ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY KOMATU INTEGRAL OPERATOR

B. Venkateswarlu[1], P. Thirupathi Reddy[2], R. Madhuri Shilpa[3] and B. Madhavi[4]

[1,3] Department of Mathematics, GSS, GITAM University,
Doddaballapur- 561 203, Bengaluru Rural, Karnataka, India.
[1] *bvlmaths@gmail.com*   [3] *madhuri.4073@gmail.com*

[2] Department of Mathematics, Kakatiya University,
Warangal- 506 009, Telangana, India.
reddypt2@gmail.com

[4] Department of Mathematics, Pingle Govt. Degree College for Women
Waddepally, Hanamkonda, Telangana, India.
shivsairaj1@gmail.com

Abstract: The main purpose of this paper, is to introduce a new subclass of analytic functions involving Komatu integral operator and obtained coefficient inequalities, distortion theorem, convex linear combination and partial sums for this class .

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1   Introduction

Let A denote the class of analytic functions f defined on the unit disk $E = \{z : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$, such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Denoted by S, the subclass of A consisting of functions that are univalent in E. Then a function $f(z)$ belonging to A is said to be satrlike of order ϑ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \vartheta, \quad z \in E, \tag{1.2}$$

for some $\vartheta(0 \le \vartheta < 1)$. We denote by $S^*(\vartheta)$ the subclass of A consisting of function are starlike of order ϑ in E. Also, a function $f(z)$ belonging to A is said to be convex of order ϑ if it satsfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \vartheta, \quad z \in E, \tag{1.3}$$

for some $\vartheta(0 \le \vartheta < 1)$. We denote by $K(\vartheta)$ the subclass of A consisting of function are convex of order ϑ in E. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in E respectively.

Also, Denote by T the subclass of A consisting of functions of the the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \ge 0, \ (z \in E) \tag{1.4}$$

and let $T^*(\vartheta) = T \cap S^*(\vartheta), C(\vartheta) = T \cap K(\vartheta)$. The classes $T^*(\vartheta)$ and $C(\vartheta)$ posses some interesting properties and have been extensively studied by Silverman [14] and others.

Recently, Komatu [5] introduced a certain integral operator $\mathscr{I}_c^{\delta}$ denoted by

$$\mathscr{I}_c^{\delta} f(z) = \frac{c^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-2} \left(log\frac{1}{t}\right)^{\delta-1} f(zt)dt, \tag{1.5}$$

where $c > 0, \delta \ge 0$ and $z \in E$.

Thus, $f \in A$ is of the form (1.1), then it is easily seen from (1.5) that (see [5])

$$\mathscr{I}_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} \psi_k(c,\delta) a_k z^k, \tag{1.6}$$

where $\psi_k(c,\delta) = \left(\frac{c}{c+k-1}\right)^{\delta}$.

From (1.5), it is clear that

$$z\left(\mathscr{I}_c^{\delta} f(z)\right)' = (c+1)\mathscr{I}_c^{\delta-1} f(z) - c\mathscr{I}_c^{\delta} f(z)$$

and

$$z^2 \left(\mathscr{I}_c^{\delta} f(z)\right)'' = (c+1)^2 \mathscr{I}_c^{\delta-2} f(z) - (2c+1)(c+1)\mathscr{I}_c^{\delta-1} f(z) + c(c+1)\mathscr{I}_c^{\delta} f(z).$$

We note that

(i). for $c = 1$ and $\delta = \gamma(\gamma$ is an integer), the multiplier transformation $\mathscr{I}_1^{\gamma} f(z) = I^{\gamma} f(z)$ was studied by Flett [2]

(ii). for $c = 1$ and $\delta = -\gamma(\gamma \in N)$, the differential operator $\mathscr{I}_1^{-\gamma} f(z) = D^{\gamma} f(z)$ was studied by Salagean [12]

(iii). for $c = 2$ and $\delta = \gamma(\gamma$ is an integer), the operator $\mathscr{I}_2^\delta f(z) = \mathscr{I}^\gamma f(z)$ was studied by Uralegaddi and Somanatha [16]

(iv). for $c = 2$, the multiplier transformation $\mathscr{I}_2^\delta f(z) = I^\delta f(z)$ was studied by Jung et al. [4].

