be int-soft quasi- Γ -ideal of S if:

- (1) $(f_S \Gamma \chi_S; S) \tilde{\cap} (\chi_S \Gamma f_S; S) \tilde{\subseteq} f_S$,
- (2) $(\forall p, q \in S) (\text{If } p \leq q \implies f_S(p) \supseteq f_S(q))$.

Example 4.11. Let $S = P$ (set of prime numbers) and $1 \in S$. Let Γ be any non-empty set. Let $U = E$ (set of even numbers) be a universal set. Define $S \times \Gamma \times S \rightarrow S$ by

$$p\lambda q = \begin{cases} p, & \text{if } p = q, \\ 1, & \text{otherwise.} \end{cases}$$

for all $p, q \in S$ and $\lambda \in \Gamma$.

Define an order relation by $p \leq q$ iff p/q . Then S is an ordered Γ -semigroup. Let f_S be a soft set is define by:

$$f_S(t) = \begin{cases} 2E, & \text{if } t = 1, \\ 4E, & \text{if } t \in [2, 50], \\ 8E, & \text{if } t \in [51, \infty]. \end{cases}$$

The soft set f_S is an int-soft quasi- Γ -ideal over U .

Theorem 4.12. *Let (S, Γ, \leq) be an ordered Γ -semigroup and $\emptyset \neq C \subseteq S$, then the given statements are equivalent:*

- (1) *C is a quasi- Γ -ideal of S .*
- (2) *The characteristic soft set (χ_C, S) is an int-soft quasi- Γ -ideal of S .*

Proof. (1) \implies (2). Let a be any element of S . Then we have

$$\begin{aligned} ((\chi_C \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma \chi_C))(a) &= (\chi_C \Gamma \chi_S)(a) \cap (\chi_S \Gamma \chi_C)(a) \\ &= \chi_{(CTS]}(a) \cap \chi_{(STC]}(a) = \chi_{(CTS] \cap (STC]}(a). \end{aligned}$$

Since C is a quasi- Γ -ideal of S , then $(CTS] \cap (STC] \subseteq C$, then we have $\chi_{(CTS] \cap (STC]} \subseteq \chi_C$. Therefore, $(\chi_C \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma \chi_C)(a) \subseteq \chi_C(a)$.

Let $p, q \in S$ be such that $p \leq q$. If $q \in C$, then $p \in C$. Indeed: if $q \in C$, then $\chi_C(q) = U$. Since $p \leq q$ and C is a quasi- Γ -ideal of S , we have $p \in C$. Hence $\chi_C(p) = U = \chi_C(q)$. If $q \notin C$, then $\chi_C(q) = \emptyset \subseteq \chi_C(p)$ or $\chi_C(p) \supseteq \chi_C(q)$. Hence in any case, $\chi_C(p) \supseteq \chi_C(q)$. Therefore χ_C is an int-soft quasi- Γ -ideal over U .

(2) \implies (1). Let χ_C is an int-soft quasi- Γ -ideal over U . Let $a \in (CTS] \cap (STC]$, then $a \leq b\lambda p$ and $a \leq q\mu c$ for some $b, c \in C$, $\lambda, \mu \in \Gamma$ and $p, q \in S$. Hence $(b, p), (q, c) \in A_a$. Since χ_C is an int-soft quasi- Γ -ideal over U , then we have

$$\begin{aligned} \chi_C(a) &\supseteq ((\chi_C \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma \chi_C))(a) \\ &= (\chi_C \Gamma \chi_S)(a) \cap (\chi_S \Gamma \chi_C)(a) \\ &= \left(\bigcup_{(s,t) \in A_a} \{\chi_C(s) \cap \chi_S(t)\} \right) \cap \left(\bigcup_{(s,t) \in A_a} \{\chi_S(s) \cap \chi_C(t)\} \right) \\ &\supseteq \{\chi_C(b) \cap \chi_S(p)\} \cap \{\chi_S(q) \cap \chi_C(c)\} \\ &= \chi_C(b) \cap \chi_C(c) = U. \end{aligned}$$

Thus $\chi_C(a) \supseteq U$ but $\chi_C(a) \subseteq U$. Hence $\chi_C(a) = U \implies a \in C$. Hence we have $(CTS] \cap (STC] \subseteq C$.

Now let $S \ni s \leq r \in C$. Then $s \in C$. Indeed: Since $s \leq r$, then by hypothesis, $\chi_C(s) \supseteq \chi_C(r)$. Also $r \in C$, then $\chi_C(r) = U$. Then $\chi_C(s) \supseteq \chi_C(r) = U \implies \chi_C(s) \supseteq U$. Also $U \supseteq \chi_C(s)$, then $\chi_C(s) = U \implies s \in C$. Therefore C is a quasi- Γ -ideal of S . \square

Theorem 4.13. *Every one sided int-soft Γ -ideal (f_S, S) of an ordered Γ -semigroup S is an int-soft quasi- Γ -ideal over U .*

Proof. Assume that the soft set f_S is an int-soft right Γ -ideal over U . Let $a \in S$, then

$$((f_S \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma f_S))(a) = (f_S \Gamma \chi_S)(a) \cap (\chi_S \Gamma f_S)(a) \subseteq (f_S \Gamma \chi_S)(a).$$

Since f_S is a int-soft right Γ -ideal over U , then we have $(f_S \Gamma \chi_S)(a) \subseteq f_S(a)$ (by proposition 4.3). Therefore $((f_S \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma f_S))(a) \subseteq f_S(a)$. Also for all $p, q \in S$ such that $p \leq q \implies f_S(p) \supseteq f_S(q)$. Thus (f_S, S) is an int-soft quasi- Γ -ideal of S . Similarly, we can prove it for int-soft left Γ -ideal of S over U .

The converse is not true. This is supported by the next example. \square

Example 4.14. Let $S = \{0, p, q, r\}$ and $\Gamma = \{\lambda\}$. Let $U = \{1, 2, 3\}$ is a universal set. Define a binary operation on S in the table given below:

| λ | 0 | p | q | r |
|-----------|---|-----|-----|-----|
| 0 | 0 | 0 | 0 | 0 |
| p | 0 | p | q | 0 |
| q | 0 | 0 | 0 | 0 |
| r | 0 | r | 0 | 0 |

The order relation " \leq " is define by:

$$\leq := \{(0, 0), (p, p), (q, q), (r, r)\}$$

This is an ordered Γ -semigroup. Define a soft set f_S over U such that $f_S(0) = \{1, 2, 3\}$, $f_S(p) = \{1, 2\}$, $f_S(q) = \{1\}$ and $f_S(r) = \{2, 3\}$. Clearly it is an int-soft quasi- Γ -ideal of S but it is not int-soft left Γ -ideal nor int-soft right Γ -ideal. Since $f_S(p\lambda q) = f_S(q) = \{1\} \not\supseteq f_S(p)$ and $f_S(r\lambda p) = f_S(r) = \{2, 3\} \not\supseteq f_S(p)$.

