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ON THE COMPUTABILITY OF ORDERED FIELDS

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ABSTRACT. In this paper we develop general techniques for structures of computable real numbers generated by classes of total computable (recursive) functions with special restrictions on basic operations in order to investigate the following problems: whether a generated structure is a real closed field and whether there exists a computable presentation of a generated structure. We prove a series of theorems that lead to the result that there are no computable presentations neither for polynomial time computable no even for \mathcal{E}_n -computable real numbers, where \mathcal{E}_n is a level in Grzegorzcyk hierarchy, $n \geq 2$. We also propose a criterion of computable presentability of an archimedean ordered field.

Keywords: computability, index set, computable model theory, computable analysis, complexity.

1. INTRODUCTION

In the framework of computable model theory there have been investigated conditions on the existences of computable copies for countable homogeneous boolean algebras [14], for superatomic boolean algebras [6, 7], for ordered abelian groups [19] among others and established several negative results for archimedean ordered fields [13, 11].

Nevertheless, till now there where no natural criteria on the existence of computable copies of ordered fields even in an archimedean case. In this paper we fill this gap. We are also going dipper to revile relations between a class of computable (recursive) functions K and a structure \tilde{K} of real K -numbers generated by K . We propose

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natural restrictions on a class K under which the structure \tilde{K} is a real closed field. It is worth noting duality: for any computable archimedean ordered field $(F, +, \cdot, \leq)$ one can create the corresponding class K such that $F = \tilde{K}$.

As an motivating example we can consider the dyadic numbers $Dyad$. In this case $(Dyad, +, \cdot, \leq)$ is a computable structure, the corresponding K is the set of computable almost constant functions and $Dyad = \tilde{K}$.

In this direction we investigate whether there exist computable copies of generated structures for popular classes of computable functions such as the polynomial time computable functions P and Grzegorzcyk classes \mathcal{E}_n , $n \geq 2$. We establish that the corresponding real closed fields do not have computable copies and moreover the polynomial time computable real numbers as an abelian group does not have a computable copy as well.

In order to do that we develop techniques of index sets and multiple m -completeness. On this way we have to establish a criterion of m -completeness for tuples of c.e. sets and Σ_2^0 -sets. From our point of view this criterion is an interesting result itself and can be used for different purposes.

The paper is organised as follows:

Section 2 contains preliminaries and basic background.

In Section 3 we show under which natural restrictions on K the corresponding generated structure \tilde{K} is a real closed field.

In Section 4 we propose a criterion of the computable presentability of an archimedean ordered field.

In Section 5 we define 3-tuple of index sets (A_0, A_1, A_2) depending on K such that $A_i \in \Sigma_2^0$ with the following embedding property $A_0 \subseteq A_1 \subseteq A_2$. In the Theorem 2 we show that if the corresponding \tilde{K} as a structure, in particular as an abelian group, has a computable presentation then $A_0 \cup (A_2 \setminus A_1) \in \Sigma_2^0$.

In Sections 6, 7 and 8 we develop techniques to establish that under natural assumptions on a class K the 3-tuple (A_0, A_1, A_2) is m -complete in the class of 3-tuples of Σ_2^0 -sets with the embedding property. It is well-known that in this case $A_0 \cup (A_2 \setminus A_1) \notin \Sigma_2^0$ and therefore for the corresponding \tilde{K} there is no a computable copy. It is worth noting than these classes contain polynomial time computable real numbers, computable real numbers generated by Grzegorzcyk classes, in particular \mathcal{E}_2 and beyond.

2. PRELIMINARIES

We refer the reader to [15] and [18] for basic definitions and fundamental concepts of recursion theory [10, 20] for computable analysis, [5] for computable model theory, [8] for Grzegorzcyk classes of computable (recursive) functions and [1] for complexity classes. We recall that, in particular, φ_e denotes the partial computable (recursive) function with an index e in the Kleene numbering. For simplicity of descriptions we identify a function with its graph. We also use notations $W_e = \text{dom}(\varphi_e)$, $\overline{W}_e = \omega \setminus W_e$, $\pi_e = \text{im}(\varphi_e)$ and for Cantor 3-tuple (c, l, r) we associate a number n with the corresponding pair $\langle l(n), r(n) \rangle$. We fix the set BF of standard basic functions $\lambda x.0$, $s(x)$ and I_m^n , where $I_m^n(x_1, \dots, x_n) = x_m$ for $1 \leq m \leq n$. We fix a standard computable numbering $q : \omega \rightarrow \mathbb{Q}$ of the rational numbers and when it is clear from a context we use the notation q_n for $q(n)$. For tuples $z_1 \dots, z_k$ of numbers or functions we use the notation \bar{z} if it is clear from a context. For the dyadic numbers we use the notation $Dyad = \{\frac{m}{2^{i+1}} \mid m \in \mathbb{Z}, i \geq 0\}$, $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$

and $B(\alpha, r) = \{x \in \mathbb{R} \mid |x - \alpha| < r\}$ for a basic open ball with the center $\alpha \in \mathbb{Q}$ and the radius $r \in \mathbb{Q}^+$.

2.1. Primitive Computable Reals. Let K be a class of total computable numerical functions with the following restrictions: it contains the basic functions BF and $+$, \cdot , closed under composition and the following bounded primitive recursion scheme: If $\alpha, g, \psi \in K$ and f is defined by

$$f(\bar{x}, y) = \begin{cases} \alpha(\bar{x}) & \text{if } y = 0 \\ \psi(\bar{x}, y, f(\bar{x}, y - 1)) & \text{if } y \geq 1 \end{cases}$$

and

$$f(\bar{x}, y) \leq g(\bar{x}, y)$$

then $f \in K$.

Then \tilde{K} denotes the set of computable real numbers

$$\{x \mid (\exists \Phi \in K)(\forall n > 0) |x - q(\Phi(n))| \leq \frac{1}{2^n}\}.$$

This is equivalent to the existence of $F \in K$ s.th. $(\forall n \geq 0)(\forall N > n) |a_N - a_n| \leq \frac{1}{2^n} \wedge \lim_{n \rightarrow \infty} a_n = x$, where $a_n = q(F(n))$. If K contains $\lambda x.2^x$ then an element of \tilde{K} is called a K -number. In particular, if K is the class of all primitive computable functions then \tilde{K} is called the primitive computable real numbers. The same for \mathcal{E}_n , $n \geq 3$. For \mathcal{E}_2 and polynomial time computable reals see Section 5.

Remark 1. *In Section 5 we are going to consider not so rich classes as above. Therefore the definition of \tilde{K} will be modified. In particular, in Section 5 and further the notion $x \in \tilde{K}$ will differ from the notion K -number.*

2.2. Computable Presentations. We say that a structure $\mathcal{A} = \langle A, \sigma \rangle$ admits an computable presentation if there is a numbering $\nu : \omega \rightarrow A$ such that the relations and operations from σ including equality are computable with respect to the numbering ν . The pair (\mathcal{A}, ν) is called a computable structure and the numbering ν is called its computable presentation (constructivisation). If only operations are computable with respect to the numbering ν , a structure (\mathcal{A}, ν) is called a numbered (effective) algebra.

3. WHEN \tilde{K} IS A REAL CLOSED FIELD

Let us fix K with the restrictions from Section 2.1.