Motivated by the works of see ([1, 6, 7, 9, 10]), we introduce a new subclass $\phi_c^\delta(\lambda, \vartheta)$ of A involving Komatu integral operator [5] as below:

Definition 1.1. For $0 \leq \lambda < 1, 0 \leq \vartheta < 1, c > 0, \delta > 0$, we say $f(z) \in A$ is in the class $\phi_c^\delta(\lambda, \vartheta)$ if it satisfies the condition

$$\mathfrak{Re}\left(\frac{z\left(\mathscr{I}_c^\delta f(z)\right)' + \lambda z^2 \left(\mathscr{I}_c^\delta f(z)\right)''}{\mathscr{I}_c^\delta f(z)}\right) > \vartheta, \ (z \in U). \tag{1.7}$$

Also we denote by $T\phi_c^\delta(\lambda, \vartheta) = \phi_c^\delta(\lambda, \vartheta) \cap T$.

2   Coefficient Inequalities

In this section we obtain a sufficient condition for a function f given by (1.1) to be in $\phi_c^\delta(\lambda, \vartheta)$.

Theorem 2.1. *A function $f \in A$ belongs to the class $\phi_c^\delta(\lambda, \vartheta)$ if*

$$\sum_{k=2}^{\infty}[k + \lambda k(k-1) - \vartheta]\psi_k(c, \delta)|a_k| \leq 1 - \vartheta. \tag{2.1}$$

Proof. Since $0 \leq \vartheta < 1$ and $\lambda \geq 0$, now if we put

$$\varrho(z) = \frac{z\left(\mathscr{I}_c^\delta f(z)\right)' + \lambda z^2 \left(\mathscr{I}_c^\delta f(z)\right)''}{\mathscr{I}_c^\delta f(z)}, \ (z \in U)$$

then it is sufficient to prove that $|\varrho(z) - 1| < 1 - \vartheta, (z \in U)$.
Indeed if $f(z) = z \ (z \in U)$, then we have $\varrho(z) = z(z \in U)$.
This implies that the desired in equality (2.1) holds.
If $f(z) \neq z \ (|z| = r < 1)$, then there exist a coefficient $\psi_k(c, \delta)a_k \neq 0$ for some $k \geq 2$. It follows that $\sum_{k=2}^{\infty} \psi_k(c, \delta)|a_k| > 0$. Further note that

$$\sum_{k=2}^{\infty}[k + \lambda k(k-1) - \vartheta]\psi_k(c, \delta)|a_k| > (1 - \vartheta)\sum_{k=2}^{\infty}\psi_k(c, \delta)|a_k|$$

$$\Rightarrow \quad \sum_{k=2}^{\infty}\psi_k(c, \delta)|a_k| < 1.$$

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By (2.1), we obtain

$$
|\varrho(z) - 1| = \left| \frac{\sum\limits_{k=2}^{\infty} [k + \lambda k(k-1) - 1]\psi_k(c,\delta)a_k z^{k-1}}{1 + \sum\limits_{k=2}^{\infty} \psi_k(c,\delta)a_k z^{k-1}} \right|
$$

$$
< \frac{\sum\limits_{k=2}^{\infty} [k + \lambda k(k-1) - 1]\psi_k(c,\delta)|a_k|}{1 - \sum\limits_{k=2}^{\infty} \psi_k(c,\delta)|a_k|}
$$

$$
\leq \frac{\sum\limits_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)|a_k| - (1-\vartheta)\psi_k(c,\delta)|a_k|}{1 - \sum\limits_{k=2}^{\infty} \psi_k(c,\delta)|a_k|}
$$

$$
\leq \frac{(1-\vartheta) - (1-\vartheta)\sum\limits_{k=2}^{\infty} \psi_k(c,\delta)|a_k|}{1 - \sum\limits_{k=2}^{\infty} \psi_k(c,\delta)|a_k|}
$$

$$
= 1 - \vartheta, \quad (z \in U).
$$

Hence we obtain

$$
\Re \left(\frac{z\left(\mathscr{I}_c^\delta f(z)\right)' + \lambda z^2 \left(\mathscr{I}_c^\delta f(z)\right)''}{\mathscr{I}_c^\delta f(z)} \right) = \Re(\varrho(z)) > 1 - (1-\vartheta) = \vartheta, (z \in U).
$$

Then $f \in \phi_c^\delta(\lambda, \vartheta)$. This completes the proof. $\qquad\square$

In the next theorem, we prove that the condition (2.1) is also necessary for a function $f \in T\phi_c^\delta(\lambda, \vartheta)$.

