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FILTRATION IN COMPOSITE INCOMPRESSIBLE MEDIA WITH VARIABLE PORE SPACE STRUCTURE

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Abstract. This article is devoted to filtration in poroelastic media with variable pore space structure. Unlike previous works here the poroelastic medium has a variable pore space structure. The problem consists of two different settings and based on the problem for two different poroelastic media with a common boundary.

Keywords:continuous medium, heterogeneous medium, periodic structure, non-periodic structure, filtration, homogenization, two-scale convergence.

1. Introduction

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ perforated by pores. A pore space (a liquid domain) Ω_f is filled with a viscous liquid and there is a solid skeleton $\Omega_s = \Omega\backslash\overline{\Omega_f}$ which is supposed to be an elastic body. As usual, we use homogenization as the most appropriate method to get a practically significant mathematical model. To get something solvable and still reasonable, we use the scheme suggested in [4], [26] and linearize the basic system. That is, we approximate the characteristic function $\tilde{\chi}$ of the liquid domain Ω_f by its value at the initial time moment

$$\tilde{\chi} \simeq \chi_0(\mathbf{x}),$$

and the free boundary $\Gamma(t)$ by its initial position Γ_0.

Various particular cases of the linearization have been intensively studied by many authors: Buchanan – Gilbert [3], Burridge – Keller [4], Levy [13], Nguetseng [24], Sanchez-Hubert [25], Sanchez-Palencia [26]. We are based on ideas [15] – [20].

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To apply the well-known homogenization results ([23, 2]), we must consider special liquid domains Ω_f and impose the following constraints.

Assumption 1. *1) Let $\chi(\mathbf{y})$ be some 1-periodic function, $Y_s = \{\mathbf{y} \in Y : \chi(\mathbf{y}) = 0\}$ be the "solid part" of the unit cube $Y = (0,1)^3 \subset \mathbb{R}^3$, and let the "liquid part" $Y_f = \{\mathbf{y} \in Y : \chi(\mathbf{y}) = 1\}$ of Y be its open complement. We write $\gamma = \partial Y_f \cap \partial Y_s$ and assume that γ is a Lipschitz continuous surface.*

2) The domain E^ε_f is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y^\varepsilon_f = \varepsilon Y_f$ and the domain E^ε_s is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y^\varepsilon_s = \varepsilon Y_s$.

3) The pore space $\Omega^\varepsilon_f \subset \Omega = \Omega \cap E^\varepsilon_f$ is a periodic repetition in Ω of the elementary cell εY_f, and the solid skeleton $\Omega^\varepsilon_s \subset \Omega = \Omega \cap E^\varepsilon_s$ is a periodic repetition in Ω of the elementary cell εY_s. The Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon_s \cap \partial\Omega^\varepsilon_f$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$.

4) Y_s and Y_f are connected sets.

Under this assumption

$$\chi_0(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}) = \varsigma(\mathbf{x})\chi(\frac{\mathbf{x}}{\varepsilon}),$$

where $\varsigma(\mathbf{x})$ is the characteristic function of the domain Ω.

In dimensionless variables

$$\mathbf{x} \to \frac{\mathbf{x}}{L},\ \mathbf{w} \to \frac{\mathbf{w}}{L},\ t \to \frac{t}{\tau},\ \mathbf{F} \to \frac{\mathbf{F}}{g},\ \rho \to \frac{\rho}{\rho^0},$$

where L is the characteristic size of the physical domain in consideration, τ is the characteristic time of the physical process, ρ^0 is the mean density of water, and g is acceleration due gravity.

$\varepsilon = \dfrac{l}{L}$ is the dimensionless pore size, l is the average size of pores,

$$\alpha^\varepsilon_p = \alpha_{p,f}\chi^\varepsilon + \alpha_{p,s}(1-\chi^\varepsilon), \quad \varrho^\varepsilon = \varrho_f\,\chi^\varepsilon + \varrho_s\,(1-\chi^\varepsilon),$$

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\mu = \frac{2\mu}{\tau L g\,\rho^0}, \quad \alpha_\lambda = \frac{2\lambda}{L g\,\rho^0},$$

$$\alpha_\nu = \frac{2\nu}{\tau L g\,\rho^0}, \quad \alpha_{p,f} = \frac{\varrho_f\,c_f^2}{Lg}, \quad \alpha_{p,s} = \frac{\varrho_s\,c_s^2}{Lg},$$

ϱ_f and ϱ_s are the respective mean dimensionless densities of the liquid in pores and the solid skeleton, correlated with the mean density of water ρ^0.

We introduce the following criteria

$$\tau_0 = \lim_{\varepsilon\searrow 0}\alpha_\tau(\varepsilon), \quad \mu_0 = \lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon), \quad \lambda_0 = \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon),$$

$$c^2_{f,0} = \lim_{\varepsilon\searrow 0}\alpha_{p,f}(\varepsilon), \quad c^2_{s,0} = \lim_{\varepsilon\searrow 0}\alpha_{p,s}(\varepsilon),$$

$$\mu_1 = \lim_{\varepsilon\searrow 0}\frac{\alpha_\mu}{\varepsilon^2}, \quad \lambda_1 = \lim_{\varepsilon\searrow 0}\frac{\alpha_\lambda}{\varepsilon^2}.$$

For filtration processes $\tau_0 = 0$.

There are some particular results obtained by W. Jäger and A. Mikelić [7] – [9] for special geometry of pore space (disconnected solid skeleton) and only for domains in $\mathbb{R}^2$. We study the complete problem in $\mathbb{R}^3$ for the arbitrary geometry of corresponding pore spaces. We have considered liquid filtration in at least two domains with a common boundary, and with different properties in works [21], [22], where local heterogeneity has a periodic structure.

As is known, the most rigorous results of homogenization theory are obtained for very special physical media, when the local heterogeneity has a periodic structure. Therefore, the objections from opponents of such a method of mathematical modeling - that the models have a low practical value - are quite reasonable. The situation is similar to the situation with differential equations with constant coefficients, and differential equations with variable coefficients. The practical value of the first is not comparable to the practical value of the latter. But the history of mathematics shows that the theory of equations with variable coefficients cannot be constructed without a complete theory of equations with constant coefficients. The above analogy suggests a way to solve the problem of mathematical modeling of physical processes in macroscopic inhomogeneous media. The detailed analysis of the homogenized problems in composite domains permits the derivation of homogenized models allowing for the variable geometry and elasticity of the solid component.

Expressly, let Ω be a domain in consideration and $\Pi^{(\delta)} = \{K_1^{(\delta)}, ..., K_{N_\delta}^{(\delta)}\}$ be a partition of Ω into nonintersecting subdomains with a diameter δ. All physical and geometrical characteristics of the medium are assumed to be constant in the given subdomain $K_n^{(\delta)}$. The problem as formulated for a fixed δ is defined by the characteristic function of the pore space $\chi^{(\delta)}(\mathbf{x}, \mathbf{y})$. This function is 1-periodic in $\mathbf{y}$ and piecewise-constant in $\mathbf{x}$. The homogenized model obtained, depending on the parameter δ, has been already studied above and admits a subsequent limit as $\delta \to 0$, which leads to the final homogenized model,taking into account the macroscopic inhomogeneity of the continuum.