Proposition 1. *The corresponding structure $\tilde{K} = (\tilde{K}, +, \cdot, \leq)$ is a real closed field.*

Proof. The claim that \tilde{K} is closed under addition, subtraction, multiplication and division is straightforward. To complete the proof, we show that the roots of polynomials with coefficients in \tilde{K} are also in the class \tilde{K} . Assume contrary there exists a polynomial $p(x) = \sum_{i=0}^n a_i x^i \in \tilde{K}[x]$ of minimal degree which has a root x in \mathbb{R} but not in \tilde{K} . The polynomial p does not have multiple roots since in opposite case it is possible to compute $g = \text{G.C.D.}(p(x), p'(x))$ provided by exact knowledge

of zero coefficients of q with $\deg(q) < \deg(p)$. So the coefficients $\bar{a} = (a_0, \dots, a_n)$ of p satisfy the following formula:

$$\psi(\bar{a}) = \psi_1(\bar{a}) \vee \psi_2(\bar{a}),$$

where

$$\psi_1(\bar{a}) \Leftrightarrow (\exists A)(\exists B)(\exists \epsilon > 0) \left(A < B \wedge p(A) < -\epsilon \wedge p(B) > \epsilon \wedge (\forall x \in [A, B]) p'(x) > 0 \right)$$

and

$$\psi_2(\bar{a}) \Leftrightarrow (\exists A)(\exists B)(\exists \epsilon > 0) \left(A < B \wedge p(A) > \epsilon \wedge p(B) < -\epsilon \wedge (\forall x \in [A, B]) p'(x) < 0 \right).$$

W.l.o.g. we assume $\mathbb{R} \models \psi_1(\bar{a})$. By the Uniformity Principal [12], the formula ψ_1 can be effectively transformed to a formula

$$\bigvee_{A, B \in \mathbb{Q}, A < B} \bigvee_{\epsilon \in \mathbb{Q}^+} \Theta_{A, B, \epsilon}(\bar{a}),$$

where $\Theta_{A, B, \epsilon}(\bar{x})$ is a uniformly computable disjunctions of \exists -formulas without equality. Therefore ψ_1 defines an effectively enumerable subset of \mathbb{R}^{n+1} . Suppose $\mathbb{R} \models \Theta_{A, B, \epsilon}(\bar{a})$. For simplicity of the further reasoning we fix the product of open balls containing \bar{a} :

$$\prod_{i=0}^n B(\alpha_i, r_i) \subseteq \{\bar{x} \mid \mathbb{R} \models \Theta_{A, B, \epsilon}(\bar{x})\}$$

and the corresponding $\alpha_i \in \mathbb{Q}$ and $r_i \in \mathbb{Q}^+$, $0 \leq i \leq n$. Since $a_i \in \tilde{K}$ for $0 \leq i \leq n$, in the framework of K one can effectively find rational tuple b_0, \dots, b_n such that for all $0 \leq i \leq n$,

- $b_i \in B(\alpha_i, r_i)$ and
- $|b_i - a_i| \leq \frac{1}{2^{3m}}$,

where m is an argument of this computations. Let $M \in \mathbb{Q}^+$ be a bound on a_i and B , i.e., $|a_i| < M$ for $0 \leq i \leq n$ and $|B| < M$. Having m and the required precision $\frac{1}{2^m}$, in the framework of K one can effectively find $y \in \mathbb{Q} \cap [A, B]$ such that

$$\left| \sum_{i=0}^n b_i y^i \right| < \frac{1}{3m}$$

One can assume that m is sufficiently big, i.e., $2^m > M^n + \dots + 1$. It is worth noting that $2^m \cdot \epsilon > 1$ and $|p(y)| \leq \frac{1}{2^{3m}} \cdot (1 + \dots + |y|^n) \leq \frac{1}{2^{3m}} \cdot 2^m = \frac{1}{2^{2m}}$. By the mean value theorem, for all $x, y \in [A, B]$ there exists $c \in [A, B]$ such that $p(x) - p(y) = (x - y) \cdot p'(c)$. If x is the root of p in the interval $[A, B]$ then

$$|x - y| \leq \frac{|p(y)|}{\epsilon} \leq \frac{1}{2^{2m}} \cdot 2^m \leq \frac{1}{2^m}.$$

So $y \in \mathbb{Q}$ is an approximation of the root x with the precision $\frac{1}{2^m}$. Therefore, $x \in \tilde{K}$, a contradiction. \square

An discussion that the previous proposition is an refinement of the result in [10] for P-numbers one can find in Section 5.

4. CRITERION OF COMPUTABLE PRESENTABILITY AND ARCHIMEDEAN PART

Let $L = (L, \leq)$ be linearly ordered and $\mathbb{Q} \subseteq L$. Assume $\mu : \omega \rightarrow L$ is a numbering. With L we associate 2 families:

$$\begin{aligned} A_k &= \{n \mid q_n \leq \mu(k)\} \\ B_k &= \{n \mid q_n \geq \mu(k)\} \end{aligned}$$

and naturally define $S_k = A_k \oplus B_k = \{2n \mid n \in A_k\} \cup \{2n + 1 \mid n \in B_k\}$ and $S_L = \{S_k \mid k \in \omega\}$.

Remark 2. *It is worth noting that S_L does not depend on the choice of μ . By the way, it is easy to see that if μ is a computable presentation of an ordered field $F = (F, +, \cdot, \leq)$ then $\mu \geq q$, i.e., $q_n = \mu(h(n))$ for a computable function $h : \omega \rightarrow \omega$ and the family S_F is computable.*

The following theorem provides a criterion of computable presentability of archimedean ordered fields.

Theorem 1. *Let $F = (F, +, \cdot, \leq)$ be an archimedean ordered field, $\mu : \omega \rightarrow F$ be its numbering such that (F, μ) is a numbered algebra. Then the family S_F is computable if and only if (F, μ) is a computable.*

Proof. The claim \rightarrow follows from Remark 2.

For the claim \leftarrow we assume that S_F is computable. Let $0 = q_i$. Since for some $a \in \omega$ we have $-1 = \mu(a)$, the subtraction is defined as $\mu(n) - \mu(m) = \mu(n) + \mu(a) \cdot \mu(m)$. So, it is clear that $\mu(n) = \mu(m)$ iff $\mu(n) - \mu(m) = \mu(k) \wedge 2i \in S_k \wedge 2i + 1 \in S_k$. Therefore equality is computably enumerable. Since positive fields are computable, equality is computable.

It is easy to see that order is also computable. Indeed,

$$\begin{aligned} \mu(n) < \mu(m) &\text{ iff } (\exists k)(\exists l) \mu(n) < q_k < q_l \leq \mu(m), \\ \mu(n) \leq \mu(m) &\text{ iff } \mu(n) < \mu(m) \vee \mu(n) = \mu(m), \\ \mu(n) \not\leq \mu(m) &\text{ iff } \mu(m) < \mu(n). \end{aligned}$$

□

Now we show that the requirement that (F, μ) is a numbered algebra one can not avoid to establish that F has a computable copy. For that we start with general definitions and observations on the archimedean part of an ordered field.

Definition 1. *For $x \in F$, where F is an ordered field we define $\text{Sp}(x) : F \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ as follows:*

$$\text{Sp}(x) = \begin{cases} +\infty & \text{if } x > \mathbb{Q} \\ y \in \mathbb{R} & \text{if } |x - y| < \mathbb{Q}^+ \\ -\infty & \text{if } x < \mathbb{Q}. \end{cases}$$

It is worth noting that y is uniquely defined and if $\text{Sp}(x) \notin \{-\infty, +\infty\}$ we say about a finite spectrum of x .

It is worth noting that Zorn lemma provides the fact that any ordered field $(F, +, \cdot, 0, 1, \leq)$ has some maximal archimedean subfield $F_0 \leq F$. The main issue of the following proposition is that all maximal archimedean subfield are isomorphic each other.

Proposition 2. *Let $(F, +, \cdot, 0, 1, \leq)$ be an ordered real closed field and F_0 be its maximal archimedean subfield. Then F_0 is a real closed field and the greatest archimedean subfield with respect the following pre-order on subfields of F :*

$F_1 \leq F_2$ iff there is an isomorphic embedding $\varphi : F_1 \rightarrow F_2$ as ordered fields.

Actually, $F_0 \simeq \text{Sp}(F_0) = \text{Sp}(F) \setminus \{-\infty, +\infty\}$ considered as subfields of the reals while for any archimedean $F_1 \leq F$ it holds $F_1 \cong \text{Sp}(F_1) \leq \text{Sp}(F_0)$. In particular, all maximal archimedean subfields of F are isomorphic each other.