Definition 4.15. Let C be any non-empty subset of an ordered Γ -semigroup S . Let $\delta \subseteq U$. Then, upper δ -inclusion of a soft set f_C of S over U , is denoted by $\mathcal{U}(f_C; \delta)$, is defined as

$$\mathcal{U}(f_C; \delta) = \{w \in C \mid f_C(w) \supseteq \delta\}.$$

Theorem 4.16. Let (S, Γ, \leq) is an ordered Γ -semigroup. Then for any soft set f_S over U , the given statements are equivalent:

- (1) $(\forall \delta \subseteq U) \mathcal{U}(f_S; \delta) \neq \emptyset, \mathcal{U}(f_S; \delta)$ is a quasi- Γ -ideal of S .
- (2) $(f_S; S)$ is an int-soft quasi- Γ -ideal of S over U .

Proof. (1) \implies (2). Assume that for every $\delta \subseteq U$, the set $\emptyset \neq \mathcal{U}(f_S; \delta)$ is a quasi- Γ -ideal of S . If $A_p = \emptyset$, then $(f_S \Gamma \chi_S)(p) = \emptyset = (\chi_S \Gamma f_S)(p)$, then $(f_S \Gamma \chi_S)(p) \cap (\chi_S \Gamma f_S)(p) = \emptyset \subseteq f_S(p)$.

Let $A_p \neq \emptyset$. Let for every $(s, t) \in A_p$, there exists some $\beta_i \subseteq U$ such that $f_S(s) \cap f_S(t) = \beta_i$, then $f_S(s) \supseteq \beta_i$ and $f_S(t) \supseteq \beta_i$ for some i . Then $s, t \in \mathcal{U}(f_S; \beta_i)$. Since $(s, t) \in A_p$, then $p \leq s\lambda t$ for some $\lambda \in \Gamma$, so $p \in (\mathcal{U}(f_S; \beta_i) \Gamma S]$ and $p \in (S \Gamma \mathcal{U}(f_S; \beta_i))$ for each i . Then $p \in (\mathcal{U}(f_S; \beta_i) \Gamma S) \cap (S \Gamma \mathcal{U}(f_S; \beta_i)) \subseteq \mathcal{U}(f_S; \beta_i)$, we have $p \in \mathcal{U}(f_S; \beta_i)$ for each i . Then $f_S(p) \supseteq \beta_i$ for each i . We have

$$\begin{aligned} ((f_S \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma f_S))(p) &= (f_S \Gamma \chi_S)(p) \cap (\chi_S \Gamma f_S)(p) \\ &= \left(\bigcup_{(s,t) \in A_p} \{f_S(s) \cap \chi_S(t)\} \right) \cap \left(\bigcup_{(s,t) \in A_p} \{\chi_S(s) \cap f_S(t)\} \right) \\ &= \bigcup_{(s,t) \in A_p} \{f_S(s) \cap f_S(t)\} = \cup \beta_i \subseteq f_S(p). \end{aligned}$$

Thus $((f_S \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma f_S))(p) \subseteq f_S(p)$. Let $p, q \in S$ be such that $p \leq q$, then $f_S(p) \supseteq f_S(q)$. Indeed: Let $f_S(p) \subset f_S(q)$. Then there exist $\alpha \subseteq U$ such that $f_S(p) \subseteq \alpha \subsetneq f_S(q) \implies q \in \mathcal{U}(f_S; \alpha)$ and $p \notin \mathcal{U}(f_S; \alpha)$, which is a contradiction because $\mathcal{U}(f_S; \alpha)$ is a quasi- Γ -ideal of S . Hence $f_S(p) \supseteq f_S(q)$. Thus $(f_S; S)$ is a int-soft quasi- Γ -ideal over U .

(2) \implies (1) Let $a \in (\mathcal{U}(f_S; \delta) \Gamma S) \cap (S \Gamma \mathcal{U}(f_S; \delta)) \implies a \in (\mathcal{U}(f_S; \delta) \Gamma S)$ and $a \in (S \Gamma \mathcal{U}(f_S; \delta))$. Then there exists some $m, n \in \mathcal{U}(f_S; \delta)$, $p, q \in S$ and $\lambda, \mu \in \Gamma$ such that $a \leq m\lambda p$ and $a \leq q\mu n$. Then $(m, p), (q, n) \in A_a$. Since $m, n \in \mathcal{U}(f_S; \delta)$, so $f_S(m) \supseteq \delta$ and $f_S(n) \supseteq \delta$. Then we have

$$(f_S \Gamma \chi_S)(a) = \bigcup_{(s,t) \in A_a} f_S(s) \cap \chi_S(t) \supseteq f_S(m) \cap \chi_S(p) \supseteq \delta \cap U = \delta.$$

Hence $(f_S \Gamma \chi_S)(a) \supseteq \delta$. Furthermore, we have

$$(\chi_S \Gamma f_S)(a) = \bigcup_{(s,t) \in A_a} \chi_S(t) \cap f_S(s) \supseteq \chi_S(q) \cap f_S(n) \supseteq U \cap \delta = \delta.$$

Hence $(\chi_S \Gamma f_S)(a) \supseteq \delta$ and $(f_S \Gamma \chi_S)(a) \cap (\chi_S \Gamma f_S)(a) \supseteq \delta$. But f_S is an int-soft quasi- Γ -ideal over U , so $f_S(a) \supseteq (f_S \Gamma \chi_S)(a) \cap (\chi_S \Gamma f_S)(a) \supseteq \delta \implies f_S(a) \supseteq \delta$ and so $a \in \mathcal{U}(f_S; \delta)$. Thus $(\mathcal{U}(f_S; \delta) \Gamma S] \cap (S \Gamma \mathcal{U}(f_S; \delta))] \subseteq \mathcal{U}(f_S; \delta)$.