Note that the formal justification of the symmetry of the diagram(the limit as $\delta \to 0$ for fixed ε and then the limit as $\varepsilon \to 0$) is not physically rigorous, because the diameter δ of the subdomain $K_n^{(\delta)}$ cannot be less than the characteristic size ε of pores in the solid skeleton. Nevertheless, for sufficiently reasonable agreements the limit as $n \to \infty$ leads to the homogenization problem with a characteristic function of the pore space $\chi(\mathbf{x}, \mathbf{y}) = \lim_{\delta \to 0} \chi^{(\delta)}(\mathbf{x}, \mathbf{y})$ (in each $\mathbf{x}_0 \in \Omega$ there is a proper pore space, defined by the characteristic function $\chi(\mathbf{x}_0, \mathbf{y})$). The homogenization of this problem coincides with the final homogenized model, obtained before. This proves the correctness of our approach.

The problem consists of two different settings and based on the problem for two different poroelastic media with a common boundary. In the first setting we consider an arbitrary partition $\Pi^{(\delta)} = \{K_1^{(\delta)}, ..., K_{N_\delta}^{(\delta)}\}$ of the domain Ω, where the structure of $K_j^{(\delta)}$ is defined by the characteristic function $\chi^{(\varepsilon,\delta)}$, and pass to the limit as $\varepsilon \to 0$. After that pass to the limit as $\delta \to 0$. In the second setting we consider the same partition, but firstly pass to the limit as $\delta \to 0$ and after that pass to the limit as $\varepsilon \to 0$. In both cases we obtain the same result.

2. The problem statements and main results

We will try to model non-periodic poroelastic media with a variable structure in the domain Ω, described for $t > 0$ with the characteristic function $\chi_0(\mathbf{x})$ of the liquid domain Ω_f. To do this we use the standard procedure of approximation of variable coefficients by means of step functions.

Suppose that for some small positive δ

$$\chi_0(\mathbf{x}) = \chi_n(\frac{\mathbf{x}}{\delta}), \;\; \lambda_0 = \lambda_0^n \;\; \varrho_s = \varrho_s^n, \text{ for } \mathbf{x} \in K_n^{(\delta)},$$

where $\chi_n(\mathbf{y})$ is a 1-periodic in $\mathbf{y}$ function,

$$\lambda_0^n = const,\ \varrho_s^n = const,\ \Omega = \bigcup_{n=1}^{N} K_n^{(\delta)},$$

and for $\delta > 0$ the cube $K_n^{(\delta)}$ is an intersection of the domain Ω with the cube δK, $K = [0,1]^3 \subset \mathbb{R}^3$, $Int K_n^{(\delta)} \cap Int K_m^{(\delta)} = \emptyset$ for $m \neq n$.

Let

$$\chi^{(\delta)}(\mathbf{x},\mathbf{y}) = \chi_n(\mathbf{y}),\ \lambda_0^{(\delta)}(\mathbf{x}) = \lambda_0^n,\ \varrho_s^{(\delta)}(\mathbf{x}) = \varrho_s^n \ \text{ for } \mathbf{x} \in K_n^{(\delta)}$$

be step functions in the variable $\mathbf{x}$. Then $\chi^{(\delta)}(\mathbf{x},\mathbf{y})$ is a 1-periodic function in the variable $\mathbf{y}$.

Now, as usual, we consider in the domain Ω for $t > 0$ the problem

(1) $$\nabla \cdot \mathbf{w}^{\delta,\varepsilon} = 0,$$

(2) $$\nabla \cdot \big(\chi^{\delta,\varepsilon}\alpha_\mu \mathbb{D}\Big(x, \frac{\partial \mathbf{w}^{\delta,\varepsilon}}{\partial t}\Big) + (1-\chi^{\delta,\varepsilon})\lambda_0^{(\delta)}\mathbb{D}(x,\mathbf{w}^{\delta,\varepsilon}) - p^{\delta,\varepsilon}\,\mathbb{I}\big) + \varrho^{\delta,\varepsilon}\mathbf{F} = 0,$$

with the characteristic function

$$\chi^{\delta,\varepsilon}(\mathbf{x}) = \chi^{(\delta)}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})$$

of the pore space $\Omega_f^{\delta,\varepsilon}$, the solid density $\varrho_s^{(\delta)}(\mathbf{x})$, and the elasticity coefficient $\lambda_0^{(\delta)}(\mathbf{x})$, depending on the variable $\mathbf{x} \in \Omega$ and the small parameter $\varepsilon < \delta$.

The problem is completed with the boundary condition

(3) $$\mathbf{w}^{\delta,\varepsilon}(\mathbf{x},0) = 0,\ \ \mathbf{x} \in S = \partial\Omega,\ t > 0,$$

and initial and normalization conditions

(4) $$\chi^{\delta,\varepsilon}(\mathbf{x})\mathbf{w}^{\delta,\varepsilon}(\mathbf{x},0) = 0,\ \ \mathbf{x} \in \Omega,$$

(5) $$\int_\Omega \frac{\chi^{\delta,\varepsilon}(\mathbf{x})}{m^{(\delta)}(\mathbf{x})}\, p^{\delta,\varepsilon}(\mathbf{x},t)dx = 0,\ m^{(\delta)}(\mathbf{x}) = \int_Y \chi^{(\delta)}(\mathbf{x},\mathbf{y})dy.$$

In (2)

$$\varrho^{\delta,\varepsilon}(\mathbf{x}) = \chi^{\delta,\varepsilon}(\mathbf{x})\varrho_f + \big(1-\chi^{\delta,\varepsilon}(\mathbf{x})\big)\varrho_s^{(\delta)}(\mathbf{x}).$$

Differential equation (2) is understood as an integral identity

(6)
$$\int_0^T\int_\Omega \big(-\alpha_\mu\chi^{\delta,\varepsilon}\mathbb{D}(x,\mathbf{w}^{\delta,\varepsilon}) : \mathbb{D}(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}) + \lambda_0^{(\delta)}(1-\chi^{\delta,\varepsilon})\mathbb{D}(x,\mathbf{w}^{\delta,\varepsilon}) : \mathbb{D}(x,\boldsymbol{\varphi})\big)dxdt = \int_0^T\int_\Omega \big(p^{\,\delta,\varepsilon}(\nabla\cdot\boldsymbol{\varphi}) + \varrho^{\,\delta,\varepsilon}\mathbf{F}\cdot\boldsymbol{\varphi}\big)dxdt$$

for all functions $\boldsymbol{\varphi}$ vanishing at $t = T$, such that $\boldsymbol{\varphi},\ \dfrac{\partial\boldsymbol{\varphi}}{\partial t} \in \overset{\circ}{\mathbf{W}}{}_2^{1,0}(G_T)$.

Throughout this paper we impose Assumptions 1, for the structures, defined by characteristic functions $\chi_n(\mathbf{y})$. We additionally impose

Assumption 2. *The solid skeleton Ω_s^ε is a connected domain.*

Let's add one more supposition to these assumptions.

Assumption 3. *Let $\Omega = \bigcup_{n=1}^{N} K_n^{(\delta)}$, where $K_n^{(\delta)}$ is an intersection of Ω with the cube δK, $K = [0,1]^3 \subset \mathbb{R}^3$, $\mathrm{Int} K_n^{(\delta)} \cap \mathrm{Int} K_m^{(\delta)} = \emptyset$ for $m \neq n$, and*

$$\chi^{(\delta)}(\mathbf{x}, \mathbf{y}) = \chi_n(\mathbf{y}), \ \textit{for } \mathbf{x} \in K_n^{(\delta)}$$

be a characteristic function of the pore space in Ω.

