A MIXED PROBLEM FOR THE THIRD EQUATION ORDER WITH BRIEF CHARACTERISTICS

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ABSTRACT. The unique solvability of the mixed problem in a rectangular domain is proven for a third-order equation with multiple characteristics containing the second derivative of time.

Keywords: third-order equations with multiple characteristics, mixed problem, uniqueness, existence, series, regular convergence.

1. INTRODUCTION

In working [1], taking into account the properties of viscosity and thermal conductivity of a gas, a third-order equation with multiple characteristics containing second time derivatives was obtained from the Navier-Stokes system

$$u_{xxx} + u_{yy} - \frac{\nu}{y}u_y = u_x u_{xx}, \quad \nu = const. \tag{1}$$

Equation (1) under $\nu = 1$ describes an axisymmetric flow, and under $\nu = 0$ describes a plane-parallel flow [2].

Third-order partial differential equations are encountered in the solution of problems of the theory of nonlinear acoustics, hydrodynamic theory of cosmic plasmas, and filtration of liquids in porous media. In the collection of all third-order equations, an important place is occupied by so-called equations with multiple characteristics. The first results on a third-order equation with multiple characteristics were obtained by H. Block [3], E. Del Vecchio [4]. Catabriga [5] constructed a fundamental solution of the equation $D_x^{2n+1}u - D_y^2 u = 0$ in the form of a double improper integral and studied the properties of the potential. In [6, 7], the fundamental solutions expressed via hypergeometric functions were constructed, their properties were studied, and

the estimates as $|t| \to \infty$ were established. We also mention the works [8–19] in which boundary-value problems for third-order equations were considered. In the present paper, we study the third-order boundary-value problem for a third-order equation with multiple characteristics in a rectangular domain. Consider an equation

$$U_{xxx} - U_{yy} + AU_{xx} + BU_x + CU_y + DU = 0$$

where

$$A, B, C, D \in R$$

note that by replacing variables

$$U(x, y) = u(x, y) e^{-\frac{A}{3}x + \frac{C}{2}y}$$

this equation is reduced to the form:

$$u_{xxx} - u_{yy} + au_x + cu = 0, \tag{2}$$

where

$$a = -\frac{A^2}{3} + B, \qquad c = \frac{2A^3}{27} + \frac{C^2}{4} - \frac{AB}{3} + D.$$

In what follows, we assume that $c > 0$, $a > 0$.

In the domain $D = \{(x, y) : 0 < x, y < 1\}$ we consider the following problem:

Problem B_3. In the domain D, to find the solution of Eq. (2) from the class $C_{x,y}^{3,2}(D) \cap C_{x,y}^{2,1}(\overline{D})$, satisfying the boundary condition:

$$\begin{cases} \alpha u(x, 0) + \beta u_y(x, 0) = 0 \\ \gamma u(x, 1) + \delta u_y(x, 1) = 0 \end{cases}, \qquad 0 \leqslant x \leqslant 1 \tag{3}$$

$$u(0, y) = \varphi_1(y), \quad u_x(1, y) = \varphi_2(y), \quad u_{xx}(1, y) = \varphi_3(y) \tag{4}$$

where

$\varphi_i(y) \in C^3[0, 1]$, $i = 1, 3$, are given functions and, moreover:

$$\beta \varphi_i'(0) = 0, \qquad \gamma \varphi_i(1) + \delta \varphi_i'(1) = 0, \qquad \varphi''(0) = \varphi''(1) = 0.$$

Here $\alpha, \beta, \gamma, \delta$ - are constants, and $\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$.

Note that, similar problems were investigated in [20] for $a = c = 0$.

2. Uniqueness of results

Theorem 1. If $\alpha\beta \leqslant 0$, $\delta\gamma \geqslant 0$, then problem B_3 is uniquely solvable.

