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SOBOLEV-TYPE FUNCTIONS ON NONHOMOGENEOUS METRIC SPACES

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Abstract. We consider analogs of classical embedding theorems for function classes of Sobolev type on nonhomogeneous metric measure spaces.

Keywords: metric, measure, embedding theorems.

Analysis on metric structures has been actively developing since the 1990s. On metric measure spaces, various classes of functions with generalized "smoothness", which are in a sense a generalization of the Sobolev spaces, were considered. The definitions of such spaces are based on the existence of alternative descriptions of Sobolev spaces not using the linear structure of the Euclidean space and admitting a statement in terms of the measure and metric.

The article deals with different types of an embedding theorem for Sobolev-type function classes $M^1_p(X, d, \mu)$, introduced by P. Hajłasz in [1]. The greatest analogy of the properties of the functions of the classical Sobolev spaces $W^1_p(G)$ and the functions of the spaces $M^1_p(X, d, \mu)$ is observed on homogeneous metric spaces (X, d) with the measure μ satisfying the estimate

$$C_1\, r^s \le \mu(B(x, r)) \le C_2\, r^s. \tag{1}$$

for $x \in X$ and $0 < r \le diam\, X$.

A quite informative theory of the spaces $M^1_p(X, d, \mu)$ is obtained in the more general situation, when the measure μ satisfies the "doubling condition", which is weaker than (1), which implies only a bound for the measure of a ball from below. In this case, it is possible to obtain analogs of various classical results including embedding theorems, which play an exceptional role in the theory of Sobolev spaces [1, 2, 3, 4]. Moreover, the metric results and the methods used for obtaining them turn out to be useful also in studying the properties of Sobolev functions in a Euclidean space.

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We will be interested in the properties of function classes of Sobolev type on nonhomogeneous metric measure spaces whose local properties depend on a point $x \in X$. In this case, it is possible to obtain embedding theorems with variable integrability exponent.

Sobolev-Type Spaces

In an arbitrary metric space (X, d) with measure μ, P. Hajłasz (see [1]) introduced the classes $M^1_p(X, d, \mu)$ of Sobolev type, the definition of which is based on a Lipschitz estimate of a special kind.

For an arbitrary μ-measurable function $u : X \to \overline{R}$, call a function $g : X \to [0, \infty)$ admissible if there exists a set $E \subset X$ such that $\mu(E) = 0$ and the inequality

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \tag{2}$$

holds for all $x, y \in X \setminus E$.

Denote the set of all admissible functions for a function u by $D(u)$ and, for $p \geq 1$, put $D_p(u) = D(u) \cap L_p(X, \mu)$.

The function classes $S^1_p(X, d, \mu)$ and $M^1_p(X, d, \mu)$ are defined by the conditions:

$$S^1_p(X, d, \mu) = \{u : X \to \overline{R} \mid D_p(u) \neq \emptyset\},$$

$$M^1_p(X, d, \mu) = \{u \in L_p(X, \mu) \mid u \in S^1_p(X, \rho, \mu)\}.$$

The seminorm in $S^1_p(X, d, \mu)$ and the norm in $M^1_p(X, d, \mu)$ are defined by the equalities

$$||u \mid S^1_p|| = \inf_{g \in D_p(u)} ||g \mid L_p||, \quad ||u \mid M^1_p|| = ||u \mid L_p|| + ||u \mid S^1_p||.$$

In Euclidean domains $G \subset R^n$ with a sufficiently smooth (for instance, smooth or Lipschitz) boundary, the classical Sobolev space $W^1_p(G)$ and the space $M^1_p(G, |*|, m_n)$, considered with the standard Euclidean metric and the Lebesgue measure, coincide as sets of functions, and their norms are equivalent (see [1]).

