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A SEMILATTICE OF DEGREES OF COMPUTABLE METRICS

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ABSTRACT. We study the ordering $\mathcal{CM}_c(\mathbf{X})$ of c-degrees of computable metrics on a Polish space $\mathbf{X}$ with a distinguished dense subset. It is proved that this ordering forms a lower semilattice. If, for a computable metric ρ on $\mathbf{X}$, there is a computable limit point in (X, ρ), it is possible to construct a computable metric $\rho' <_c \rho$. Under the same assumption, there exists a computable metric $\widehat{\rho}$ such that $\deg_c(\rho)$ and $\deg_c(\widehat{\rho})$ have no common upper bounds in $\mathcal{CM}_c(\mathbf{X})$; thus, in this case $\mathcal{CM}_c(\mathbf{X})$ is not upward directed and is not an upper semilattice.

KEYWORDS: computable metric space, Cauchy representation, reducibility of representations, computable analysis

In computable analysis, a concept of computability on spaces of cardinality of at most continuum is commonly introduced via *representations,* i. e., partial surjections from the Cantor space or the Baire space. The notion of a representation generalizes the notion of a numbering of a countable set. Just like in the countable case, one aims to find the optimal representation with the most suitable effective properties and wants to somehow compare different representations with each other. For instance, it was pointed out by Turing [1] that the decimal representation of the real numbers is not suitable for developing a reasonable computability theory on the reals since addition and multiplication are not computable (not even continuous) with respect to this representation. This lead to an extensive study of representations of real numbers from the perspective of their topological and effective properties (see e. g. [2, 3, 4, 5, 6, 7, 8]).

To see whether two given representations of a space lead to the same concept of computability on this space, one should check whether they are reducible to each other. Reducibility of representations is defined just in the same manner as for numberings of a countable set. Representation δ_1 of a set X is *computably (continuously)* reducible to a representation δ_2 of X, written $\delta_1 \leq_c \delta_2$ ($\delta_1 \leq_t \delta_2$), if there is a Turing functional (a continuous functional) Φ that uniformly translates δ_1-names of elements of X into δ_2-names of these elements [4]. Intuitively, continuously equivalent representations induce the same notion of continuity on X, and computably equivalent representations induce the same notion of computability on X. It is clear that continuous and computable reducibilities are reflexive and transitive. Classical numbering theory is heavily concerned with the structural properties of the reducibility of numberings. Likewise, it is interesting to examine the properties of the orderings of degrees of representations under the mentioned reducibilities. For example, similarly to the classical fact that the equivalence classes of numberings form an upper semilattice, it turns out that the structure of c-degrees of representations forms a lattice under the natural *join* $\vee$ and *meet* $\wedge$ operations.

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Topological properties of operations $\vee$ and $\wedge$ have been studied in the literature; for example, it is known that $\delta_1 \wedge \delta_2$ is an admissible representation, given that δ_1 and δ_2 are admissible.

The study of representations is closely related to the problem of uniqueness of a computable presentation of a given space that is in turn inspired by the classical notions of computable categoricity and computable dimension of an algebraic structure. This problem was first considered by Pour-El and Richards in their book [9]. Following Melnikov [10], we say that a computably presentable metric space (X, ρ) is *computably categorical* if it has, up to computable isometry, a unique countable dense subset that makes it a computable metric space. Pour-El and Richards proved that a separable Hilbert space is computably categorical in the signature of Banach spaces, while the space ℓ^1 also contains "nonstandard" structures. Melnikov strengthened these results by showing that ℓ^2 is computably categorical as a metric space, while ℓ^1 is not; he also proved that Cantor and Urysohn spaces are computably categorical, while $C[0, 1]$ is not (see also [11]). McNicholl [12] showed that ℓ^2 is the only computably categorical space among the spaces ℓ^p. For more results in this direction, we refer the reader to [13, 14, 15, 16].

Our work is motivated by both these directions of research. In contrast to the approach of Pour-El and Richards, we want to see whether a topological space $\mathbf{X}$ with a fixed countable dense subset admits nonequivalent presentations induced by different metrics, where the equivalence can be understood in two different ways: presentations can be considered equivalent if either they are equivalent under the Weihrauch's computable reducibility or the corresponding metric spaces are computably homeomorphic. This gives rise to the definitions of reducibilities $\leq_c$ and $\leq_{ch}$ of metrics on $\mathbf{X}$ that were introduced in [17] for the case when $\mathbf{X}$ is the space of real numbers with the usual topology. It was proved in [17, 18] that there are infinitely many ch-nonequivalent metrics below the standard metric on $\mathbf{R}$ and above it. It was also proved that the countable atomless Boolean algebra can be embedded into the degree structure of $\leq_c$ above the degree of the standard metric.