Proof. Assume the contrary, let $u_1(x, y)$ and $u_2(x, y)$ be solutions of B_3. Then $u(x, y) = u_1(x, y) - u_2(x, y)$ is a solution of homogenous problem B_3 in domain D. Consider the identity

$$u(u_{xxx} - u_{yy} + au_x + cu) = 0,$$

or

$$\frac{\partial}{\partial x}(uu_{xx}) - \frac{1}{2}\frac{\partial}{\partial x}(u_x^2) - \frac{\partial}{\partial y}(uu_y) + u_y^2 + \frac{a}{2}\frac{\partial}{\partial x}(u^2) + cu^2 = 0.$$

Integrating this identity over the domain D, we get

$$\iint\limits_D \frac{\partial}{\partial x}(uu_{xx})\,dxdy - \frac{1}{2}\iint\limits_D \frac{\partial}{\partial x}(u_x^2)\,dxdy - \iint\limits_D \frac{\partial}{\partial y}(uu_y)\,dxdy + \iint\limits_D u_y^2\,dxdy +$$

$$+ \frac{a}{2}\iint\limits_D \frac{\partial}{\partial x}(u^2)\,dxdy + c\iint\limits_D u^2\,dxdy = 0.$$

Using the Green formula, we obtain

$$\int\limits_0^1 \left[u\left(1,y\right) u_{xx}\left(1,y\right) - u\left(0,y\right) u_{xx}\left(0,y\right) \right] dy - \frac{1}{2} \int\limits_0^1 \left[u_x^2\left(1,y\right) - u_x^2\left(0,y\right) \right] dy -$$

$$- \int\limits_0^1 \left[u\left(x,1\right) u_y\left(x,1\right) - u\left(x,0\right) u_y\left(x,0\right) \right] dx + \iint\limits_D u_y^2\left(x,y\right) dxdy +$$

$$+ \frac{a}{2} \int\limits_0^1 \left[u^2\left(1,y\right) - u^2\left(0,y\right) \right] dy + c \iint\limits_D u^2\left(x,y\right) dxdy = 0.$$

In view of the boundary conditions (3), under the assumption that $\alpha \neq 0, \ \gamma \neq 0$ we obtain

$$\frac{\delta}{\gamma} \int\limits_0^1 u_y^2\left(x,1\right) dx - \frac{\beta}{\alpha} \int\limits_0^1 u_y^2\left(x,0\right) dx + \frac{a}{2} \int\limits_0^1 u^2\left(1,y\right) dy +$$

$$+ \frac{1}{2} \int\limits_0^1 u_x^2\left(0,y\right) dy + \iint\limits_D u_y^2\left(x,y\right) dxdy + c \iint\limits_D u^2\left(x,y\right) dxdy = 0.$$

Takin into account the conditions of theorem and $a > 0, \ c > 0$, we obtain $u\left(x,y\right) \equiv 0$, in $\left(x,y\right) \in \overline{D}$.
In the cases $\beta \neq 0, \ \delta \neq 0; \quad \alpha \neq 0, \ \delta \neq 0; \ \ \gamma \neq 0, \ \beta \neq 0$ in a similar way, we obtain the equality $u\left(x,y\right) \equiv 0$, in $\left(x,y\right) \in \overline{D}$. $\qquad\qquad\square$

3. Solution existence

We seek the solution of the problem by the method of separation of variables:

$$u(x,y) = X(x)Y(y). \tag{5}$$

Thus, equation (2) implies that

$$\frac{X''' + aX'}{X} - \frac{Y''}{Y} = -c, \qquad -c = -v + \mu.$$

This yields

$$X''' + aX' + \nu X = 0, \tag{6}$$

$$Y'' + \mu Y = 0. \tag{7}$$

To determine we obtain problems of the Sturm-Louvilly type

$$\begin{cases} Y'' + \mu Y = 0, \\ \alpha Y(0) + \beta Y'(0) = 0, \\ \gamma Y(1) + \delta Y'(1) = 0. \end{cases} \tag{8}$$

Problem (8) has eigenvalues [20] in the form of positive roots of the transcendental

$$ctg\sqrt{\mu} = \frac{\alpha\gamma + \delta\beta\mu}{\sqrt{\mu}\left(\gamma\beta - \alpha\delta\right)} \tag{9}$$

Which implies that $\sqrt{\mu_n} = \pi n + \varepsilon_n$, where $\lim\limits_{n\to\infty} \varepsilon_n = 0$, or $\mu_n = O\left(n^2\right), \ \ n \to \infty$.
The corresponding eigenfunctions take the form

$$Y_n(y) = \left(\alpha \sin \sqrt{\mu_n}y - \beta\sqrt{\mu_n} \cos \sqrt{\mu_n}y\right). \tag{10}$$