Informative results for the spaces $M^1_p(X, d, \mu)$ can be obtained under rather natural assumptions on the relationship between the metric d and the measure μ. We will assume that the metric space (X, d) and the finite Borel measure μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)), \tag{3}$$

i.e., the measure of the ball of doubled radius is estimated in terms of the measure of the initial ball. This simple geometric condition guarantees the fulfillment of the Vitali covering lemma and its standard consequences for the measure μ. For a locally integrable function, Lebesgue's theorem on the differentiation of an integral remains valid, and, as a consequence, almost all points of the set X are Lebesgue points of u. The property important to us is the boundedness of the Hardy–Littlewood maximal operator in the Lebesgue spaces $L_p(X, \mu)$ for $p > 1$. The doubling condition for $r \leq \mathit{diam}\, X$ implies the estimate

$$\mu(B(x, r)) \geq C_1 r^s, \tag{4}$$

where $s = \log_2 C_d$. The degree s is called the regularity exponent of the measure μ with respect to the measure d and plays the role of the "dimension" of the metric space (X, d) in the embedding theorems.

Henceforth, we assume that the measure μ satisfies the doubling condition and is regular with exponent s.

Denote by u_E the mean value of a function u on the set E

$$u_E = \fint_E u \, d\mu = \frac{1}{\mu(E)} \int_E u \, d\mu.$$

Define the maximal Hardy–Littlewood operator in a standard manner, by setting

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |u| \, d\mu.$$

As was already observed,

$$\|\mathcal{M}f \mid L_p(X,\mu)\| \le C\|f \mid L_p(X,\mu)\|$$

for $p > 1$.

If $0 < \gamma \le 1$ then $d_\gamma(x,y) = (d(x,y))^\gamma$ is again a metric. This makes it possible to introduce the Hölder function classes M_p^γ by replacing the estimate in the initial definition of the spaces M_p^1 with

$$|u(x) - u(y)| \le (d(x,y))^\gamma (g(x) + g(y)).$$

It is easy to notice here that

$$M_p^\gamma(X,d,\mu) = M_p^1(X,d_\gamma,\mu),$$

i.e., the Hölder classes with respect to the initial metric can be regarded as the space with "unit smoothness" but with respect to the Hölder metric. This is often convenient since, in obtaining results for the spaces M_p^γ, it suffices to recalculate the regularity exponent of the measure μ with respect to the Hölder metric and use the assertion for the spaces M_p^1.

In accordance with [2], the inequality

$$|u(x) - u(y)| \le C(d(x,y))^\gamma (u_\gamma^\#(x) + u_\gamma^\#(y)) \tag{5}$$

holds for all the Lebesgue points of a locally integrable function u, where $0 < \gamma \le 1$ and $u_\gamma^\#$ is the sharp maximal function of order γ, defined uniquely at all points of X by the equality

$$u_\gamma^\#(x) = \sup_{r>0} r^{-\gamma} \fint\limits_{B(x,r)} |u - u_{B(x,r)}| \, d\mu.$$

For measure with doubling condition, function classes of Sobolev type admit different equivalent descriptions (see [2]).

Lemma 1. *Let $1 < p \le \infty$. The following three conditions are equivalent:*

(1) $u \in S_p^1(X,d,\mu)$;

(2) there exists a function $h \in L_p(X,\mu)$ such that the Poincaré inequality

$$\fint\limits_{B(x,r)} |u - u_{B(x,r)}| d\mu \le r \fint\limits_{B(x,r)} h \, d\mu \tag{6}$$

holds for arbitrary $x \in X$ and $r > 0$;

(3) $u_1^\# \in L_p(X,\mu)$.

Moreover,

$$||u \mid S_p^1(X,d,\mu)|| \sim \inf_h ||h \mid L_p(X,\mu)|| \sim ||u_1^\# \mid L_p(X,\mu)||.$$

If $u_\gamma^\# \in L_p(X,\mu)$ then, by (5), $u \in S_p^1(X,d_\gamma,\mu)$.

Among all the numerous results related to function classes of Sobolev type, we will first of all be interested in analogs of the Sobolev embedding theorems. We will now confine ourselves to formulating two assertions concerning the compactness of the embedding operators.

Lemma 2. *Let $1 < p < \infty$. Then the embedding operator*

$$I : M_p^1(X,d,\mu) \to L_q(X,\mu)$$

is compact for

1. $1 \le q < \dfrac{ps}{s-p}$ *if* $p < s$;
2. $1 \le q < \infty$ *if* $p = s$;
3. $1 \le q \le \infty$ *if* $p > s$.

The first two items are consequences of the results of [1, 3], and the proof of the third is given in [4].