Let $a, b \in S$ be such that $a \leq b$ and $b \in \mathcal{U}(f_S; \delta)$, then $a \in \mathcal{U}(f_S; \delta)$. Indeed: As $b \in \mathcal{U}(f_S; \delta)$, then $f_S(b) \supseteq \delta$. Since f_S is an int-soft quasi- Γ -ideal over U , so $f_S(a) \supseteq f_S(b) \supseteq \delta \implies f_S(a) \supseteq \delta$. Hence $a \in \mathcal{U}(f_S; \delta)$. Therefore, $\mathcal{U}(f_S; \delta)$ is a quasi- Γ -ideal of S . \square

Theorem 4.17. *Let (S, Γ, \leq) be an ordered Γ -semigroup, then the soft intersection of any int-soft right Γ -ideal (f_S, S) and any int-soft left Γ -ideal (g_S, S) over U is an int-soft quasi- Γ -ideal of S over U .*

Proof. Let f_S be any int-soft right Γ -ideal and g_S be any int-soft left Γ -ideal over U . Let $a \in S$. Then

$$((f_S \widetilde{\cap} g_S) \Gamma \chi_S)(a) \subseteq (f_S \Gamma \chi_S)(a) \subseteq f_S(a) \quad (\text{by Proposition 4.3})$$

Thus $((f_S \widetilde{\cap} g_S) \Gamma \chi_S)(a) \subseteq f_S(a)$. Moreover, we have

$$(\chi_S \Gamma (f_S \widetilde{\cap} g_S))(a) \subseteq (\chi_S \Gamma g_S)(a) \subseteq g_S(a) \quad (\text{by Proposition 4.3})$$

Thus $(\chi_S \Gamma (f_S \widetilde{\cap} g_S))(a) \subseteq g_S(a)$. Then $((f_S \widetilde{\cap} g_S) \Gamma \chi_S)(a) \cap (\chi_S \Gamma (f_S \widetilde{\cap} g_S))(a) \subseteq f_S(a) \cap g_S(a) = (f_S \widetilde{\cap} g_S)(a)$. Thus $((f_S \widetilde{\cap} g_S) \Gamma \chi_S) \widetilde{\cap} (\chi_S \Gamma (f_S \widetilde{\cap} g_S)) \subseteq f_S \widetilde{\cap} g_S$.

Let $s, t \in S$ be such that $s \leq t$, then $(f_S \widetilde{\cap} g_S)(s) \supseteq (f_S \widetilde{\cap} g_S)(t)$. Indeed: Since f_S and g_S are int-soft right Γ -ideal and int-soft left Γ -ideal over U . Then $f_S(s) \supseteq f_S(t)$ and $g_S(s) \supseteq g_S(t)$, so $f_S(s) \cap g_S(s) \supseteq f_S(t) \cap g_S(t)$. Therefore, $(f_S \widetilde{\cap} g_S)(s) = f_S(s) \cap g_S(s) \supseteq f_S(t) \cap g_S(t) = (f_S \widetilde{\cap} g_S)(t)$. Hence $(f_S \widetilde{\cap} g_S; S)$ is an int-soft quasi- Γ -ideal over U . \square

Theorem 4.18. *(see [1]) In an ordered Γ -semigroup S , the following are equivalent:*

- (1) S is regular.
- (2) The set $(R \Gamma L]$ is a quasi- Γ -ideal of S for each right Γ -ideal R and for each left Γ -ideal L of S .

Theorem 4.19. *In an ordered Γ -semigroup S , the given statements are equivalent:*

- (1) S is regular.
- (2) for any int-soft right Γ -ideal (f_S, S) and for any int-soft left Γ -ideal (g_S, S) of S , the product $(f_S \Gamma g_S, S)$ is an int-soft quasi- Γ -ideal over U .

Proof. (1) \implies (2) Since S is regular, then by Corollary 4.9, we have $(f_S \widetilde{\cap} g_S, S) = (f_S \Gamma g_S, S)$. By Theorem 4.17, we have $(f_S \widetilde{\cap} g_S, S)$ is an int-soft quasi- Γ -ideal over U and so $(f_S \Gamma g_S, S)$ is an int-soft quasi- Γ -ideal of S over U .

(2) \implies (1) Let R be any right Γ -ideal and L be any left Γ -ideal of S , so by using Proposition 4.4 and assumption $(\chi_R \Gamma \chi_L, S)$ is an int-soft quasi- Γ -ideal over U . Moreover, $(\chi_R \Gamma \chi_L, S) = (\chi_{(R \Gamma L)}, S)$, so $\chi_{(R \Gamma L)}$ is an int-soft quasi- Γ -ideal over U . Thus $(R \Gamma L]$ is a quasi- Γ -ideal of S (by Theorem 4.12). Hence by Theorem 4.18, S is regular. \square

5. CHARACTERIZATIONS OF COMPLETELY REGULAR ORDERED Γ -SEMIGROUP BY USING ITS INT-SOFT QUASI- Γ -IDEALS

In this section, characterizations of completely regular ordered Γ -semigroups by using int-soft quasi- Γ -ideals and semiprime int-soft quasi- Γ -ideals are provided.

Definition 5.1. [19] An ordered Γ -semigroup (S, Γ, \leq) is called completely regular if S is left regular, right regular and regular. Or in other words, an element $s \in S$ is called completely regular if there exist some element $r \in S$ and $\lambda, \mu, \rho, \sigma \in S$ such that $s \leq s\lambda\mu\rho\sigma s$.

If every element of an ordered Γ -semigroup S is completely regular, then S is called completely regular ordered Γ -semigroup. Equivalent definitions are:

- (1) For every $K \subseteq S$, $K \subseteq (K\Gamma K\Gamma S\Gamma K\Gamma K)$.
- (2)) For every element $s \in S$, $s \in (s\Gamma s\Gamma S\Gamma s\Gamma s)$.

Lemma 5.2. *Every int-soft quasi- Γ -ideal (f_S, S) of an ordered Γ -semigroup S , is an int-soft Γ -semigroup over U .*

Proof. Let (f_S, S) is an int-soft quasi- Γ -ideal of an ordered Γ -semigroup S over U . Let $p, q \in S$ and $\lambda \in S$. Since $(p, q) \in A_{p\lambda q}$ and (f_S, S) is an int-soft quasi- Γ -ideal of S , Then we have

$$\begin{aligned} f_S(p\lambda q) &\supseteq ((f_S\Gamma\chi_S) \widetilde{\cap} (\chi_S\Gamma f_S))(p\lambda q) \\ &= (f_S\Gamma\chi_S)(p\lambda q) \cap (\chi_S\Gamma f_S)(p\lambda q) \\ &= \left(\bigcup_{(s,t) \in A_{p\lambda q}} \{f_S(s) \cap \chi_S(t)\} \right) \cap \left(\bigcup_{(s,t) \in A_{p\lambda q}} \{\chi_S(s) \cap f_S(t)\} \right) \\ &\supseteq \{f_S(p) \cap \chi_S(q)\} \cap \{\chi_S(p) \cap f_S(q)\} \\ &= f_S(p) \cap f_S(q). \end{aligned}$$

Thus $f_S(p\lambda q) \supseteq f_S(p) \cap f_S(q)$ for all $p, q \in S$ and $\lambda \in \Gamma$. Therefore, (f_S, S) is a int-soft Γ -semigroup of S . \square

Lemma 5.3. ([19]) *Let (S, Γ, \leq) be an ordered Γ -semigroup, then the statements given below are equivalent:*

- (1) S is completely regular.
- (2) Every bi- Γ -ideal B of S is semiprime.