The orthogonality of the system of functions (10) is proved as in [20].
We now find the norm of the eigenfunctions $Y_n(y)$ in $L_2[0,1]$:

$$\|Y_n(y)\|^2 = \int\limits_0^1 Y_n^2(y)dy = \frac{1}{2}\left(\alpha^2 + \beta^2\mu_n - \alpha\beta\right) +$$

$$+\left(\frac{\beta^2\sqrt{\mu_n}}{4} - \frac{\alpha^2}{4\sqrt{\mu_n}}\right)\sin 2\sqrt{\mu_n} + \frac{\alpha\beta}{2}\cos 2\sqrt{\mu_n} = \frac{1}{2}\beta^2\mu_n\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{\mu_n}}\right)\right). \quad (11)$$

The characteristic equation (7) takes the form:

$$m^3 + am + \nu = 0 \qquad (12)$$

Equation (12) has the roots [21]:

$$m_1 = -2\tau, \qquad m_{2,3} = \tau \pm i\rho, \qquad \tau = -\frac{1}{2}(u+v), \qquad \rho = \frac{\sqrt{3}}{2}(u-v),$$

where

$$u = \sqrt[3]{-\frac{\nu}{2} + \sqrt{\frac{\nu^2}{4} + \frac{a^3}{27}}}, \qquad v = \sqrt[3]{-\frac{\nu}{2} - \sqrt{\frac{\nu^2}{4} + \frac{a^3}{27}}},$$

since $a > 0$, $\nu > 0$ то $\frac{\nu^2}{4} + \frac{a^3}{27} > 0$.
We write the general solution of Eq. (6) in the form

$$X_n(x) = C_{1n}e^{-2\tau_n x} + e^{\tau_n x}\left(C_{2n}\cos\rho_n x + C_{3n}\sin\rho_n x\right),$$

where are arbitrary constants. According to (5), the functions

$$u_n(x,y) = \left[C_{1n}e^{-2\tau_n x} + e^{\tau_n x}\left(C_{2n}\cos\rho_n x + C_{3n}\sin\rho_n x\right)\right]Y_n(y),$$

are particular solutions of Eq. (2) satisfying conditions (3).
In view of the linearity and homogeneity of Eq. (2), the sum of particular solutions
is also a particular solution
  of Eq. (2). This is also true for the series

$$u(x,y) = \sum_{n=1}^{\infty}\left[C_{1n}e^{-2\tau_n x} + C_{2n}e^{\tau_n x}\cos\rho_n x + C_{3n}e^{\tau_n x}\sin\rho_n x\right]Y_n(y). \qquad (13)$$

In view of the fact that the function u(x, y) satisfies the boundary conditions (4),
we arrive at the system of Equations

$$\begin{cases} A_{1n} = C_{1n} + C_{2n}, \\ A_{2n} = -2\tau_n e^{-2\tau_n}C_{1n} + e^{\tau_n}\left(\tau_n cos\rho_n - \rho_n \sin\rho_n\right)C_{2n} + e^{\tau_n}\left(\tau_n \sin\rho_n + \rho_n cos\rho_n\right)C_{3n}, \\ A_{3n} = 4\tau_n^2 e^{-2\tau_n}C_{1n} + e^{\tau_n}\left[\left(\tau_n^2 - \rho_n^2\right)cos\rho_n - 2\tau_n\rho_n \sin\rho_n\right]C_{2n} + \\ \qquad\qquad\qquad + e^{\tau_n}\left[\left(\tau_n^2 - \rho_n^2\right)\sin\rho_n + 2\tau_n\rho_n cos\rho_n\right]C_{3n}, \end{cases}$$
$$(14)$$

where

$$A_{in} = \frac{1}{\|Y_n\|^2}\int\limits_0^1 \varphi_i(\eta)Y_n(\eta)\,d\eta, \qquad i = 1,2,3.$$

This system has the determinant

$$\Delta = \tau_n e^{2\tau_n}\left\{\left(\frac{1}{2}\tau_n^2 - \frac{3}{2}\rho_n^2\right)\sin 2\rho_n + \rho_n\left(\frac{3}{2}\tau_n - \rho_n\right)\cos 2\rho_n + \right.$$

$$+2e^{-3\tau_n} \left[\left(\tau_n^2 - \rho_n^2\right) \sin \rho_n + 2\tau_n\rho_n \cos \rho_n\right]\right\},$$

Assume that $\Delta \neq 0$. We prove the following lemma:

Lemma 1. *The boundary-value problem*

$$\begin{cases} X''' + aX' + \nu X = 0, \\ X(0) = X'(1) = X''(1) = 0, \end{cases} \tag{16}$$

possesses solely the trivial solution.