Then the common pore space in Ω is connected (see [22, 21]), that is, for any $K_n^{(\delta)}$ and $K_m^{(\delta)}$, having a common boundary,

$$Y_f^{(n)} \bigcap Y_f^{(m)} \neq \emptyset, \quad Y_s^{(n)} \bigcap Y_s^{(m)} \neq \emptyset,$$

where $Y_f^{(n)}$ and $Y_f^{(m)}$ are elementary liquid domains and $Y_s^{(n)}$ and $Y_s^{(m)}$ are elementary solid domains, defined by characteristic functions $\chi_n(\mathbf{y})$ and $\chi_m(\mathbf{y})$ respectively.

Next we introduce an extension

$$\mathbf{w}_s^{\delta,\varepsilon}(\mathbf{x}, t) = \mathbb{E}^{(\delta)}_{\Omega_s^\varepsilon}\big(\mathbf{w}^{\delta,\varepsilon}\big)$$

from the solid part

$$\Omega_s^\varepsilon = \{\mathbf{x} \in \Omega : \chi^{(\delta)}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) = 0\}$$

of the domain Ω onto the whole domain Ω, with the following properties:

$$\big(1 - \chi^{\delta,\varepsilon}(\mathbf{x})\big)\big(\mathbf{w}^{\delta,\varepsilon}(\mathbf{x}, t) - \boldsymbol{w}_s^{\delta,\varepsilon}(\mathbf{x}, t)\big) = 0, \quad \mathbf{x} \in \Omega, \ t \in (0, T),$$

and

$$\int_\Omega |\boldsymbol{w}_s^{\delta,\varepsilon}(\mathbf{x}, t)|^2 dx \leqslant C_0 \int_{\Omega_s^\varepsilon} |\mathbf{w}^{\delta,\varepsilon}(\mathbf{x}, t)|^2 dx,$$
$$\int_\Omega |\mathbb{D}\big(x, \boldsymbol{w}_s^{\delta,\varepsilon}(\mathbf{x}, t)\big)|^2 dx \leqslant C_0 \int_{\Omega_s^\varepsilon} |\mathbb{D}\big(x, \mathbf{w}^{\delta,\varepsilon}(\mathbf{x}, t)\big)|^2 dx, \ \ t \in (0, T),$$

where C_0 is independent of ε, δ, and $t \in (0, T)$.

The existence of such an extension for domains Ω_s^ε with a non- periodic structure is proved as well as the existence of the extension for domains Ω_s^ε with periodic structure (see "Extension results"section).

Under these assumptions for solutions $\{\mathbf{w}^{\delta,\varepsilon}, p^{\delta,\varepsilon}\}$ of the problem (1) – (5) all statements of our previous works [21], [22] hold true, which we reformulate as the following theorems.

Theorem 1. *For all $\varepsilon > 0$ and for arbitrary time interval $[0, T]$ there exists a unique generalized solution of problem (1) – (5) and*

$$\text{(7)} \quad \max_{0<t<T} \int_\Omega \chi^{\delta,\varepsilon}\Big(\alpha_\mu |\mathbb{D}\big(x, \mathbf{w}^{\delta,\varepsilon}(\mathbf{x}, t)\big)|^2 + \frac{\alpha_\mu}{\varepsilon^2} |\mathbf{w}^{\delta,\varepsilon}(\mathbf{x}, t) - \mathbf{w}_s^{\delta,\varepsilon}(\mathbf{x}, t)|^2\Big) dx +$$
$$\int_0^T \int_\Omega \Big(|\pi^{\delta,\varepsilon}|^2 + \lambda_0^\delta |\mathbb{D}(x, \mathbf{w}^{\delta,\varepsilon})|^2\Big) dx dt \leqslant C_0 \mathfrak{P}^2,$$

where C_0 is independent of ε, λ_0^δ for $\lambda_0^{(\delta)} > \lambda^-$, and

$$\pi^{\delta,\varepsilon}(\mathbf{x}, t) = \int_0^t p^{\delta,\varepsilon}(\mathbf{x}, \tau) d\tau,$$

$$\mathfrak{P}^2 = \max_{0<t<T} \int_\Omega |\mathbf{F}(\mathbf{x}, t)|^2 dx < \infty.$$

Theorem 2. *Let*

$$\mu_0 = 0,\ \mu_1 = \infty,\ 0 < \lambda^- < \lambda_0^{(\delta)}(\mathbf{x}) < \lambda^+ < \infty,$$

$\{\mathbf{w}^{\delta,\varepsilon},\, p^{\,\delta,\varepsilon}\}$ *be the weak solution of the problem (1) – (5),*

$$\pi^{\,\delta,\varepsilon}(\mathbf{x},t) = \int_0^t p^{\delta,\varepsilon}(\mathbf{x},\tau)d\tau,$$

and $\mathbf{w}_s^{\delta,\varepsilon} = \mathbb{E}_{Q_s^\varepsilon}\big(\mathbf{w}^{\delta,\varepsilon}\big)$ *be an extension from the domain* $\Omega_s^{\delta,\varepsilon} = \{\mathbf{x}\in\Omega : \chi^{\delta,\varepsilon}(\mathbf{x}) = 0\}$ *onto the domain* Ω.

Then the sequences $\{\mathbf{w}^{\delta,\varepsilon}\}$ *and* $\{\chi^{\delta,\varepsilon}\pi^{\delta,\varepsilon}\}$ *converge weakly in* $\mathbf{L}_2(\Omega_T)$ *and* $L_2(\Omega_T)$ *as* $\varepsilon\to 0$ *to the functions* $\mathbf{w}_s^{(\delta)}$ *and* $m^{(\delta)}\,\pi_f^{(\delta)}$ *respectively, and the sequence* $\{\mathbf{w}_s^{\delta,\varepsilon}\}$ *converges weakly in* $W_2^{1,0}(\Omega_T)$ *as* $\varepsilon\to 0$ *to the function* $\mathbf{w}_s^{(\delta)}$.

The limiting functions solve in the domain Ω *for* $t>0$ *the homogenized system, consisting of the homogenized momentum balance equation*

$$\nabla\cdot\mathbb{P}_1^{(\delta)}(\mathbf{x}) + \hat{\varrho}^{(\delta)}(\mathbf{x})\,\mathbf{F} = 0, \tag{8}$$

$$\mathbb{P}_1^{(\delta)}(\mathbf{x}) = \lambda_0^{(\delta)}(\mathbf{x})\,\mathfrak{N}_1^{s,\delta}(\mathbf{x}) : \mathbb{D}(x,\mathbf{w}_s^{(\delta)}) - p_f^{(\delta)}\,\mathbb{I}, \tag{9}$$

and the continuity equation

$$\nabla\cdot\mathbf{w}_s^{(\delta)} = 0. \tag{10}$$

The problem is completed with the normalization condition

$$\int_\Omega p_f^{(\delta)}(\mathbf{x},t)dx = 0 \tag{11}$$

and the boundary condition

$$\mathbf{w}_s^{(\delta)} = 0 \tag{12}$$

on the outer boundary S *for* $t>0$.