Lemma 5.4. ([20]). *Every quasi- Γ -ideal Q of an ordered Γ -semigroup S is a bi- Γ -ideal of S .*

Lemma 5.5. (see [1]) *In a regular ordered Γ -semigroup S , every bi- Γ -ideal B of S is a quasi- Γ -ideal of S .*

Theorem 5.6. *In an ordered Γ -semigroup S , the given statements are equivalent:*

- (1) S is completely regular.
- (2) For all int-soft quasi- Γ -ideal (f_S, S) of S , we have, $(f_S(p), S) = \left(\bigcap_{\theta \in \Gamma} f_S(p\theta p), S \right)$ for all $p \in S$ and S is regular.

Proof. (1) \implies (2). Let f_S be an int-soft quasi- Γ -ideal over U and $p \in S$. Since S is left regular as well as right regular, then $p \leq u\theta p\mu p$ and $p \leq p\rho p\sigma v$ for some

$u, v \in S$ and $\theta, \mu, \rho, \sigma \in \Gamma$. Then $(u, p\mu\rho), (p\rho\rho, v) \in A_p$ and we have

$$\begin{aligned} f_S(p) &\supseteq ((f_S\Gamma\chi_S)\tilde{\cap}(\chi_S\Gamma f_S))(p) \\ &= (f_S\Gamma\chi_S)(p) \cap (\chi_S\Gamma f_S)(p) \\ &= \left(\bigcup_{(s,t) \in A_p} \{f_S(s) \cap \chi_S(t)\}\right) \cap \left(\bigcup_{(s,t) \in A_p} \{\chi_S(s) \cap f_S(t)\}\right) \\ &\supseteq \{f_S(p\rho\rho) \cap \chi_S(v)\} \cap \{\chi_S(u) \cap f_S(p\mu\rho)\} \\ &= f_S(p\rho\rho) \cap f_S(p\mu\rho) \\ &\supseteq \bigcap_{\theta \in \Gamma} f_S(p\theta\rho). \end{aligned}$$

Thus $f_S(p) \supseteq \bigcap_{\theta \in \Gamma} f_S(p\theta\rho)$. Since f_S is an int-soft quasi- Γ -ideal over U , then by Lemma 5.2, f_S is a int-soft Γ -semigroup of S . Hence $\bigcap_{\theta \in \Gamma} f_S(p\theta\rho) \supseteq f_S(p)$. Thus $f_S(p) = \bigcap_{\theta \in \Gamma} f_S(p\theta\rho)$ for all $p \in S$. Since S is completely regular, then S is regular.

(2) \implies (1). Let $p \in S$. Let A be any quasi- Γ -ideal of S , then by Lemma 5.4, A is a bi- Γ -ideal of S . Let $p\theta\rho \in A$ for all $\theta \in \Gamma$. Then by Theorem 4.12 and (2), we have, $\chi_A(p) = \bigcap_{\theta \in \Gamma} \chi_A(p\theta\rho)$. Since $p\theta\rho \in A$, then $\chi_A(p\theta\rho) = U$ imply $\bigcap_{\theta \in \Gamma} \chi_A(p\theta\rho) = U = \chi_A(p)$. Thus $p \in A$. Hence A is semiprime. Since S is regular, then by Lemma 5.5, A is any bi- Γ -ideal of S and is semiprime. Thus by Theorem 5.3, S is completely regular. \square

Definition 5.7. Let (S, Γ, \leq) be an ordered Γ -semigroup and $\emptyset \neq C \subseteq S$. Then C is called semiprime if for all $p \in S$ and $\lambda \in \Gamma$, $p\lambda p \in C$ implies $p \in C$. Equivalently, for every $D \subseteq S$, if $D\Gamma D \subseteq C$ implies $D \subseteq C$.

Definition 5.8. Let a be any element of an ordered Γ -semigroup (S, Γ, \leq) . Let $\emptyset \neq \lambda \subseteq U$. We define a soft set called critical soft point, denoted by $((a]_\lambda, S)$, is defined as

$$(a]_\lambda : S \longrightarrow P(U), \quad a_\lambda(p) = \begin{cases} \lambda, & \text{if } p \in (a], \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 5.9. Let (f_S, S) be an int-soft quasi- Γ -ideal of an ordered Γ -semigroup S over U . If $(f_S(p), S) \supseteq (f_S(p\theta\rho), S)$ for all $p \in S$ and $\theta \in \Gamma$, then (f_S, S) is called semiprime int-soft quasi- Γ -ideal of S . Equivalently, if for every critical soft point $(a]_\lambda$ of S such that $(a]_\lambda \Gamma (a]_\lambda \subseteq f_S$ implies $(a]_\lambda \subseteq f_S$.

For any soft set (f_S, S) of S over U , we define, $((f_S]](x) = \bigcup_{x \leq y} f_S(y)$ for any $x \in S$.

If for any soft set f_S of S , we have $((f_S]], S) = (f_S, S)$, then f_S is called strongly convex.

Lemma 5.10. Let (f_S, S) be a soft subset of an ordered Γ -semigroup S . Then, f_S is a strongly convex soft subset of S if and only if $x \leq y$ implies $f_S(y) \subseteq f_S(x)$, for all $x, y \in S$.

Proof. Straightforward. \square

Lemma 5.11. Let $(a]_\lambda, (a]_\mu$ be critical soft points of an ordered Γ -semigroup S and let f_S be a soft set of S over U , then the following are true:

- (1) If f_S is strongly convex soft subset of S , then $(a]_\lambda \subseteq f_S$ iff $f_S(a) \supseteq \lambda$.
- (2) $(a]_\lambda \Gamma (b]_\mu = \bigcup_{c \in a\Gamma b} (c]_{\lambda \cap \mu}$.

Proof. (1) \implies Let f_S be a strongly convex soft subset of S . Let $(a]_\lambda \subseteq f_S$ implies $(a]_\lambda(a) \subseteq f_S(a)$ or $f_S(a) \supseteq (a]_\lambda(a) = \lambda$ implies $f_S(a) \supseteq \lambda$.