Proof. Assume the contrary, let $X(x) \neq 0$.

Consider the identity

$$X(X''' + aX' + \nu X) = 0,$$

or

$$\left(XX'' - \frac{1}{2}(X')^2 + \frac{a}{2}X^2\right)' + \nu X^2 = 0,$$

Integrating over the domain $(0 < x < 1)$, we get

$$\int_0^1 \left(XX'' - \frac{1}{2}(X')^2 + \frac{a}{2}X^2\right)' dx + \nu \int_0^1 X^2 dx = 0,$$

$$X(1)X''(1) - X(0)X''(0) - \frac{1}{2}(X'(1))^2 + \frac{1}{2}(X'(0))^2 +$$

$$+\frac{a}{2}X^2(1) - \frac{a}{2}X^2(0) + \nu \int_0^1 X^2 dx = 0.$$

Given the boundary conditions, we obtain

$$\frac{1}{2}(X'(0))^2 + \frac{a}{2}X^2(1) + \nu \int_0^1 X^2 dx = 0,$$

since $a > 0$, $\nu > 0$, then $X(x) \equiv 0$. $\square$

Assume that $\Delta = 0$, then there exist constants C_1^*, C_2^*, C_3^* that are not simultaneously equal to zero and satisfy the following system of equations:

$$\begin{cases} 0 = C_1^* + C_2^*, \\ 0 = -2\tau_n e^{-2\tau_n}C_1^* + e^{\tau_n}(\tau_n cos\rho_n - \rho_n \sin \rho_n)C_2^* + e^{\tau_n}(\tau_n \sin \rho_n + \rho_n cos\rho_n)C_3^*, \\ 0 = 4\tau_n^2 e^{-2\tau_n}C_1^* + e^{\tau_n}\left[\left(\tau_n^2 - \rho_n^2\right)cos\rho_n - 2\tau_n\rho_n \sin \rho_n\right]C_2^* + \\ \qquad\qquad\qquad\qquad +e^{\tau_n}\left[\left(\tau_n^2 - \rho_n^2\right)\sin \rho_n + 2\tau_n\rho_n cos\rho_n\right]C_3^*, \end{cases}$$

This means that the function

$$V_n(x) = C_1^* e^{-2\tau_n x} + e^{\tau_n x}(C_2^* \cos \rho_n x + C_3^* \sin \rho_n x),$$

is a solution of the boundary-value problem (16), however, by the proved lemma:

$$C_1^* e^{-2\tau_n x} + C_2^* e^{\tau_n x} \cos \rho_n x + C_3^* e^{\tau_n x} \sin \rho_n x \equiv 0,$$

which is impossible in view of the linear independence of the functions

$$e^{-2\tau_n x}, \quad e^{\tau_n x} \cos \rho_n x, \quad e^{\tau_n x} \sin \rho_n x.$$

So $\Delta \neq 0$.

The system of equations (14) possesses a unique solution of the form

$$C_{1n} = \frac{1}{\Delta} \left[e^{2\tau_n} \left(2\tau_n^2 \rho_n - \rho_n^3 \right) A_{1n} - e^{\tau_n} \left[\left(\tau_n^2 - \rho_n^2 \right) \sin \rho_n + 2\tau_n \rho_n cos\rho_n \right] A_{2n} + \right.$$

$$\left. + e^{\tau_n} \left(\tau_n \sin \rho_n + \lambda_n cos\rho_n \right) A_{3n} \right],$$

$$C_{2n} = \frac{1}{\Delta} \left[2\tau_n e^{-\tau_n} \left[\left(3\tau_n^2 - \rho_n^2 \right) \sin \rho_n + 4\tau_n \rho_n \cos \rho_n \right] A_{1n} + \right.$$

$$\left. + e^{\tau_n} \left[\left(\tau_n^2 - \rho_n^2 \right) \sin \rho_n + 2\tau_n \rho_n \cos \rho_n \right] A_{2n} - e^{\tau_n} \left(\tau_n \sin \rho_n + \rho_n cos\rho_n \right) A_{3n} \right],$$

$$C_{3n} = \frac{1}{\Delta} \left[-2\tau_n e^{-\tau_n} \left(\left(3\tau_n^2 - \rho_n^2 \right) \cos \rho_n - 4\tau_n \rho_n \sin \rho_n \right) A_{1n} + \right.$$

$$+ \left(4\tau_n^2 e^{-2\tau_n} - e^{\tau_n} \left[\left(\tau_n^2 - \rho_n^2 \right) cos\rho_n - 2\tau_n \rho_n \sin \rho_n \right] \right) A_{2n} +$$

$$\left. + \left[e^{\tau_n} \left(\tau_n cos\rho_n - \rho_n \sin \rho_n \right) + 2\tau_n e^{-2\tau_n} \right] A_{3n} \right].$$