Lemma 1. F_0 is a real closed field.

Proof. (Lemma 1) We are going to check that monic (unitary)odd polynomials and $x^2 - b$, ($b > 0$) from $F_0[x]$ have roots in F_0 . For the case $x^2 - b$, ($b > 0$) the claim is straightforward. Indeed, since F is real closed there exists $x_0 > 0$ such that $x_0^2 = b$ and $x_0 \in F_0$. If $x_0 \notin F_0$ then $F_0(x_0)$ is non-archimedean, so, for some $a \in F_0$, $|x_0 - a| < \mathbb{Q}^+$. Then $|x_0 - a| < \mathbb{Q}^+$ and $|a^2 - b| < \mathbb{Q}^+$, a contradiction.

Assume $p(x)$ is a monic polynomial of the least odd degree that does not have roots in F_0 . Fix $x_0 \in F$ such that $p(x_0) = 0$, so $\text{Sp}(x_0)$ is finite. Moreover, it follows that $p(x)$ is irreducible over F_0 since the equality $p = p_1 \cdot p_2$, where p_1, p_2 are monic polynomial of degree greater than 0 leads to a contradiction to the choice of p .

If $x_0 \notin F_0$ then $F_0(x_0)$ is non-archimedean, so, for some $q \in F_0[x]$, $\deg(q) < \deg(p)$ we have $|q(x_0)| < \mathbb{Q}^+$. Since p and q do not have a common factor, $\text{Res}(p, q) \in F_0 \setminus \{0\}$. It is well-known that $\text{Res}(p, q) = p(x)A(x) + q(x)B(x)$ for some $A, B \in F_0[x]$. Therefore, $|\text{Res}(p, q)| < \mathbb{Q}^+$, a contradiction. \square

To finish the proof of the proposition first let us note that $\text{Sp} \upharpoonright F_0$ is an isomorphic embedding. It is sufficient to show that $\text{Sp}(F) \setminus \{-\infty, +\infty\} = \text{Sp}(F_0)$. Let $x_0 \in F$ such that $\text{Sp}(x_0)$ is finite and $x_0 \notin F_0$. Since $F_0(x_0)$ is non-archimedean, for some $a, b \in F_0[x]$ we have $|\frac{a(x_0)}{b(x_0)}| < \mathbb{Q}^+$. W.l.o.g. assume that the fraction is irreducible. We have two cases:

1) $|b(x_0)| > \mathbb{Q}$. This contradicts the finiteness of $\text{Sp}(x_0)$.

2) $|a(x_0)| < \mathbb{Q}^+$. Then $\text{Sp}(a)(\text{Sp}(x_0)) = 0$. That means that $\text{Sp}(x_0)$ is algebraic over the field $\text{Sp}(F_0)$. Since, by Lemma 1, F_0 and $\text{Sp}(F_0)$ are real closed we have $\text{Sp}(x_0) \in \text{Sp}(F_0)$, that finishes the proof of Proposition 2.

Now F_0 is called an archimedean part of F .

In [13] R. Miller and V. O. Gonzales constructed a computable ordered field F' such that no maximal archimedean subfields of F' have computable copies. In our terms it means precisely that the archimedean part of F' does not have a computable copy. We take this example to illustrate that the requirement that (F, μ) is a numbered algebra one can not avoid to establish that F has a computable copy. This note highlights the importance of the results in [13].

Assume now that F' is computable and its archimedean part $F = F'_0$ does not have a computable copy. By Theorem 1 the family $S_{F'}$ is computable. While $S_{F'} \neq S_F$, the following formula defines S_F :

$$A \in S_F \leftrightarrow (\exists q_i)(\exists q_j)(A \in S_{F'} \wedge q_i < q_j \wedge 2i \in A \wedge 2j + 1 \in A).$$

Therefore S_F is computable, however F does not have a computable copy.

5. INDEX SETS VS. COMPUTABLE PRESENTABILITY

In this section we assume K is a class of total computable numerical functions. We associate with K the class $K^{\{0,1,2\}} = \{f \in K \mid \text{im}(f) \subseteq \{0,1,2\}\}$. By AC we denote the almost constant functions, i.e., $\text{AC} = \{f : \omega \rightarrow \omega \mid (\exists c \in \omega)(\exists x \in \omega)(\forall y > x)f(y) = c\}$ and use the following notation $f =^* c$ for a function $f \in \text{AC}$ with the evidence $c \in \omega$.

We assume that $\text{AC} \subset K$. We proceed with the definition of the corresponding structure \widetilde{K} . It is worth noting that for classes with the restriction from Section 2.1 the previous and the modified definition are equivalent. First the set \widetilde{K} is defined as follows:

$$x \in \widetilde{K} \leftrightarrow x = \sum_{i=0}^{\infty} \frac{f(i) - 1}{2^{i+1}},$$

where $f \in K^{\{0,1,2\}}$. It is easy to see that \widetilde{K} contains $\text{Dyad} \cap [-1, 1]$ since $\text{AC} \subset K$ and $1 = \sum_{i=0}^{\infty} \frac{2-1}{2^{i+1}} \in \widetilde{K}$. Then $\widetilde{K} = \{m + x \mid m \in \mathbb{Z}, x \in \widetilde{K}\}$.

Motivations why we from right now do not identify K -numbers and elements of \widetilde{K} are as follows.

Let us consider the following example of the P-numbers computed in polynomial time, where a polynomial is applied to the length of an argument. These numbers form a real closed field [10] and denoted there as P_{CF} . It turns out that for $x \in [-1, 1]$ nobody can state that

$$x \text{ is P-number iff } (\exists f \in \text{P}^{\{0,1,2\}}) x = \sum_{i \in \omega} \frac{f(i) - 1}{2^{i+1}}.$$

Actually, in these terms one can define a new class K_{P} containing functions that are computed in polynomial time, where a polynomial is applied to an argument instead of its length.

It is easy to see that the class K_{P} is closed under the bounded primitive recursion scheme, in particular, Proposition 3 is applicable to \widetilde{K}_{P} . For this class we can state that, for $x \in [-1, 1]$,

$$x \text{ is P-number iff } (\exists f \in K_{\text{P}}^{\{0,1,2\}}) x = \sum_{i \in \omega} \frac{f(i) - 1}{2^{i+1}}.$$

That means that the P-numbers are exactly \widetilde{K}_{P} (c.f. [10]). Further it makes sense to analyse the set $\widetilde{\text{P}}$ that consists from elements x that the computation of i -th sign of x after the comma requires not more than N steps, where N is a polynomial on the length of i . Unfortunately we can not state that $\widetilde{\text{P}}$ is a subgroup of $(\mathbb{R}, +)$ since this question is still an open problem. This leads us to introduce a new definition of a structure, see below. For right now it is worth noting that the SP-numbers (i.e., $\widetilde{\text{P}}$) is naturally associated with the class SP of computable functions which computations require no more steps than $\lceil \log_2(\lceil \log_2(x_i) \rceil) \rceil$, $i \leq s$, where $\bar{x} = (x_1, \dots, x_s)$ is an s -tuple of arguments. The corresponding structure looks like small however in Section 8 we show that neither the P-numbers no SP-numbers have computable copies.