Theorem 2.2. *Let f be given by (1.4). Then the function $f \in T\phi_c^\delta(\lambda, \vartheta)$ if and only if*

$$
\sum_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)|a_k| \leq 1 - \vartheta. \tag{2.2}
$$

Proof. In view of Theorem 2.1, we need only to prove that $f \in T\phi_c^\delta(\lambda, \vartheta)$ satisfies the coefficient inequality (2.1). If $f \in T\phi_c^\delta(\lambda, \vartheta)$ then the function

$$
\varrho(z) = \frac{z\left(\mathscr{I}_c^\delta f(z)\right)' + \lambda z^2 \left(\mathscr{I}_c^\delta f(z)\right)''}{\mathscr{I}_c^\delta f(z)}, \quad (z \in U)
$$

satisfies $Re(\varrho(z)) > \vartheta (z \in U)$. This implies that

$$
\mathscr{I}_c^\delta f(z) = z - \sum_{k=2}^{\infty} \psi_k(c,\delta)|a_k|z^k \neq 0; (z \in U \setminus \{0\}).
$$

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Noting that $\frac{\mathscr{I}_c^\delta f(r)}{r}$ is the real continuous function in the open interval $(0,1)$ with $f(0) = 1$, we have

$$\frac{\mathscr{I}_c^\delta f(r)}{r} = 1 - \sum_{k=2}^{\infty} \psi_k(c,\delta)|a_k|r^{k-1} > 0, (0 < r < 1). \qquad (2.3)$$

Now $\vartheta < \varrho(r) = \dfrac{1 - \sum\limits_{k=2}^{\infty} [k + \lambda k(k-1)]\psi_k(c,\delta)|a_k|r^{k-1}}{1 - \sum\limits_{k=2}^{\infty} \psi_k(c,\delta)|a_k|r^{k-1}}$ and consequently by (2.3),

we get $\sum\limits_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)|a_k|r^{k-1} \leq 1 - \vartheta$.

Letting $r \to 1$, we get $\sum\limits_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)|a_k| \leq 1 - \vartheta$.

This proves the converse part. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 2.3. If a function f of the form (1.2) belongs to the class $T\phi_c^\delta(\lambda, \vartheta)$ then

$$|a_k| \leq \frac{1 - \vartheta}{[k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)}, k \geq 2.$$

The equality holds for the functions

$$f_k(z) = z - \frac{1 - \vartheta}{[k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)}z^n, (z \in U, k \geq 2). \qquad (2.4)$$

3 Distortion Theorem

In the section distortion bounds for functions belonging to the class $T\phi_c^\delta(\lambda, \vartheta)$.

Theorem 3.1. *Let f be in the class $T\phi_c^\delta(\lambda, \vartheta)$ and $|z| = r < 1$. Then*

$$r - \frac{1 - \vartheta}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)}r^2 \leq |f(z)| \leq r + \frac{1 - \vartheta}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)}r^2 \qquad (3.1)$$

and

$$1 - \frac{2(1 - \vartheta)}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)}r \leq |f'(z)| \leq 1 + \frac{2(1 - \vartheta)}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)}r. \qquad (3.2)$$

The result is sharp with the extremal function $f_2(z)$ is given by (2.4).

Proof. Since $f \in T\phi_c^\delta(\lambda, \vartheta)$, we apply Theorem 2.2 to obtain

$$[2\lambda - \vartheta + 2]\psi_k(c,\delta) \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta]\psi_k(c,\delta)|a_k|$$
$$\leq 1 - \vartheta.$$

Thus $|f(z)| \leq |z| + |z|^2 \sum\limits_{k=2}^{\infty} |a_k| \leq r + \frac{1 - \vartheta}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)} r^2.$

Also we have, $|f(z)| \leq |z| - |z|^2 \sum\limits_{k=2}^{\infty} |a_k| \leq r - \frac{1 - \vartheta}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)} r^2,$

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and (3.1) follows. In similar manner for f', the inequalities

$$|f'(z)| \leq 1 + \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \leq 1 + |z| \sum_{k=2}^{\infty} k|a_k|$$

and

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{2(1-\vartheta)}{[2\lambda - \vartheta + 2]\psi_k(c,\delta)}$$

are satisfied, which leads to (3.2). This completes the proof. $\qquad\square$

4   Radii of close-to-convexity and starlikeness

In this section the radii of close-to-convex and starlikeness of this class $T\phi_c^\delta(\lambda, \vartheta)$ will be obtained.

