

# On equations over direct powers of algebraic structures

Artem N. Shevlyakov

August 6, 2020

## Abstract

We study systems of equations over graphs, posets and matroids. We give the criteria, when a direct power of such algebraic structures is equationally Noetherian. Moreover we prove that any direct power of a finite algebraic structure is weakly equationally Noetherian.

## 1 Introduction

Let  $\mathbf{K}$  be an arbitrary class of mathematical objects. One of the main problem of mathematics is to describe “simple” and “hard” objects in  $\mathbf{K}$ . One can do it in different ways using various technic of algebra, geometry, calculus etc. In the current paper we make an attempt to classify “simple” and “hard” algebraic structures by universal algebraic geometry (UAG).

Following [1], UAG is a discipline of model theory, and it deals with equations over arbitrary algebraic structures. There are many notions of UAG which allow us to separate algebraic structures with “simple” and “hard” equational properties. The main feature here is the equationally Noetherian property. Recall that an algebraic structure  $\mathcal{A}$  is equationally Noetherian if any system of equations  $\mathbf{S}$  is equivalent over  $\mathcal{A}$  to a finite subsystem. Roughly speaking, if an algebraic structure  $\mathcal{A}$  is equationally Noetherian, then its equational properties are said to be “simple”. Otherwise, we assume that  $\mathcal{A}$  has a complicated equational theory.

Indeed, the Noetherian property is a central notion of UAG, and papers [1, 2, 3] contain the series of results which establish nice properties of equationally Noetherian algebraic structures. However, for *finite* algebraic structures the Noetherian property gives the trivial partition into “simple” and “hard” classes, since all finite algebraic structures are equationally Noetherian.

Thus, we have to propose an alternative approach in the division of finite algebraic structures into the classes with “simple” and “hard” equational properties. Our approach satisfies the following:

1. we deal with lattices of algebraic sets over a given algebraic structures (a set  $Y$  is algebraic over an algebraic structure  $\mathcal{A}$  if  $Y$  is the solution set of an appropriate system of equations);
2. we use the common operations of UAG (direct products, substructures, ultra-products etc.);
3. the partition into “simple” and “hard” algebraic structures is implemented by a list of first-order formulas  $\Phi$  such that

$$\mathcal{A} \text{ is “simple”} \Leftrightarrow \mathcal{A} \text{ satisfies } \Phi.$$

In other words, the “simple” class is axiomatizable by the formulas  $\Phi$ .

Namely, we offer to consider infinite direct powers  $\Pi\mathcal{A}$  of an algebraic structure  $\mathcal{A}$  and study of Diophantine equations over  $\Pi\mathcal{A}$  instead of Diophantine equations over  $\mathcal{A}$  (an equation  $E(X)$  is said to be Diophantine over an algebraic structure  $\mathcal{B}$  if  $E(X)$  may contain occurrences of any element from  $\mathcal{B}$ ). The decision rule in our approach is the following:

$$\mathcal{A} \text{ is "simple"} \Leftrightarrow \text{all direct powers of } \mathcal{A} \text{ are equationally Noetherian ;} \quad (1)$$

otherwise, an algebraic structure  $\mathcal{A}$  is said to be “hard”.

Some results of the type (1) were obtained in [4], where we describe all groups, rings and monoids satisfying (1). For example, a group (ring) satisfies (1) iff it is abelian (respectively, with zero multiplication).

The current paper continues the study of [4], and in Sections 3–5 we consider equations over the important classes of relational algebraic structures: graphs, partial orders and matroids. For each of these classes we describe algebraic structures that satisfies (1).

However, the most complicated and nontrivial part of our paper is Section 6. It contains the series of general results that hold for any direct power of any finite algebraic structure  $\mathcal{A}$ . In particular, we prove that any infinite system of equations  $\mathbf{S}$  over  $\Pi\mathcal{A}$  is equivalent to a finite system  $\mathbf{S}'$  (here we do not claim  $\mathbf{S}' \subseteq \mathbf{S}$ ). Thus, we prove that any direct power of a finite algebraic structure is weakly equationally Noetherian (see the definition in Section 2).

## 2 Basic definitions

Following [1, 2, 3], we give the main definitions of universal algebraic geometry.

Let  $\mathcal{L}$  be a language and  $\mathcal{A}$  be an algebraic structure of the language  $\mathcal{L}$  ( $\mathcal{L}$ -structure). In the current paper we consider languages of the following types:  $\mathcal{L}_g = \{E^{(2)}\}$  (graph language),  $\mathcal{L}_p = \{\leq^{(2)}\}$  (partial order language),  $\mathcal{L}_m = \{P_1^{(1)}, P_2^{(2)}, \dots\}$  (matroid language). An *equation over  $\mathcal{L}$*  ( $\mathcal{L}$ -equation) is an atomic formula over  $\mathcal{L}$ . The examples of equations in various languages are the following:  $E(x, y)$ ,  $E(x, x)$ ,  $x = y$  (language  $\mathcal{L}_g$ );  $x \leq y$ ,  $x \leq x$ ,  $x = y$  (language  $\mathcal{L}_p$ )  $P_1(x)$ ,  $P_2(x, y)$   $x = y$  (language  $\mathcal{L}_m$ ).

