Stress-Sum Index for Graphs

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Abstract

The stress of a vertex is a node centrality index, which has been introduced by Shimbel (1953). The stress of a vertex in a graph is the number of geodesics (shortest paths) passing through it. A topological index of a chemical structure (graph) is a number that correlates the chemical structure with chemical reactivity or physical properties. In this paper, we introduce a new topological index for graphs called stress-sum index using stresses of vertices. Further, we establish some inequalities, prove some results and compute stress-sum index for some standard graphs.

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1 Introduction

For standard terminology and notion in graph theory, we follow the textbook of Harary [5]. The non-standard will be given in this paper as and when

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required.

Let $G = (V, E)$ be a graph (finite and undirected). The distance between two vertices u and v in G, denoted by $d(u, v)$ is the number of edges in a shortest path (also called a graph geodesic) connecting them. We say that a graph geodesic P is passing through a vertex v in G if v is an internal vertex of P (i.e., v is a vertex in P, but not an end vertex of P). For two vertices u and v in G, $g(u, v)$ denotes the number of geodesics whose end vertices are u and v.

The concept of stress of a node (vertex) in a network (graph) has been introduced by Shimbel as centrality measure in 1953 [9]. This centrality measure has applications in biology, sociology, psychology, etc., (See [8, 6]). The stress of a vertex v in a graph G, denoted by $\mathrm{str}_G(v)$ $\mathrm{str}(v)$, is the number of geodesics passing through it. We denote the maximum stress among all the vertices of G by Θ_G and minimum stress among all the vertices of G by θ_G. Further, the concepts of stress number of a graph and stress regular graphs have been studied by K. Bhargava, N.N. Dattatreya, and R. Rajendra in their paper [1]. The stress number of G, denoted by $N_{\mathrm{str}}(G)$, is defined as:

$$N_{\mathrm{str}}(G) = \sum_{v \in V} \mathrm{str}(v).$$

A graph G is k-stress regular if $\mathrm{str}(v) = k$ for all $v \in V(G)$.

The Zagreb indices have been defined using degrees of vertices in a graph to explain some properties of chemical compounds at molecular level [2, 3]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a simple graph G are defined as:

$$M_1(G) = \sum_{v \in V(G)} d(v)^2 \tag{1}$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v). \tag{2}$$

By the motivation of these indices, Rajendra et al. [7] have introduced two topological indices of for graphs called first stress index and second stress index, using stresses of vertices. The first stress index $\mathcal{S}_1(G)$ and the second stress index $\mathcal{S}_2(G)$ of a simple graph G are defined as

$$\mathcal{S}_1(G) = \sum_{v \in V(G)} \mathrm{str}(v)^2 \tag{3}$$

$$\mathcal{S}_2(G) = \sum_{uv \in E(G)} \mathrm{str}(u)\,\mathrm{str}(v). \tag{4}$$

We note that the first Zagreb index $M_1(G)$ satisfies the identity

$$M_1(G) = \sum_{uv \in E(G)} d(u) + d(v) \tag{5}$$

but $\mathcal{S}_1(G)$ does not satisfy such identity. For instance, consider the path P_3 on 3 vertices.

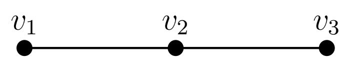

Figure 1: The path P_3.

The stresses of the vertices of P_3 are as follows: $\mathrm{str}(v_1) = \mathrm{str}(v_3) = 0$ and $\mathrm{str}(v_2) = 1$. The first stress index of P_3 is,

$$\mathcal{S}_1(P_3) = \mathrm{str}(v_1)^2 + \mathrm{str}(v_2)^2 + \mathrm{str}(v_3)^2 = 0^2 + 1^2 + 0^2 = 1.$$

But

$$\sum_{uv \in E(P_3)} \mathrm{str}(u) + \mathrm{str}(v) = \mathrm{str}(v_1) + \mathrm{str}(v_2) + \mathrm{str}(v_2) + \mathrm{str}(v_3) = 0 + 1 + 1 + 0 = 2.$$

Therefore there is a scope for introducing a new topological index using stress on vertices which is motivated by the identity (5). In this paper we introduce such topological index for graphs using stress on vertices called stress-sum index. Further, we establish some inequalities and compute stress-sum index for some standard graphs.