Theorem 4.1. *Let f be given by (1.4) is in $T\phi_c^\delta(\lambda, \vartheta)$. Then f is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < t_1$, where*

$$t_1 = \inf_{k \geq 2} \left[\frac{(1-\delta)[k + k\lambda(k-1) - \vartheta]\psi_k(c,\delta)}{k(1-\vartheta)} \right]^{\frac{1}{k-1}} \qquad k \geq 2. \qquad (4.1)$$

The result is sharp with the extremal function $f(z)$ is given by (2.4).

Proof. If $f \in T$ and f is close-to-convex of order δ then we get

$$|f'(z) - 1| \leq 1 - \delta. \qquad (4.2)$$

For the left and side of (4.2), we obtain

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} ka_k|z|^{k-1} < 1 - \delta$$

$$\Rightarrow \quad \sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k|z|^{k-1} \leq 1.$$

We know that $f(z) \in T\phi_c^\delta(\lambda, \vartheta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[k + k\lambda(k-1) - \vartheta]\psi_k(c,\delta)}{(1-\vartheta)} a_k \leq 1.$$

Thus (4.2) holds true if

$$\frac{k}{1-\delta}|z|^{k-1} \leq \frac{[k + k\lambda(k-1) - \vartheta]\psi_k(c,\delta)}{(1-\vartheta)}$$

or equivalently

$$|z| \leq \left[\frac{(1-\delta)[k + k\lambda(k-1) - \vartheta]\psi_k(c,\delta)}{k(1-\vartheta)} \right]^{\frac{1}{k-1}}$$

and hence the proof is complete. $\qquad\square$

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Theorem 4.2. *Let $f \in T\phi_c^\delta(\lambda, \vartheta)$. Then f is starlike of order δ, $(0 \le \delta < 1)$ in the disc $|z| < t_2$, where*

$$t_2 = \inf_{k \ge 2} \left[\frac{(1-\delta)[k + k\lambda(k-1) - \vartheta]\psi_k(c, \delta)}{(k-\delta)(1-\vartheta)} \right]^{\frac{1}{k-1}} \qquad k \ge 2. \qquad (4.3)$$

The result is sharp with the extremal function $f(z)$ is given by (2.4).

Proof. We have $f \in T$ and f is starlike of order δ, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta. \qquad (4.4)$$

For the left hand side of (4.4), we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{\sum\limits_{k=2}^{\infty} (k-1)a_k|z|^{k-1}}{1 - \sum\limits_{k=2}^{\infty} a_k|z|^{k-1}}$$

$(1 - \delta)$ is bigger than the right hand side of the left relation if

$$\sum_{k=2}^{\infty} \frac{k-\delta}{1-\delta} a_k |z|^{k-1} < 1.$$

We know that $f \in T\phi_c^\delta(\lambda, \vartheta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[k + k\lambda(k-1) - \vartheta]\psi_k(c, \delta)}{(1-\vartheta)} a_k \le 1.$$

Thus (4.4) is true if

$$\frac{k-\delta}{1-\delta}|z|^{k-1} \le \frac{[k + k\lambda(k-1) - \vartheta]\psi_k(c, \delta)}{(1-\vartheta)}$$

or equivalently

$$|z| \le \left[\frac{(1-\delta)[k + k\lambda(k-1) - \vartheta]\psi_k(c, \delta)}{(k-\delta)(1-\vartheta)} \right]^{\frac{1}{k-1}}.$$

It yield starlikeness of the family. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5 Convex Linear combinations

Theorem 5.1. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\vartheta}{[k + \lambda k(k-1) - \vartheta]\psi_k(c, \delta)} z^k, \qquad (z \in U, k \ge 2). \qquad (5.1)$$

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Then $f \in T\phi_c^\delta(\lambda, \vartheta)$ if and only if f can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \mu_k \geq 0 \tag{5.2}$$

and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. If a function f is of the form $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \mu_k \geq 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$ then

$$\sum_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta] \psi_k(c, \delta) |a_k|$$

$$= \sum_{k=2}^{\infty} [k + \lambda k(k-1) - \vartheta] \psi_k(c, \delta) \frac{(1-\vartheta)\mu_k}{[k + \lambda k(k-1) - \vartheta] \psi_k(c, \delta)}$$

$$= \sum_{k=2}^{\infty} (1-\vartheta)\mu_k = (1 - \mu_1)(1 - \vartheta)$$

$$\leq (1 - \vartheta)$$

which provides (2.2), hence $f \in T\phi_c^\delta(\lambda, \vartheta)$, by Theorem 2.2.