In the present paper, we discuss the elementary properties of the degree structure of the computable reducibility $\leq_c$ on arbitrary Polish spaces. Let us give an informal definition of this reducibility. Consider a Polish space $\mathbf{X}$ with a distinguished dense subset W and fix a numbering of this subset; consider the set $M(\mathbf{X})$ of all complete metrics compatible with the topology of $\mathbf{X}$. For $\rho_1, \rho_2 \in M(\mathbf{X})$, we say that $\rho_1 \leq_c \rho_2$ if $\delta_{\rho_1} \leq_c \delta_{\rho_2}$, where δ_{ρ_i} is the *Cauchy representation* that puts points of X into correspondence with fast Cauchy sequences of elements of W converging to these points. Then the reduction $\rho_1 \leq_c \rho_2$ intuitively means that there exists an effective procedure that, given a fast Cauchy sequence in (X, ρ_1), outputs a fast Cauchy sequence in (X, ρ_2) converging to the same point. By $\mathcal{M}_c(\mathbf{X})$ we denote the degree structure of this reducibility, and by $\mathcal{CM}_c(\mathbf{X})$ we denote the structure of degrees of computable elements of $M(\mathbf{X})$. Our results can be formulated as follows. First of all, $\mathcal{CM}_c(\mathbf{X})$ forms a lower semilattice under a very natural operation: c-degree of a pointise maximum of two metrics is the meet of the degrees of these metrics. Moreover, $\mathcal{CM}_c(\mathbf{X})$ canonically embeds into $\mathcal{D}_c(X)$ as a lower semilattice. Secondly, in Theorem 3.1 we prove that if $\rho \in M(\mathbf{X})$ is a computable metric such that the corresponding computable metric space contains at least one computable limit point, then there exists a computable metric $\rho' <_c \rho$, thus $\deg_c(\rho)$ is not minimal in $\mathcal{CM}_c(\mathbf{X})$. This result is then relativized to show that, if $\mathbf{X}$ contains at least one limit point (with no computability-theoretical assumptions), then the ordering $\mathcal{M}_c(\mathbf{X})$ of c-degrees of all complete metrics on $\mathbf{X}$ contains no minimal elements and has cardinality $2^{\aleph_0}$. When $\mathbf{X}$ does not contain limit points, i. e., is discrete, $\mathcal{M}_c(\mathbf{X})$ does have a least element (in fact, when $\mathbf{X}$ is finite, $\mathcal{M}_c(\mathbf{X})$ consists of a single degree). Thirdly, in Theorem 4.1 we show that if $\rho \in M(\mathbf{X})$ is a computable metric such that (X, ρ) contains a computable limit point, then there exists a computable metric $\widehat{\rho}$ such that $\rho \nleq_c d$ or $\widehat{\rho} \nleq_c d$ for every computable metric d on $\mathbf{X}$. As a consequence, $\{\deg_c(\rho), \deg_c(\widehat{\rho})\}$ has no upper bound in $\mathcal{CM}_c(\mathbf{X})$, and $\mathcal{CM}_c(\mathbf{X})$ is not upward directed.

The paper is organized as follows. Section 1 contains necessary definitions. In Section 2 we prove that $\mathcal{CM}_c(\mathbf{X})$ is a lower semilattice. In Section 3 we prove Theorem 3.1 and the related results. Three remaining sections of the paper are devoted to the proof of our main result, Theorem 4.1. In Section 4 we state the theorem, break the proof into a series of requirements and discuss strategies for these requirements. Each requirement can be satisfied via a small deformation of the space (X, ρ). Section 5 contains an analytical argument showing that each deformation can be performed preserving the topology of $\mathbf{X}$. In fact, there are explicit formulas for the metric on the deformed space, which will later permit us to show that this metric is computable. The deformations are then assembled together into the resulting metric $\widehat{\rho}$. In the final section we write down the formal construction and show that $\widehat{\rho}$ is computable and satisfies all requirements.

1. Preliminaries

The *Baire space* is the set ω^ω of all countable sequences of natural numbers endowed with the product topology of countably many copies of ω with discrete topology.

Fix a standard pairing function $\langle \cdot, \cdot \rangle \colon \omega^2 \to \omega$ and left and right projections of this function, $\langle \cdot \rangle_0$ and $\langle \cdot \rangle_1$. As usual, the pairing function can be continued to a bijection $\langle \cdot \rangle \colon \omega^{<\omega} \to \omega$.

Partial computable functions and Turing functionals are denoted by uppercase letter Φ, and corresponding use functions are denoted by lowercase letter φ. For a partial computable function Φ_e, $\Phi_{e,s}(n)$ is the result of computation of $\Phi_e(n)$ in s steps with use $\varphi_{e,s}(n)$; similarly for a Turing functional.

A *numbering* of a set X is a partial surjection $\nu \colon \omega \to X$. A *representation* of X is a partial surjection $\delta \colon \omega^\omega \to X$. Let $\delta_X \colon \omega^\omega \to X$, $\delta_Y \colon \omega^\omega \to Y$ be representations of sets X and Y. Partial mapping $\Phi \colon \omega^\omega \to \omega^\omega$ is called a (δ_X, δ_Y)-*realization* of a partial function $F \colon X \to Y$ if

$$F \cdot \delta_X(f) = \delta_Y \cdot \Phi(f) \text{ for } f \in \operatorname{dom}(F \cdot \delta_X).$$

Function F is called (δ_X, δ_Y)-*computable* if it has a computable (δ_X, δ_Y)-realization, i. e., is realized by a Turing functional. Let $\nu_1 \colon \omega \to X_1, \dots, \nu_k \colon \omega \to X_k$ be numberings. Partial function $\Phi \colon \omega^{k+1} \to \omega$ is called a $(\nu_1, \dots, \nu_k, \delta)$-*realization* of a partial function $F \colon X_1 \times \ldots \times X_k \to Y$ if, whenever $(\nu_1 n_1, \dots, \nu_k n_k) \in \operatorname{dom}(F)$, then $g(m) = \Phi(n_1, \dots n_k, m) \downarrow$ for all m and $\delta(g) = F(x)$. F is called $(\nu_1, \dots, \nu_k, \delta)$-*computable* if it has a computable $(\nu_1, \dots, \nu_k, \delta)$-realization.

We refer the reader to Weihrauch's book [8] for a general definition of a computable function $F \colon X_1 \times \ldots \times X_k \to Y$, where each of X_i, Y can be either a numbered set or a represented set. We will not need that definition in its entirety.

Let δ, δ' be representations of a set X. Representation δ is *computably reducible* to δ' (written $\delta \leq_c \delta'$) if there exists a Turing functional Φ_z such that

$$\delta(f) = \delta'\Phi_z(f) \text{ for } f \in \operatorname{dom}(\delta),$$

or, equivalently, if the identity mapping id_X is (δ, δ')-computable. Binary relation $\leq_c$ is a preordering on the set of all representations of X. Define $\delta \equiv_c \delta'$ if $\delta \leq_c \delta'$ and $\delta' \leq_c \delta$. Equivalence class of δ under $\equiv_c$ is called the *c-degree* of δ, written $\deg_c(\delta)$. We have obtained a partial ordering $\mathcal{D}_c(X)$ of c-degrees of representations of X. As was already mentioned, $\mathcal{D}_c(X)$ is a lattice.