2 Stress-Sum Index for Graphs

Definition 2.1. The stress-sum index $\mathcal{SS}(G)$ of a simple graph G is defined as

$$\mathcal{SS}(G) = \sum_{uv \in E(G)} \mathrm{str}(u) + \mathrm{str}(v) \tag{6}$$

Observation: From the Definition 2.1, it follows that, for any graph G,

$$2m\theta_G \le \mathcal{SS}(G) \le 2m\Theta_G$$

where m is the number of edges in G.

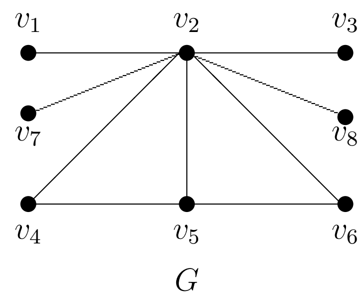

Figure 2: A graph G

Example 2.2. Consider the graph G given in Figure 2.

The stresses of the vertices of G are as follows:
$\mathrm{str}(v_1) = \mathrm{str}(v_3) = \mathrm{str}(v_7) = \mathrm{str}(v_8) = 0$,
$\mathrm{str}(v_2) = 19$,
$\mathrm{str}(v_5) = 1$,
$\mathrm{str}(v_4) = \mathrm{str}(v_6) = 0$.

The stress-sum index of G is:

$$\begin{aligned}\mathcal{S}\,\mathcal{S}(G) =&(\mathrm{str}(v_2) + \mathrm{str}(v_1)) + (\mathrm{str}(v_2) + \mathrm{str}(v_3)) + (\mathrm{str}(v_2) + \mathrm{str}(v_7))\\ &+ (\mathrm{str}(v_2) + \mathrm{str}(v_8) + (\mathrm{str}(v_2) + \mathrm{str}(v_4)) + (\mathrm{str}(v_2) + \mathrm{str}(v_5))\\ &+ (\mathrm{str}(v_2) + \mathrm{str}(v_6)) + (\mathrm{str}(v_4) + \mathrm{str}(v_5)) + (\mathrm{str}(v_5) + \mathrm{str}(v_6))\\ =&(19 + 0) + (19 + 0) + (19 + 0) + (19 + 0) + (19 + 0) + (19 + 1)\\ &+ (19 + 0) + (0 + 1) + (1 + 0)\\ =&136.\end{aligned}$$

Proposition 2.3. Let N be the number of geodesics of length ≥ 2 in a graph G. Then

$$0 \leq \mathcal{S}\,\mathcal{S}(G) \leq 2N(|E| - t), \tag{7}$$

where t is the number of edges with end vertices having zero stress in G.

Proof. If N is the number of all geodesics of length ≥ 2 in a graph G, then by the definition of stress of a vertex, for any vertex v in G, $0 \leq \mathrm{str}(v) \leq N$. Hence by the Definition 2.1, we have

$$0 \leq \mathcal{S}\,\mathcal{S}(G) \leq 2N(|E| - t), \tag{8}$$

where t is the number of edges with end vertices having zero stress in G. □

Corollary 2.4. If there is no geodesic of length ≥ 2 in a graph G, then $\mathcal{S}\,\mathcal{S}(G) = 0$. Moreover, for a complete graph K_n, $\mathcal{S}\,\mathcal{S}(K_n) = 0$.

Proof. If there is no geodesic of length ≥ 2 in a graph G, then $N = 0$. Hence, by the Proposition 2.3, we have $\mathcal{S}\mathcal{S}(G) = 0$.

In K_n, there is no geodesic of length ≥ 2 and so $\mathcal{S}\mathcal{S}(K_n) = 0$. □

Theorem 2.5. For a graph G, $\mathcal{S}\mathcal{S}(G) = 0$ if and only if neighbours of every vertex induce a complete subgraph of G.