In (8), (9)

$$p_f^{(\delta)} = \frac{\partial\pi_f^{(\delta)}}{\partial t},\ \hat{\varrho}^{(\delta)}(\mathbf{x}) = m^{(\delta)}(\mathbf{x})\,\varrho_f + \big(1-m^{(\delta)}(\mathbf{x})\big)\,\varrho_s^{(\delta)}(\mathbf{x}),$$

the symmetric strictly positively definite fourth-rank tensor $\mathfrak{N}_1^{s,\delta}(\mathbf{x})$ *is given at point* $\mathbf{x}\in\Omega$ *by the formula*

$$\mathfrak{N}_1^{s,\delta}(\mathbf{x}) = \mathfrak{N}^s + \langle\mathbb{D}(y,\mathbf{U}_0^{(0)})\rangle_{Y_s}\otimes\mathbb{I} - \langle\sum_{i,j=1}^{3} P_{\,0}^{(ij)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{J}^{ij} - \langle P_{\,0}^{(0)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{I}. \tag{13}$$

for the pore space with the characteristic function $\chi^{(\delta)}(\mathbf{x},\mathbf{y})$

Proof. The proof is similar to the proof of the theorems from works [21], [22], so we only give the main formulas here, omitting index δ.

A priori estimates of the previous theorem) guarantees the boundedness of sequences $\{\mathbf{w}^\varepsilon\}$, $\{\varepsilon\,\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$, $\{\chi^\varepsilon\,\mathbf{w}^\varepsilon\}$, $\{\boldsymbol{w}_s^\varepsilon\}$, $\{\mathbb{D}(x,\mathbf{w}_s^\varepsilon)\}$, $\{\chi^\varepsilon\,\pi^{\,\varepsilon}\}$, $\{(1-\chi^\varepsilon)\pi^{\,\varepsilon}\}$ and $\{\pi^{\,\varepsilon}\}$ in $\mathbf{L}_2(\Omega_T)$ and $L_2(\Omega_T)$. Therefore, these sequences, except $\{\varepsilon\,\chi^\varepsilon\,\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$, weakly converge in $\mathbf{L}_2(\Omega_T)$ and $L_2(\Omega_T)$ to functions $\mathbf{w}$, $\mathbf{w}^{(f)}$, $\mathbf{w}_s$, $\mathbb{D}(x,\mathbf{w}_s)$, $m\,\pi_f$, $(1-m)\pi_s$, and $\pi = m\,\pi_f + (1-m)\pi_s$ respectively.

Owing to Nguetseng's theorem, there exist 1-periodic in $\mathbf{y}$ functions

$$\mathbf{W}(\mathbf{x},t,\mathbf{y}),\ \mathbb{D}\big(y,\mathbf{W}(\mathbf{x},t,\mathbf{y})\big),\ \mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y}),\ \mathbb{D}\big(y,\mathbf{U}(\mathbf{x},t,\mathbf{y})\big),$$

$$\Pi_f(\mathbf{x},t,\mathbf{y}),\ \Pi_s(\mathbf{x},t,\mathbf{y}),\ \textit{and}\ \Pi(\mathbf{x},t,\mathbf{y})=\Pi_f+\Pi_s$$

such that the above mentioned sequences, including $\{\varepsilon\,\chi^\varepsilon\,\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$, two-scale converge in $\mathbf{L}_2(\Omega_T)$ and $L_2(\Omega_T)$ as $\varepsilon\to 0$ respectively to

$$\mathbf{W},\ \chi(\mathbf{y})\,\mathbb{D}\big(y,\mathbf{W}^{(f)}\big),\ \mathbf{W}^{(f)},\ \mathbf{w}_s,\ \mathbb{D}(x,\mathbf{w}_s)+\mathbb{D}\big(y,\mathbf{U}\big),\ \Pi_f,\ \Pi_s,\ \textit{and}\ \Pi.$$

The same theorem of Nguetseng states that

$$\mathbf{W}^{(f)}=\chi(\mathbf{y})\,\mathbf{W},\ \mathbf{W}=\mathbf{W}^{(f)}+\big(1-\chi(\mathbf{y})\big)\,\mathbf{w}_s(\mathbf{x},t),$$
$$\Pi_f=\chi(\mathbf{y})\,\Pi,\ \Pi_s=\big(1-\chi(\mathbf{y})\big)\,\Pi,$$

and

$$\mathbf{W},\,\mathbb{D}\big(y,\mathbf{W}\big),\ \mathbb{D}\big(y,\mathbf{U}\big),\,\Pi\in L_2(\Omega_T\times Y).$$

The two-scale limit in the continuity equation in the form

$$\int_{\Omega_T}\mathbf{w}^\varepsilon\cdot\nabla\xi dxdt=0 \tag{14}$$

with test function $\xi=\varepsilon\,\xi_0(\dfrac{\mathbf{x}}{\varepsilon})h(\mathbf{x},t)$, where functions $\xi_0(\mathbf{y})$ are finite in Y_f, results in the microscopic continuity equation

$$\nabla_y\cdot\mathbf{W}=0,\ \ \mathbf{y}\in Y_f. \tag{15}$$

The two-scale limit in the continuity equation gives us the missing microscopic continuity equation

$$\big(1-\chi(\mathbf{y})\big)\big(\nabla\,\cdot\,\mathbf{w}_s+\nabla_y\cdot\,\mathbf{U}\big)=0,\ \mathbf{y}\in Y. \tag{16}$$

The microscopic problem is completed with the normalization condition

$$\langle\mathbf{U}\rangle_{Y_s}=\int_{Y_s}\mathbf{U}dy=0.$$

$$\mathbb{D}(y,\mathbf{U})=\mathbb{B}_0^s(\mathbf{y}):\mathbb{D}(x,\mathbf{w}_s),$$

and

$$\mathfrak{N}^s=(1-m)\sum_{i,j=1}^{3}\mathbb{J}^{ij}\otimes\mathbb{J}^{ij}+\langle\mathbb{B}_0^s\rangle_{Y_s},$$

where

$$\mathbb{J}^{ij}=\frac{1}{2}\big(\mathbf{e}_i\otimes\mathbf{e}_j+\mathbf{e}_j\otimes\mathbf{e}_i\big),$$

$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a standard Cartesian basis, and the fourth-rank tensor $\mathbb{A}\otimes\mathbb{B}$ is the tensor (direct) product of the second-rank tensors $\mathbb{A}$ and $\mathbb{B}$: $(\mathbb{A}\otimes\mathbb{B}):\mathbb{C}=\mathbb{A}(\mathbb{B}:\mathbb{C})$ for any second-rank tensor $\mathbb{C}$.

$$\nabla_y\cdot\,\Big((1-\chi)\big(\mathbb{D}(x,\mathbf{w}_s)+\mathbb{D}(y,\mathbf{U})-\frac{1}{\lambda_0}(P_s-p_f)\mathbb{I}\big)\Big)=0, \tag{17}$$

and will look for a solution of (17) in the form

$$\mathbf{U}=\sum_{i,j=1}^{3}\mathbf{U}_0^{(ij)}(\mathbf{y})D_{ij}(\mathbf{x},t),$$

$$P_s-p_f=\lambda_0\sum_{i,j=1}^{3}P_0^{(ij)}(\mathbf{y})D_{ij}(\mathbf{x},t),$$

where

$$D_{ij}(\mathbf{x},t)=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}(\mathbf{x},t)+\frac{\partial u_j}{\partial x_i}(\mathbf{x},t)\Big),\quad \mathbf{w}_s=(u_1,\,u_2,\,u_3),$$

$$\mathbb{D}(x,\mathbf{w}_s)=\sum_{i,j=1}^{3}D_{ij}\,\mathbb{J}^{ij},$$

and

$$\left.\begin{array}{l}\nabla_y\cdot\Big((1-\chi)\big(\mathbb{D}(y,\mathbf{U}_0^{(ij)})+\mathbb{J}^{ij}-P_0^{(ij)}\mathbb{I}\big)\Big)=0,\ \mathbf{y}\in Y,\\ (1-\chi)\nabla_y\cdot\mathbf{U}_0^{(ij)}=0,\ \mathbf{y}\in Y,\ \langle\mathbf{U}_0^{(ij)}\rangle_{Y_s}=0.\end{array}\right\}\tag{18}$$

$$\mathfrak{N}^s=(1-m)\sum_{i,j=1}^{3}\mathbb{J}^{ij}\otimes\mathbb{J}^{ij}+\sum_{i,j=1}^{3}\langle\mathbb{D}(y,\mathbf{U}_0^{(ij)})\rangle_{Y_s}\otimes\mathbb{J}^{ij}.\tag{19}$$