A *system of  $\mathcal{L}$ -equations* ( $\mathcal{L}$ -system for shortness) is an arbitrary set of  $\mathcal{L}$ -equations. Notice that we consider only systems in a finite set of variables  $X = \{x_1, x_2, \dots, x_n\}$ . The set of all solutions of  $\mathbf{S}$  in an  $\mathcal{L}$ -structure  $\mathcal{A}$  is denoted by  $V_{\mathcal{A}}(\mathbf{S}) \subseteq \mathcal{A}^n$ . A set  $Y \subseteq \mathcal{A}^n$  is said to be an *algebraic over  $\mathcal{A}$*  if there exists an  $\mathcal{L}$ -system  $\mathbf{S}$  with  $Y = V_{\mathcal{A}}(\mathbf{S})$ . If the solution set of an  $\mathcal{L}$ -system  $\mathbf{S}$  is empty,  $\mathbf{S}$  is said to be *inconsistent*. Two  $\mathcal{L}$ -systems  $\mathbf{S}_1, \mathbf{S}_2$  are called *equivalent over an  $\mathcal{L}$ -structure  $\mathcal{A}$*  if  $V_{\mathcal{A}}(\mathbf{S}_1) = V_{\mathcal{A}}(\mathbf{S}_2)$ . This equivalence relation is denoted by  $\mathbf{S}_1 \sim \mathbf{S}_2$ .

An  $\mathcal{L}$ -structure  $\mathcal{A}$  is  *$\mathcal{L}$ -equationally Noetherian* if any infinite  $\mathcal{L}$ -system  $\mathbf{S}$  is equivalent over  $\mathcal{A}$  to a finite subsystem  $\mathbf{S}' \subseteq \mathbf{S}$ . The class of equationally Noetherian  $\mathcal{L}$ -structures is denoted by  $\mathbf{N}$ .

In [3] it was introduced generalizations of the Noetherian property. An  $\mathcal{L}$ -structure  $\mathcal{A}$  is *weakly  $\mathcal{L}$ -equationally Noetherian* if any infinite  $\mathcal{L}$ -system  $\mathbf{S}$  is equivalent over  $\mathcal{A}$  to a finite system  $\mathbf{S}'$  (here we do not claim  $\mathbf{S}' \subseteq \mathbf{S}$ ). The class of weakly equationally Noetherian  $\mathcal{L}$ -structures is denoted by  $\mathbf{N}'$ . Obviously,  $\mathbf{N} \subseteq \mathbf{N}'$ .

Let  $\mathcal{A}$  be an  $\mathcal{L}$ -structure. By  $\mathcal{L}(\mathcal{A})$  we denote the language  $\mathcal{L} \cup \{a \mid a \in \mathcal{A}\}$  extended by new constant symbols which correspond to elements of  $\mathcal{A}$ . The language extension allows us to use constants in equations. The examples of equations in extended languages are the following (below  $\mathcal{G}, \mathcal{M}$  are graph and matroid respectively):  $E(x, a)$  (language  $\mathcal{L}_g(\mathcal{G})$  and  $a \in \mathcal{G}$ );  $P_2(a, x)$ ,  $P_3(x, b, c)$ ,  $P_4(a, x, y, b)$

(language  $\mathcal{L}_m(\mathcal{M})$  and  $a, b, c \in \mathcal{M}$ ). Obviously, the class of  $\mathcal{L}(\mathcal{A})$ -equations is wider than the class of  $\mathcal{L}$ -equations, so an  $\mathcal{L}$ -equationally Noetherian  $\mathcal{L}$ -algebra may lose this property in the language  $\mathcal{L}(\mathcal{A})$ .

Let  $\mathcal{A}$  be an  $\mathcal{L}$ -structure. An element of a direct power  $\Pi\mathcal{A} = \prod_{i \in I} \mathcal{A}$  is denoted by a sequence in square brackets  $[a_i \mid i \in I]$ . Functions and relations over  $\Pi\mathcal{A}$  have the coordinate-wise definition. For example, any relation  $R^m \in \mathcal{L}$  is defined on  $\Pi\mathcal{A}$  as follows:

$$R([a_i^{(1)} \mid i \in I], [a_i^{(2)} \mid i \in I], \dots, [a_i^{(m)} \mid i \in I]) \Leftrightarrow R(a_i^{(1)}, a_i^{(2)}, \dots, a_i^{(m)}) \text{ for each } i \in I.$$

The map  $\pi_k: \Pi\mathcal{A} \rightarrow \mathcal{A}$  is called a *projection onto the  $i$ -th coordinate* if  $\pi_k([a_i \mid i \in I]) = a_k$ .