If series (13) and its derivatives u_{xxx}, u_{yy} are uniformly convergent in both variables in the domain D, then this series is the classical solution of problem B_3. We now prove the uniform convergence of series (13) in the domain D.

By using (13) we get

$$|u(x,y)| \leqslant \sum_{n=1}^{\infty} \left[|C_{1n}| e^{-2\tau_n x} + |C_{2n}| e^{\tau_n x} + |C_{3n}| e^{\tau_n x} \right] \cdot |Y(y)|. \qquad (17)$$

Given that

$$\Delta = \beta e^{2\alpha_n} \bar{\Delta}, \qquad \bar{\Delta} = 1 - e^{-3\alpha_n} \left(\frac{3\alpha_n}{\beta_n} \sin \beta_n + \cos \beta_n \right),$$

we have

$$\lim_{n \to \infty} \bar{\Delta}_n = 1.$$

Estimate the first term

$$|C_{1n}| e^{-2\tau_n x} \leqslant \frac{e^{-2\tau_n x} e^{2\tau_n}}{\tau_n e^{2\tau_n} |\bar{\Delta}|} \left\{ \left| 2\tau_n^2 - \rho_n^3 \right| |A_{1n}| + e^{-\tau_n} \left(\left| \tau_n^2 - \rho_n^2 \right| + 2 |\tau_n \rho_n| \right) |A_{2n}| + \right.$$

$$\left. + e^{-\tau_n} \left(|\tau_n| + |\rho_n| \right) |A_{3n}| \right\} \leqslant \left(|A_{1n}| + |A_{2n}| + |A_{3n}| \right) M_1,$$

similarly

$$|C_{2n}| e^{\tau_n x} \leqslant \left(|A_{1n}| + |A_{2n}| + |A_{3n}| \right) M_2,$$

$$|C_{3n}| e^{\tau_n x} \leqslant \left(|A_{1n}| + |A_{2n}| + |A_{3n}| \right) M_3.$$

Equation (17) yields

$$|u(x,y)| \leqslant \sum_{n=1}^{\infty} \left(|A_{1n} Y_n(y)| + |A_{2n} Y_n(y)| + |A_{3n} Y_n(y)| \right) M_4. \qquad (18)$$

Further, we estimate the quantity $|A_{in} Y_n(y)|$

$$|A_{in} Y_n(y)| \leqslant \frac{|Y_n(y)|}{\|Y_n\|^2} \int_0^1 \varphi_i(\eta) Y_n(\eta) \, d\eta.$$

By using the inequality

$$|Y_n(y)| \leqslant |\alpha \sin \sqrt{\mu_n} y - \beta \sqrt{\mu_n} \cos \sqrt{\mu_n} y| \leqslant |\alpha| + |\beta| \sqrt{\mu_n},$$

we obtain

$$|A_{in}Y_n(y)| \leqslant \frac{\left(|\alpha| + |\beta|\sqrt{\mu_n}\right)^2}{\|Y_n\|^2} \int\limits_0^1 \varphi_i(\eta)d\eta.$$

The expression $\frac{\left(|\alpha| + |\beta|\sqrt{\mu_n}\right)^2}{\|Y_n\|^2}$ is bounded as $n \to \infty$, i.e. as $\mu_n \to \infty$, because it follows from (11) that

$$\frac{\left(|\alpha| + |\beta|\sqrt{\mu_n}\right)^2}{\|Y_n\|^2} = \frac{\alpha^2 + 2|\alpha\beta|\sqrt{\mu_n} + \beta^2\mu_n}{\|Y_n\|^2} \to 2.$$

We conclude that, starting from a certain number n, the following inequality is true:

$$|A_{in}|\,|Y(y)| < \int\limits_0^1 |\varphi_i(\eta)|d\eta, \qquad i = \overline{1,3},\ where \quad B > 2.$$