Let (X, ρ) be a metric space. We will use the following standard notation for open and closed balls in (X, ρ):

$$B_\rho(x, \varepsilon) = \{y \in X \mid \rho(x, y) < \varepsilon\},\ B_\rho[x, \varepsilon] = \{y \in X \mid \rho(x, y) \leqslant \varepsilon\}.$$

Balls B_1 and B_2 (either can be open or closed) with centers x_1 and x_2 and radii $\varepsilon_1, \varepsilon_2$, respectively, are called *formally disjoint* if $\rho(x_1, x_2) > \varepsilon_1 + \varepsilon_2$. This property implies that $B_\rho[x_1, \varepsilon_1] \cap B_\rho[x_2, \varepsilon_2] = \varnothing$. $B_\rho[x_1, \varepsilon_1]$ is said to be *formally contained in* $B_\rho(x_2, \varepsilon_2)$ if $\rho(x_1, x_2) < \varepsilon_2 - \varepsilon_1$. It implies that $B_\rho[x_1, \varepsilon_1] \subseteq B_\rho(x_2, \varepsilon_2)$. Usually, these notions are defined only for *rational* balls in computable metric spaces, i. e., balls with center in a special point and a rational radius; see definitions below. However, it will be more convenient for us to use them for arbitrary balls.

Let (X, ρ, W, ν) be a complete separable metric space with a distinguished countable dense subset W enumerated by integers via numbering $\nu\colon \omega \to W$. Elements of W will be referred to as *special points*. Throughout the paper we assume that ν is total. Denote $w_n = \nu n$. A *Cauchy name* for $x \in X$ is an element $f \in \omega^\omega$ such that

$$w_{f(n)} \underset{n\to\infty}{\longrightarrow} x \text{ and } \rho(w_{f(n)}, x) < 2^{-n} \text{ for all } n.$$

Define the *Cauchy representation* $\delta_{(X,\rho,W,\nu)}\colon \omega^\omega \to X$ by letting $\delta_{(X,\rho,W,\nu)}(f) = x$ if f is a Cauchy name for x. When X, W and ν are clear from context, we will denote the Cauchy representation by δ_ρ or just by ρ.

Remark 1. The requirement $\rho(w_{f(n)}, x) < 2^{-n}$ in the definition of a Cauchy name can be changed to $\rho(w_{f(n)}, w_{f(m)}) \leqslant 2^{-n}$ for $m > n$ or $\rho(w_{f(n)}, w_{f(m)}) < 2^{-n}$ for $m > n$, yielding a notion of a Cauchy representation that is c-equivalent to ours. The main property is quick convergence of $w_{f(n)}$ to x, with exact definition being a matter of convenience.

Space (X, ρ, W, ν) is called a *computable metric space,* and the underlying metric ρ is called *computable,* if the distance $\rho(w_n, w_m)$ is a computable real number uniformly in n and m, that is, there is a computable function $g\colon \omega^3 \to \omega$ such that $|q_{g(i,j,k)} - \rho(w_i, w_j)| < 2^{-k}$ for all $i, j, k \in \omega$, where q_n is the nth rational number under a fixed Gödel numbering of the rationals. This is equivalent to saying that $\rho \restriction W^2$ is a $(\nu, \nu, \delta_{\mathbf{R}})$-computable function, where $\delta_{\mathbf{R}}$ is the standard Cauchy representation of $\mathbf{R}$. More generally, for a Turing degree $\mathbf{d}$, metric ρ will be called $\mathbf{d}$-computable if $\rho \restriction W^2$ has a $(\nu, \nu, \delta_{\mathbf{R}})$-realization g with $\deg_T(g) = \mathbf{d}$.

Let $\mathbf{X} = (X, \tau, W, \nu)$ be a Polish space with a distinguished dense subset W and a numbering ν of W. Let $M(\mathbf{X})$ be the set of all complete metrics on X inducing topology τ. Our aim is to study a reducibility of elements of $M(\mathbf{X})$ derived from c-reducibility of representations in the following manner. For $\rho, \rho' \in M(\mathbf{X})$ let $\rho \leq_c \rho'$ if $\delta_{(X,\rho,W,\nu)} \leq_c \delta_{(X,\rho',W,\nu)}$. It is clear that $\leq_c$ is a preordering on $M(\mathbf{X})$.

Factorization by equivalence relation $\equiv_c$ leads to a partial ordering $\mathcal{M}_c(\mathbf{X})$ of c-degrees of metrics on $\mathbf{X}$. Ordering of c-degrees of computable metrics on $\mathbf{X}$ will be denoted by $\mathcal{CM}_c(\mathbf{X})$.

It is worth noting explicitly that these notions depend on the choice of W and ν. Our definitions are motivated by that we want to investigate the properties of c-reducibility of metrics on (X,τ) assuming that W and ν are *canonical*. Separable spaces occurring in analysis very often come with a natural countable dense subset admitting a canonical method of effective enumeration (a Gödel numbering). Common examples are the space of real numbers with rational numbers as dense subset or the space $C[0,1]$ with piecewise linear functions (or rational polynomials) as dense subset. Note that from the results of [10, 12] it follows that such a subset is, up to computable isometry, unique in such spaces as the real numbers, Cantor space and ℓ^2, while $C[0,1]$ and ℓ^p, $p \neq 2$, do not have this property.

2. $\mathcal{CM}_c(\mathbf{X})$ is a lower semilattice

Fix a Polish space $\mathbf{X} = (X, \tau, W, \nu)$. To prove our result, we will need the following useful lemma that was proved in [19]; see also [17].

Lemma 2.1. *Let $\rho_1, \rho_2 \in M(\mathbf{X})$. If $\rho_1(x,y) \leqslant \rho_2(x,y)$ for all $x, y \in X$, then $\rho_2 \leq_c \rho_1$.*

For $\rho_1, \rho_2 \in M(\mathbf{X})$ let $\rho(x,y) = \max(\rho_1(x,y), \rho_2(x,y))$. It is well-known that ρ satisfies the axioms of metric; it is easy to check that ρ induces τ and is complete. It follows from the previous lemma that $\rho \leq_c \rho_1, \rho_2$. As a consequence, $\mathcal{M}_c(\mathbf{X})$ is downward directed. We now show that $\deg_c(\rho)$ is the greatest lower bound of $\deg_c(\rho_1)$ and $\deg_c(\rho_2)$ provided that at least one of the metrics ρ_1, ρ_2 is computable; in particular, $\mathcal{CM}_c(\mathbf{X})$ is a lower semilattice. In fact, $\mathcal{CM}_c(\mathbf{X})$ is embedded into $\mathcal{D}_c(X)$ with preservation of meets.