\Leftarrow If $f_S(a) \supseteq \lambda$, then $(a]_\lambda \subseteq f_S$. Indeed: If $p \notin (a]$, then $(a]_\lambda(p) = \emptyset \subseteq f_S(p)$. If $p \in (a]$, then $p \leq a$, since f_S is strongly convex, so by Lemma 5.10, $f_S(p) \supseteq f_S(a) \supseteq \lambda = (a]_\lambda(p)$ implies $f_S(p) \supseteq (a]_\lambda(p)$ or $(a]_\lambda \subseteq f_S$.

(2). Let $p \in S$. If $p \in (c]$ for some $c \in a\Gamma b$, then $p \leq c \in a\Gamma b$, then $p \leq a\alpha b$ for some $\alpha \in \Gamma$, then $(a, b) \in A_p$ and we have ,

$$\left((a]_\lambda \Gamma (b]_\mu \right) (p) = \bigcup_{(s,t) \in A_p} (a]_\lambda(s) \cap (b]_\mu(t) \supseteq (a]_\lambda(a) \cap (b]_\mu(b) = \lambda \cap \mu.$$

Hence $((a]_\lambda \Gamma b]_\mu)(p) \supseteq \lambda \cap \mu$. Since $(a]_\lambda(s) \cap (b]_\mu(t) \subseteq \lambda \cap \mu$ for any $s, t \in S$, so we have $\left((a]_\lambda \Gamma (b]_\mu \right) (p) \subseteq \lambda \cap \mu$, thus $\left((a]_\lambda \Gamma (b]_\mu \right) (p) = \lambda \cap \mu = \bigcup_{c \in a\Gamma b} (c]_{\lambda \cap \mu} (p)$.

If $p \notin (c]$ for all $c \in a\Gamma b$, then we have $\bigcup_{c \in a\Gamma b} (c]_{\lambda \cap \mu} (p) = \emptyset$. On the other hand,

$\left((a]_\lambda \Gamma (b]_\mu \right) (p) = \emptyset$. In fact, if $\left((a]_\lambda \Gamma (b]_\mu \right) (p) \neq \emptyset$, then we have, $((a]_\lambda \Gamma b]_\mu)(p) = \bigcup_{(s,t) \in A_p} (a]_\lambda(s) \cap (b]_\mu(t) \neq \emptyset$. Then there exist $(u, v) \in A_p$ such that $(a]_\lambda(u) = \lambda$

and $(b]_\mu(v) = \mu$. Thus $u \in (a]$ and $v \in (b]$. Since $(u, v) \in A_p$, so $p \leq u\lambda v$ for some $\lambda \in \Gamma$. Thus $p \in ((a] \Gamma (b]) = (a\Gamma b]$, then $p \leq a\alpha b$ for some $\alpha \in \Gamma$ which is impossible. Thus, in this case $\left((a]_\lambda \Gamma (b]_\mu \right) (p) = \emptyset = \bigcup_{c \in a\Gamma b} (c]_{\lambda \cap \mu}$. \square

Theorem 5.12. *Let f_S be an int-soft quasi- Γ -ideal of an ordered Γ -semigroup S . Then the statements given below are equivalent:*

- (1) f_S is semiprime.
- (2) $f_S(p) \supseteq \bigcap_{q \in p\Gamma p} f_S(q)$ for any $p \in S$.
- (3) $f_S(p) = \bigcap_{q \in p\Gamma p} f_S(q)$ for any $p \in S$.

Proof. (1) \implies (2). Let $p \in S$, $f_S(p) \supseteq \bigcap_{q \in p\Gamma p} f_S(q)$. Indeed: If $f_S(p) \subset \bigcap_{q \in p\Gamma p} f_S(q)$, then there exists $\lambda \subseteq U$ such that $f_S(p) \subseteq \lambda \subseteq \bigcap_{q \in p\Gamma p} f_S(q)$. Then $f_S(q) \supseteq \lambda$ for all $q \in p\Gamma p$, so by Lemma 5.11 (1) $(q]_\lambda \subseteq f_S$ for all $q \in p\Gamma p$. Thus by Lemma 5.11 (2), $(p]_\lambda \Gamma (p]_\lambda = \bigcup_{q \in p\Gamma p} (q]_\lambda \subseteq f_S$. Since f_S is semiprime int-soft quasi- Γ -ideal of S , then $(p]_\lambda \subseteq f_S$. Thus by Lemma 5.11 (1), we have $f_S(p) \supseteq \lambda$ which is a contradiction. Therefore, $f_S(p) \supseteq \bigcap_{q \in p\Gamma p} f_S(q)$ for any $p \in S$.

(2) \implies (3). Let $p \in S$. Since f_S is an int-soft quasi- Γ -ideal, so by Lemma 5.2, f_S is a int-soft Γ -semigroup, so $\bigcap_{q \in p\Gamma p} f_S(q) \supseteq f_S(p)$ for all $q \in p\Gamma p$ and by (2), we have $f_S(p) = \bigcap_{q \in p\Gamma p} f_S(q)$ for any $p \in S$.

(3) \implies (1). Let $p \in S$ and $(p]_\lambda \Gamma (p]_\lambda \subseteq f_S$, $\lambda \subseteq U$, then by Lemma 5.11 (2), we have $(p]_\lambda \Gamma (p]_\lambda = \bigcup_{q \in p\Gamma p} (q]_\lambda \subseteq f_S$. Then $(q]_\lambda \subseteq f_S$ for every $q \in p\Gamma p$ and so by Lemma 5.11 (1) $f_S(q) \supseteq \lambda$ for every $q \in p\Gamma p$. Thus we have, $f_S(p) = \bigcap_{q \in p\Gamma p} f_S(q) \supseteq \lambda$. So $f_S(p) \supseteq \lambda$. Since f_S is strongly convex, then by Lemma 5.11 (1), we have $(p]_\lambda \subseteq f_S$. Therefore, f_S is semiprime. \square

Theorem 5.13. *In an ordered Γ -semigroup S , the following statements are equivalent:*

- (1) S is completely regular.

- (2) Each int-soft quasi- Γ -ideal (f_S, S) of S over U is semiprime int-soft quasi- Γ -ideal and S is regular.