Our next example is related to an appropriate definition of \mathcal{E}_2 -numbers. It is well-known that $\lambda x.2^x \notin \mathcal{E}_2$ and $\mathcal{E}_2 = \text{LogSpace}$, i.e., $f \in \mathcal{E}_2$ if and only if there exists $c \in \omega$ such that for all x_1, \dots, x_m a computation of $f(\bar{x})$ requires not more

that $c \cdot \max_{i \leq m} |x_i|$ cells of a tape (memory), where $|x|$ denotes the length of x and w.l.o.g it can be substituted by $\lceil \log_2(x) \rceil$. According to the approach in [10] it is natural to have the following definition:

for $x \in [-1, 1]$,

$$x \text{ is a } \mathcal{E}_2\text{-number} \leftrightarrow (\exists f \in \text{DSpace}^{\{0,1,2\}}(n)) x = \sum_{i=0}^{\infty} \frac{f(i) - 1}{2^{i+1}},$$

and for $x \in \mathbb{R}$,

$$x \text{ is a } \mathcal{E}_2\text{-number} \leftrightarrow (\exists m \in \mathbb{Z})(\exists y \in \widetilde{\text{DSpace}}(n)) x = m + y.$$

Therefore x is a \mathcal{E}_2 -number iff $x \in \widetilde{\text{DSpace}}(n)$.

According the observations above we introduce the definition of a structure on \tilde{K} .

Definition 2. Suppose K is a class of total computable numerical functions, \tilde{K} is the set of reals generated by K . Then we associate with \tilde{K} a structure $\tilde{K} = (\tilde{K}, 0, Q_+^3, Q_-^3)$, where

$$\begin{aligned} \tilde{K} \models Q_+^3(x, y, z) &\leftrightarrow x + y \leq z \\ \tilde{K} \models Q_-^3(x, y, z) &\leftrightarrow x + y \geq z. \end{aligned}$$

It is easy to see that, since the graph of addition is computable, if a structure \tilde{K} has a computable copy (\tilde{K}, μ) then the sets $\mu^{-1}(\text{Dyad})$ and $\mu^{-1}(\mathbb{Q})$ are computably enumerable. For example,

$$\mu(n) \in \text{Dyad} \leftrightarrow (\exists k \in \omega)(\exists l \in \mathbb{Z}) 2^k \cdot \mu(n) + l = 0.$$

To proceed further we define index sets

$$\begin{aligned} A_0 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge n \notin \text{Tot}\}, \\ A_1 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge (n \notin \text{Tot} \vee \varphi_n \in \text{AC})\}, \\ A_2 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge (n \notin \text{Tot} \vee \varphi_n \in K)\}. \end{aligned}$$

Theorem 2. Suppose K is a class of total computable numerical functions. If the structure \tilde{K} generated by K has a computable presentation then $A_0 \cup (A_2 \setminus A_1) \in \Sigma_2^0$.

Proof. Let $\mu : \omega \rightarrow \tilde{K}$ be a computable presentation. Since the set $E = \{n \mid -1 \leq \mu(n) \leq 1\}$ is computable there exists a computable function h such that $\text{im}(h) = E$ and $\tilde{\mu} = \mu \circ h$ is a computable numbering of $\tilde{K} \cap [-1, 1]$. Assume $x = \tilde{\mu}(n)$. Now we construct $\nu_1 : \omega \rightarrow K$ by induction.

$$\begin{aligned} \nu_1(n)(0) &= 1 \\ \nu_1(n)(s+1) &= \begin{cases} 0 & \text{if } x < x_s \\ 1 & \text{if } x = x_s \\ 2 & \text{if } x > x_s, \end{cases} \end{aligned}$$

where $x_s = \sum_{i \leq s} \frac{\nu(n)(i)-1}{2^i}$. Since $|x_s - x| \leq \frac{1}{2^s}$, $|x_{s+1} - x| \leq \frac{1}{2^{s+1}}$. From $\text{AC} \subseteq K$ it follows that $x_s \in \tilde{K}$.

We have the following properties: $\nu_1(n)$ is total, $\nu_1(n) \in K$ and $\nu_1(n)$ provides a sign-digit representation of x . If $x \notin \text{Dyad}$ then by the uniqueness of sign-digit

representation of x $\nu_1(n)$ uniquely represent x . If $x \in \text{Dyad}$ then $\nu_1(n)$ gives some sign-digit representation of x . Therefore we enumerate not all functions from K but almost. To improve that we take a computable numbering $\nu_2 : \omega \rightarrow \text{AC}$ induced by a computable numbering of the finite sets and $\nu = \nu_1 \oplus \nu_2$, i.e., $\nu(2n) = \nu_1(n)$ and $\nu(2n+1) = \nu_2(n)$.

From properties of μ it follows that $\nu(n) \in \text{AC} \leftrightarrow \mu(n) \in \text{Dyad}$ is a Σ_1^0 -condition since $\mu(n) \in \text{Dyad} \leftrightarrow (\exists k \in \omega)(\exists l \in \mathbb{Q}) 2^k \cdot \mu(n) + l = 0$. As a corollary, $\{n \mid \nu_1(n) \in \text{AC}\}$ and $Y = \{n \mid \nu(n) \in \text{AC}\}$ are computably enumerable. Now we show that $A_0 \cup A_2 \setminus A_1 \in \Sigma_2^0$.

$$n \in A_0 \cup (A_2 \setminus A_1) \leftrightarrow \left(n \in A_2 \wedge (\exists m \in \omega \setminus Y) \nu(m) \supseteq \varphi_n \right) \vee n \in A_0.$$

The direction \rightarrow is straightforward from the definition of the index sets. The direction \leftarrow follows from the following observation. Assume $\nu(m) \supseteq \varphi_n$ for $m \in \omega \setminus Y$. Then $\nu(m) \notin \text{AC}$, $\nu(m) \in K$ and either $\varphi_n = \nu(m)$ (that means $n \in A_2 \setminus A_1$) or $n \notin \text{Tot}$ (that means $n \in A_0$).

We have the following:

- The relation $n \notin Y$ is Π_1^0 .
- The relation $\nu \supseteq \varphi_n$ is Π_1^0 since

$$\nu(m) \supseteq \varphi_n \leftrightarrow (\forall k \in \omega)(\forall s \in \omega) \left(\varphi_n^s(k) \downarrow \rightarrow \varphi_n^s(k) = \nu(m)(k) \right)$$

- The relation $n \notin \text{Tot}$ is Σ_2^0 .

Therefore $A_0 \cup (A_2 \setminus A_1) \in \Sigma_2^0$. \square

It is worth noting that in many cases \tilde{K} is an abelian ordered group, in particular $\tilde{\mathcal{E}}_n$, $n \geq 2$, P_{CF} -numbers. The same proof is valid when one consider just a computable presentation μ of a linear ordered (\tilde{K}, \leq) with the requirement that $\mu \geq d$, where d is a standard computable presentation of (Dyad, \leq) .

6. CRITERION OF M-COMPLETENESS FOR TUPLES OF Σ_1^0 AND Σ_2^0 SETS

In this section for $s \geq 1$ we consider s -tuples (A_0, \dots, A_{s-1}) , where all A_i are either Σ_1^0 -sets or all A_i are Σ_2^0 -sets.

For uniformity of a presentation we introduce a symbol l where $l \in \{1, 2\}$, a relation \sim_l on sets and an oracle z_l that have the following interpretation. If $l = 1$ then $A \sim_l B$ means that A and B are equal. The oracle $z_l = \emptyset$. If $l = 2$ then $A \sim_l B$ means that $(A \setminus B) \cup (B \setminus A)$ is finite, i.e., A and B are almost equal, denoted $A \sim^* B$. The oracle $z_l = Kw$, where $Kw = \{n \mid \varphi_n(n) \downarrow\}$ or could be any creative set. We generalise ideas of the criterion of m -completeness of Σ_1^0 -sets and Σ_2^0 -sets in [2] to fit m -completeness of s -tuples of Σ_1^0 -sets and Σ_2^0 -sets that requires modifications of concepts and definitions.