Conversely, if f is in the class $f \in T\phi_c^\delta(\lambda, \vartheta)$, then we may set

$$\mu_k = \frac{[k + \lambda k(k-1) - \vartheta] \psi_k(c, \delta)}{1 - \vartheta} |a_k|, k \geq 2,$$

and $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$.

Then the function f is of the form (5.2) and this completes the proof. $\square$

6   Partial Sums of Functions in the Class $\phi_c^\delta(\lambda, \vartheta)$

For a function $f \in A$ given by (1.1), Silverman [14] investigated the partial sums f defined by

$$f_1(z) = z \text{ and } f_m(z) = z + \sum_{k=2}^{\infty} a_k z^k, m = 2, 3, 4, \cdots. \tag{6.1}$$

In [14], Silverman examined sharp lower bounds on the real part of the quotients between the normalized convex or starlike functions and their sequences of partial sums. Also, Srivastava et al. [15] and Owa et al. [8] have investigated an interesting results on the partial sums. In this section, we consider partial sums of functions in the class $\phi_c^\delta(\lambda, \vartheta)$ and obtain sharp lower bounds for the ratios of real part of f to f_m and f' to f_m'.

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Theorem 6.1. *Let be a function f of the form (1.1) belong to the class $\phi_c^\delta(\lambda, \vartheta)$ and satisfy (2.1). Then*

$$\Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1}{d_{m+1}}, (z \in U, m \in N), \qquad (6.2)$$

where

$$d_k = \frac{[k + \lambda k(k-1) - \vartheta]}{1 - \vartheta}. \qquad (6.3)$$

Proof. Clearly, $d_{k+1} > d_k > 1, k = 2, 3, 4, \cdots$.

Thus by Theorem 2.1 we get,

$$\sum_{k=2}^{\infty} |a_k| + d_{m+1} \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k |a_k| \leq 1. \qquad (6.4)$$

$$\text{Setting } g(z) = d_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right) \right\}$$

$$g(z) = 1 + \frac{d_{m+1} \sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{m} a_k z^{k-1}} \qquad (6.5)$$

it suffices to show $Re(g(z)) > 0, z \in U$. Applying (6.4) we find that

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{d_{m+1} \sum_{k=2}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{m} |a_k| - d_{m+1} \sum_{k=m+1}^{\infty} |a_k|}$$

$$\leq 1,$$

which gives,

$$\Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1}{d_{m+1}},$$

and the proof is complete. $\qquad \square$

Theorem 6.2. *Let be a function f of the form (1.1) belong to the class $T\phi_c^\delta(\lambda, \vartheta)$ and satisfy (2.1). Then*

$$\Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{d_{m+1}}{1 + d_{m+1}}, z \in U, m \in N, \qquad (6.6)$$

where

$$d_k = \frac{[k + \lambda k(k-1) - \vartheta]}{1 - \vartheta}. \qquad (6.7)$$

Proof. Clearly, $d_{k+1} > d_k > 1, k = 2, 3, 4, \cdots$.

Thus by Theorem 2.1 we get,

$$\sum_{k=2}^{\infty} |a_k| + d_{m+1} \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k |a_k| \leq 1. \tag{6.8}$$

Setting $h(z) = (1 + d_{m+1}) \left\{ \dfrac{f_m(z)}{f(z)} - \left(\dfrac{d_{m+1}}{1 + d_{m+1}} \right) \right\}$

$$h(z) = 1 - \frac{(1 + d_{m+1}) \sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{m} a_k z^{k-1}} \tag{6.9}$$

it suffices to show $Re(h(z)) > 0, z \in U$. Applying (6.8) we find that

$$\left| \frac{h(z) - 1}{h(z) + 1} \right| \leq \frac{(1 + d_{m+1}) \sum\limits_{k=2}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=2}^{m} |a_k| - (1 + d_{m+1}) \sum\limits_{k=m+1}^{\infty} |a_k|}$$

$$\leq 1,$$

which gives,

$$\Re \left(\frac{f_m(z)}{f(z)} \right) \geq \frac{d_{m+1}}{1 + d_{m+1}},$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.3. *Let be a function f of the form (1.1) belong to the class $\phi_c^\delta(\lambda, \vartheta)$ and satisfy (2.1). Then*

$$\Re \left(\frac{f'(z)}{f_m'(z)} \right) \geq 1 - \frac{m+1}{d_{m+1}}, z \in U, m \in N, \tag{6.10}$$