Proof. (1) \implies (2). Let f_S be an int-soft quasi- Γ -ideal of an ordered Γ -semigroup S over U . Let $p \in S$. Since S is a left as well as right regular, then $p \leq u\lambda\rho\mu p$ and $p \leq p\rho p\sigma v$ for some $u, v \in S$ and $\lambda, \mu, \rho, \sigma \in \Gamma$. Then $(u, p\mu p), (p\rho p, v) \in A_p$ and we have

$$\begin{aligned} f_S(p) &\supseteq ((f_S\Gamma\chi_S) \tilde{\cap} (\chi_S\Gamma f_S))(p) \\ &= (f_S\Gamma\chi_S)(p) \cap (\chi_S\Gamma f_S)(p) \\ &= \left(\bigcup_{(s,t) \in A_p} \{f_S(s) \cap \chi_S(t)\} \right) \cap \left(\bigcup_{(s,t) \in A_p} \{\chi_S(s) \cap f_S(t)\} \right) \\ &\supseteq \{f_S(p\rho p) \cap \chi_S(v)\} \cap \{\chi_S(u) \cap f_S(p\mu p)\} \\ &= f_S(p\rho p) \cap f_S(p\mu p) \\ &\supseteq \bigcap_{\lambda \in \Gamma} f_S(p\lambda p). \end{aligned}$$

Thus $f_S(p) \supseteq \bigcap_{\lambda \in \Gamma} f_S(p\lambda p)$. Then by Theorem 5.12, (f_S, S) is semiprime int-soft quasi- Γ -ideal over U . Since S is completely regular, so S is regular.

(2) \implies (1). Let $p \in S$. Let A be any quasi- Γ -ideal of S , then by lemma 5.4, A is a bi- Γ -ideal of S . Let $p\theta p \in A$ for all $\theta \in \Gamma$. Then by Theorem 4.12 and (2), we have, $\chi_A(p) = \bigcap_{\theta \in \Gamma} \chi_A(p\theta p)$. Since $p\theta p \in A$, so $\chi_A(p\theta p) = U$ imply $\bigcap_{\theta \in \Gamma} \chi_A(p\theta p) = U = \chi_A(p)$. Thus $p \in A$. Hence A is semiprime. Since S is regular, so by Lemma 5.5, A is any bi- Γ -ideal of S and is semiprime. Thus by Theorem 5.3, S is a completely regular. \square

6. CHARACTERIZATIONS OF SEMILATTICES OF LEFT AS WELL AS RIGHT SIMPLE ORDERED Γ -SEMIGROUPS IN TERMS OF INT-SOFT QUASI- Γ -IDEALS

In this section, we define semilattice of left and right simple semigroups of ordered Γ -semigroups and characterize semilattice of left and right simple semigroups in terms of their int-soft quasi- Γ -ideals.

Definition 6.1. ([12]) A sub- Γ -semigroup F of an ordered Γ -semigroup S is said to be a filter of S if:

- (1) $(\forall u, v \in S \text{ and } \forall \lambda \in \Gamma) (u\lambda v \in F \implies u \in F \text{ and } v \in F)$.
- (2) $(\forall u \in F \text{ and } \forall v \in S) (u \geq v \implies v \in F)$.

For $a \in S$, the least filter of S generated by a , is denoted by $N(a)$. We denote, the relation on S by \mathcal{N} , is defined by $\mathcal{N} := \{(u, v) \in S \times S \mid N(u) = N(v)\}$. The relation \mathcal{N} on S is an equivalence relation on S .

Definition 6.2. Let (S, Γ, \leq) be an ordered Γ -semigroup. Then

- (1) A congruence (see [26]) on S is an equivalence relation σ on S such that, if $(u, v) \in \sigma \implies (u\lambda s, v\lambda s) \in \sigma$ and $(s\lambda u, s\lambda v) \in \sigma$ for every $s \in S$ and $\lambda \in \Gamma$.
- (2) A semilattice congruence (see [26]) on S is a congruence σ on S such that $(u, u\lambda u) \in \sigma$ and $(u\lambda v, v\lambda u) \in \sigma$ for every $u, v \in S$ and $\lambda \in \Gamma$.

Note if σ is a semilattice congruence on S , then for every $s \in S$, the σ -class $(s)_\sigma$ of S is a sub- Γ -semigroup of S . Moreover, \mathcal{N} is a semilattice congruence on S (see [26]).

Definition 6.3. Let (S, Γ, \leq) be an ordered Γ -semigroup. Then S is called a semilattice of left as well as right simple semigroups if there exists a semilattice X and a family $\{S_\beta\}_{\beta \in X}$ of left as well as right simple sub- Γ -semigroup of S such that:

- (1) $(\forall \alpha, \beta \in X, \alpha \neq \beta) (S_\alpha \cap S_\beta) = \emptyset$,
- (2) $S = \bigcup_{\alpha \in X} S_\alpha$,
- (3) $(\alpha, \beta \in X) (S_\alpha \Gamma S_\beta \subseteq S_{\alpha\beta})$.

Equivalently, if there exists a semilattice congruence σ on S such that every σ -class $(s)_\sigma$ of S is a left simple as well as right simple sub- Γ -semigroup of S , for every $s \in S$.

Lemma 6.4. ([17]) *For every $s \in S$, the \mathcal{N} -class $(s)_\mathcal{N}$ of an ordered Γ -semigroup S is a right (resp., left) simple sub- Γ -semigroup of S if and only if every right (resp., left) ideal is a left (resp., right) ideal of S and is semiprime.*

Theorem 6.5. *Let (S, Γ, \leq) be an ordered Γ -semigroup, then the statements given below are equivalent:*

- (1) S is a semilattice of left as well as right simple semigroups.
- (2) For every int-soft quasi- Γ -ideal (f_S, S) over U , we have,

$$(i) (f_S(u), S) = \left(\bigcap_{\lambda \in \Gamma} f_S(u\lambda u), S \right),$$

$$(ii) (f_S(u\theta v), S) = (f_S(v\theta u), S) \text{ for all } u, v \in S \text{ and } \theta \in \Gamma.$$

Proof. (1) \implies (2). Let S is a semilattice of left as well as right simple semigroups. Let (f_S, S) is an int-soft quasi- Γ -ideal over U .

To prove (i), by Theorem 5.6, it is enough to prove that S is completely regular. Let $u \in S = \bigcup_{\alpha \in X} S_\alpha$, then $u \in S_\alpha$ for some $\alpha \in X$. Since each S_α is left as well as right simple, then $S_\alpha = (u\Gamma S_\alpha]$ and $S_\alpha = (S_\alpha\Gamma u]$. So we have,

$$\begin{aligned} (u\Gamma S_\alpha] &= (u\Gamma (S_\alpha\Gamma u]) \\ &= ((u] \Gamma ((S_\alpha\Gamma u])) \\ &= ((u] \Gamma (S_\alpha\Gamma u]) \\ &= (u\Gamma (S_\alpha\Gamma u]) \end{aligned}$$

Thus $u \in (u\Gamma S_\alpha] = (u\Gamma (S_\alpha\Gamma u])$ implies $u \in (u\Gamma (S_\alpha\Gamma u])$, then there exists $v \in S_\alpha$ and $\theta, \mu \in \Gamma$ such that $u \leq u\theta v\mu u$. Since $v \in S_\alpha = (u\Gamma S_\alpha\Gamma u]$ implies $v \in (u\Gamma S_\alpha\Gamma u]$, then there exists $s \in S_\alpha$ and $\rho, \lambda \in \Gamma$ such that $v \leq u\rho s\lambda u$. Thus we have, $u \leq u\theta v\mu u \leq u\theta(u\rho s\lambda u)\mu u \leq (u\theta u)\rho s\lambda(u\mu u) \in u\Gamma u\Gamma S\Gamma u\Gamma u$ implies $u \in (u\Gamma u\Gamma S\Gamma u\Gamma u]$. Therefore, S is completely regular.