To find $\mathfrak{N}_1^s$ we have to calculate expressions $\langle\mathbb{D}(y,\mathbf{U})\rangle_{Y_s}$ and p as operators on $\mathbb{D}(x,\mathbf{w}_s)$ and $(\nabla\cdot\mathbf{w}_s)$:

$$(1-m)\mathbb{D}(x,\mathbf{w}_s)+\langle\mathbb{D}(y,\mathbf{U})\rangle_{Y_s}=$$

$$\mathfrak{N}^s:\mathbb{D}(x,\mathbf{w}_s)+\langle\mathbb{D}(y,\mathbf{U}_0^{(0)})\rangle_{Y_s}\big(\nabla\cdot\mathbf{w}_s\big)=\Big(\mathfrak{N}^s+\langle\mathbb{D}(y,\mathbf{U}_0^{(0)})\rangle_{Y_s}\otimes\mathbb{I}\Big):\mathbb{D}(x,\mathbf{w}_s),$$

$$p=\langle P\rangle_Y=\langle\chi\,p_f+(1-\chi)P_s\rangle_Y=\langle p_f+(1-\chi)(P_s-p_f)\rangle_Y=$$

$$p_f+\langle(P_s-p_f)\rangle_{Y_s}=p_f+\lambda_0\langle\sum_{i,j=1}^{3}P_0^{(ij)}\rangle_{Y_s}D_{ij}+\lambda_0\langle P_0^{(0)}\rangle_{Y_s}\big(\nabla\cdot\mathbf{w}_s\big),$$

and

$$p\mathbb{I}-p_f\mathbb{I}=$$

$$\Big(\lambda_0\langle\sum_{i,j=1}^{3}P_0^{(ij)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{J}^{ij}\Big):\mathbb{D}(x,\mathbf{w}_s)+\Big(\lambda_0\langle P_0^{(0)}\rangle_{Y_s}\mathbb{I}\Big)\big(\nabla\cdot\mathbf{w}_s\big)=$$

$$\Big(\lambda_0\langle\sum_{i,j=1}^{3}P_0^{(ij)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{J}^{ij}+\lambda_0\langle P_0^{(0)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{I}\Big):\mathbb{D}(x,\mathbf{w}_s).$$

Therefore,

$$\mathfrak{N}_1^s=\mathfrak{N}^s+\langle\mathbb{D}(y,\mathbf{U}_0^{(0)})\rangle_{Y_s}\otimes\mathbb{I}-\langle\sum_{i,j=1}^{3}P_0^{(ij)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{J}^{ij}-\langle P_0^{(0)}\rangle_{Y_s}\mathbb{I}\otimes\mathbb{I}.$$

□

We refer to the problem (8) – (12) as the homogenized **model** $(\mathbb{FCM})_{13}^{(\delta)}$.

Theorem 3. *Let*

$$\mu_0=0,\ 0<\lambda^-<\mu_1,\ \lambda_0^{(\delta)}(\mathbf{x})<\lambda^+<\infty,$$

$\{\mathbf{w}^{\delta,\varepsilon},\,p^{\delta,\varepsilon}\}$ *be the weak solution of the problem (1) – (5),* $\mathbf{w}_s^{\delta,\varepsilon}=\mathbb{E}_{Q_s^\varepsilon}\big(\mathbf{w}^{\delta,\varepsilon}\big)$ *be an extension from the domain* $\Omega_s^\varepsilon=\{\mathbf{x}\in\Omega:\chi^{\delta,\varepsilon}(\mathbf{x})=0\}$ *onto the domain* Ω*, and*

$$\pi^{\,\delta,\varepsilon}(\mathbf{x},t)=\int_0^t p^{\delta,\varepsilon}(\mathbf{x},\tau)d\tau.$$

Then the sequences $\{\chi^{\delta,\varepsilon}\,\pi^{\delta,\varepsilon}\}$, *and* $\{\chi^{\delta,\varepsilon}\,\mathbf{w}^{\delta,\varepsilon}\}$ *converge weakly in* $L_2(\Omega_T)$ *and* $\mathbf{L}_2(\Omega_T)$ *as* $\varepsilon\to 0$ *to the functions* $m^{(\delta)}\,\pi_f^{(\delta)}$ *and* $\mathbf{w}^{(\delta,f)}$ *respectively, and the sequence* $\{\mathbf{w}_s^{\delta,\varepsilon}\}$ *converges weakly in* $\mathbf{W}_2^{1,0}(\Omega_T)$ *as* $\varepsilon\to 0$ *to the function* $\mathbf{w}_s^{(\delta)}$.

The limiting functions $\pi_f^{(\delta)}$, $\mathbf{w}^{(\delta,f)}$, *and* $\mathbf{w}_s^{(\delta)}$, *where* $\nabla\pi_f^{(\delta)}\in\mathbf{L}_2(\Omega_T)$, $\dfrac{\partial\pi_f^{(\delta)}}{\partial t}\in L_2(\Omega_T)$, *solve in the domain* Ω *for* $t>0$ *the homogenized system, consisting of the continuity equation*

$$\nabla\cdot\big(\mathbf{w}^{(\delta,f)}+\big(1-m^{(\delta)}(\mathbf{x})\big)\,\boldsymbol{w}_s^{(\delta)}\big)=0, \tag{20}$$

the homogenized momentum balance equation

$$\nabla\cdot\mathbb{P}_1^{(\delta)}(\mathbf{x})+\hat{\varrho}^{(\delta)}(\mathbf{x})\,\mathbf{F}=0, \tag{21}$$

$$\mathbb{P}_1^{(\delta)}(\mathbf{x})=\lambda_0^{(\delta)}(\mathbf{x})\,\mathfrak{N}_1^{s,\delta}(\mathbf{x}):\mathbb{D}(x,\mathbf{w}_s^{(\delta)})-p_f^{(\delta)}\,\mathbb{I} \tag{22}$$

for the solid component, and Darcy's law in the form

$$\mathbf{w}^{(\delta,f)}=m^{(\delta)}(\mathbf{x})\,\boldsymbol{w}_s^{(\delta)}+\frac{1}{\mu_1}\mathbb{B}^{(\delta)}(\mathbf{x})\cdot\big(-\nabla\,\pi_f^{(\delta)}+\varrho_f\int_0^t\mathbf{F}(\mathbf{x},\tau)d\tau\big) \tag{23}$$

for the liquid component.

The problem is completed with the normalization condition (11), the boundary condition (12) for the solid displacements $\mathbf{w}_s^{(\delta)}$, *and the boundary condition*

$$\mathbf{w}^{(\delta,f)}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})=0 \tag{24}$$

for the liquid displacements on the outer boundary S *for* $t>0$.