Proposition 2.2. *Suppose that $\rho_1, \rho_2 \in M(\mathbf{X})$ and ρ_1 is computable. Let $\rho(x,y) = \max(\rho_1(x,y), \rho_2(x,y))$. Then $\deg_c(\delta_\rho) = \deg_c(\delta_{\rho_1}) \wedge \deg_c(\delta_{\rho_2})$ in the lattice $\mathcal{D}_c(X)$.*

Proof. By Lemma 2.1, $\delta_\rho \leq_c \delta_{\rho_1}, \delta_{\rho_2}$. We need to show that for any representation δ of X such that $\delta \leq_c \delta_{\rho_1}, \delta_{\rho_2}$ we also have $\delta \leq_c \delta_\rho$. Let δ be an arbitrary representation of X, c-reducible to δ_{ρ_1} and δ_{ρ_2} (say, by functionals Φ_e and Φ_z, respectively). Let $\delta(g) = x$ for some $g \in \omega^\omega$, $x \in X$. Then $f_1 = \Phi_e(g)$ and $f_2 = \Phi_z(g)$ are a δ_{ρ_1}-and a δ_{ρ_2}-name for x. We show that f_1 and f_2 can be effectively translated into a ρ-name f for x, this will imply that $\delta \leq_c \delta_\rho$.

Define f as follows. For $n \in \omega$, let $f(n) = f_2(m)$, where $m \geqslant n$ is such that $\rho_1(w_{f_2(m)}, w_{f_1(k)}) < 2^{-n-1}$ for some $k > n$. Since $w_{f_1(k)} \to x$ and $w_{f_2(k)} \to x$, such an m always exists. Then $\rho_2(w_{f(n)}, x) = \rho_2(w_{f_2(m)}, x) < 2^{-m} \leqslant 2^{-n}$ and $\rho_1(w_{f(n)}, x) \leqslant \rho_1(w_{f_2(m)}, w_{f_1(k)}) + \rho_1(w_{f_1(k)}, x) < 2^{-n-1} + 2^{-k} \leqslant 2^{-n}$. Thus, $\rho(w_{f(n)}, x) < 2^{-n}$ for all n, and f is a ρ-name for x. Since ρ_1 is computable, f is constructed effectively from f_1 and f_2. As a result, $\delta \leq_c \delta_\rho$. □

Corollary 2.3. *$\mathcal{CM}_c(\mathbf{X})$ is a lower semilattice. Order embedding of $\mathcal{CM}_c(\mathbf{X})$ into $\mathcal{D}_c(X)$ given by the rule $\rho \mapsto \delta_\rho$ is actually a lower semilattice embedding.*

Proof. If metrics ρ_1 and ρ_2 are computable, then so is $\max(\rho_1, \rho_2)$. The rest is by definitions and the previous proposition. □

Corollary 2.4. *In $\mathcal{M}_c(\mathbf{X})$, degree of a computable metric has a greatest lower bound with any other degree.*

3. Existence of a metric below the given one

3.1. **The computable case.** In this subsection we prove a theorem stating that, given a computable metric ρ and a ρ-computable limit point in $\mathbf{X}$, it is possible to construct a computable metric $\rho' <_c \rho$. The proof is essentially a generalization of [17, Theorem 4] to the case of an arbitrary metric space with at least one computable limit point.

Theorem 3.1. *Let $\mathbf{X} = (X, \tau, W, \nu)$ be a Polish space. Let $\rho \in M(\mathbf{X})$ be a computable metric such that the space (X, ρ, W, ν) contains a computable limit point. Then there exists a computable metric $\rho' \in M(\mathbf{X})$ such that $\rho' <_c \rho$. In particular, $\deg_c(\rho)$ is not minimal in $\mathcal{CM}_c(\mathbf{X})$.*

Proof. Fix a computable metric $\rho \in M(\mathbf{X})$ and a limit point $\lambda \in X$ that has a computable ρ-name. First of all, we show that there is a computable sequence of distinct points converging to λ.

Lemma 3.2. *There exists a computable $f \in \omega^\omega$ with the following properties:*

(1) $\rho(f) = \lambda$;
(2) $\rho(w_{f(n+1)}, \lambda) < \rho(w_{f(n)}, \lambda)$ *for all* $n \in \omega$.

Proof. Suppose that g is a computable ρ-name for λ. Let $w_{f(0)}$ be any special point not equal to λ; to obtain such a point, it suffices to make sure that $\rho(w_{f(0)}, g(k)) > 2^{-k+1}$ for some k, which can be done effectively. Similarly, let $w_{f(n+1)}$ be any point not equal to λ such that $\rho(w_{f(n+1)}, g(k_n)) < 2^{-k_n}$, where k_n is such that $\rho(w_{f(n)}, g(k_n)) > 2^{-k_n+1}$. Then $\rho(w_{f(n)}, \lambda) \geqslant |\rho(w_{f(n)}, g(k_n)) - \rho(g(k_n), \lambda)| > 2^{-k_n+1} - 2^{-k_n} = 2^{-k_n} > \rho(w_{f(n+1)}, \lambda)$. ☐

Metric ρ' should satisfy the following requirements:

$\mathcal{R}_e$: Φ_e does not reduce ρ to ρ',
$\mathcal{S}$: $\rho' \leq_c \rho$.

As in [17], requirement $\mathcal{S}$ will be satisfied by making $\rho'(x, y) \geqslant \rho(x, y)$ for all $x, y \in X$ and using Lemma 2.1. Strategy for $\mathcal{R}_e$ in isolation is based on the following simple corollary of the Use Principle. For $e, k \in \omega$ and $f \in \omega^\omega$, denote

$$C_\rho^{e,f,k} = \bigcap_{i=0}^{\varphi_e(f)(k+1)-1} B_\rho(w_{f(i)}, 2^{-i})$$

(recall that $\varphi_e(f)(n)$ is the use of $\Phi_e(f)(n)$).