Proof. Suppose that $\mathcal{S}\mathcal{S}(G) = 0$. Then by the Definition 2.1(Eq.(3)), $\mathrm{str}(u)+\mathrm{str}(v) = 0$, $\forall uv \in E(G)$. Hence $\mathrm{str}(v) = 0$, $\forall v \in V(G)$. Let $v \in V(G)$. We need to show that neighbors of v induce a complete subgraph of G. If v is a pendant vertex, then there is nothing to prove. Suppose that v is not a pendant vertex. We claim that any two neighbouring vertices are adjacent in G. If there are two neighbours u and w of v that are not adjacent in G, then uvw is a graph geodesic passing through v, which implies $\mathrm{str}(v) \geq 1$, a contradiction. Hence our claim holds. Thus neighbours of v induce a complete subgraph of G. Since v is arbitrary in $V(G)$, the neighbours of every vertex induce a complete subgraph of G.

Conversely, suppose that neighbours of every vertex in G induce a complete subgraph of G. Let $v \in V(G)$. Since neighbors of v induce a complete subgraph of G, any two neighbouring vertices are adjacent and so there is no geodesic of length ≥ 2 passing through v. Since v is an arbitrary vertex in G, by the Corollary 2.4, it follows that $\mathcal{S}\mathcal{S}(G) = 0$. □

Proposition 2.6. For the complete bipartite $K_{m,n}$,

$$\mathcal{S}\mathcal{S}(K_{m,n}) = \frac{mn}{2}\left[n(n-1) + m(m-1)\right].$$

Proof. Let $V_1 = \{v_1, \ldots, v_m\}$ and $V_2 = \{u_1, \ldots, u_n\}$ be the partite sets of $K_{m,n}$. We have,

$$\mathrm{str}(v_i) = \frac{n(n-1)}{2} \text{ for } 1 \leq i \leq m \tag{9}$$

and

$$\mathrm{str}(u_j) = \frac{m(m-1)}{2} \text{ for } 1 \leq j \leq n. \tag{10}$$

Using (9) and (10) in the Definition 2.1, we have

$$\begin{aligned}\mathcal{S}\mathcal{S}(K_{m,n}) &= \sum_{uv\in E(G)} \mathrm{str}(u) + \mathrm{str}(v)\\ &= \sum_{1\leq i\leq m,\ 1\leq j\leq m} \mathrm{str}(v_i) + \mathrm{str}(u_j)\end{aligned}$$

$$= \sum_{1\le i\le m,\ 1\le j\le n} \left[\frac{n(n-1)}{2} + \frac{m(m-1)}{2}\right]$$
$$= mn\left[\frac{n(n-1)}{2} + \frac{m(m-1)}{2}\right]$$
$$= \frac{mn}{2}\left[n(n-1) + m(m-1)\right]. \qquad \square$$

Proposition 2.7. If $G = (V, E)$ is a k-stress regular graph, then

$$\mathcal{S}\,\mathcal{S}(G) = 2k|E|.$$

Proof. Suppose that G is a k-stress regular graph. Then
$$\text{str}(v) = k \text{ for all } v \in V(G).$$
By the Definition 2.1, we have

$$\mathcal{S}\,\mathcal{S}(G) = \sum_{uv\in E(G)} \text{str}(u) + \text{str}(v)$$
$$= \sum_{uv\in E(G)} k + k$$
$$= 2k|E|. \qquad \square$$

Corollary 2.8. For a cycle C_n,

$$\mathcal{S}\,\mathcal{S}(C_n) = \begin{cases} \frac{n(n-1)(n-3)}{4}, & \text{if } n \text{ is odd} \\ \frac{n^2(n-2)}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. For any vertex v in C_n, we have,

$$\text{str}(v) = \begin{cases} \frac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd} \\ \frac{n(n-2)}{8}, & \text{if } n \text{ is even.} \end{cases}$$

Hence C_n is

$$\begin{cases} \frac{(n-1)(n-3)}{8}\text{-stress regular,} & \text{if } n \text{ is odd} \\ \frac{n(n-2)}{8}\text{-stress regular,} & \text{if } n \text{ is even.} \end{cases}$$

Since C_n has n vertices and n edges, by the Proposition 2.7, we have

$$\mathcal{SS}(C_n) = 2n \times \begin{cases} \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd} \\ \dfrac{n(n-2)}{8}, & \text{if } n \text{ is even.} \end{cases}$$
$$= \begin{cases} \dfrac{n(n-1)(n-3)}{4}, & \text{if } n \text{ is odd} \\ \dfrac{n^2(n-2)}{4}, & \text{if } n \text{ is even.} \end{cases} \qquad \square$$