In (20) – (24) $\mathbf{n}(\mathbf{x})$ *is a unit normal to* S *at* $\mathbf{x}\in S$,

$$p_f^{(\delta)}=\frac{\partial\pi_f^{(\delta)}}{\partial t},\ \hat{\varrho}^{(\delta)}(\mathbf{x})=m^{(\delta)}(\mathbf{x})\,\varrho_f+\big(1-m^{(\delta)}(\mathbf{x})\big)\,\varrho_s^{(\delta)}(\mathbf{x}),$$

the symmetric strictly positively definite fourth-rank tensor $\mathfrak{N}_1^{s,\delta}(\mathbf{x})$ *is given for almost all points* $\mathbf{x}\in\Omega$ *by (13) for the pore space with the characteristic function* $\chi^{(\delta)}(\mathbf{x},\mathbf{y})$, *the symmetric strictly positive definite matrix* $\mathbb{B}^{(\delta)}(\mathbf{x})$ *is given for almost all points* $\mathbf{x}\in\Omega$ *by*

$$\mathbb{B}^{(\delta)}(\mathbf{x})=2\sum_{i=1}^{3}\big(\int_{Y_f}\mathbf{V}^{(i)}(\mathbf{y})dy\big)\otimes\mathbf{e}_i=2\sum_{i=1}^{3}\langle\mathbf{V}^{(i)}\rangle_{Y_f}\otimes\mathbf{e}_i, \tag{25}$$

for the pore space with the characteristic function $\chi^{(\delta)}(\mathbf{x},\mathbf{y})$

We refer to the problem (11), (12) – (24) as a homogenized **model** $(\mathbb{FCM})_{14}^{(\delta)}$.

Proof. As in the previous theorem, we will give a brief scheme of the proof, omitting index δ.

The two-scale limit in the integral identity corresponding to the continuity equation with special test function gives us the microscopic continuity equation

$$\nabla_y\cdot\mathbf{V}=0,\ \mathbf{y}\in Y. \tag{26}$$

The following equalities hold true

$$P(\mathbf{x},t,\mathbf{y})=\chi(\mathbf{y})\,p(\mathbf{x},t),\ Q(\mathbf{x},t,\mathbf{y})=\chi(\mathbf{y})\,q(\mathbf{x},t). \tag{27}$$

In the same way we arrive at the equation

$$\int_{\Omega_T}\Big(\varphi_0(\mathbf{x},t)\int_{Y_f}\big(\mu_1\mathbb{D}(y,\mathbf{V}):\mathbb{D}(y,\varphi_1)-\varrho_f\mathbf{F}\cdot\varphi_1(\mathbf{y})\big)dy- \nabla\varphi_0(\mathbf{x},t)\cdot\big(\int_{Y_f}\varphi_1(\mathbf{y})\chi(\mathbf{y})(\mathbf{y})dy\big)\,q\Big)dxdt = \int_{\Omega_T}\big(\varphi_0(\mathbf{x},t)a(\mathbf{x},t)+q\,\nabla\varphi_0(\mathbf{x},t)\cdot\mathbf{b}\big)dxd=0, \tag{28}$$

where

$$a(\mathbf{x},t)=\int_{Y_f}\big(\mu_1\mathbb{D}(y,\mathbf{V}):\mathbb{D}(y,\varphi_1)-\varrho_f\mathbf{F}\cdot\varphi_1(\mathbf{y})\big)dy,$$

$$\mathbf{b}=-\int_{Y_f}\varphi_1(\mathbf{y})dy=const.$$

Due to Lemma 5 we first may choose φ_1 such that $\mathbf{b}=\mathbf{e}_i$, $i=1,2,3$, where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a standard Cartesian basis. Nguetseng's Theorem guarantees that $a\in L_2(\Omega_T)$. Therefore,

$$\nabla q\in\mathbf{L}_2(\Omega_T).$$

Next we reintegrate (28) with respect the variables $(\mathbf{x},t)$ and arrive at the microscopic equation

$$\frac{\mu_1}{2}\triangle_y\mathbf{V}-\nabla_y Q-\nabla q+\varrho_f\mathbf{F}=0 \tag{29}$$

in the domain Y_f, which is understood in the sense of distributions. Here we have used the equality

$$\nabla\cdot\mathbb{D}(y,\mathbf{V})=\frac{1}{2}\triangle\mathbf{V}+\frac{1}{2}\nabla\big(\nabla\cdot\mathbf{V}\big),$$

and the continuity equation (26).

The term $\nabla_y\Pi(\mathbf{x},t,\mathbf{y})$ in (29) appears due to the orthogonality in $\mathbf{L}_2(Y_f)$ of the set of all divergence free vectors φ_1 to the set of gradients $\nabla_y\Pi$ of scalar functions Π.

The two-scale limit in the equality

$$(1-\chi^\varepsilon)\mathbf{v}^\varepsilon=0$$

gives us

$$\big(1-\chi(\mathbf{y})\big)\mathbf{V}(\mathbf{x},t,\mathbf{y})=0,\ \text{or}\ \mathbf{V}(\mathbf{x},t,\mathbf{y})=0,\ \mathbf{y}\in Y_s.$$

The last condition and the regularity condition (**??**) result in the boundary condition

$$\mathbf{V}(\mathbf{x},t,\mathbf{y})=0,\ \mathbf{y}\in\gamma=\partial Y_s\cap\partial Y_s. \tag{30}$$

Let $\mathbf{e}_i$, $i=1,2,3$ be the usual Cartesian basis in $\mathbb{R}^3$ and

$$\frac{2}{\mu_1}\Big(-\nabla q+\varrho_f\mathbf{F}\Big)=\sum_{i=1}^{3}z_i(\mathbf{x},t)\mathbf{e}_i.$$

Then the solution $\mathbf{V}$ of the problem (26), (29), and (30) has a form

$$\mathbf{V}=\sum_{i=1}^{3}z_i\mathbf{V}^{(i)}(\mathbf{y})=\frac{2}{\mu_1}\Big(\sum_{i=1}^{3}\mathbf{V}^{(i)}\otimes\mathbf{e}_i\Big)\cdot\big(-\nabla q+\varrho_f\mathbf{F}\big), \tag{31}$$

where the 1-periodic function $\mathbf{V}^{(i)}(\mathbf{y})$ solves the periodic boundary value problem

$$\left.\begin{aligned} &\triangle_y \mathbf{V}^{(i)} - \nabla Q^{(i)} = -\mathbf{e}_i, \quad \mathbf{y} \in Y_f, \\ &\nabla \cdot \mathbf{V}^{(i)} = 0, \quad \mathbf{y} \in Y_f, \\ &\mathbf{V}^{(i)} = 0, \quad \mathbf{y} \in \gamma. \end{aligned}\right\} \tag{32}$$

The existence and uniqueness results for the problem (32) and the properties of the matrix

$$\mathbb{B} = 2\sum_{i=1}^{3}\big(\int_{Y_f}\mathbf{V}^{(i)}(\mathbf{y})dy\big)\otimes\mathbf{e}_i = 2\sum_{i=1}^{3}\langle\mathbf{V}^{(i)}\rangle_{Y_f}\otimes\mathbf{e}_i,$$

follow from the energy equality

$$\int_{Y_f}\nabla\mathbf{V}^{(i)} : \nabla\mathbf{V}^{(j)}dy = \int_{Y_f}\mathbf{e}_i\cdot\mathbf{V}^{(j)}dy \tag{33}$$

□

We have not formulated all statements for a medium with a variable structure in this paper. In our future work, we will prove several more theorems and complete the construction of a mathematical model with a variable properties of the medium.