The following assertion is an internal embedding theorem in the scale of the spaces M_p^γ and shows the relationship between the classes of functions defined by various metrics (see [4]).

Lemma 3. *Let* $1 < p < \infty$, $0 < \gamma < 1$. *The embedding operator*

$$I : M_p^1(X, d, \mu) \to M_\omega^1(X, d_\gamma, \mu)$$

is compact for
1. $1 \le \omega < \dfrac{ps}{s-(1-\gamma)p}$ if $(1-\gamma)p < s$;
2. $1 \le \omega < \infty$ if $(1-\gamma)p = s$;
3. $1 \le \omega \le \infty$ if $(1-\gamma)p > s$.

Note that item 3 implies the Hölder continuity of the function understood in the usual sense $|u(x) - u(y)| \le C\ (d(x,y))^\gamma$.

Measures with Variable Integrability Exponent

Lemmas 2 and 3 are very universal but they do not depend on the specific nature of the metric space and are completely defined by the regularity exponent s characterizing the relationship between the measure and metric.

In the case of homogeneous metric spaces, when the measure of an arbitrary ball $B(x,r)$ admits a two-sided estimate (1) via r^s, the claims of lemmas 2 and 3 concerning the integrability exponents q and ω are accurate and unimprovable.

In accordance with [5], every compact set in R^n can be endowed with a metric satisfying the doubling condition (3). The doubling condition implies a bound for the measure of a ball of the form (4) but there is no upper bound in the general case.

It is not hard to give examples when the homogeneity condition (1) fails.

1. The simplest situation is when the set $X \subset R^m$ is the union of sets E_k having different Euclidean dimensions n_k.

2. If the interval $[0,1] \subset R$ is endowed with the weight metric $d\mu = 2x\,dx$ then $\mu(B(0,r)) = r^2$, $r \le 1$. Moreover, $\mu(B(1,r)) = 1 - (1-r)^2 = 2r - r^2 = r(2-r)$. Consequently,

$$r \le \mu(B(1,r)) \le 2r.$$

For arbitrary $x \in (0,1)$ and $0 < r < 1$, we have an inequality with variable regularity exponent $\mu(B(x,r)) \ge Cr^{s(x)}$, where $s(x) \nearrow 2$ as $x \to 0$ and $s(x) \searrow 1$ as $x \to 1$. Rough estimates show that, as $x \to 0$, the inequality $\mu(B(x,r)) \ge Cr^{\omega(x)}$ holds for the function $\omega(x) = 2\left(1 - \log_2(1+2x)\right)$.

Even for Euclidean domain and the Lebesgue measure, the exponent in the lower bound (4) can be different for different points of the domain, i.e., be a function of a point. Standard examples of such a kind are peaks with Hölder singularities at the vertex.

3. Let $1 < \lambda < \infty$; define the "zero peak" $G_\lambda \subset R^2$ by the condition:

$$G_\lambda = \{(x,y) \in R^2 \mid 0 \le x \le 1,\ 0 \le y \le x^\lambda\}.$$

As the measure μ, consider the restriction of the Lebesgue measure in R^2 to the peak G_λ. If a is the vertex of the peak G_λ then $\mu(B(a,r)) = \Lambda^{-1} r^\Lambda$, where $\Lambda = 1 + \lambda > 2$.

If a point $b = (x, y)$ different from the vertex belongs to G_λ then the balls of a sufficiently small radius satisfy $\mu(B(b, r) > C_1 r^2$. For a suitable choice of the constant C^*, for an arbitrary point $c = (x, y) \in G_\lambda$ and $0 < r < 1$, we have the estimate $\mu(B(c, r)) \geq C^* r^{s(x,y)}$, where $s(x, y) \nearrow \Lambda > 2$ as $x \to 0$ and $s(x, y) \searrow 2$ as $x \to 1$.

The estimate $\mu(B(a, r)) > C\, r^\Lambda$ holds for all points $a \in G_\lambda$; moreover, the uniform regularity exponent of the measure μ cannot be less than Λ. By Lemma 2, for $p < \Lambda$, the embedding operator

$$I : M_p^1(G_\lambda, |*|, \mu) \to L_q(X, \mu)$$

is compact then

$$q < \frac{p\Lambda}{\Lambda - p}.$$

Moreover, local embedding into the Lebesgue space holds with a greater integrability exponent.