Proposition 2.9. Let T be a tree on n vertices. Then

$$\mathcal{SS}(T) = \sum_{uv\in J}\left[\sum_{1\le i<j\le m(u)} |C_i^u||C_j^u| + \sum_{1\le i<j\le m(v)} |C_i^v||C_j^v|\right] + \sum_{w\in Q}\sum_{1\le i<j\le m(w)} |C_i^w||C_j^w|.$$

where J is the set of internal(non-pendant) edges in T, Q denotes the set of all vertices adjacent to pendent vertices in T, and the sets $C_1^v, \ldots, C_m^v$ denotes the vertex sets of the components of $T - v$ for an internal vertex v of degree $m = m(v)$.

Proof. We know that a pendant vertex in T has zero stress. Let v be an internal vertex of T of degree $m = m(v)$. Let $C_1^v, \ldots, C_m^v$ be the components of $T - v$. Since there is only one path between any two vertices in a tree, it follows that,

$$\text{str}(v) = \sum_{1\le i<j\le m} |C_i^v||C_j^v| \tag{11}$$

Let J denotes the set of internal(non-pendant) edges, and P denotes pendant edges and Q denotes the set of all vertices adjacent to pendent vertices in T. Then using (11) in the Definition 2.1 ((6)), we have

$$\begin{aligned}\mathcal{SS}(T) &= \sum_{uv\in J} \text{str}(u) + \text{str}(v) + \sum_{uv\in P} \text{str}(u) + \text{str}(v) \\ &= \sum_{uv\in J} \text{str}(u) + \text{str}(v) + \sum_{w\in Q} \text{str}(w) \\ &= \sum_{uv\in J}\left[\sum_{1\le i<j\le m(u)} |C_i^u||C_j^u| + \sum_{1\le i<j\le m(v)} |C_i^v||C_j^v|\right] \\ &\quad + \sum_{w\in Q}\sum_{1\le i<j\le m(w)} |C_i^w||C_j^w|. \end{aligned} \qquad \square$$

Corollary 2.10. For the path P_n on n vertices

$$\mathcal{S}\,\mathcal{S}(P_n) = \frac{1}{3}n(n-1)(n-2).$$

Proof. The proof of this corollary follows by above Proposition 2.9. We follow the proof of the Proposition 2.9 to compute the index. Let P_n be the path with vertex sequence $v_1, v_2, \ldots, v_n$ (shown in Figure 3).

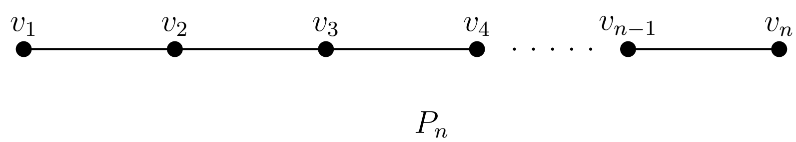

Figure 3: The path P_n on n vertices.

We have,

$$\mathrm{str}(v_i) = (i-1)(n-i), \qquad 1 \le i \le n.$$

Then

$$\begin{aligned}
\mathcal{S}\,\mathcal{S}(P_n) &= \sum_{uv \in E(P_n)} \mathrm{str}(u) + \mathrm{str}(v) \\
&= \sum_{i=1}^{n-1} \mathrm{str}(v_i) + \mathrm{str}(v_{i+1}) \\
&= \sum_{i=1}^{n-1} [(i-1)(n-i) + (i)(n-i-1)] \\
&= \frac{1}{3}n(n-1)(n-2).
\end{aligned}$$