3. Auxiliary statements

3.1. **Two-scale convergence.** The method of two-scale convergence was proposed by G. Nguetseng [24] and has been applied to a wide range of homogenization problems (see, for example, the survey [14]).

A sequence $\{w^\varepsilon\} \subset L_2(\Omega_T)$ is said to be *two-scale convergent* to a function $\widetilde{W}(\mathbf{x},t,\mathbf{y},\tau) \in L_2(\Omega_T\times Y)$, 1-periodic in the variables $(\mathbf{y},\tau)\in Y\times(0,1)$, if and only if for any function $\sigma=\sigma(\mathbf{x},t,\mathbf{y},\tau)$, 1-periodic in $(\mathbf{y},\tau)$

$$\int_{\Omega_T} w^\varepsilon(\mathbf{x},t)\sigma(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon},\frac{t}{\varepsilon})dxdt \to \int_{\Omega_T}\big(\int_0^1\int_Y \widetilde{W}(\mathbf{x},t,\mathbf{y},\tau)\sigma(\mathbf{x},t,\mathbf{y},\tau)dyd\tau\big)dxdt \tag{34}$$

as $\varepsilon\to 0$.

In what follows we restrict ourself to the test functions $\sigma=\sigma(\mathbf{x},t,\mathbf{y})$. Then the relation 34 takes the form

$$\int_{\Omega_T} w^\varepsilon(\mathbf{x},t)\sigma(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})dxdt \to \int_{\Omega_T}\big(\int_Y W(\mathbf{x},t,\mathbf{y})\sigma(\mathbf{x},t,\mathbf{y})dy\big)dxdt, \tag{35}$$

where

$$W(\mathbf{x},t,\mathbf{y}) = \int_0^1 \widetilde{W}(\mathbf{x},t,\mathbf{y},\tau)d\tau.$$

The existence and main properties of weakly convergent sequences are established by the following fundamental theorem [24, 14]:

Theorem 4. *(**Nguetseng's theorem** for vector functions)*

1. *Any sequence $\{\mathbf{w}^\varepsilon\}$ bounded in $\mathbf{L}_2(\Omega_T)$ contains a subsequence, two-scale convergent to some function $\mathbf{W}\in\mathbf{L}_2(\Omega_T\times Y)$, 1- periodic in $\mathbf{y}$.*

2. *Let sequences $\{\mathbf{w}^\varepsilon\}$ and $\{\varepsilon\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$ be uniformly bounded in $\mathbf{L}_2(\Omega_T)$.*

Then there exists a function $\mathbf{W}=\mathbf{W}(\mathbf{x},t,\mathbf{y})$, 1-periodic in $\mathbf{y}$, and a subsequence $\{\mathbf{w}^\varepsilon\}$ such that $\mathbf{W}$, $\nabla_y\mathbf{W}\in\mathbf{L}_2(\Omega_T\times Y)$, and the subsequences $\{\mathbf{w}^\varepsilon\}$ and $\{\varepsilon\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$ two-scale converge in $\mathbf{L}_2(\Omega_T$ to $\mathbf{W}$ and $\mathbb{D}(y,\mathbf{W})$ respectively.

3. *Let sequences $\{\mathbf{w}^\varepsilon\}$ and $\{D(x,\mathbf{w}^\varepsilon)\}$ be bounded in $\mathbf{L}_2(\Omega_T$.*

Then there exist the functions $\mathbf{w}\in\mathbf{L}_2(\Omega_T)$ and $\mathbf{W}\in\mathbf{L}_2(\Omega_T\times Y)$ and a subsequence from $\{\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$ such that the function $\mathbf{W}$ is 1-periodic in $\mathbf{y}$, $\{\mathbb{D}(x,\mathbf{w})\in\mathbf{L}_2(\Omega_T)$, $D(y,\mathbf{W})\in\mathbf{L}_2(\Omega_T\times Y)$, and the subsequence $\{\mathbb{D}(x,\mathbf{w}^\varepsilon)\}$ two-scale converges to the function $\mathbb{D}(x,\mathbf{w})+D(y,\mathbf{W})$.

4. *Let $\sigma\in L_2(Y)$ and $\sigma^\varepsilon(\mathbf{x})=\sigma(\dfrac{\mathbf{x}}{\varepsilon})$. Assume that a sequence $\{w^\varepsilon\}\subset L_2(\Omega_T)$ two-scale converges to $W\in L_2(\Omega_T\times Y)$. Then the sequence $\{\sigma^\varepsilon w^\varepsilon\}$ two-scale converges to the function σW.*

3.2. **Extension results.** The following statements are valid due to the well-known results from [1, 5, 6, 24]. We formulate them in the forms that are appropriate for us.

Lemma 1. *(**Extension lemma for the scalar functions** [1, 6])*

Suppose that Assumptions 1 and 2 regarding the geometry of a periodic structure hold true (the domain Ω_s is a connected set) and $w\in W^1_2(\Omega)$.

Then there exists an extension

$$w_s=E_{\Omega^\varepsilon_s}(w),\quad \mathbb{E}_{\Omega^\varepsilon_s}:\,W^1_2(\Omega^\varepsilon_s)\to W^1_2(\Omega),\tag{36}$$

from the domain Ω^ε_s onto the whole domain Ω such that

$$\big(1-\chi^\varepsilon(\mathbf{x})\big)\big(w(\mathbf{x},t)-w_s(\mathbf{x},t)\big)=0,\quad \mathbf{x}\in\Omega,\ t\in(0,T),\tag{37}$$

and

$$\int_\Omega|w_s(\mathbf{x},t)|^2dx\leqslant C_0\int_{\Omega^\varepsilon_s}|w(\mathbf{x},t)|^2dx,$$
$$\int_\Omega|\nabla w_s(\mathbf{x},t)|^2dx\leqslant C_0\int_{\Omega^\varepsilon_s}|\nabla w(\mathbf{x},t)|^2dx,\ \ t\in(0,T),\tag{38}$$

where C_0 is independent of ε and $t\in(0,T)$.

Lemma 2. *(**Extension lemma for the vector functions** [5, 24])*

Suppose that Assumptions 1 and 2 on the geometry of periodic structure hold true (the domain Ω_s is a connected set) and $\mathbf{w}\in\mathbf{W}^1_2(\Omega)$.

Then there exists an extension

$$\mathbf{w}_s=\mathbb{E}_{\Omega^\varepsilon_s}(\mathbf{w}),\quad \mathbb{E}_{\Omega^\varepsilon_s}:\,\boldsymbol{W}^1_2(\Omega^\varepsilon_s)\to\boldsymbol{W}^1_2(\Omega),\tag{39}$$

from the domain Ω^ε_s onto the whole domain Ω such that

$$\big(1-\chi^\varepsilon(\mathbf{x})\big)\big(\mathbf{w}(\mathbf{x},t)-\mathbf{w}_s(\mathbf{x},t)\big)=0,\quad \mathbf{x}\in\Omega,\ t\in(0,T),\tag{40}$$

and

$$\int_{\Omega}|\mathbf{w}_s(\mathbf{x},t)|^2dx \leqslant C_0\int_{\Omega_s^\varepsilon}|\mathbf{w}(\mathbf{x},t)|^2dx, \tag{41}$$
$$\int_{\Omega}|\mathbb{D}(x,\mathbf{w}_s)|^2dx \leqslant C_0\int_{\Omega_s^\varepsilon}|\mathbb{D}(x,\mathbf{w})|^2dx,\ \ t\in(0,T),$$

where C_0 is independent of ε and $t\in(0,T)$.