Lemma 3.3. *Suppose that $\rho, \rho' \in M(\mathbf{X})$, Φ_e c-reduces ρ to ρ' and f is a ρ-name of some element $y \in X$. Then $C_\rho^{e,f,k} \subseteq B_{\rho'}(y, 2^{-k})$ for all $k \in \omega$.*

Proof. Since $C_\rho^{e,f,k}$ is open, together with every element $x \in C_\rho^{e,f,k}$ it contains a sequence $(w_{g(n)})_{n\in\omega}$ of special points converging to x. We can assume that this sequence converges quickly enough so that

$$g' = \big(f(0), \ldots, f(\varphi_e(f)(k+1) - 1), g(0), g(1), \ldots\big)$$

is a ρ-name for x. By the Use Principle, $\Phi_e(f)(k+1) = \Phi_e(g')(k+1)$ as f and g' agree on initial segment of length $\varphi_e(f)(k+1)$. Since Φ_e reduces ρ to ρ', then $\rho'(\Phi_e(f)) = y$ and $\rho'(\Phi_e(g')) = x$, so

$$\rho'(x,y) \leqslant \rho'(x, w_{\Phi_e(f)(k+1)}) + \rho'(w_{\Phi_e(f)(k+1)}, y) =$$
$$= \rho'(x, w_{\Phi_e(g')(k+1)}) + \rho'(w_{\Phi_e(f)(k+1)}, y) < 2^{-k-1} + 2^{-k-1} = 2^{-k}. \qquad \square$$

Let f be the name of λ constructed in Lemma 3.2. Define a computable function $\widehat{f}$ as follows: let $q_{\widehat{f}(n)} = r$ be a rational number such that

$$r < \min\bigl(\rho(w_{f(n)}, w_{f(n+1)}), \rho(w_{f(n)}, w_{f(n-1)})\bigr)/2.$$

Then the balls $B_\rho[w_{f(n)}, q_{\widehat{f}(n)}]$ are pairwise formally disjoint and do not contain λ. Let $f_0(n) = f(2n)$ and $f_1(n) = f(2n+1)$. Note that f_0, f_1 also are computable ρ-names for λ. Finally, let $f_2(n) = \widehat{f}(2n+1)$.

Lemma 3.3 says that, in order to meet $\mathcal{R}_e$, it is sufficient to show that there is at least one point $x_e \in C_\rho^{e,f_0,k} - B_{\rho'}(\lambda, 2^{-k})$ for some $k \in \omega$; we can let $k = e$. Proceed as follows. When $\Phi_e(f_0)(k+1)\downarrow$, pick any point $x_e \neq \lambda$ in $C_\rho^{e,f_0,e}$; this is possible since λ is a limit point and $C_\rho^{e,f_0,e}$ is its open neighbourhood. Let $r > 0$ be such that $\rho(x_e, \lambda) > r$. Define a "peak" function $\Gamma_e\colon X \to [0,1]$ by

$$\Gamma_e(x) = \begin{cases} 2^{-e} \cdot \frac{r - \rho(x, x_e)}{r}, & \text{if } x \in B_\rho(x_e, r), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\rho_e(x,y) = \rho(x,y) + |\Gamma_e(x) - \Gamma_e(y)|$. It is easy to see that the function Γ_e is continuous and ρ_e is a complete metric on X inducing the same topology as ρ. We have $\rho_e(x_e, \lambda) = \rho(x_e, \lambda) + \Gamma_e(x_e) > 2^{-e}$, so ρ_e satisfies the requirement $\mathcal{R}_e$.

In order to satisfy all requirements $\mathcal{R}_e$, we should assign a follower x_e to each $\mathcal{R}_e$, choose radii r_e and repeat the diagonalization process for each e. Followers x_e and radii r_e can be chosen, using functions f_1 and f_2. The resulting metric will have form

$$\rho'(x,y) = \rho(x,y) + \sum_{e\in\omega} \bigl|\Gamma_e(x) - \Gamma_e(y)\bigr|.$$

Axioms of metric are easily verified for ρ'. Note that, because $B_\rho[x_e, r_e]$ are pairwise disjoint, for each $x \in X$ there is at most one $e = e_x$ such that $\Gamma_{e_x}(x) \neq 0$; in particular, the definition of ρ' is correct. If $\Gamma_e(x) = 0$ for all e, it is convenient to say that $e_x = e$ for all e. Because of this, ρ' can be expressed as follows:

$$\rho'(x,y) = \begin{cases} \rho(x,y) + \bigl|\Gamma_{e_x}(x) - \Gamma_{e_y}(y)\bigr|, & \text{if } e_x = e_y, \\ \rho(x,y) + \Gamma_{e_x}(x) + \Gamma_{e_y}(y), & \text{otherwise.} \end{cases} \tag{3.1}$$

Let us show that the metric ρ' is complete and induces topology τ on X. Since $\rho'(x,y) \geqslant \rho(x,y)$ for all x, y, then any sequence converging in (X, ρ') converges in (X, ρ) to the same limit. By the same reason and by completeness of ρ, ρ' is complete. Suppose now that a sequence x_n converges to a point x in metric ρ. If $x \neq \lambda$, then it is not hard to see that $e_{x_n} = e_x$ for all sufficiently large n. Since Γ_{e_x} is continuous, then $\rho'(x_n, x) \to 0$ by (3.1). If $x = \lambda$, note that by the choice of balls $B_\rho[x_e, r_e]$ we have $\Gamma_e(\lambda) = 0$ for all e. Since $x_e \to \lambda$ and the height 2^{-e} of Γ_e tends to 0 as e increases, it is clear that $\Gamma_{e_{x_n}}(x_n) \to 0$. By (3.1), $\rho'(x_n, x) \to 0$.