Let  $E(X)$  be an  $\mathcal{L}(\Pi\mathcal{A})$ -equation over the direct power  $\Pi\mathcal{A}$ . We may rewrite  $E(X)$  in the form  $E(X, \vec{\mathbf{C}})$ , where  $\vec{\mathbf{C}}$  is an array of constants occurring in the equation  $E(X)$ . One can introduce the *projection of an equation* onto the  $i$ -th coordinate as follows:

$$\pi_i(E(X)) = \pi_i(E(X, \vec{\mathbf{C}})) = E(X, \pi_i(\vec{\mathbf{C}})),$$

where  $\pi_i(\vec{\mathbf{C}})$  is an array of the  $i$ -th coordinates of the elements from  $\vec{\mathbf{C}}$ . For example, the  $\mathcal{L}_g(\Pi\mathcal{G})$ -equation  $E(x, [a_1, a_2, a_3, \dots])$  has the following projections

$$\begin{aligned} E(x, a_1), \\ E(x, a_2), \\ E(x, a_3), \\ \dots \end{aligned}$$

Similarly, the matroid equation  $P_4(x, [a_1, a_2, a_3, \dots], y, [b_1, b_2, b_3, \dots])$  has the projections

$$\begin{aligned} P_4(x, a_1, y, b_1), \\ P_4(x, a_2, y, b_2), \\ P_4(x, a_3, y, b_3), \\ \dots \end{aligned}$$

Let us take an  $\mathcal{L}(\Pi\mathcal{A})$ -system  $\mathbf{S} = \{E_j(X) \mid j \in J\}$ . The  $i$ -th projection of  $\mathbf{S}$  is the  $\mathcal{L}(\mathcal{A})$ -system  $\pi_i(\mathbf{S}) = \{\pi_i(E_j(X)) \mid j \in J\}$ . The projections of an  $\mathcal{L}(\Pi\mathcal{A})$ -system  $\mathbf{S}$  allow to describe the solution set of  $\mathbf{S}$  by

$$V_{\Pi\mathcal{A}}(\mathbf{S}) = \{[P_i \mid i \in I] \mid P_i \in V_{\mathcal{A}}(\pi_i(\mathbf{S}))\}. \quad (2)$$

In particular, if one of the projections  $\pi_i(\mathbf{S})$  is inconsistent, so is  $\mathbf{S}$ .

The following statement immediately follows from (2).

**Lemma 2.1.** *Let  $\mathbf{S} = \{E_j(X) \mid j \in J\}$  be an  $\mathcal{L}(\Pi\mathcal{A})$ -system over  $\Pi\mathcal{A}$ . The system  $\mathbf{S}$  is consistent iff so are all projections  $\pi_i(\mathbf{S})$ . Moreover, if  $\mathcal{A}$  is  $\mathcal{L}$ -equationally Noetherian, then an inconsistent  $\mathcal{L}(\Pi\mathcal{A})$ -system  $\mathbf{S}$  is equivalent to a finite subsystem.*

*Proof.* The first assertion directly follows from (2). Suppose  $\mathcal{A}$  is  $\mathcal{L}$ -equationally Noetherian, and  $\pi_i(\mathbf{S})$  is inconsistent. Hence,  $\pi_i(\mathbf{S})$  is equivalent to its finite inconsistent subsystem  $\{\pi_i(E_j(X)) \mid j \in J'\}$ ,  $|J'| < \infty$ , and the subsystem  $\mathbf{S}' = \{E_j(X) \mid j \in J'\}$  of  $\mathbf{S}$  is also inconsistent.  $\square$

### 3 Graphs

Recall that a *graph* is an algebraic structure of the language  $\mathcal{L}_g = \{E^{(2)}\}$  satisfying the following axioms:

$$\begin{aligned} \forall x \neg E(x, x) & \text{ (no loops),} \\ \forall x \forall y E(x, y) \rightarrow E(y, x) & \text{ (symmetry).} \end{aligned}$$

**Theorem 3.1.** *A graph  $\Pi\mathcal{G} = \prod_{i \in I} \mathcal{G}$  is  $\mathcal{L}_g(\Pi\mathcal{G})$ -equationally Noetherian iff  $\mathcal{G}$  satisfies the quasi-identity*

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 (E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4) \rightarrow E(x_4, x_1)). \quad (3)$$

*Proof.* Let us prove the “if” part of the statement.

Let  $\mathbf{S}$  be an  $\mathcal{L}_g(\Pi\mathcal{G})$ -system over  $\Pi\mathcal{G}$  in variables  $X = \{x_1, \dots, x_n\}$ . One can rewrite  $\mathbf{S}$  as a finite union of systems

$$\mathbf{S} = \bigcup_{j=1}^n \mathbf{S}_j \cup \mathbf{S}_0,$$

where  $\mathbf{S}_j = \{E(x_j, \mathbf{c}_k) \mid k \in K_j\}$  and  $\mathbf{S}_0$  is the system of equations of the following types:  $E(x_i, x_j)$ ,  $x_i = x_j$ ,  $x_i = \mathbf{c}_j$ . Obviously, the system  $\mathbf{S}_0$  is equivalent to a finite subsystem. Hence, it is sufficient to prove that each system  $\mathbf{S}_j$  in one variable  $x_j$  is equivalent to its finite subsystem.

Let us write the coordinate-wise versions of the system  $\mathbf{S}_j$ :

$$\pi_i(\mathbf{S}_j) = \{E(x_j, \pi_i(\mathbf{c}_k)) \mid k \in K_j\}, \quad i \in I,$$

where  $\pi_i(\mathbf{c}_k)$  is the  $i$ -th coordinate of an element  $\mathbf{c}_k$ .