Put

$$s_{a,r} = \sup_{B(a,r)} s(x, y).$$

If a point a is different from the vertex of the peak then, for $1 < p < 2$, on the ball $B(a, r)$, the space $M_p^1(B(a, r), |*|, \mu)$ is compactly embedded in the Lebesgue space $L_{q(a,r)}(X, \mu)$, where

$$q(a, r) < \frac{2s_{a,r}}{s_{a,r} - p} > \frac{p\Lambda}{\Lambda - p}.$$

This enables us to assume that, in this situation, for refining the result, it is appropriate to consider the embedding of Sobolev-type spaces into spaces with variable integrability exponents.

Spaces with Variable Integrability Exponents

Consider a metric space (X, d) with finite diameter, a finite Borel measure μ with support in X, and fix a positive measurable function $p : X \to [1, \infty)$. On the set of measurable functions $f : X \to R$, introduce the functional

$$\rho_{p(\cdot)}(f) = \int_X |f(x)|^{p(x)} d\mu.$$

Define the Lebesgue space with variable integrability exponent $L_{p(\cdot)}(X, \mu)$ as the class of all such functions f such that $\rho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$.

If $p(x) \geq 1$ then the norm in $L_{p(\cdot)}(X, \mu)$ is introduced by the equality

$$\|f\|_{p(\cdot)} = \|f \mid L_{p(\cdot)}(X, \mu)\| = \inf\left\{\alpha > 0 \mid \ \rho_{p(\cdot)}(f/\alpha) \leq 1\right\}.$$

For $p(x) = p = const$, the above-introduced norm coincides with the standard norm in L_p. The norm in $L_{p(\cdot)}(X, \mu)$ is monotone, i.e., the condition $|f| \leq |g|$ almost everywhere implies the inequality $\|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}$.

Reckoning with the finiteness of the measure μ, it is easy to show that, for $1 \leq p(x) \leq q(x) < \infty$, the space $L_{q(\cdot)}(X, \mu)$ is continuously embedded in $L_{p(\cdot)}(X, \mu)$.

For $1 < p(x) < \infty$, define the conjugate exponent $p'(x)$ by the standard equality $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. It is rather easy to prove an analog of the classical Hölder inequality (see [6])

$$\int_X |f(x)g(x)|\, d\mu \leq C\, \|f\|_{p(\cdot)} \cdot \|g\|_{p'(\cdot)}.$$

We will need the following assertion, proved in [7].

Lemma 4. *Let $1 < p_- \leq p(x) \leq p^+ < \infty$. Then*
(1) the conditions $\|f\|_{p(\cdot)} \leq 1$ and $\rho_{p(\cdot)}(f) \leq 1$ are equivalent;

(2) $\|f_n\|_{p(\cdot)} \to 0$ *if and only if* $\rho_{p(\cdot)}(f_n) \to 0$;
(3) *if* $\rho_{p(\cdot)}(f) \le L < \infty$ *then* $\|f\|_{p(\cdot)} \le K = K\,(p(x), L) < \infty$.

Replacing in the definition of the spaces $S^1_p(X, d_\gamma, \mu)$ and $M^1_p(X, d_\gamma, \mu)$ the usual Lebesgue space $L_p(X, \mu)$ by the Lebesgue space with variable integrability exponent $L_{p(\cdot)}(X, \mu)$, we obtain the classes of functions with variable integrability exponent — $S^1_{p(\cdot)}(X, d_\gamma, \mu)$ and $M^1_{p(\cdot)}(X, d_\gamma, \mu)$.

Embedding Theorems for Nonhomogeneous Measures

Henceforth, we will consider the properties of functions of the Sobolev-type spaces $M^1_p(X, d, \mu)$ in the case when the measure μ satisfies:
(i) the doubling condition;
(ii) the estimate $\mu(B(x, r)) \ge C_0 r^{s(x)}$ for $x \in X,\ r \le diam\, X$.
For $p < s(x)$, put $q(x) = \dfrac{ps(x)}{s(x) - p}$.