Integrating by parts and taking into account condition (15), we arrive at the estimates

$$|\varphi_i| \leqslant N_i \frac{|\varphi_{in}{}'''|}{n^3}, \quad i = \overline{1,3},$$

where

$$\varphi_{in}''' = \frac{1}{\|Y_n\|^2} \int\limits_0^l \varphi'''_{in}(\eta)Y_n(\eta)\,d\eta, \quad i = \overline{1,3}.$$

Then

$$|A_{in}Y_n| \leqslant 2N_i B \frac{|\varphi_{in}{}'''|}{n^3}, \quad i = \overline{1,3}. \tag{19}$$

By using (19), we rewrite estimate (18) in the form

$$|u(x,y)| \leqslant BM_4 N_i \sum_{n=1}^{\infty} \frac{|\varphi'''_{in}|}{n^3} < \infty, \quad i = \overline{1,3}.$$

This implies that series (13) is absolutely and uniformly convergent. We prove the uniform convergence of $\frac{\partial^2 u}{\partial y^2}$. It follows from (13) that

$$\frac{\partial^2 u}{\partial y^2} = \sum_{n=1}^{\infty} \left[C_{1n}e^{-2\tau_n x} + C_{2n}e^{\tau_n x}\cos\rho_n x + C_{3n}e^{\tau_n x}\sin\rho_n x\right] Y''_n(y),$$

where

$$Y''_n(y) \leqslant -\mu_n\left(\alpha\sin\sqrt{\mu_n}y - \beta\sqrt{\mu_n}\cos\sqrt{\mu_n}y\right)A_n = -\mu_n Y_n(y).$$

Thus, we get

$$\frac{\partial^2 u}{\partial y^2} = \sum_{n=1}^{\infty} \left[C_{1n}e^{-2\tau_n x} + C_{2n}e^{\tau_n x}\cos\rho_n x + C_{3n}e^{\tau_n x}\sin\rho_n x\right](-\mu_n)Y_n(y),$$

$$\left|\frac{\partial^2 u}{\partial y^2}\right| = \sum_{n=1}^{\infty} \left[|C_{1n}|\,e^{-2\tau_n x} + |C_{2n}|\,e^{\tau_n x} + |C_{3n}|\,e^{\tau_n x}\right]|\mu_n|\,|Y_n(y)| \leqslant$$

$$\leqslant M_4 \sum_{n=1}^{\infty} \left[|A_{1n}Y_n(y)| + |A_{2n}Y_n(y)| + |A_{3n}Y_n(y)|\right]|\mu_n|.$$

Since $\mu_n = \mathrm{O}\left(n^2\right)$, we find

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leqslant 2BM_4 N_i \sum_{i=1}^{\infty} \frac{|\varphi'''_{in}|}{n}.$$

By using the Cauchy–Buniakowski and Bessel inequalities [22], we get

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leqslant 2BM_4 N_i \sqrt{\sum_{i=1}^{\infty} |\varphi'''_{in}|^2} \sqrt{\sum_{i=1}^{\infty} \frac{1}{n^2}} \leqslant 2BM_4 N_i \sqrt{2\|\varphi'''_{in}\|^2} \sqrt{\frac{\pi^2}{6}} = 2M_4 N_i \frac{\pi}{\sqrt{3}} \|\varphi'''_{in}\| < \infty,$$

where

$$\sum_{i=1}^{\infty} |\varphi'''_{in}|^2 = 2 \|\varphi'''_{in}\|^2_{L_2(0,1)}, \ \ i = \overline{1,3}, \ \ \sum_{i=1}^{n} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In view of Eq. (1), we obtain a similar estimate for $\frac{\partial^3 u}{\partial x^3}$. This implies that $\left|\frac{\partial^3 u}{\partial x^3}\right|$ and $\left|\frac{\partial^2 u}{\partial y^2}\right|$ are absolutely and uniformly convergent.

The theorem is proved.

Theorem 2. *If $\varphi_i(y) \in C^3[0,1]$, and $\alpha\,\varphi_i(0) + \beta\,\varphi'_i(0) = 0$, $\gamma\,\varphi_i(1) + \delta\,\varphi'_i(1) = 0$, $\varphi''(0) = \varphi''(1) = 0$, then the solution to the problem exists and presented in the form (13).*

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