Remark 3. *It is well known (see c.f. [3]) that on the set of all computable enumerable sets of ω , given any (partial) Σ_2^0 -function f one can effectively construct a total computable function F such that $(\forall x \in \text{dom}(f)) W_{f(x)} =^* W_{F(x)}$, moreover $\varphi_{f(x)} =^* \varphi_{F(x)}$.*

Definition 3. *Let (F_0, \dots, F_{s-1}) be an s -tuple of functions, where $F_i : \omega^s \rightarrow \omega$, $0 \leq i \leq s-1$ and (A_0, \dots, A_{s-1}) be an s -tuple of Σ_l^0 -sets. We say that (F_0, \dots, F_{s-1})*

is m -reducible to (A_0, \dots, A_{s-1}) , denoted as $(F_0, \dots, F_{s-1}) \leq_m (A_0, \dots, A_{s-1})$, if there exist computable functions $h : \omega^s \rightarrow \omega$, $a_i : \omega^s \rightarrow \omega$, $b_i : \omega^s \rightarrow \omega$, $0 \leq i \leq s-1$, such that

$$F_i(\bar{x}) = \begin{cases} a_i(\bar{x}) & \text{if } h(\bar{x}) \in A_i \\ b_i(\bar{x}) & \text{if } h(\bar{x}) \notin A_i. \end{cases}$$

It is easy to see that this definition is a generalisation of the common m -reducibility of c.e. sets [15].

Lemma 2. For s -tuples X and A of Σ_1^0 -sets if $(F_0, \dots, F_{s-1}) \leq_m (X_0, \dots, X_{s-1})$ and $(X_0, \dots, X_{s-1}) \leq_m (A_0, \dots, A_{s-1})$ then $(F_0, \dots, F_{s-1}) \leq_m (A_0, \dots, A_{s-1})$.

Proposition 3. Let (A_0, \dots, A_{s-1}) be a s -tuple of Σ_1^0 -sets. The following claims are equivalent.

- (1) (A_0, \dots, A_{s-1}) is m -complete in the class of s -tuples of Σ_1^0 -set.
- (2) There exists a computable function (its productive function) $H : \omega^s \rightarrow \omega$, i.e., for all $x_0, \dots, x_{s-1} \in \omega$

$$H(x_0, \dots, x_{s-1}) \in \bigcap_{i=0}^{s-1} \left((A_i \cap W_{x_i}) \cup (\overline{A_i} \cap \overline{W_{x_i}}) \right)$$

- (3) There exists a s -tuple of functions (F_0, \dots, F_{s-1}) , where $F_i : \omega^s \rightarrow \omega$, $0 \leq i < s$, such that
 - (a) $(F_0, \dots, F_{s-1}) \leq_m (A_0, \dots, A_{s-1})$,
 - (b) $W_{F_i(\bar{x})} \not\leq_l W_{x_i}$ for all $0 \leq i < s$.
- (4) There exists a s -tuple of functions (F_0, \dots, F_{s-1}) , where $F_i : \omega^s \rightarrow \omega$, $0 \leq i < s$, such that
 - (a) $(F_0, \dots, F_{s-1}) \leq_m (A_0, \dots, A_{s-1})$,
 - (b) $\varphi_{F_i(\bar{x})} \not\leq_l \varphi_{x_i}$ for all $0 \leq i < s$.

Proof. 1) \leftrightarrow 2). For $l = 1$ the equivalents of the statements can be found in [4]. The existence of m -complete s -tuple of computably enumerable sets has been also established there. For $l = 2$ we only need a relativisation to the oracle z_2 which also could be found in [4].

1) \rightarrow 3). Without loss of generality we assume $s = 3$.

Case 1: $l=1$. First we take the following m -complete 3-tuple:

$$\begin{aligned} X_0 &= \{n \mid \varphi_n(0) \downarrow\}, \\ X_1 &= \{n \mid \varphi_n(1) \downarrow\}, \\ X_2 &= \{n \mid \varphi_n(2) \downarrow\}. \end{aligned}$$

Further on by Lemma 2 the considerations below will hold for any m -complete 3-tuple. By Graph theorem [15] we construct a computable sequence $B = \{B_{x_0 x_1 x_2}\}_{x_0 x_1 x_2 \in \omega}$ of partial computable functions such that

$$\begin{aligned} B_{x_0 x_1 x_2}(i) &= \varphi_{x_i}(i) \text{ if } i = 0, 1, 2 \\ B_{x_0 x_1 x_2}(k) &\uparrow \text{ if } k > 2. \end{aligned}$$

Then there exists a computable function $h : \omega^3 \rightarrow \omega$ such that $B_{x_0 x_1 x_2} = \varphi_{h(x_0, x_1, x_2)}$. By definition of B we have $h(x_0, x_1, x_2) \in X_i \leftrightarrow x_i \in X_i$ for $i \leq 2$. To finish the construction of F_i we take a_i and b_i for $i \leq 2$ as follows.

a_0 is an index of \perp ;
 b_0 is an index of the function $\{ \langle 0, 0 \rangle \}$;
 a_1 is an index of \perp ;
 b_1 is an index of the function $\{ \langle 1, 0 \rangle \}$;
 a_2 is an index of \perp ;
 b_2 is an index of the function $\{ \langle 2, 0 \rangle \}$;

We define for $i \leq 2$

$$F_i(\bar{x}) = \begin{cases} a_i & \text{if } h(\bar{x}) \in X_i \\ b_i & \text{if } h(\bar{x}) \notin X_i. \end{cases}$$

By construction (F_0, F_1, F_2) is m-reducible to (X_0, X_1, X_2) . Let us show that $W_{F_i(\bar{x})} \neq W_{x_i}$. Fix i . Assume $h(\bar{x}) \in X_i$. Since $F_i(\bar{x}) = a_i$, $W_{F_i(\bar{x})} = \emptyset$. At the same time $W_{x_i} \neq \emptyset$ since $x_i \in X_i$ and $\varphi_{x_i}(i) \downarrow$. Assume $h(\bar{x}) \notin X_i$. Since $F_i(\bar{x}) = b_i$, $\varphi_{b_i}(i) \downarrow$ and $i \in W_{F_i(\bar{x})} \neq \emptyset$. At the same time $\varphi_{x_i}(i) \uparrow$, i.e., $i \notin W_{x_i}$.

Therefore (F_0, F_1, F_2) is a required 3-tuple.

Case 2: 1=2. First we take the following 3-tuple:

$$\begin{aligned}
 Z_0 &= \{n \mid W_n \cap 3\omega \text{ is finite}\}, \\
 Z_1 &= \{n \mid W_n \cap (3\omega + 1) \text{ is finite}\}, \\
 Z_2 &= \{n \mid W_n \cap (3\omega + 2) \text{ is finite}\}.
 \end{aligned}$$

By analogy to the case 1 we chose $h : \omega^3 \rightarrow \omega$ such that for all $\bar{x} = (x_0, x_1, x_2)$ and every $k \in \omega$ we have $\varphi_{h(\bar{x})}(3k+1) = \varphi_{x_i}(3k+1)$ for $i \leq 2$. To finish the construction of F_i we take a_i and b_i for $i \leq 2$ as follows.

a_0 is an index of constant zero function;
 b_0 is an index of \perp ;
 a_1 is an index of constant zero function;
 b_1 is an index of \perp ;
 a_2 is an index of constant zero function;
 b_2 is an index of \perp ;

We define for $i \leq 2$

$$F_i(\bar{x}) = \begin{cases} a_i & \text{if } h(\bar{x}) \in Z_i \\ b_i & \text{if } h(\bar{x}) \notin Z_i. \end{cases}$$

By construction (F_0, F_1, F_2) is m-reducible to (Z_0, Z_1, Z_2) . By analogy to the case 1 we have $W_{F_i(\bar{x})} \neq^* W_{x_i}$. Therefore (F_0, F_1, F_2) is a required 3-tuple.