If for each  $i$  the equations  $\{E(x_j, \pi_i(\mathbf{c}_k)) \mid k \in K_j\}$  have the same solution sets, then  $\mathbf{S}_j$  is equivalent to a single equation  $E(x_j, \mathbf{c}_k) \in \mathbf{S}_j$  for arbitrary  $k \in K_j$ . Otherwise, there exists an index  $i$  such that

$$Y_1 = V_{\mathcal{G}}(E(x_j, \pi_i(\mathbf{c}_{k_1}))) \neq V_{\mathcal{G}}(E(x_j, \pi_i(\mathbf{c}_{k_2}))) = Y_2 \quad (4)$$

for some  $k_1, k_2 \in K_j$ .

If  $Y_1 \cap Y_2 = \emptyset$ , then  $\mathbf{S}_j$  is inconsistent and it is obviously equivalent to the subsystem  $\{E(x_j, \mathbf{c}_{k_1}), E(x_j, \mathbf{c}_{k_2})\}$ . Thus, we may assume  $Y_1 \not\subseteq Y_2$  and one can take elements  $b_1, b_2 \in \mathcal{G}$  such that  $b_1 \in Y_1 \setminus Y_2$ ,  $b_2 \in Y_1 \cap Y_2$ , i.e.  $E(b_1, \pi_i(\mathbf{c}_{k_1}))$ ,  $E(b_2, \pi_i(\mathbf{c}_{k_1}))$  and  $E(b_2, \pi_i(\mathbf{c}_{k_2}))$ .

Since the quasi-identity (3) is true in  $\Pi\mathcal{G}$ , we have  $E(b_1, \pi_i(\mathbf{c}_{k_2}))$  that contradicts the choice of the element  $b_1$ .

Let us prove the “only if” part of the statement. Assume the quasi-identity (3) does not hold in  $\mathcal{G}$ , i.e. there exists elements  $a_1, a_2, a_3, a_4$  with  $E(a_1, a_2)$ ,  $E(a_2, a_3)$ ,  $E(a_3, a_4)$ ,  $\neg E(a_4, a_1)$ . Consider the  $\mathcal{L}_g(\Pi\mathcal{G})$ -system  $\mathbf{S}$  of the following equations:

$$\begin{aligned} E(x, [a_2, a_2, a_2 \dots]), \\ E(x, [a_4, a_2, a_2 \dots]), \\ E(x, [a_4, a_4, a_2 \dots]), \\ \dots \end{aligned}$$

Let  $\mathbf{S}_n$  be the subsystem of  $\mathbf{S}$  formed by the first  $n$  equations of  $\mathbf{S}$ .

The point  $\mathbf{a} = \underbrace{[a_3, a_3, \dots, a_3, a_1, a_1, \dots]}_{n-1 \text{ times}}$  satisfies  $\mathbf{S}_n$  but  $\mathbf{a}$  does not satisfy the  $(n+1)$ -th equation of  $\mathbf{S}$ . Thus,  $\mathbf{S}_n$  is not equivalent to  $\mathbf{S}$  for any  $n$ , and  $\Pi\mathcal{G}$  is not  $\mathcal{L}_g(\Pi\mathcal{G})$ -equationally Noetherian.  $\square$

**Corollary 3.2.** *If a graph  $\mathcal{G}$  contains a triangle (i.e. there exist vertices  $x_1, x_2, x_3 \in \mathcal{G}$  with  $E(x_1, x_2)$ ,  $E(x_2, x_3)$ ,  $E(x_3, x_1)$ ) then  $\Pi\mathcal{G}$  is not  $\mathcal{L}_g(\Pi\mathcal{G})$ -equationally Noetherian.*

*Proof.* Obviously, the condition of Theorem 3.1 fails for such graphs, since there are not loops in  $\mathcal{G}$ .  $\square$

Let  $\mathbf{K} = \{\mathcal{G} \mid \Pi\mathcal{G} \in \mathbf{N}\}$  be the set of all graphs with equationally Noetherian direct powers. Theorem 3.1 gives that the class  $\mathbf{K}$  is axiomatizable. The class  $\mathbf{K}$  may be also described by forbidden graphs and distance functions.

**Corollary 3.3.** *A graph  $\Pi\mathcal{G}$  is  $\mathcal{L}_g(\Pi\mathcal{G})$ -equationally Noetherian iff  $\mathcal{G}$  is triangular-free and the distance between any pair of vertices  $x, y$  is either  $\infty$  (if  $x, y$  belong to different connected components) or less than 4.*

*Proof.* First, we prove the “only if” part of the statement. By Corollary 3.2  $\mathcal{G}$  is triangular-free. Let us take two vertices  $x, y$  with the distance  $4 \leq d(x, y) = d < \infty$  and the shortest path  $x = x_1, x_2, \dots, x_d = y$  between  $x$  and  $y$ . However, the quasi-identity (3) provides that  $E(x_1, x_4)$ , and the minimal path between  $x, y$  has the length less than  $d$ , a contradiction.