For proving the boundedness of the embedding operator into the Lebesgue space with limit exponent $q(x)$, make us of the description of Sobolev-type spaces in terms of maximal order functions.

Theorem 1. *Suppose that a measure* μ *on a metric space* (X, d) *satisfies conditions (i), (ii). If* $1 < p < s_- = \inf\limits_{x\in X} s(x)$ *and* $q(x) = \dfrac{ps(x)}{s(x) - p}$ *then the embedding operator*

$$I : M^1_p(X, d, \mu) \to L_{q(\cdot)}(X, \mu)$$

is bounded. Moreover,

$$\|u - u_X \mid L_{q(\cdot)}(X, \mu)\| \le C\|u \mid S^1_p(X, d, \mu)\|.$$

Proof. By the finiteness of the measure and the boundedness of the exponent $q(x)$, it suffices to prove the last inequality. If $u \equiv const$ then the theorem is obvious. If the function u is nonconstant then $u^\#_1 > 0$ everywhere and u is finite almost everywhere.

Consider a Lebesgue point $x \in X$ of the function u at which $u^\#_1(x) = \lambda < \infty$ and estimate the deviation of the value of u at the point x from the mean value u_X.

Put $r_k = 2^{-k} \cdot diamX$ and consider the sequence of balls $\{B_k = B(x, r_k)\}$. Since $u_X = u_{B_0}$ and $u_{B_k} \to u(x)$ as $k \to \infty$, we get

$$|u_X - u(x)| = |(u_{B_0} - u_{B_1}) + (u_{B_1} - u_{B_2}) + \dots + (u_{B_k} - u_{B_{k+1}}) + \dots| \le$$

$$\sum_{k=0}^{\infty} \frac{\mu(B_k)}{\mu(B_{k+1})} \fint_{B_k} |u - u_{B_k}|d\mu \le C_d \sum_{k=0}^{\infty} \fint_{B_k} |u - u_{B_k}|d\mu. \tag{7}$$

Consider two cases.

I. Let $diamX \le (\|h \mid L_p\|/\lambda)^{p/s(x)}$, where h is the function of the Poincaré inequality (6). Using (7), we infer

$$|u(x) - u_X| \le C_d \sum_{k=0}^{\infty} r_k \left(r_k^{-1} \fint_{B_k} |u - u_{B_k}|d\mu \right) \le C_1\, \lambda\, diamX \le$$

$$C_1\, \|h \mid L_p\|^{p/s(x)} \lambda^{p/q(x)} = C_1\, \|h \mid L_p\| \left(\|h \mid L_p\|^{-p} \lambda^p\right)^{1/q(x)}$$

or

$$\left(\frac{|u(x) - u_X|}{\|h \mid L_p\|} \right)^{q(x)} \le C_2\, (u^\#_1(x))^p\ \|h \mid L_p\|^{-p}. \tag{8}$$

II. If $(\|h \mid L_p\|/\lambda)^{p/s(x)} \le diamX$ then fix m such that

$$r_m \le (\|h \mid L_p\|/\lambda)^{p/s(x)} < 2r_m.$$

By (7),

$$|u_X - u(x)|C_d \left(\sum_{k=0}^{m} \fint_{B_k} |u - u_{B_k}| d\mu + \sum_{k=m+1}^{\infty} \fint_{B_k} |u - u_{B_k}| d\mu \right).$$

Estimate the first sum as in item I:

$$\sum_{k=0}^{m} \fint_{B_k} |u - u_{B_k}| d\mu \le \lambda \sum_{k=0}^{m} r_k \le C_3 \|h \mid L_p\|^{p/s(x)} \lambda^{p/q(x)} =$$

$$C_3 \|h \mid L_p\| \left(\|h \mid L_p\|^{-p} \lambda^p\right)^{1/q(x)}. \tag{9}$$

For estimating the second sum, we consecutively use the Poincaré inequality and Hölder's inequality:

$$\sum_{k=m+1}^{\infty} \fint_{B_k} |u - u_{B_k}|\, d\mu \le \sum_{k=m+1}^{\infty} r_k \fint_{B_k} h\, d\mu \le r_k \|h \mid L_p(X,\mu)\| \mu(B_k)^{-1/p} \le$$

$$C_4 \|h \mid L_p(X,\mu)\|\, r_m^{1-s(x)/p} = C_4 \|h \mid L_p(X,\mu)\|\, r^{-s(x)/q(x)} \le$$

$$C_5 \|h \mid L_p\| \left(\|h \mid L_p\|^{-p} \lambda^p\right)^{1/q(x)}. \tag{10}$$