(ii) Let $u, v \in S = \bigcup_{\alpha \in X} S_\alpha$. Then $u \in S_\alpha$ and $v \in S_\beta$ for some $\alpha, \beta \in X$. Thus for all $\theta \in \Gamma$, $u\theta v \in S_\alpha\Gamma S_\beta \subseteq S_{\alpha\beta}$ implies $u\theta v \in S_{\alpha\beta}$. Also $v\theta u \in S_\beta\Gamma S_\alpha = S_{\beta\alpha}$ implies $v\theta u \in S_{\beta\alpha} = S_{\alpha\beta}$ (because X is a semilattice). Thus $v\theta u \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left as well as right simple sub- Γ -semigroup of S , then $S_{\alpha\beta} = (S_{\alpha\beta}\Gamma c]$ and $S_{\alpha\beta} = (c\Gamma S_{\alpha\beta}]$ for all $c \in S_{\alpha\beta}$. Since $u\theta v \in S_{\alpha\beta}$, then $u\theta v \in (v\theta u\Gamma S_{\alpha\beta}]$ and $u\theta v \in (S_{\alpha\beta}\Gamma v\theta u]$ implies $u\theta v \in (v\theta u\Gamma S_{\alpha\beta}] \cap (S_{\alpha\beta}\Gamma v\theta u] \subseteq (v\theta u\Gamma S] \cap (S\Gamma v\theta u]$. Thus $u\theta v \in (v\theta u\Gamma S] \cap (S\Gamma v\theta u]$ implies $u\theta v \in (v\theta u\Gamma S]$ and $u\theta v \in (S\Gamma v\theta u]$, then there exists $\rho, \lambda \in \Gamma$ and $x, y \in S$ such that $u\theta v \leq (v\theta u)\rho x$ and $u\theta v \leq y\lambda(v\theta u)$. So

$(y, v\theta u), (v\theta u, x) \in A_{u\theta v}$. Since f_S is an int-soft quasi- Γ -ideal over U , so we have

$$\begin{aligned} f_S(u\theta v) &\supseteq ((f_S\Gamma\chi_S) \tilde{\cap} (\chi_S\Gamma f_S))(u\theta v) \\ &= (f_S\Gamma\chi_S)(u\theta v) \cap (\chi_S\Gamma f_S)(u\theta v) \\ &= \left(\bigcup_{(s,t) \in A_{u\theta v}} \{f_S(s) \cap \chi_S(t)\} \right) \cap \left(\bigcup_{(s,t) \in A_{u\theta v}} \{\chi_S(s) \cap f_S(t)\} \right) \\ &\supseteq \{f_S(v\theta u) \cap \chi_S(x)\} \cap \{\chi_S(y) \cap f_S(v\theta u)\} \\ &= f_S(v\theta u) \cap f_S(v\theta u) = f_S(v\theta u). \end{aligned}$$

Hence $f_S(u\theta v) \supseteq f_S(v\theta u)$. Similary we can proof that $f_S(v\theta u) \supseteq f_S(u\theta v)$. Hence $f_S(u\theta v) = f_S(v\theta u)$ for all $u, v \in S$ and $\theta \in \Gamma$.

(2) \implies (1) Let for every int-soft quasi- Γ -ideals (f_S, S) of S , (i) and (ii) hold. By using Lemma 6.4, we prove (1). Let L is a left Γ -ideal of S and $u \in L, v \in S$, so by Proposition 4.4 and Theorem 4.13, we have χ_L is a int-soft quasi- Γ -ideal of S . Then by (2), $\chi_L(u\theta v) = \chi_L(v\theta u)$. Since $v\theta u \in S\Gamma L \subseteq L$ implies $v\theta u \in L$, then $\chi_L(v\theta u) = U = \chi_L(u\theta v)$. Thus $u\theta v \in L$. Hence $L\Gamma S \subseteq L$. Also for all $u \in L$ and for all $v \in S$, if $v \leq u \implies v \in L$. Therefore L is right Γ -ideal of S . Let $u \in S$ such that $u\theta u \in L$ for all $\theta \in \Gamma$. Since L is a quasi- Γ -ideal of S , so by using theorem 4.12 and (2), we have $\chi_L(u) = \bigcap_{\theta \in \Gamma} \chi_L(u\theta u)$. Since $u\theta u \in L$ for all $\theta \in \Gamma$, we have $\bigcap_{\theta \in \Gamma} \chi_L(u\theta u) = U = \chi_L(u)$. Thus $u \in L$. Therefore L is semiprime. Similarly we can proof this for every right Γ -ideal of S . \square

Lemma 6.6. *Let (S, Γ, \leq) be an ordered Γ -semigroup. If $u \leq u\theta u, \forall u \in S$ and $\theta \in \Gamma$, then for every int-soft quasi- Γ -ideal f_S over U , we have,*

$$(f_S(u), S) = \left(\bigcap_{\theta \in \Gamma} f_S(u\theta u), S \right) \text{ for all } u \in S.$$

Proof. Reader can easily prove this by using Lemma 5.2. \square

Theorem 6.7. *Let (S, Γ, \leq) be an ordered Γ -semigroup. If $u \leq u\theta u, \forall u \in S$ and $\theta \in \Gamma$, then the following statements are equivalent:*

- (1) $u\lambda v \in (v\lambda u\Gamma S] \cap (S\Gamma v\lambda u]$ for each $u, v \in S$ and $\lambda \in \Gamma$.
- (2) For every int-soft quasi- Γ -ideal (f_S, S) of S , we have, $(f_S(u\lambda v), S) = (f_S(v\lambda u), S)$ for every $u, v \in S$ and $\lambda \in \Gamma$.