Let us prove the “if” part of the statement and take arbitrary  $x_1, x_2, x_3, x_4 \in \mathcal{G}$  such that  $E(x_1, x_2)$ ,  $E(x_2, x_3)$ ,  $E(x_3, x_4)$ . Since the distance between vertices of the same connected component is less or equal than 3, then there exists an edge between the vertices  $x_i$ . If there exists one of the edges  $E(x_1, x_3)$ ,  $E(x_2, x_4)$  then  $\mathcal{G}$  contains a triangle. Thus,  $\mathcal{G}$  has the edge  $E(x_1, x_4)$  and the quasi-identity (3) holds in  $\mathcal{G}$ .  $\square$

Let us give the explicit examples of graphs  $\mathcal{G} \in \mathbf{K}$ .

One can directly prove that the disjoint union  $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$  has an equationally Noetherian direct power  $\Pi\mathcal{G}$  if both graphs satisfy the quasi-identity (3). Thus, there arises a question: is there a *connected* graph  $\mathcal{G}$  with  $n$  vertices such that any direct power  $\Pi\mathcal{G}$  is  $\mathcal{L}_g(\Pi\mathcal{G})$ -equationally Noetherian?

The answer is positive. Let us define the following graph  $\mathcal{G}$  with the vertex set  $\{x_0, x_1, \dots, x_n, x_{n+1}\}$  and edges  $\{E(x_0, x_i), E(x_i, x_{n+1}) \mid 1 \leq i \leq n\}$ . The direct check gives that  $\mathcal{G}$  satisfies (3), contains  $n+1$  vertices and  $\mathcal{G}$  is connected.

## 4 Partial orders

A *partial order*  $\mathcal{P}$  is an algebraic structure of the language  $\mathcal{L}_p = \{\leq^{(2)}\}$  such that  $\mathcal{P}$  satisfies the following axioms

$$\begin{aligned} \forall x (x \leq x), \\ \forall x \forall y (x \leq y) \wedge (y \leq x) \rightarrow (x = y), \\ \forall x \forall y (x \leq y) \wedge (y \leq z) \rightarrow (x \leq z). \end{aligned}$$

A partial order  $\mathcal{P}$  is said to be *non-trivial* if there exists a pair  $a, b \in \mathcal{P}$  such that  $a < b$  (i.e.  $a \leq b$  and  $a \neq b$ ).

**Theorem 4.1.** *Let  $\mathcal{P}$  be a non-trivial partial order, and  $\Pi\mathcal{P}$  be an infinite direct power of  $\mathcal{P}$ . Then  $\Pi\mathcal{P}$  is not  $\mathcal{L}_p(\Pi\mathcal{P})$ -equationally Noetherian.*

*Proof.* Since  $\mathcal{P}$  is non-trivial, there exists  $a, b \in \mathcal{P}$  with  $a < b$ . It is sufficient to show that an infinite direct power  $\Pi\mathcal{E} \subseteq \Pi\mathcal{P}$  of the partial order  $\mathcal{E} = \{a, b\}$  is not  $\mathcal{L}_p(\Pi\mathcal{E})$ -equationally Noetherian.

Indeed, one should consider the following infinite  $\mathcal{L}_p(\Pi\mathcal{E})$ -system  $\mathbf{S}$ :

$$\begin{aligned} x &\leq [b, b, b, \dots,] \\ x &\leq [a, b, b, \dots,] \\ x &\leq [a, a, b, \dots,] \\ &\dots \end{aligned}$$

Obviously, the unique solution of  $\mathbf{S}$  is  $[a, a, a, \dots,]$ . However the solution set of any finite subsystem of  $\mathbf{S}$  contain a point  $\underbrace{[a, a, a, \dots, a, b, b, b, \dots,]}_{n \text{ times}}$  for sufficiently large  $n$ . Thus,  $\mathbf{S}$  is not equivalent to any finite subsystem.  $\square$

## 5 Matroids

One can consider a *matroid*  $\mathcal{M}$  as an algebraic structure of an infinite language  $\mathcal{L}_m = \{P_1^{(1)}, P_2^{(2)}, P_3^{(3)}, \dots\}$ , where each predicate symbol  $P_n$  have the following interpretation:

$$P_n(x_1, \dots, x_n) \Leftrightarrow \text{the set } \{x_i\} \text{ is independent in } \mathcal{M}.$$

Moreover, any matroid satisfies the following axioms:

$$\begin{aligned} &\forall x_1 \dots \forall x_n \left( \bigvee_{i \neq j} (x_i = x_j) \rightarrow \neg P_n(x_1, \dots, x_n) \right) \\ &\forall x_1 \dots \forall x_n \left( P_n(x_1, \dots, x_n) \rightarrow \bigwedge_{i=1}^n P_{n-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right) \quad (n > 1), \\ &\forall x_1 \dots \forall x_n \left( P_n(x_1, \dots, x_n) \wedge P_{n+1}(y_1, \dots, y_{n+1}) \rightarrow \bigvee_{i=1}^{n+1} P_{n+1}(x_1, \dots, x_n, y_i) \right). \end{aligned}$$

*Notice that a direct power  $\Pi\mathcal{M}$  of a matroid  $\mathcal{M}$  is not necessarily a monoid itself. However, here we study direct powers of matroids, since the algebraic geometry over  $\Pi\mathcal{M}$  may clarify algebraic and geometric properties of the original matroid  $\mathcal{M}$ .*