and

$$\Re \left(\frac{f_m'(z)}{f'(z)} \right) \geq \frac{d_{m+1}}{m+1+d_{m+1}}, z \in U, m \in N \tag{6.11}$$

where

$$d_k = \frac{[k + \lambda k(k-1) - \vartheta]}{1 - \vartheta}. \tag{6.12}$$

Proof. By Setting

$$g(z) = d_{m+1} \left\{ \frac{f('z)}{f_m'(z)} - \left(1 - \frac{m+1}{d_{m+1}} \right) \right\}, z \in U$$

and $h(z) = (m + 1 + d_{m+1}) \left\{ \dfrac{f_m'(z)}{f('z)} - \left(\dfrac{d_{m+1}}{m+1+d_{m+1}} \right) \right\}, z \in U,$

the proof is similar to that of Theorems 6.1 and 6.2, and so we omit the details. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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7 Convolution properties

In this section we will prove that the class $T\phi_c^\delta(\lambda, \vartheta)$ is closed under convolution.

Theorem 7.1. *Let $g(z)$ of the form*

$$g(z) = z - \sum_{k=2}^\infty b_k z^k$$

be analytic in U. If $f \in T\phi_c^\delta(\lambda, \vartheta)$ then the function $f * g$ is in the class $T\phi_c^\delta(\lambda, \vartheta)$. Here the symbol $*$ denoted to the Hadmard product (or convolution).

Proof. Since $f \in T\phi_c^\delta(\lambda, \vartheta)$, we have

$$\sum_{k=2}^\infty [k + \lambda k(k - 1) - \vartheta]\psi_k(c, \delta)|a_k| \leq 1 - \vartheta.$$

By utilizing the last inequality and the fact that

$$f(z) * g(z) = z - \sum_{k=2}^\infty a_k b_k z^k.$$

We obtain

$$\sum_{k=2}^\infty [k + \lambda k(k - 1) - \vartheta]\psi_k(c, \delta)|a_k||b_k|$$

$$\leq \sum_{k=2}^\infty [k + \lambda k(k - 1) - \vartheta]\psi_k(c, \delta)|a_k|$$

$$\leq 1 - \vartheta$$

and hence, in view of Theorem 2.1, the result follows. $\qquad\square$

8 Neighbourhood for the class $T\phi_c^\delta(\lambda, \vartheta)$

Following [3, 11], we defined the $\alpha-$neighbourhood of the function $f(z) \in T$ by

$$N_\alpha(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^\infty b_k z^k \text{ and } \sum_{k=2}^\infty k|a_k - b_k| \leq \alpha \right\}. \quad (8.1)$$

Definition 8.1. A function $f \in A$ is said to in the class $T\phi_c^\delta(\lambda, \vartheta)$ if there exists a function $h \in T\phi_c^\delta(\lambda, \vartheta)$ such that

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \gamma, \quad (z \in U, \ 0 \leq \gamma < 1). \quad (8.2)$$

Theorem 8.2. *If $h \in T\phi_c^\delta(\lambda, \vartheta)$ and*

$$\gamma = 1 - \frac{\alpha(2\lambda - \vartheta + 2)\psi_2(c, \lambda)}{2(2\lambda - \vartheta + 2)\psi_2(c, \lambda) - (1 + \vartheta)}$$

then $N_\alpha(h) \subseteq T\phi_c^{\delta, \gamma}(\lambda, \vartheta)$.

Proof. Let $f \in N_\alpha(h)$. We then find from that

$$\sum_{k=2}^\infty k|a_k - b_k| \leq \alpha,$$

which is easily implies the coefficient inequality

$$\sum_{k=2}^\infty |a_k - b_k| \leq \frac{\alpha}{k}, \quad (n \in N).$$

Since $h \in T\phi_c^\delta(\lambda, \vartheta)$, we have from equation (2.1) that

$$\sum_{k=2}^\infty |a_k| \leq \frac{1 - \vartheta}{(2\lambda - \vartheta + 2)\psi_2(c, \lambda)}$$

and

$$\left| \frac{f(z)}{h(z)} - 1 \right| < \frac{\sum_{k=2}^\infty k|a_k - b_k|}{1 - \sum_{k=2}^\infty b_k}$$

$$\leq \frac{\alpha}{2} \frac{(2\lambda - \vartheta + 2)\psi_2(c, \lambda)}{(2\lambda - \vartheta + 2)\psi_2(c, \lambda) - (1 + \vartheta)}$$

$$= 1 - \gamma.$$

This completes the proof of the theorem. $\square$

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