□

Proposition 2.11. Let $Wd(n,m)$ denotes the windmill graph constructed for $n \ge 2$ and $m \ge 2$ by joining m copies of the complete graph K_n at a shared universal vertex v. Then

$$\mathcal{S}\,\mathcal{S}(Wd(n,m)) = \frac{n^3 m(m-1)}{2}.$$

Hence, for the friendship graph F_k on $2k+1$ vertices,

$$\mathcal{S}\,\mathcal{S}(F_k) = 4k(k-1).$$

Proof. Clearly the stress of any vertex other than universal vertex is zero in $Wd(n,m)$, because neighbors of that vertex induces a complete subgraph of $Wd(n,m)$. Also, since there are m copies of K_n in $Wd(n,m)$ and their vertices are adjacent to v, it follows that, the only geodesics passing through v are of length 2 only. So, $\mathrm{str}(v) = n^2(m-1)/2$. Note that there are mn edges incident on v and the edges that are not incident on v have end vertices of stress zero. Hence by the Definition 2.1, we have

$$\begin{aligned}\mathcal{S}\,\mathcal{S}(Wd(n,m)) &= mn\,\mathrm{str}(v)\\ &= mn[n^2(m-1)/2]\\ &= \frac{n^3m(m-1)}{2}.\end{aligned}$$

Since the friendship graph F_k on $2k+1$ vertices is nothing but $Wd(2,k)$, it follows that

$$\mathcal{S}\,\mathcal{S}(F_k) = \frac{2^3k(k-1)}{2} = 4k(k-1).$$ □

Proposition 2.12. Let W_n denotes the wheel graph constructed on $n \geq 4$ vertices. Then

$$\mathcal{S}\,\mathcal{S}(W_n) = \begin{cases} \dfrac{(n-1)\,(7n-10)\,(n-4)}{8}, & \text{if } n \text{ is even;}\\ \dfrac{(n-1)^2\,(7n-25)}{8}, & \text{if } n \text{ is odd.}\end{cases}$$

Proof. In W_n with $n \geq 4$, there are $(n-1)$ peripheral vertices and one central vertex, say v. It is easy to see that

$$\mathrm{str}(v) = \frac{(n-1)(n-4)}{2} \tag{12}$$

Let p be a peripheral vertex. Since v is adjacent to all the peripheral vertices in W_n, there is no geodesic passing through p and containing v. Hence contributing vertices for $\mathrm{str}(p)$ are the rest peripheral vertices. So, by denoting the cycle $W_n - p$ (on $n-1$ vertices) by C_{n-1}, we have

$$\begin{aligned}\mathrm{str}_{W_n}(p) &= \mathrm{str}_{W_n - v}(p)\\ &= \mathrm{str}_{C_{n-1}}(p)\\ &= \begin{cases} \dfrac{(n-2)(n-4)}{8}, & \text{if } n-1 \text{ is odd;}\\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n-1 \text{ is even,}\end{cases}\end{aligned}$$

$$= \begin{cases} \dfrac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases} \tag{13}$$

Let us denote the set of all radial edges in W_n by R, and the set of all peripheral edges by Q. Note that there are $(n-1)$ radial edges and $(n-1)$ peripheral edges in W_n. Using (12) and (13) in the Definition 2.1, we have

$$\begin{aligned} \mathcal{S}\,\mathcal{S}(W_n) =& \sum_{xy\in R}[\mathrm{str}(x)+\mathrm{str}(y)] + \sum_{xy\in Q}[\mathrm{str}(x)+\mathrm{str}(y)] \\ =&(n-1)[\mathrm{str}(v)+\mathrm{str}(p)] + (n-1)\cdot 2\cdot \mathrm{str}(p) \\ =&(n-1)\left[\frac{(n-1)(n-4)}{2} + \begin{cases} \dfrac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases}\right] \\ &+ 2(n-1)\times \begin{cases} \dfrac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases} \\ =& \begin{cases} \dfrac{(n-1)^2(n-4)}{2} + \dfrac{3(n-1)(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)^2(n-4)}{2} + \dfrac{3(n-1)^2(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases} \\ =& \begin{cases} \dfrac{(n-1)\,(7n-10)\,(n-4)}{8}, & \text{if } n \text{ is even;} \\ \dfrac{(n-1)^2\,(7n-25)}{8}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

□

Conclusion

We have introduced a new topological index for graphs called stress-sum index using stresses of vertices. Further, we established some inequalities, proved some results and computed the stress-difference index for some standard graphs. A large number of molecular-graph-based structure descriptors (topological indices) have been defined, depending on vertex degrees. But in this paper, we have defined the new topological index for graphs without using the degrees of vertices. This index can be used to determine $\mathcal{S}\,\mathcal{S}(G)$ for other classes of graphs and results in this direction will be reported in a subsequent paper.

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