Estimates (8), (9), (10) imply that the inequality

$$\left(\frac{|u(x) - u_X|}{\|h \mid L_p\|} \right)^{q(x)} \le \tilde{C}\, (u_1^{\#}(x))^p\ \|h \mid L_p\|^{-p}$$

holds for almost all $x \in X$. Integrating this inequality and taking into account the equivalence of the different norms observed in Lemma 1, we obtain

$$\rho_{q(\cdot)}(\left(\frac{|u(x) - u_X|}{\|h \mid L_p\|} \right) \le \tilde{C}\, \|u_1^{\#} \mid L_p\|^p\ \|h \mid L_p\|^{-p} \le M < \infty, \tag{11}$$

where the constant M does not depend on the function $u \in M_p^1(X, d, \mu)$.

By Lemma 4, inequality (11) implies

$$\|u - u_X\|_{q(\cdot)} \le K \|h \mid L_p\| \le C \|u \mid S_p^1(X, d, \mu)\|.$$

■

Like for a constant integrability exponent, in this case, we can consider the embedding into spaces of Sobolev type defined by the Hölder metrics d_γ.

For $0 < \gamma < 1$ and $(1 - \gamma)p < s_-$, we put

$$q_\gamma(x) = \frac{ps(x)}{s(x) - (1-\gamma)p}.$$

The proof of the boundedness of the embedding operator into the space $M^1_{q_\gamma}(\cdot)(X, d_\gamma \mu)$ is based on the estimates of the sharp maximal functions of fractional order $u_\gamma^{\#}$.

Theorem 2. *Suppose that a measure μ on a metric space (X, d) satisfies conditions (i), (ii), $0 < \gamma < 1$, and $(1 - \gamma)p < s_-$. Then the embedding operator*

$$I : S_p^1(X, d, \mu) \to S^1_{q_\gamma(\cdot)}(X, d_\gamma, \mu)$$

is continuous.

Proof. By Lemma 1 and inequality (5), it suffices to demonstrate that the condition $u_1^{\#} \in L_p(X, \mu)$ implies $u_\gamma^{\#} \in L_{q_\gamma(\cdot)}(X, \mu)$.

Since

$$u_\gamma^{\#}(x) = \sup_{r>0} r^{1-\gamma}\ r^{-1} \fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le (diam X)^{1-\gamma}\ u_1^{\#}(x),$$

we have $u_\gamma^{\#}(x) < \infty$ for almost all $x \in X$.

Let $u^{\#}_{\gamma}(x) = \lambda < \infty$. Choose r_0 so that

$$\lambda \le 2\, r_0^{-\gamma} \fint_{B(x,r_0)} |u - u_{B(x,r_0)}|\, d\mu;$$

then

$$\lambda\, r_0^{\gamma-1} \le 2\, r_0^{-1} \fint_{B(x,r_0)} |u - u_{B(x,r_0)}|\, d\mu \le 2\, u^{\#}_{1}(x). \tag{12}$$

Using the Poincaré inequality (6), Hölder's inequality, and condition (ii), we infer

$$\lambda \le 2\, r_0^{1-\gamma}\, r_0^{-1} \fint_{B(x,r_0)} |u - u_{B(x,r_0)}|\, d\mu \le 2 r_0^{1-\gamma} \fint_{B(x,r_0)} h\, d\mu \le$$

$$2 r_0^{1-\gamma}\, \|h \mid L_p\|\, \left(\mu(B(x,r_0)\right)^{-1/p} \le C_1\, r_0^{-s(x)/q_\gamma(x)}\, \|h \mid L_p\|.$$

This gives an estimate for r_0:

$$r_0^{-1} \ge C_2 \lambda^{q_\gamma(x)/s(x)}\, \|h \mid L_p\|^{-q_\gamma(x)/s(x)}. \tag{13}$$