3) \rightarrow 4). The implication is straightforward since $\varphi_{F_i(\bar{x})} \not\sim_l \varphi_{x_i}$ follows from $W_{F_i(\bar{x})} \not\sim_l W_{x_i}$

4) \rightarrow 2). The following construction is uniform for both l . First for $i \leq 2$ we define functions G_i and T_i which are computable with the oracle z_l .

$$\begin{aligned} G_i(\bar{x}, \bar{y}) &= b_i(\bar{y}) \text{ if only } h(\bar{y}) \in W_{x_i}^{z_l} \\ T_i(\bar{x}, \bar{y}) &= a_i(\bar{y}) \text{ if only } h(\bar{y}) \in A_i. \end{aligned}$$

By Reduction principle for function graphs [15] we find a function $E_i(\bar{x}, \bar{y})$ with the following properties:

- $E_i(\bar{x}, \bar{y})$ is computable with the oracle z_l , therefore for some computable function $g_i : \omega^3 \rightarrow \omega$, $E_i(\bar{x}, \bar{y}) = K^{4, z_l}(g_i(\bar{x}), \bar{y})$, where K^{4, z_l} is Kleene universal function for 3-arity functions computable with the oracle z_l .
- If $h(\bar{y}) \in W_{x_i}^{z_l} \setminus A_i$ then $E_i(\bar{x}, \bar{y}) = b_i(\bar{y})$. If $h(\bar{y}) \in A_i \setminus W_{x_i}^{z_l}$ then $E_i(\bar{x}, \bar{y}) = a_i(\bar{y})$.

If $l = 1$ by Smullyan Theorem [17] there exist three computable functions $n_0, n_1, n_2 : \omega^3 \rightarrow \omega$ such that

$$\varphi_{K^{4, z_1}(g_i(\bar{x}), n_0(\bar{g}(\bar{x}), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x}))))} = \varphi_{n_i(\bar{g}(\bar{x}))},$$

If $l = 2$ there exist three computable functions $n_0, n_1, n_2 : \omega^3 \rightarrow \omega$ such that

$$\varphi_{K^{4, z_2}(g_i(\bar{x}), n_0(\bar{g}(\bar{x}), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x}))))} =^* \varphi_{n_i(\bar{g}(\bar{x}))},$$

under the condition that $K^{4, z_2}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x}))) \downarrow$. The last statement requires more details which we show below. It is easy to see that, for $i \leq 2$, $f_i(\bar{z}, \bar{y}) \equiv K^{4, z_2}(z_i, \bar{y})$ is Σ_2^o -function. Therefore by Remark 3 there exist computable functions λ_i such that for $(\bar{z}, \bar{y}) \in \text{dom}(f_i)$

$$\varphi_{f_i(\bar{z}, \bar{y})} =^* \varphi_{\lambda_i(\bar{z}, \bar{y})}.$$

By Smullyan Theorem,

$$\varphi_{\lambda_i(\bar{z}, \bar{n}(\bar{z}))} = \varphi_{n_i(\bar{z})}$$

Therefore

$$\varphi_{f_i(\bar{z}, \bar{n}(\bar{z}))} =^* \varphi_{n_i(\bar{z})}$$

for $(\bar{z}, \bar{n}(\bar{z})) \in \text{dom}(f_i)$. Now we are ready do define $H : \omega^3 \rightarrow \omega$:

$$H(\bar{x}) = h(n_0(\bar{g}(\bar{x})), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x}))).$$

Let us show that $H(\bar{x}) = h(n_0(\bar{g}(\bar{x})), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x})))$ is a required computable function. Assume contrary that for some x_0, x_1 and x_2

$$H(x_0, x_1, x_2) \notin \left((A_i \cap W_{x_i}^{z_l}) \cup (\overline{A_i} \cap \overline{W_{x_i}^{z_l}}) \right)$$

for some $i \in \{0, 1, 2\}$. We have two cases:

- $H(\bar{x}) \in A_i \setminus W_{x_i}^{z_l}$
- $H(\bar{x}) \in W_{x_i}^{z_l} \setminus A_i$.

It is worth noting that in both cases $K^{4, z_l}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x}))) \downarrow$. In the case (a),

$$\begin{aligned} \varphi_{F_i(\bar{n}(\bar{g}(\bar{x})))} &= \varphi_{a_i(\bar{n}(\bar{g}(\bar{x})))} \text{ by the definition of } F_i \\ \varphi_{a_i(\bar{n}(\bar{g}(\bar{x})))} &= \varphi_{K^{4, z_l}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x})))} \text{ by the definition of } g_i \\ \varphi_{K^{4, z_l}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x})))} &= \varphi_{n_i(\bar{g}(\bar{x}))} \text{ by the choice of } \bar{n} \end{aligned}$$

This contradicts the condition on F_i . In the case (b),

$$\begin{aligned}\varphi_{F_i(\bar{n}(\bar{g}(\bar{x})))} &= \varphi_{b_i(\bar{n}(\bar{g}(\bar{x})))} \text{ by the definition of } F_i \\ \varphi_{b_i(\bar{n}(\bar{g}(\bar{x})))} &= \varphi_{K^{4,z_1}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x})))} \text{ by the definition of } g_i \\ \varphi_{K^{4,z_1}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x})))} &= \varphi_{n_i(\bar{g}(\bar{x}))} \text{ by the choice of } \bar{n}\end{aligned}$$

This contradicts the condition on F_i . Therefore H is a required productive function. \square

7. INDEX SETS THAT WE NEED

Let us fix the following index sets:

$$\begin{aligned}E_0 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge ((W_n \text{ is finite}) \vee \varphi_n \cap \omega \times \{0\} =^* \emptyset)\}; \\ E_1 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge ((W_n \text{ is finite}) \vee \varphi_n \cap \omega \times \{1\} =^* \emptyset)\}; \\ E_2 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge ((W_n \text{ is finite}) \vee \varphi_n \cap \omega \times \{2\} =^* \emptyset)\}.\end{aligned}$$

Using Proposition 3 we show that the 3-tuple (E_0, E_1, E_2) is m-complete in the class of 3 -tuples of Σ_2^0 -set.

In order to define (F_0, F_1, F_2) we first take a computable function $h : \omega^3 \rightarrow \omega$ such that

$$\begin{aligned}\varphi_{h(x_0, x_1, x_2)}(3k) &= \begin{cases} 0 & \text{if } \varphi_{x_0}(k) \downarrow \\ \uparrow & \text{otherwise,} \end{cases} \\ \varphi_{h(x_0, x_1, x_2)}(3k+1) &= \begin{cases} 1 & \text{if } \varphi_{x_1}(k) \downarrow \\ \uparrow & \text{otherwise,} \end{cases} \\ \varphi_{h(x_0, x_1, x_2)}(3k+2) &= \begin{cases} 2 & \text{if } \varphi_{x_2}(k) \downarrow \\ \uparrow & \text{otherwise.} \end{cases}\end{aligned}$$

Now we pick up appropriate functions a_i and b_i for $i \leq 2$. The functions b_i is defined by $\varphi_{b_i(\bar{x})}(k) = \varphi_{x_i}(k) + 1$, a_i is an index of id function. Then we define for $i \leq 2$

$$F_i(\bar{x}) = \begin{cases} a_i & \text{if } h(\bar{x}) \in E_i \\ b_i(\bar{x}) & \text{if } h(\bar{x}) \notin E_i. \end{cases}$$

Let us show that $\varphi_{F_i(\bar{x})} \not\sim_2 \varphi_{x_i}$ for $i \leq 2$. Without loss of generality it is sufficient to consider $i = 0$. Assume $h(\bar{x}) \notin E_0$. Then $W_{h(\bar{x})}$ is infinite and infinitely often a value of $\varphi_{h(\bar{x})}$ is zero. By construction, W_{x_0} is infinite. Since for $k \in \omega$,

$$\varphi_{F_0(\bar{x})}(k) = \varphi_{b_0(\bar{x})}(k) = \varphi_{x_0}(k) + 1,$$

there exist infinitely many $k \in \omega$ such that $\varphi_{F_0(\bar{x})}(k) \neq \varphi_{x_0}(k)$. As a corollary, $\varphi_{F_0(\bar{x})} \neq^* \varphi_{x_0}$. Assume $h(\bar{x}) \in E_0$. Then φ_{x_0} is a finite function. Since for $k \in \omega$,

$$\varphi_{F_0(\bar{x})}(k) = \varphi_{a_0(\bar{x})}(k) = k,$$

we have $\varphi_{F_0(\bar{x})} \neq^* \varphi_{x_0}$.