**Lemma 5.1.** *Let  $\mathcal{M}$  be a matroid with  $P_3(a, b, c)$  for some  $a, b, c \in \mathcal{M}$ . Then any infinite direct power  $\Pi\mathcal{M}$  is not  $\mathcal{L}_m(\Pi\mathcal{M})$ -equationally Noetherian.*

*Proof.* Let us consider a system  $\mathbf{S}$  of  $\mathcal{L}_m(\Pi\mathcal{M})$ -equations

$$\begin{aligned} &P_2(x, [a, a, a, \dots]), \\ &P_2(x, [b, a, a, \dots]), \\ &P_2(x, [b, b, a, \dots]), \\ &\dots \end{aligned}$$

Denote by  $\mathbf{S}_n$  the first  $n$  equations of  $\mathbf{S}$ . Clearly,  $\mathbf{S}_n$  is satisfied by the point

$$\underbrace{[c, c, \dots, c, b, b, \dots, ]}_{n \text{ times}}.$$

However this point does not belong to the solution set of  $\mathbf{S}$ , since the predicate

$$P_2(\underbrace{[c, c, \dots, c, b, b, \dots, ]}_{n \text{ times}}, \underbrace{[b, b, \dots, b, a, a, \dots, ]}_{n+1 \text{ times}})$$

is not true for the  $(n+1)$ -th coordinate.  $\square$

According to Lemma 5.1, any matroid  $\mathcal{M}$  with  $\Pi\mathcal{M} \in \mathbf{N}$  may be represented by a graph  $\mathcal{G}(\mathcal{M})$  such that

1. the vertex set of  $\mathcal{G}$  coincides with the set  $\mathcal{M}$ ;
2.  $P_2(a, b) \Leftrightarrow E(a, b)$ .

Hence, such matroids may be classified by the analogue of Theorem 3.1.

**Theorem 5.2.** *A direct power  $\Pi\mathcal{M}$  of a matroid  $\mathcal{M}$  is  $\mathcal{L}_m(\mathcal{M})$ -equationally Noetherian iff  $\mathcal{M}$  satisfies the following axioms*

$$\begin{aligned} & \forall x \forall y \forall z \neg P_3(x, y, z), \\ & \forall x_1 \forall x_2 \forall x_3 \forall x_4 (P_2(x_1, x_2) \wedge P_2(x_2, x_3) \wedge P_2(x_3, x_4) \rightarrow P_2(x_4, x_1)). \end{aligned}$$

*Proof.* The proof immediately follows from Lemma 5.1, Theorem 3.1 and the correspondence  $\mathcal{M} \leftrightarrow \mathcal{G}(\mathcal{M})$ .  $\square$

## 6 Direct powers of finite structures

Let us prove a general fact about direct powers of arbitrary finite algebraic structures. The proof of the following theorem is complicated enough, so its main steps are explained in Example 6.2.

**Theorem 6.1.** *Let  $\mathcal{A}$  be a finite  $\mathcal{L}$ -structure. Then any direct power  $\Pi\mathcal{A} = \Pi_{i \in I} \mathcal{A}$  is weakly  $\mathcal{L}(\Pi\mathcal{A})$ -equationally Noetherian.*

*Proof.* Let  $\mathbf{S} = \{E_j(X, \vec{\mathbf{C}}_j) \mid j \in J\}$  be an infinite  $\mathcal{L}(\Pi\mathcal{A})$ -system over  $\Pi\mathcal{A}$ , and  $\pi_i(\mathbf{S}) = \{E_j(X, \pi_i(\vec{\mathbf{C}}_j)) \mid j \in J\}$  ( $i \in I$ ) be the projections of  $\mathbf{S}$  onto all coordinates of  $\Pi\mathcal{A}$ . Notice that any system  $\pi_i(\mathbf{S})$  is a system of  $\mathcal{L}(\mathcal{A})$ -equations over  $\mathcal{A}$ .

Since  $\mathcal{A}$  is finite, then there exists a finite number of equations  $M = \{E_j(X, \pi_i(\vec{\mathbf{C}}_j)) \mid (i, j) \in K\}$  ( $|K| < \infty$ ) such that any  $E_j(X, \pi_i(\vec{\mathbf{C}}_j)) \in \bigcup_{i \in I} \pi_i(\mathbf{S})$  is equivalent over  $\mathcal{A}$  to an appropriate equation from  $M$ . Hence, each  $\pi_i(\mathbf{S})$  is equivalent to a subsystem  $\mathbf{S}'_i \subseteq M$  over  $\mathcal{A}$ . The idea of the further proof is the following: we try to wrap all systems  $\mathbf{S}'_i$  into a finite number of equations  $\mathbf{S}'$  over  $\Pi\mathcal{A}$ .

Let us define an  $\mathcal{L}(\Pi\mathcal{A})$ -system  $\mathbf{S}'$  by the following procedure.