As in other constructions of this kind [17, 18], at each stage s we output a finite set $A_s \subseteq W$. Sets A_s form an increasing sequence satisfying the following properties:

(1) $\Gamma(x) = 0$ for all $x \in A_s$ and all mappings Γ defined after stage s,
(2) $A_s \subseteq A_t$ for $s \leqslant t$,
(3) $\bigcup_{i\in\omega} A_s = W$.

This ensures that ρ' is computable, since for each w_n we are able to compute e_{w_n} as follows. Go to stage n. Then $\Gamma(w_n) = 0$ for all Γ defined after this stage. Let $\Gamma_{e_1}, \dots, \Gamma_{e_k}$ be all peaks defined so far. Since their supports $B_\rho[x_{e_1}, r_{e_1}], \dots, B_\rho[x_{e_k}, r_{e_k}]$ are disjoint, we can effectively determine the index i such that $w_n \notin B_\rho[x_{e_j}, r_{e_j}]$, $j \neq i$. Then $e_{w_n} = e_i$, and $\Gamma_{e_{w_n}}(w_n)$ is computable uniformly in n by definition of Γ_{e_i}. If no mapping Γ has been defined so far, then $\Gamma(w_n) = 0$ for all Γ that will ever be defined. Applying Formula (3.1), we see that the distance $\rho'(w_n, w_m)$ is computable uniformly in n, m.

Element $w_{f_1(i)}$ is called *fresh* at stage s if $i > j$ for all $w_{f_1(j)}$ seen in the construction so far.

Construction. *Stage 0.* Let $A_0 = \varnothing$, $\gamma_0 = 0$, $\rho_0 = \rho$.
Stage $s+1$. Let $A_{s+1} = A_s \cup \{w_s\}$. We work with requirement $\mathcal{R}_e$, $e = \langle s+1\rangle_0$, if this requirement has not been satisfied yet. Compute $\Phi_{e,s+1}(f_0)(e)$. If this computation halts, pick a fresh element $x_e = w_{f_1(i)} \in C_\rho^{e,f_0,e}$ such that $\rho(x_e, y) > 0$ for all $y \in A_{s+1}$. Choose rational r_e such that $0 < r_e \leqslant q_{f_2(i)}$ and $\rho(x_e, y) > r_e$ for all $y \in A_{s+1}$. Define a function $\Gamma_e \colon X \to [0,1]$ by

$$\Gamma_e(x) = \begin{cases} 2^{-e} \cdot \frac{r_e - \rho(x, x_e)}{r_e}, & \text{if } x \in B_\rho(x_e, r_e), \\ 0, & \text{otherwise.} \end{cases}$$

Verification. We have shown above that $\rho'(x,y) = \rho(x,y) + \sum_e \left|\Gamma_e(x) - \Gamma_e(y)\right|$, where the sum is taken over e such that Γ_e has been defined, is a complete metric inducing topology τ on X. By Lemma 2.1, $\rho' \leq_c \rho$.

To see that every requirement $\mathcal{R}_e$ is satisfied, consider two possibilities. If $\Phi_e(f_0)(e)\uparrow$, then Φ_e clearly fails to reduce ρ to ρ'. Otherwise, there is a stage s at which we find out that $\Phi_e(f_0)(e)\downarrow$, pick an element $x_e \in C_\rho^{e,f_0,e}$ and define a mapping Γ_e such that $\Gamma_e(x_e) = 2^{-e}$. Since $\Gamma(\lambda) = 0$ for all Γ, then $\rho'(x_e, \lambda) = \rho(x_e, \lambda) + 2^{-e} > 2^{-e}$, and $x_e \notin B_{\rho'}(\lambda, 2^{-e})$. Lemma 3.3 guarantees that Φ_e does not reduce ρ to ρ'.

At stage s we choose the radius r_e so that $B_\rho(x_e, r_e) \cap A_s = \varnothing$. Thus, $\Gamma(x) = 0$ for $x \in A_s$ and all mappings Γ defined after stage s. We have shown above that it gives a method of computation of $\rho'(w_n, w_m)$ uniformly in n and m. □

Theorem 3.4. *Under the assumptions of Theorem 3.1, the ordering* $(P(\omega), \subseteq)$ *of subsets of* ω *is (anti-)isomorphically embeddable into* $\mathcal{M}_c(\mathbf{X})$ *below* $\deg_c(\rho)$.

Proof. For any set $A \subseteq \omega$ we will construct a metric $\rho_A \leq_c \rho$ in such a way that, for all $A, B \subseteq \omega$, $A \subseteq B$ if and only if $\rho_B \leq_c \rho_A$; this will give us the embedding in question. The first step of the proof is to construct metrics $\rho_{\{i\}}$ for $i \in \omega$, satisfying the following series of requirements:

$\mathcal{R}_{ie}$: Φ_e does not reduce $\rho_{\{j\}}$ to $\rho_{\{i\}}$ for all $j \neq i$.

Note that requirement $\mathcal{R}_{ie}$ says that Φ_e fails to reduce an infinite amount of metrics $\rho_{\{j\}}$ to $\rho_{\{i\}}$. Let f_0, f_1, f_2 be computable functions from the proof of Theorem 3.1.

Construction runs in the same way as before. In order to meet $\mathcal{R}_{ie}$, proceed as follows. Let $n = \langle i, e\rangle$. When $\Phi_e(f_0)(n)\downarrow$, choose a fresh element $x_n = w_{f_1(k)} \in C^{e,f_0,n}_{\rho}$ and define a peak Γ_n of height 2^{-n}. Metric $\rho_{\{i\}}$ has form $\rho_{\{i\}}(x,y) = \rho(x,y)+\sum_n\left|\Gamma_n(x)-\Gamma_n(y)\right|$, where the sum is taken over $n = \langle i,e\rangle$ such that Γ_n has been defined in the construction. Let x_n be a follower of a requirement $\mathcal{R}_{ie}$. Since the supports of different peaks are pairwise disjoint, it is clear that $\Gamma_m(x_n) = 0$ for all $m = \langle j, e'\rangle$, $j \neq i$, thus $\rho_{\{j\}}(x_n, \lambda) = \rho(x_n, \lambda)$ and $x_n \in C^{e,f_0,n}_{\rho_{\{j\}}} - B_{\rho_{\{i\}}}(\lambda, 2^{-n})$. By definition of f_1 and f_2, points $w_{f_0(n)}$ are not affected by the construction, i. e., $\Gamma(w_{f_0(n)}) = 0$ for all Γ, so f_0 is a name for λ in all metrics $\rho_{\{k\}}$. By Lemma 3.3, Φ_e does not reduce $\rho_{\{j\}}$ to $\rho_{\{i\}}$, and $\mathcal{R}_{ie}$ is satisfied.