Lemma 3. *Let a 3-tuple (E_1, E_1, E_2) be m -complete in the class of 3-tuples of Σ_2^0 -set. Then the following 3-tuple*

$$\begin{aligned} Y_0 &= E_1 \cap E_1 \cap E_2 \\ Y_1 &= E_1 \cap E_2 \\ Y_2 &= E_2 \end{aligned}$$

is m -complete in the class of 3-tuples (X_0, X_1, X_2) of Σ_2^0 -set with the additional condition $X_0 \subseteq X_1 \subseteq X_2$.

By the choice of (E_0, E_1, E_2) at the beginning of this section we have $Y_0 \subseteq Y_1 \subseteq Y_2$, where

$$\begin{aligned} Y_0 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge W_n \text{ is finite}\}; \\ Y_1 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge (W_n \text{ is finite} \vee (\exists N)(\forall k \geq N)(\varphi_n(k) \downarrow \rightarrow \varphi_n(k) = 0))\}; \\ Y_2 &= \{n \mid \pi_n \subseteq \{0, 1, 2\} \wedge (W_n \text{ is finite} \vee \varphi_n \cap \omega \times \{2\} =^* \emptyset)\}. \end{aligned}$$

Remark 4. *It is worth noting that if 3-tuple (Z_0, Z_1, Z_2) is m -complete in the class of 3-tuple (X_0, X_1, X_2) of Σ_2^0 -sets with the additional condition $X_0 \subseteq X_1 \subseteq X_2$ then the set $Z_0 \cup (Z_2 \setminus Z_1) \notin \Sigma_2^0$ since this combination is the m -greatest among the similar combinations of Σ_2^0 -sets.*

8. CLASSES K WITHOUT COMPUTABLY PRESENTABLE \tilde{K}

Below we list important restrictions on a class K of total computable numerical functions:

- (1) Among the basic functions BF the class contains $+$, \div , \cdot , $[\frac{x}{2}]$ and $[\sqrt{x}]$.
- (2) The class has a computable universal function for all unary functions i.e. the sequence $\{\mathcal{F}_n\}_{n \in \omega}$ of all unary functions from K is computable.
- (3) There exists a computable function $H : \omega \rightarrow \omega$ such that for all $n \in \omega$ $\text{im}(\mathcal{F}_{H(n)}) = \{0\} \cup \{x + 1 \mid W_n\}$. Moreover, for all $i \in W_n$, $\mathcal{F}_{H(n)}^{-1}(i + 1)$ is an infinite set.
- (4) The class is closed under superposition and either under the standard bounded recursion scheme or the following recursion scheme 2: If $\alpha, g, \psi \in K$ and f is defined by

$$f(\bar{x}, y) = \begin{cases} \alpha(\bar{x}) & \text{if } y = 0 \\ \psi(\bar{x}, y, f(\bar{x}, [\frac{y}{2}])) & \text{if } y \geq 1 \end{cases}$$

and

$$f(\bar{x}, y) \leq g(\bar{x}, y)$$

then $f \in K$.

Remark 5. *It is worth noting that the class K satisfying the restrictions above contains all almost constant functions, Cantor 3-tuple (c, l, r) , $\text{sg}|x - y|$.*

Theorem 3. *Let K satisfy the requirements above and \tilde{K} be a structure generated by K . Then \tilde{K} does not have a computable presentation.*

The proof follows from Theorem 2 and the claim $A_0 \cup (A_2 \setminus A_1) \notin \Sigma_2^0$ which is based on the following proposition and Section 7.

Proposition 4. $(Y_0, Y_1, Y_2) \leq_m (A_0, A_1, A_2)$.

Proof. We are going to construct a computable function f such that $n \in Y_i \leftrightarrow f(n) \in A_i$. In order to do that we will construct a computable sequence $\{F_n\}_{n \in \omega}$ of computable functions by steps and then effectively find a required reduction f .

We take

- a standard computable reduction function $\alpha : \omega \rightarrow \omega$ for $\text{Fin} \leq_m \omega \setminus \text{Tot}$ (c.f. [16]) with the following properties:
 - if W_n is finite then $W_{\alpha(n)}$ is finite;
 - if W_n is infinite then $\varphi_{\alpha(n)}$ is total;
 - $\pi_n = \pi_{\alpha(n)}$;
 - If $(\exists^\infty a) \varphi_n(a) = x$ then $(\exists^\infty b) \varphi_{\alpha(n)}(b) = x$ and vice versa.
- a computable function t such that
 - $W_{t(n)} = \{ \langle k, d \rangle \mid \varphi_{\alpha(n)}(k) = d \}$.

Now we point out the requirements on a step s which we want to meet in our construction:

- $F_n^{s+1} \supseteq F_n^s$
- $\text{dom}(F_n^s) = [0, \dots, m_n^s]$ is a proper initial segment of ω .
- If $\mathcal{F}_{H(t_s(n))}(s) = \langle c, d \rangle + 1 \wedge d > 2$ then $(\exists j) F_n(j) > 2$.
- If $\mathcal{F}_{H(t_s(n))}(s+1) = \langle i, 2 \rangle + 1$ then in the process of the construction we provide the following: $F_n \notin \{\mathcal{F}_0, \dots, \mathcal{F}_i\}$.
- $F_n = \bigcup_{s \in \omega} F_n^s$.

W.l.o.g. we assume now that K is closed under the recursion scheme 2 since the case when K is closed under the primitive recursion is much more easy so it is left to a reader.

Description of a construction of m_n^s , $\{F_n^s\}_{n, s \in \omega}$, $t_s(n)$ and I_n^s :

Step 0

$$m_0^0 = [0], F_n^0(0) = 0, t_0(n) = t(n) \text{ and } I_n^0 = 0.$$

Step s+1

Case 1 If $\mathcal{F}_{H(t_s(n))}(s+1) = \langle i, 0 \rangle + 1$ or $\mathcal{F}_{H(t_s(n))}(s+1) = 0$ then we proceed as follows:

$$m_n^{s+1} = \begin{cases} m_n^s & \text{if } m_n^s > 0 \\ 1 & \text{if } m_n^s = 0. \end{cases}$$

and for all $j \leq m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$, for all $m_n^s \leq j \leq m_n^{s+1}$ we put $F_n^{s+1}(j) = F_n^s(m_n^s)$, $t_{s+1}(n) = t_s(n)$ and $I_n^{s+1} = I_n^s$.

Case 2 $\mathcal{F}_{H(t_s(n))}(s+1) = \langle i, 1 \rangle + 1$ we consider the following subcases:

Subcase 2.1 $i > s+1$. We proceed as in Case 1.

Subcase 2.2 $i \leq s+1$ and $i \leq I_n^s$. We proceed as in Case 1.

Subcase 2.3 $I_n^s < i \leq s+1$ We proceed as follows:

$$m_n^{s+1} = \begin{cases} m_n^s & \text{if } m_n^s > 0 \\ 1 & \text{if } m_n^s = 0. \end{cases}$$

and for all $j \leq m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$, for all $m_n^s \leq j < m_n^{s+1}$ we put $F_n^{s+1}(j) = F_n^s(m_n^s)$ and for $F_n^{s+1}(m_n^{s+1})$ we chose the least value from

from $\{0, 1\}$ such that $F_n^{s+1}(m_n^{s+1}) \neq F_n^s(m_n^s)$, $t_{s+1}(n) = t_s(n)$ and $I_n^{s+1} = i$.