Proof. (1) \implies (2). As $u\lambda v \in (v\lambda u\Gamma S] \cap (S\Gamma v\lambda u]$, then $u\lambda v \in (v\lambda u\Gamma S]$ and $u\lambda v \in (S\Gamma v\lambda u]$. If $u\lambda v \in (v\lambda u\Gamma S]$, then there exists $\rho \in \Gamma$ and $p \in S$ such that $u\lambda v \leq (v\lambda u)\rho p$. By (1), we have $(v\lambda u)\rho p \in (p\rho(v\lambda u)\Gamma S] \cap (S\Gamma p\rho(v\lambda u)]$ implies $(v\lambda u)\rho p \in (S\Gamma p\rho(v\lambda u)]$, then there exists $\theta \in \Gamma$ and $q \in S$ such that $(v\lambda u)\rho p \leq (q\theta p)\rho(v\lambda u)$. Thus $u\lambda v \leq (v\lambda u)\rho p \leq (q\theta p)\rho(v\lambda u) \implies u\lambda v \leq (q\theta p)\rho(v\lambda u)$. So $(q\theta p, v\lambda u) \in A_{u\lambda v}$. If $u\lambda v \in (S\Gamma v\lambda u]$, then there exists $m, \in S$ and $\alpha \in \Gamma$ such that $u\lambda v \leq m\alpha(v\lambda u)$. By (1), we have $m\alpha(v\lambda u) \in ((v\lambda u)\alpha m\Gamma S] \cap (S\Gamma(v\lambda u)\alpha m]$ implies $m\alpha(v\lambda u) \in ((v\lambda u)\alpha m\Gamma S]$, then there exists $n \in S$ and $\beta \in \Gamma$ such that $m\alpha(v\lambda u) \leq (v\lambda u)\alpha(m\beta n)$ and so $u\lambda v \leq m\alpha(v\lambda u) \leq (v\lambda u)\alpha(m\beta n) \implies u\lambda v \leq (v\lambda u)\alpha(m\beta n)$. So $(v\lambda u, m\beta n) \in A_{u\lambda v}$.

Now let (f_S, S) is an int-soft quasi-ideal over U and $A_{u\lambda v} \neq \emptyset$, Then,

$$\begin{aligned} f_S(u\lambda v) &\supseteq ((f_S \Gamma \chi_S) \tilde{\cap} (\chi_S \Gamma f_S))(u\lambda v) \\ &= (f_S \Gamma \chi_S)(u\lambda v) \cap (\chi_S \Gamma f_S)(u\lambda v) \\ &= \bigcup_{(s,t) \in A_{u\lambda v}} \{f_S(s) \cap \chi_S(t)\} \cup \bigcup_{(s,t) \in A_{u\lambda v}} \{\chi_S(s) \cap f_S(t)\} \\ &\supseteq \{f_S(v\lambda u) \cap \chi_S(m\beta n)\} \cup \{\chi_S(q\theta p) \cap f_S(v\lambda u)\} \\ &= f_S(v\lambda u) \cup f_S(v\lambda u) = f_S(v\lambda u). \end{aligned}$$

Thus $f_S(u\lambda v) \supseteq f_S(v\lambda u)$. Similarly we can prove that $(f_S(v\lambda u), S) \supseteq (f_S(u\lambda v), S)$. Therefore, $(f_S(v\lambda u), S) = (f_S(u\lambda v), S)$ for all $u, v \in S$ and $\lambda \in \Gamma$.

(2) \implies (1). Let f_S be a int-soft quasi- Γ -ideal over U . Since $u \leq u\lambda u$ for all $u \in S$ and $\lambda \in \Gamma$, by Lemma 6.6, we have $f_S(u) = \bigcap_{\lambda \in \Gamma} f_S(u\lambda u)$ for all $u \in S$.

By (2), we have $(f_S(v\lambda u), S) = (f_S(u\lambda v), S)$ for every $u, v \in S$ and $\lambda \in \Gamma$. By Theorem 6.5, implies that S is a semilattice of left simple as well as right simple sub- Γ -semigroups of S .

Let $u, v \in S = \bigcup_{\alpha \in X} S_\alpha$, then $u \in S_\alpha$ and $v \in S_\beta$ for some $\alpha, \beta \in X$. Then for all $\lambda \in \Gamma$, $u\lambda v \in S_\alpha \Gamma S_\beta \subseteq S_{\alpha\beta}$ implies $u\lambda v \in S_{\alpha\beta}$. Also $v\lambda u \in S_\beta \Gamma S_\alpha = S_{\beta\alpha}$ implies $v\lambda u \in S_{\beta\alpha} = S_{\alpha\beta}$ (because X is a semilattice). Hence $v\lambda u \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left as well as right simple sub- Γ -semigroup of S , so $S_{\alpha\beta} = (S_{\alpha\beta} \Gamma w]$ and $S_{\alpha\beta} = (w \Gamma S_{\alpha\beta}]$ for all $w \in S_{\alpha\beta}$. Since $u\lambda v \in S_{\alpha\beta}$, then $u\lambda v \in (v\lambda u \Gamma S_{\alpha\beta}]$ and $u\lambda v \in (S_{\alpha\beta} \Gamma v\lambda u]$ implies $u\lambda v \in (v\lambda u \Gamma S_{\alpha\beta}] \cap (S_{\alpha\beta} \Gamma v\lambda u] \subseteq (v\lambda u \Gamma S] \cap (S \Gamma v\lambda u]$. Thus $u\lambda v \in (v\lambda u \Gamma S] \cap (S \Gamma v\lambda u]$. This completes the proof. \square

7. CONCLUSION

In this paper, we have extended int-soft ideal theory of ordered semigroups to ordered Γ -semigroups. The results that have been proved in po-semigroups by using their fuzzy ideals, fuzzy quasi-ideals and uni-soft quasi ideals, we can prove the same results for ordered Γ -semigroups by using their int-soft quasi- Γ -ideals. The results that relates int-soft quasi- Γ -ideals and int-soft one sided Γ -ideals have been proved. Using the notion of int-soft quasi- Γ -ideals in ordered Γ -semigroups, we characterized completely regular ordered Γ -semigroups by using their int-soft quasi- Γ -ideal. The notion of critical soft point in po- Γ -semigroups have been defined. By using the definition of critical soft point, we have defined the notion of semiprime int-soft quasi- Γ -ideal and characterized completely regular ordered Γ -semigroups by using their semiprime int-soft quasi- Γ -ideals. Moreover, semilattice of left as well as right simple semigroups of ordered Γ -semigroups have been defined and some characterizations of semilattice of left as well as right simple semigroups by using their int-soft quasi- Γ -ideals have been done. From above results, we can conclude that the theory of int-soft ideals can be extended to other structures such as groups, rings etc. The structure of ordered Γ -semigroup can be applied to int-soft ideals.

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