For a set $A \subseteq \omega$, let

$$\rho_A(x,y) = \rho(x,y) + \sum_{n=\langle i,e\rangle,\ i\in A} \left|\Gamma_n(x) - \Gamma_n(y)\right|.$$

It is clear that if $A \subseteq B$, then $\rho_A(x,y) \leqslant \rho_B(x,y)$ for all x, y, thus $\rho_B \leq_c \rho_A$. On the other hand, suppose that $A \not\subseteq B$. Fix any $i \in A - B$. Suppose that x_n is a follower of requirement $\mathcal{R}_{ie}$. Arguing as above, we see that $\rho_B(x_n, \lambda) = \rho(x_n, \lambda)$, so $x_n \in C^{e,f_0,n}_{\rho_B} - B_{\rho_A}(\lambda, 2^{-n})$, and Φ_e does not reduce ρ_B to ρ_A. □

Corollary 3.5. *Any countable partial ordering is isomorphically embeddable into $\mathcal{CM}_c(\mathbf{X})$ under $\deg_c(\rho)$. In particular, $|\mathcal{CM}_c(\mathbf{X})| = \aleph_0$.*

Proof. Note that metric ρ_A from the proof of the previous theorem is computable if A is a computable set: to compute $\rho_A(w_j, w_k)$, use Formula (3.1), remembering that $\Gamma_n(x)$ is not counted towards $\rho_A(x,y)$ if $n = \langle i, e\rangle$ for $i \notin A$. Then we are able to embed the computable countably-universal partial ordering into $\mathcal{CM}_c(\mathbf{X})$, as in [18, Theorem 2]. □

Proofs of two theorems above rely on the existence of a ρ-computable limit point. Let us give examples of simple conditions guaranteeing the existence of such a point.

Corollary 3.6. *Suppose that there is a special limit point $\lambda \in W$. Then $\mathcal{CM}_c(\mathbf{X})$ contains no minimal elements.*

Proof. Let $\lambda = w_n$. Then $\bar{n} = (n, n, \ldots)$ is a computable ρ-name for λ in any metric ρ, and we are able to run the construction from Theorem 3.1. □

Corollary 3.7. *Suppose that $\mathbf{X}$ contains no isolated points. Then $\mathcal{CM}_c(\mathbf{X})$ contains no minimal elements.*

3.2. **The general case.** Direct relativization of results from the previous subsection gives us the following theorem.

Theorem 3.8. *Suppose that a Polish space $\mathbf{X}$ contains at least one limit point λ. Then the following hold:*

1. *For any metric $\rho \in M(\mathbf{X})$, there exists a metric $\rho' \in M(\mathbf{X})$ such that $\rho' <_c \rho$. Moreover, if ρ is $\mathbf{d}$-computable and λ has an $\mathbf{e}$-computable ρ-name, where $\mathbf{d}$ and $\mathbf{e}$ are Turing degrees, then ρ' is $\mathbf{d} \cup \mathbf{e}$-computable.*
2. *$\mathcal{M}_c(\mathbf{X})$ contains no minimal elements.*
3. *$(P(\omega), \subseteq)$ is isomorphically embeddable into $\mathcal{M}_c(\mathbf{X})$ below any degree.*
4. *$|\mathcal{M}_c(\mathbf{X})| = 2^{\aleph_0}$.*

Proof. Immediate. The last clause follows from the fact that there are exactly $2^{\aleph_0}$ metrics on a separable space $\mathbf{X}$. □

3.3. **Discrete spaces.** We have shown that $\mathcal{M}_c(\mathbf{X})$ contains no minimal elements whenever $\mathbf{X}$ contains a limit point. What can be said when $\mathbf{X}$ contains no limit points, i. e., is a discrete space? There are two different cases: $\mathbf{X}$ is a finite space, or $\mathbf{X}$ is infinite, in which case it, being separable, is necessarily countable. Note that in any discrete space $\mathbf{X}$ the only possible dense set is X itself.

Proposition 3.9. *Let $\mathbf{X} = (X, \tau, X, \nu)$ be a Polish space, where X is a finite set and $\nu\colon \omega \to X$ is an arbitrary numbering. Then $|\mathcal{M}_c(\mathbf{X})| = 1$.*

Proof. Fix arbitrary metrics $\rho_1, \rho_2 \in M(\mathbf{X})$. There are $\varepsilon_1, \varepsilon_2 > 0$ such that $\rho_i(x, y) > \varepsilon_i$ for all $x \neq y \in X$ and $i = 1, 2$. By Lemma 2.1, ρ_i is c-equivalent to a metric $\rho'_i = \frac{1}{\varepsilon_i}\rho_i$. We have $\rho'_i(x, y) > 1$ for $x \neq y \in X$. By definition of a Cauchy name, whenever $\rho'_1(f) = x \in X$, then $\nu f(n) = x$ for all n (i. e., f "enumerates" a sequence consisting of a single point x), so we also have $\rho'_2(f) = x$. Similarly, every ρ'_2-name is a ρ'_1-name for the same element. Thus, $\rho_1 \equiv_c \rho'_1 \equiv_c \rho'_2 \equiv_c \rho_2$. □

Proposition 3.10. *Let $\mathbf{X} = (X, \tau, X, \nu)$ be a discrete Polish space, where X is a countable set and $\nu\colon \omega \to X$ is an arbitrary numbering. Let ρ be the standard discrete metric on X given by $\rho(x, y) = 1$ for $x \neq y$. Then $\deg_c(\rho)$ is the least degree in $\mathcal{M}_c(\mathbf{X})$.*

Proof. As in the previous proposition, if $\rho(f) = x$, then $\nu f(n) = x$ for all n, thus $\rho'(f) = x$ for every metric $\rho' \in M(\mathbf{X})$. □

Let $\mathbf{X} = (X, \tau, X, \nu)$ be a countable discrete Polish space. W. l. o. g. we can assume that $X = \omega$. Identifying integer n with the real number n, we can view ω as a subspace of $(\mathbf{R}, \rho_R)$, where ρ_R is the standard metric on the reals. The construction of [18, Theorem 2] gives continuum many pairwise c-nonequivalent metrics on $\mathbf{X}$. The details are easy and will be left to the reader. As a consequence, we have the following.