**Step 0.** Put

$$\mathbf{S}_0 = \bigcup_{(i,j) \in K} E_j(X, \vec{\mathbf{C}}_j) \subseteq \mathbf{S}$$

( $|\mathbf{S}_0| = |K|$ ) and  $\mathbf{S}' := \mathbf{S}_0$ . The main property of  $\mathbf{S}_0$  is the following: each equation from  $M$  occurs in some projection of equations from  $\mathbf{S}_0$ . Let us arbitrarily enumerate equations in the set  $M$ , i.e. each equations from  $M$  has the number  $s \in [1, |K|]$ .

**Step  $s$**  ( $1 \leq s \leq |K|$ ). Let us take the  $s$ -th equation  $E_j(X, \pi_i(\vec{\mathbf{C}}_j))$  from  $M$  and define the following sets of indexes  $I_0 = \{l \in I \mid E_j(X, \pi_l(\vec{\mathbf{C}}_j)) \in \mathbf{S}'_l\}$ ,  $I_1 = I \setminus I_0$ . In other words,  $I_0$  is the set of all indexes  $l$  such that the given equation from  $M$  occurs in the system  $\mathbf{S}'_l$ . Define a set  $M_s = \{D_l(X) \mid l \in I\}$  of  $\mathcal{L}(\mathcal{A})$ -equations as follows:

$$D_l(X) = \begin{cases} E_j(X, \pi_l(\vec{\mathbf{C}}_j)) & \text{if } l \in I_0, \\ E_j(X, \pi_l(\vec{\mathbf{C}}_j)) & \text{if } l \in I_1 \end{cases}$$

The sense of the set  $M_s$  is the following. If the system  $\mathbf{S}'_l$  contains  $E_j(X, \pi_l(\vec{\mathbf{C}}_j)) \in M$  we take this equation as the  $k$ -th projection in  $M_s$ . Otherwise, the  $l$ -th projection in  $M_s$  is taken from the equation  $E_j(X, \vec{\mathbf{C}}_j) \in \mathbf{S}_0$ .

The  $\mathcal{L}(\mathcal{A})$ -equations  $M_s$  may be wrapped into the  $\mathcal{L}(\Pi\mathcal{A})$ -equation  $D_s(X, \vec{\mathbf{D}}_s)$ , where

$$\pi_l(\vec{\mathbf{D}}_s) = \begin{cases} \pi_l(\vec{\mathbf{C}}_j) & \text{if } l \in I_0, \\ \pi_l(\vec{\mathbf{C}}_j) & \text{if } l \in I_1 \end{cases}$$

We put  $\mathbf{S}' := \mathbf{S}' \cup D_s(X, \vec{\mathbf{D}}_s)$  and go to the following step ( $s + 1$ ).

By the definition of the system  $\mathbf{S}'$ , the  $i$ -th projection  $\pi_i(\mathbf{S}')$  contain all equations from  $\mathbf{S}'_i \sim \pi_i(\mathbf{S})$ . Hence,  $\pi_i(\mathbf{S}') \sim \pi_i(\mathbf{S})$  over  $\mathcal{A}$ , and finally  $\mathbf{S}' \sim \mathbf{S}$  over  $\Pi\mathcal{A}$ .  $\square$

The following example explains the technique and denotations from Theorem 6.1.

**Example 6.2.** Let  $\mathcal{G}$  be the graph with vertices  $\{a, b, c\}$  and edges  $E(a, b)$ ,  $E(b, c)$ ,  $E(c, a)$  (i.e.  $\mathcal{G}$  is a complete graph). Let us consider an infinite  $\mathcal{L}(\Pi\mathcal{G})$ -system  $\mathbf{S}$  of equations:

$$\begin{aligned} &E(x, [a, a, a, a, a, a, \dots]), \\ &E(x, [b, a, a, a, a, a, \dots]), \\ &E(x, [b, c, a, a, a, a, \dots]), \\ &E(x, [b, c, b, a, a, a, \dots]), \\ &E(x, [b, c, b, c, a, a, \dots]), \\ &\dots \end{aligned}$$

The projections  $\pi_i(\mathbf{S})$  are the following (we omit in the projections equations which occur earlier):

$$\begin{aligned} \pi_1(\mathbf{S}) &= \{E(x, a), E(x, b)\}, \\ \pi_2(\mathbf{S}) &= \{E(x, a), E(x, c)\}, \\ \pi_3(\mathbf{S}) &= \{E(x, a), E(x, b)\}, \\ \pi_4(\mathbf{S}) &= \{E(x, a), E(x, c)\}, \\ &\dots \end{aligned}$$

The set  $M$  consists of the equations  $E(x, a)$ ,  $E(x, b)$ ,  $E(x, c)$  (any equation from  $\bigcup_{i=1}^n \pi_i(\mathbf{S})$  is equivalent to one of the given equations). Since the third equation of  $\mathbf{S}$  contain all equations from  $M$  as projections, we may put  $\mathbf{S}_0 = \{E(x, [b, c, a, a, a, a, \dots])\}$  (the set  $K$  here is  $\{(1, 3), (2, 3), (3, 3)\}$ ). For the projections  $\pi_i(\mathbf{S})$  we have

$$\begin{aligned} \pi_{2k+1}(\mathbf{S}) &\sim \{E(x, a), E(x, b)\} = \mathbf{S}'_{2k+1}, \\ \pi_{2k}(\mathbf{S}) &\sim \{E(x, a), E(x, c)\} = \mathbf{S}'_{2k}. \end{aligned}$$

Now we construct the final system  $\mathbf{S}'$  with  $|\mathbf{S}_0| + |M| = 4$  equations. First, we put  $\mathbf{S}' = \mathbf{S}_0$  and make the following three steps.