Using (12), (13), and recalculating the exponents, we obtain the inequality

$$\left(\frac{u^{\#}_{\gamma}(x)}{\|h \mid L_p\|}\right)^{q_\gamma(x)} \le \tilde{C}\, (u^{\#}_{1}(x))^p\, \|h \mid L_p\|^{-p},$$

integrating which and taking into account the equivalence of the different norms, we obtain

$$\rho_{q_\gamma(\cdot)}\left(\frac{u^{\#}_{\gamma}(x)}{\|h \mid L_p\|}\right) \le \tilde{C}\, \|u^{\#}_{1} \mid L_p\|^p\, \|h \mid L_p\|^{-p} \le M < \infty, \tag{14}$$

where the constant M does not depend on the choice of the function $u \in M^1_p(X,d,\mu)$.

Lemma 4 and inequality (14) yield

$$\|u^{\#}_{\gamma}\|_{q(\cdot)} \le K\|h \mid L_p\| \le C\|u \mid S^1_p(X,d,\mu)\|.$$

■

Theorem 3. *Suppose that a measure μ on a compact metric space (X,d) satisfies conditions (i), (ii), where the function $s(x)$ is continuous and $1 < p < s_- = \min s(x)$. Then, for every sufficiently small $\varepsilon > 0$ and an arbitrary function $r(x)$ satisfying the estimate $1 \le r(x) \le q(x) - \varepsilon$, the embedding operator*

$$I : M^1_p(X,d,\mu) \to L_{r(\cdot)}(X,\mu)$$

is compact.

Proof. Fix $\varepsilon > 0$ and a function $r(x)$ satisfying the hypotheses of the theorem. Since $p < s_-$, there exists a constant $L < \infty$ such that $|q(x) - q(y)| \le L|s(x) - s(y)|$. Put $\varepsilon_1 = \varepsilon/2L$.

The function $s(x)$, being continuous on a compact space, is uniformly continuous. Therefore, the set X can be covered by a finite family of balls $\{B_k\}$ such that $|s(x) - s(y)| < \varepsilon_1$ for all $x, y \in B_k$.

Put $s_k = \sup_{x \in B_k} s(x)$ and $q_k = \dfrac{s_k p}{s_k - p}$, then $r(x) \le q(x) - \varepsilon \le q_k - \varepsilon/2$ for all $x \in B_k$.

Since, on the ball B_k, the measure μ is s_k-regular then the space $M^1_p(B_k,d,\mu)$ (by Lemma 2) is compactly embedded in $L_\omega(B_k,\mu)$ for all values of ω satisfying the inequality $q_k - \varepsilon/2 \le \omega < q_k$. Moreover, the space $L_\omega(B_k,\mu)$ is continuously embedded in $L_{r(\cdot)}(B_k,\mu)$. Hence, the embedding operator

$$I : M^1_p(B_k,d,\mu) \to L_{r(\cdot)}(B_k,\mu)$$

is compact because it is the composition of a compact operator and a continuous operator.

Since the family of balls $\{B_k\}$ is finite, the Sobolev-type space $M_p^1(X,d,\mu)$ is embedded in the Lebesgue space with variable integrability exponent $L_{r(\cdot)}(X,\mu)$. Moreover, from every bounded set of functions in $M_p^1(X,d,\mu)$, we can choose a sequence of functions $\{u_n\}$ whose restrictions to each of the balls B_k constitute a Cauchy sequence in the corresponding space $L_{r(\cdot)}(B_k,\mu)$. By Lemma 4,

$$\int_{B_k} |u_n(x)-u_m(x)|^{r(x)} d\mu \to 0 \quad \text{as} \quad n,m\to\infty$$

and

$$\rho_{r(\cdot)}(u_n-u_m) = \int_X |u_n(x)-u_m(x)|^{r(x)} d\mu \le$$

$$\le \sum_k \int_{B_k} |u_n(x)-u_m(x)|^{r(x)} d\mu \to 0 \quad \text{as} \quad n,m\to\infty.$$

Again by Lemma 4, we obtain $\|u_n-u_m\|_{X,r(\cdot)} \to 0$, i.e., the function sequence $\{u_n\}$ is a Cauchy sequence in $L_{r(\cdot)}(X,\mu)$, which completes the proof of the compactness of the embedding operator. ■

Theorem 4. *Suppose that a measure μ on a compact metric space (X,d) satisfies conditions (i), (ii), where the function $s(x)$ is continuous and $1<p<s_-/(1-\gamma)$. Then, for every sufficiently small $\varepsilon>0$ and an arbitrary function $r(x)$ satisfying the estimate $1\le r(x)\le q_\gamma(x)-\varepsilon$, the embedding operator*

$$I: M_p^1(X,d,\mu) \to M_{r(\cdot)}^1(X,d_\gamma,\mu)$$

is compact.