For $w \in \overset{\circ}{W}{}_2^1(\Omega)$ ($\mathbf{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$) these statements do not guarantee the inclusion $w_s \in \overset{\circ}{W}{}_2^1(\Omega)$ ($\mathbf{w}_s \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$). But for the special geometry of the pore space the extension permits this inclusion, namely, the following lemma holds true.

Let the first geometry of the pore space be connected solid and liquid parts, and the second geometry be disconnected solid part

Lemma 3. ***(Extension lemma for the special geometry** [10, 24]**)***

Let Assumption 1 hold and $\mathbf{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$.

Then for the first geometry of the pore space there exist extensions $\mathbf{w}_s = \mathbb{E}_{\Omega_s^\varepsilon}(\mathbf{w})$ from Ω_s^ε onto Ω and $\mathbf{w}_f = \mathbb{E}_{\Omega_f^\varepsilon}(\mathbf{w})$ from Ω_f^ε onto Ω such that $\mathbf{w}_s, \boldsymbol{w}_f \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ and the estimates 40 for $\mathbf{w}_s$ and $\mathbf{w}_f$ hold true.

For the second geometry there exists the extension $\mathbf{w}_f = \mathbb{E}_{\Omega_f^\varepsilon}(\mathbf{w})$ from Ω_f^ε onto Ω such that $\mathbf{w}_f \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ and the estimates 40 for $\mathbf{w}_f$ holds true.

If for the second geometry additionally $\nabla\cdot\mathbf{w}=0$ in Ω, then $\nabla\cdot\mathbf{w}_f=0$ in Ω.

Sometimes we do not need the homogeneous boundary condition $\mathbf{w}_s=0$ on $\partial\Omega$, but we do need the estimate

$$\int_{\Omega}|\mathbf{w}_s|^2dx \leqslant C\int_{\Omega}|D(x,\mathbf{w}_s)|^2dx \tag{42}$$

with the constant C independent of ε.

For this case we prove the following statement.

Lemma 4. *Under the conditions of Lemma 2 let $\mathbf{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$.*

Then the estimate (42) holds true.

Proof. Let Q be a cube, $\Omega\subset Q$, and $\mathbf{u}$ be an extension of $\mathbf{w}$ such that $\mathbf{u}=0$ for $\mathbf{u}\in Q\backslash\Omega$. The inclusion $\mathbf{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ implies $\mathbf{u} \in \overset{\circ}{\mathbf{W}}{}_2^1(Q)$. For the domain Q we may define the solid part Q_s^ε and the extension $\mathbf{u}_s$ in the same way as for the domain Ω we have defined the solid part Ω_s^ε and the extension $\mathbf{w}_s$. It is clear that $\mathbf{u}_s \in \overset{\circ}{\mathbf{W}}{}_2^1(Q)$ for sufficiently large Q_s.

Thus, we may apply the Friedrichs–Poincaré inequality

$$\int_{Q}|\mathbf{u}_s|^2dx \leqslant C\int_{Q}|D(x,\mathbf{u}_s)|^2dx.$$

It is also clear that

$$\mathbf{u}_s=\mathbf{w}_s \,\mathrm{in}\,\Omega \text{ and } \int_{\Omega}|\mathbf{w}_s|^2dx \leqslant \int_{Q}|\mathbf{u}_s|^2dx.$$

Therefore

$$\int_{\Omega} |\mathbf{w}_s|^2 dx \leqslant C \int_{Q} |D(x,\mathbf{u}_s)|^2 dx.$$

Lemma 2 states that

$$\int_{Q} |D(x,\mathbf{u}_s)|^2 dx \leqslant C \int_{Q^{\varepsilon}_s} |D(x,\mathbf{u})|^2 dx.$$

But

$$\int_{Q^{\varepsilon}_s} |D(x,\mathbf{u})|^2 dx = \int_{\Omega^{\varepsilon}_s} |D(x,\mathbf{w})|^2 dx =$$

$$\int_{\Omega^{\varepsilon}_s} |D(x,\mathbf{w}_s)|^2 dx \leqslant \int_{\Omega} |D(x,\mathbf{w}_s)|^2 dx.$$

Gathering all together we arrive at the desired estimate (42). □

Lemma 5. *For any unit vector* $\mathbf{e}$ *there exists a solenoidal vector function* $\mathbf{u}(\mathbf{y})$ *satisfying condition*

$$\langle \mathbf{u} \rangle_{Y_f} = \mathbf{e}. \tag{43}$$

and the condition $\operatorname{supp} u \subset Y_f$.

Proof. Let $B \subset Y_f$ be a ball and $\mathbf{u}(\mathbf{y})$ be a nontrivial solution of the problem

$$\triangle \mathbf{u} - \nabla p = \boldsymbol{f}, \;\; \mathbf{y} \in B, \tag{44}$$

$$\nabla \cdot \mathbf{u} = 0, \;\; \mathbf{y} \in B, \tag{45}$$

$$\mathbf{u} = 0, \;\; \mathbf{y} \in \partial B \tag{46}$$

with some fixed function $\boldsymbol{f}(\mathbf{y})$.

We may always assume that

$$\int_B \mathbf{u} dy = \mathbf{e}_0, \;\; \text{with } |\mathbf{e}_0| = 1.$$

Let $\mathbb{T}$ be the orthogonal matrix and $\mathbb{T} \cdot \mathbf{e}_0 = \mathbf{e}$.

Then in the new variables $\mathbf{z} = \mathbb{T} \cdot \mathbf{y}$ the function $\mathbf{v}(\mathbf{z}) = \mathbb{T} \cdot \mathbf{u}(\mathbf{y})$ satisfies the problem

$$\triangle_z \mathbf{v} - \nabla_z q = \boldsymbol{F}, \;\; \mathbf{z} \in B, \tag{47}$$

$$\nabla_z \cdot \mathbf{v} = 0, \;\; \mathbf{v} \in B, \tag{48}$$

$$\mathbf{v} = 0, \;\; \mathbf{z} \in \partial B \tag{49}$$

where $\boldsymbol{F}(\mathbf{z}) = \mathbb{T} \cdot \boldsymbol{f}$ and $q(\mathbf{z}) = p(\mathbf{y})$. By the construction

$$\mathbf{e} = \mathbb{T} \cdot \mathbf{e}_0 = \int_B \mathbb{T} \cdot \mathbf{u} dy = \int_B \mathbf{v} dz.$$

□

4. Designations

Notations of functional spaces and norm there are the same as in [11], [12]
$\mathbb{I}$ – identity matrix;
$\mathbb{D}(x, \boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$;
$\mathbb{B} : \mathbb{C} = tr(\mathbb{B} \cdot \mathbb{C}^T)$, where $\mathbb{B}$, $\mathbb{C}$ – second-rank tensors;
$\boldsymbol{a} \otimes \boldsymbol{b}$ – dyad, for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$: $(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})$;
$\mathbb{J}^{ij} = \frac{1}{2}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i)$, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ – standard basis in $\mathbb{R}^3$;
$\mathbb{A} \otimes \mathbb{B}$ – fourth-rank tensor;
$(\mathbb{A} \otimes \mathbb{B}) : \mathbb{C} = \mathbb{A}(\mathbb{B} : \mathbb{C})$ for any second-rank tensors $\mathbb{A}$, $\mathbb{B}$) $\mathbb{C}$.

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