Case 3 If $\mathcal{F}_{H(t_s(n))}(s+1) = \langle i, 2 \rangle + 1$ then for all $j \leq m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$ and we proceed as follows: $m_n^{s+1} = m_n^s + i + 1$ and for $k \leq i$ for we chose the least value from for $F_n^{s+1}(m_n^s + k + 1)$ from $\{0, 1\}$ such that $F_n^{s+1}(m_n^s + k + 1) \neq \mathcal{F}_k(m_n^s + k + 1)$. The equality $W_{t_{s+1}(n)} = W_{t_s(n)} \setminus \{\langle i, 2 \rangle\}$ defines the value of $t_{s+1}(n)$ and $I_n^{s+1} = I_n^s$.

Case 4 If $\mathcal{F}_{H(t_s(n))}(s+1) = \langle i, d \rangle + 1$ and $d > 2$ then for all $j \leq m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$ and we proceed as follows: $m_n^{s+1} = m_n^s + 1$, $F_n^{s+1}(m_n^{s+1}) = d$, $t_{s+1}(n) = t_s(n)$ and $I_n^{s+1} = I_n^s$.

We put $F_n = \bigcup_{s \in \omega} F_n^s$ and effectively find computable function $f : \omega \rightarrow \omega$ such that $\varphi_{f(n)} = (F_n \cap (W_{\alpha(n)} \times \omega)) \cup \{\langle x, d \rangle \mid F_n(x) = d \wedge d > 2\}$.

Now we show that f is a required reduction.

If $n \in Y_0$ then $W_{\alpha(n)}$ is finite, so is $\varphi_{f(n)}$ and $f(n) \in A_0$. If $n \notin Y_0$ then there are two cases:

- 1) $W_{\alpha(n)}$ is ω , by construction, $\varphi_{f(n)} = F_n$ and F_n is total, so $f(n) \notin A_0$.
- 2) $\pi_n \not\subseteq \{0, 1, 2\}$. Then for some $j \in \omega$ $F_n(j) > 2$, so $\pi_{f(n)} \not\subseteq \{0, 1, 2\}$. Again $f(n) \notin A_0$. **So we have** $f^{-1}(A_0) = Y_0$.

If $n \in Y_1 \setminus Y_0$ then $\varphi_{\alpha(n)}$ is total and $\varphi_{\alpha(n)} =^* 0$. In this case $\varphi_{f(n)} = F_n$ and by construction $F_n \in \text{AC}$ since $(\exists s_1 \in \omega)(\forall s \geq s_1) I_s^n = I_{s_1}^n$. So $\varphi_{f(n)} = F_n$ and $f(n) \in A_1 \setminus A_0$.

If $n \notin Y_2$ then $\varphi_{\alpha(n)}$ is total and $\varphi_{f(n)} = F_n$. By construction the case $\mathcal{F}_{H(t_s(n))}(s+1) = \langle i, 2 \rangle + 1$ arises infinitely often and the collection of numbers i is infinite too. So, for infinitely many i , $F_n \notin \{\mathcal{F}_0, \dots, \mathcal{F}_i\}$. Hence $F_n \notin K$. That means $f(n) \notin A_2$.

If $n \in Y_2 \setminus Y_1$ then $\varphi_{\alpha(n)} = F_n$ is total. By the choice of n , $(\exists N)(\forall i \geq N) \varphi_{\alpha(n)}(i) \neq 2$. Hence $(\exists^\infty i) \varphi_{\alpha(n)}(i) = 1$.

Let us note that after some step s_0 for all i we have $\langle i, 2 \rangle \notin W_{t_s(n)}$ for $s \geq s_0$. We define $t_\infty(n) = t_{s_0}(n)$. It is easy to see that, for $s \geq s_0$, $t_\infty(n) = t_s(n)$. On the step $s+1$, when $\mathcal{F}_{H(t_\infty(n))}(s+1) = \langle i, 1 \rangle + 1$, $F_n^{s+1}(m_n^{s+1}) \neq F_n^{s+1}(m_n^{s+1} - 1)$. Hence $F_n \notin \text{AC}$, so $f(n) \notin A_1$. **So we have** $f^{-1}(A_1) = Y_1$.

Using s_0 and N from above we explain that $F_n \in K$.

Let $m_n = m_n^{s_0+1}$. It is easy to see that the following functions belongs to K :

- the characteristic function of the set $A = \{m_n \cdot 2^i \mid i \geq 0\}$;
- the function $g(x)$, that computes $\max\{y \in A \mid y \leq x\}$ for $x \geq m_n$ and for $x < m_n$ it is equal to 0;
- the function $S(x) = \mu(s')(F_n^{s'}(x) \downarrow)$.

In order to meet our goal we construct the function $I(x) = I_n^{S(x)-1}$ by the following rules:

Assume $I_0 = I_n^{s_0}$. Then we define

- for $x < m_n$, $I(x) = 0$,
- for $x = m_n$, $I(x) = I_0$,
- for $x > m_n$ and $g(x) > m_n$,

$$I(x) = \begin{cases} l(\mathcal{F}_{H(t_\infty(n))}(S(x) - 1) - 1) = i & \text{if} \\ I(\lfloor \frac{x}{2} \rfloor) \leq i \leq S(x) - 1 \wedge r(\mathcal{F}_{H(t_\infty(n))}(S(x) - 1) - 1) = 1 & \\ I(\lfloor \frac{x}{2} \rfloor) \text{ otherwise} . & \end{cases}$$

for $x > m_n$ and $g(x) = m_n$, $I(x) = I_0$.

From above we can see that $\lambda x. I(x) \in K$.

We can assume that $x \geq N$ and $S(x) > s_0$.

Suppose $x \in A$.

If $\mathcal{F}_{H(t_\infty(n))}(S(x)) = \langle i, 0 \rangle + 1$ then $F_n(x) = F_n(\lfloor \frac{x}{2} \rfloor)$. The same is done if $\mathcal{F}_{H(t_\infty(n))}(S(x)) = \langle i, 1 \rangle + 1$ but $i \leq I(x)$ or $i > S(x)$.

Otherwise, i.e., if $\mathcal{F}_{H(t_\infty(n))}(S(x)) = \langle i, 1 \rangle + 1$ and $I(x) \leq i \leq S(x)$ then for the value of $F_n(x)$ we chose the first one from $\{0, 1\}$ which differs from $F_n(\lfloor \frac{x}{2} \rfloor)$.

Suppose $x \notin A$.

If $\mathcal{F}_{H(t_\infty(n))}(S(\lfloor \frac{x}{2} \rfloor + 1)) = \langle i, 0 \rangle + 1$ or 0 then $F_n(x) = F_n(\lfloor \frac{x}{2} \rfloor)$. The same is done if $\mathcal{F}_{H(t_\infty(n))}(S(\lfloor \frac{x}{2} \rfloor + 1)) = \langle i, 0 \rangle + 1$ but $i \leq I(\lfloor \frac{x}{2} \rfloor + 1)$ or $i > S(\lfloor \frac{x}{2} \rfloor + 1)$. Otherwise, i.e., if $\mathcal{F}_{H(t_\infty(n))}(S(\lfloor \frac{x}{2} \rfloor)) = \langle i, 1 \rangle + 1$ and $I(\lfloor \frac{x}{2} \rfloor) \leq i \leq S(\lfloor \frac{x}{2} \rfloor)$ then for the value of $F_n(x)$ we chose the first one from $\{0, 1\}$ which differs from $F_n(\lfloor \frac{x}{2} \rfloor)$. Therefore the scheme above shows that $F_n \in K$. **So we have** $f^{-1}(A_2) = Y_2$. \square

Corollary 1. *The structures of P-numbers, SP-numbers and \mathcal{E}_n -numbers, $n \geq 2$ do not have computable copies.*

Corollary 2. *The fields of P-numbers and \mathcal{E}_n -numbers, $n \geq 3$ do not have computable copies.*

The claim follows from real closeness of the corresponding fields.

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