As for reckoning with Lemma 3, the proof of Theorem 4 is practically identical to that of Theorem 3. All changes amount to formally replacing $q(x)$ by $q_\gamma(x)$, the space $L_{r(\cdot)}(B_k,\mu)$, by $M_{r(\cdot)}^1(B_k,d_\gamma\mu)$, and checking in $L_{r(\cdot)}(X,\mu)$ that a sequence of admissible functions $\{g_{\gamma,n}\}$ such that

$$|u_n(x)-u_n(y)| \le (d(x,y))^\gamma \left(g_{\gamma,n}(x)+g_{\gamma,n}(y)\right)$$

is a Cauchy sequence. ■

Functions with Variable "Smoothness" Exponent

In the Euclidean case, for $p>n$, the functions of the space $W_p^1(R^n)$ satisfy the Hölder condition with exponent $\gamma=1-n/p$. In the metric case, inequality (5) and the boundedness of the sharp maximal function $u_\alpha^\#$ imply the Hölder continuity of the function u with exponent α with respect to the metric d. If a nonhomogeneous measure μ satisfies condition (ii) then, for large integrability exponents, a function $u\in S_p^1(X,d,\mu)$ can have different local "smoothness" characteristics in neighborhoods of different points since they are determined by the local properties of the measure and the properties of the sharp maximal function $u_{\gamma(\cdot)}^\#$, where $\gamma(x)=1-s(x)/p$.

Theorem 5. *If a measure μ on a metric space (X,d) satisfies conditions (i), (ii), $p>s_+=\sup\limits_{x\in X} s(x)$ and $u\in S_p^1(X,d,\mu)$ then $u_{\gamma(\cdot)}^\#\in L_\infty(X,\mu)$, where $\gamma(x)=1-s(x)/p$.*

Proof. Using the Poincaré inequality, Hölder's inequality, and condition (ii), we obtain

$$\frac{1}{r^{1-s(x)/p}} \fint_{B(x,r)} |u-u_{B(x,r)}|\,d\mu \le r^{s(x)/p} \fint_{B(x,r)} h\,d\mu \le$$

$$r^{s(x)/p}\left(\mu(B(x,r))\right)^{-1/p} \ \|h \mid L_p(X,\mu)\| \le C\,\|h \mid L_p(X,\mu)\| < \infty.$$

Therefore, $u_{\gamma(\cdot)}^\#\in L_\infty(X,\mu)$. ■

References

[1] P. Hajlasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal., **5**:4 (1996), 403–415. Zbl 0859.46022

[2] P. Hajlasz, J. Kinnunen, *Hölder quasicontinuity of Sobolev functions*, Rev. Mat. Iberoam., 1998, **14**:3 (1998), 601–622. Zbl 1155.46306

[3] P. Hajlasz, P. Koskela, *Sobolev Met Poincare* Mem. Am. Math. Soc., **688**, 2000. Zbl 0954.46022

[4] A.S. Romanov, *Traces of functions of generalized Sobolev classes*, Sib. Mat. Zh., **48**:4 (2007), 848–866. Zbl 1164.46326

[5] A.L. Vol'berg, S.V. Konyagin, *There is a homogeneous measure on any compact subset in* R^n, Sov. Math., Dokl., **30** (1984), 453–456. Zbl 0598.28010

[6] O. Kovacik, J. Rakosnik, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czech. Math. J, **41**:4 (1991), 592–618. Zbl 0784.46029

[7] A.S. Romanov, *Sobolev-type functions with variable integrability exponent on metric measure spaces*, Sib. Math. J., **55**:1 (2014), 142–155. Zbl 1314.46047

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