1. We take  $E(x, a) \in M$ . Since this equation occurs in any system  $\mathbf{S}'_i$  ( $I_0 = \mathbb{N}$ ,  $I_1 = \emptyset$ ), we add to  $\mathbf{S}'$  the equation  $E(x, [a, a, a, a, a, \dots])$ .
2. Take  $E(x, b) \in M$ . Since  $E(x, b)$  occurs in the systems  $\mathbf{S}_i$  with odd  $i$  ( $I_0 = \{1, 3, \dots\}$ ,  $I_1 = \{2, 4, \dots\}$ ), we should add to  $\mathbf{S}'$  an equation of the form  $E(x, [b, *, b, *, b, *, \dots])$ . The elements for even positions are taken from the equation from  $\mathbf{S}_0$ , and we obtain the equation  $E(x, [b, c, b, a, b, a, \dots])$ . The last equation is added to  $\mathbf{S}'$ .
3. For the equation  $E(x, c) \in M$  we make dual operations. Since  $E(x, c)$  occurs in the systems  $\mathbf{S}_i$  with even  $i$  ( $I_0 = \{2, 4, \dots\}$ ,  $I_1 = \{1, 3, \dots\}$ ) then we should add to  $\mathbf{S}'$  an equation of the form  $E(x, [* , c, *, c, *, c, \dots])$ . The elements for odd positions are taken from the equation from  $\mathbf{S}_0$ , and we obtain the equation  $E(x, [b, c, a, c, a, c, \dots])$ . Also we add the last equation to  $\mathbf{S}'$ .

Thus, the final system  $\mathbf{S}'$  consists of the following equations

$$\begin{aligned}
&E(x, [b, c, a, a, a, a, \dots]), \\
&E(x, [a, a, a, a, a, a, \dots]), \\
&E(x, [b, c, b, a, b, a, \dots]), \\
&E(x, [b, c, a, c, a, c, \dots]).
\end{aligned}$$

It is easy to see that all projections  $\pi_i(\mathbf{S}')$  are equivalent over  $\mathcal{G}$  to the systems  $\mathbf{S}'_i$ . Thus,  $\mathbf{S}'$  is equivalent to  $\mathbf{S}$ .

The ideas of Theorem 6.1 allow us to estimate uniformly the minimal number of equations in the finite system  $\mathbf{S}'$ .

**Corollary 6.3.** *Let  $\mathbf{S}$  be a system of  $\mathcal{L}(\Pi\mathcal{A})$ -equations in  $n$  variables over the direct power  $\Pi\mathcal{A}$  of a finite  $\mathcal{L}$ -structure  $\mathcal{A}$ ,  $|\mathcal{A}| = k$ . Then  $\mathbf{S}$  is equivalent to a system  $\mathbf{S}'$  with at most  $2^{k^n+1}$  equations.*

*Proof.* Since we deal with equations in  $n$  variables, all algebraic sets over  $\mathcal{A}$  are the subsets of the affine space  $\mathcal{A}^n$ ,  $|\mathcal{A}^n| = k^n$ . Hence, there exists at most  $2^{k^n}$  different algebraic sets over  $\mathcal{A}$ . Since the set  $M$  in Theorem 6.1 consists of pairwise non-equivalent equations, we have  $|M| \leq 2^{k^n}$ .

The final system  $\mathbf{S}'$  consists of at most  $|M| + |M| = 2|M|$  equations ( $|\mathbf{S}_0| = |M|$ , and  $|M|$  iterations of the procedure add to  $\mathbf{S}'$  exactly  $|M|$  equations). Thus, we obtain  $|\mathbf{S}'| \leq 2 \cdot 2^{k^n} = 2^{k^n+1}$ .  $\square$

## References

- [1] E. Daniyarova, A. Myasnikov, V. Remeslennikov, Unification theorems in algebraic geometry. Algebra and Discrete Mathematics, 1 (2008), 80–112.
- [2] E. Daniyarova, A. Myasnikov, V. Remeslennikov, Algebraic geometry over algebraic structures, II: Foundations. J. Math. Sci., 185:3 (2012), 389–416.
- [3] E. Daniyarova, A. Myasnikov, V. Remeslennikov, Algebraic geometry over algebraic structures, III: Equationally noetherian property and compactness. South. Asian Bull. Math., 35:1 (2011), 35–68.
- [4] A. Shevlyakov, M. Shahryary, Direct products, varieties, and compactness conditions, Groups Complexity Cryptology, 9:2 (2017), 159–166.

The information of the author:  
Artem N. Shevlyakov  
Sobolev Institute of Mathematics  
644099 Russia, Omsk, Pevtsova st. 13  
  
Omsk State Technical University  
pr. Mira, 11, 644050  
Phone: +7-3812-23-25-51.  
e-mail: [a\\_shevl@mail.ru](mailto:a_shevl@mail.ru)