Proposition 3.11. *For a Polish space $\mathbf{X} = (X, \tau, W, \nu)$, the following are equivalent:*

(1) $|\mathcal{M}_c(\mathbf{X})| = 2^{\aleph_0}$,
(2) $|\mathcal{M}_c(\mathbf{X})| > 1$,
(3) *X is infinite.*

4. Degrees with no common upper bounds

Theorem 4.1. *Suppose that $\mathbf{X} = (X, \tau, W, \nu)$ is a Polish space, ρ is a computable metric on $\mathbf{X}$ and $\lambda \in X$ is a limit point that has a computable ρ-name. Then there exists a computable metric $\widehat{\rho} \in M(\mathbf{X})$ such that $\rho \not\leq_c d$ or $\widehat{\rho} \not\leq_c d$ for any computable metric $d \in M(\mathbf{X})$. In other words, $\{\deg_c(\rho), \deg_c(\widehat{\rho})\}$ has no upper bound in $\mathcal{CM}_c(\mathbf{X})$.*

Observe some simple consequences of the theorem.

Corollary 4.2. *If there is a computable metric $\rho \in M(\mathbf{X})$ such that the corresponding metric space contains a computable limit point, then $\mathcal{CM}_c(\mathbf{X})$ is not upward directed, is not an upper semilattice and does not have a greatest element.*

Corollary 4.3. *Suppose that there is a special limit point* $\lambda \in W$*. Then*
$$\mathcal{CM}_c(\mathbf{X}) \models \forall \mathbf{a}\, \exists \mathbf{b}\, \forall \mathbf{c}\, (\mathbf{a} \not\leq \mathbf{c} \vee \mathbf{b} \not\leq \mathbf{c}).$$
In particular, this holds when $\mathbf{X}$ *contains no isolated points.*

Let us outline the idea of the proof of the theorem. Similarly to [20], we can list all computable metrics on $\mathbf{X}$ in the following manner. By definition, for every computable metric d on $\mathbf{X}$ there exists a computable function $\Phi_e\colon \omega^3 \to \omega$ such that
$$d(w_i, w_j) = d_e(w_i, w_j) = \lim_{k\to\infty} q_{\Phi_e(i,j,k)}.$$
Since W is dense in X, d is completely determined by its values on W. Thus, the usual numbering of all partial computable functions $(\Phi_e)_{e\in\omega}$ of three variables gives us a list of all partial functions $d_e\colon X^2 \to \mathbf{R}$ that can possibly be a computable metric on $\mathbf{X}$. This list exhausts all possible computable metrics on $\mathbf{X}$, but $e \mapsto d_e$ is not a numbering of the class of all computable metrics on $\mathbf{X}$ since our list also includes "junk" functions.

We construct the metric $\widehat{\rho}$, satisfying the following requirements for $e, z, z' \in \omega$:

$\mathcal{R}_{ezz'}$: If d_e is a metric on $\mathbf{X}$ and $\rho \leq_c d_e$ via Φ_z, then $\widehat{\rho} \not\leq_c d_e$ via $\Phi_{z'}$.

Let us address the question of computability of $\widehat{\rho}$. Metric $\widehat{\rho}$ will be a pointwise limit of computable metrics ρ_s defined in the course of the construction. As in the proof of Theorem 3.1, at each stage s we output a finite set $A_s \subseteq W$. These sets will satisfy the following properties:

(1) $\rho_s(z, v) = \rho_{s+1}(z, v) = \ldots = \widehat{\rho}(z, v)$ for all $z, v \in A_s$,
(2) $A_s \subseteq A_t$ for $s \leqslant t$,
(3) $\bigcup_{i\in\omega} A_s = W$.

This permits us to prove that the metric $\widehat{\rho}$ is computable: in order to compute (an algorithm of computation of) $\widehat{\rho}(w_i, w_j)$, it suffices to wait for a stage s at which $w_i, w_j \in A_s$. Extra work will be required, of course, to make all ρ_s computable.

Strategy for $\mathcal{R}_{ezz'}$ in isolation is again based on Lemma 3.3. Fix a computable ρ-name f_λ for λ. In order to satisfy $\mathcal{R}_{ezz'}$, choose a special point $y = w_b \neq \lambda$. We believe that d_e is a metric inducing topology τ on X and Φ_z reduces ρ to d_e. Then the distance $d_e(y, \lambda)$ is a nonzero computable real number, since $\Phi_z(f_\lambda)$ is a computable d_e-name for λ. Thus, at some point we obtain a $k \in \omega$ such that $d_e(y, \lambda) > 2^{-k+1}$. If it turns out that no such k is found throughout the construction, then either d_e is a junk function and not a metric on $\mathbf{X}$, or $\Phi_z(f_\lambda)$ is not a d_e-name for λ, i. e., Φ_z does not reduce ρ to d_e. This situation is no problem as $\mathcal{R}_{ezz'}$ is automatically satisfied. But, if an appropriate k is obtained, we can proceed further and compute $\Phi_z(f_\lambda)(k+1)$. When $\Phi_z(f_\lambda)(k+1)\downarrow$, pick an element x in $C_\rho^{z,f_\lambda,k} - \{\lambda\}$. By Lemma 3.3, if Φ_z does reduce ρ to d_e, then $d_e(x, \lambda) < 2^{-k}$. Suppose that we failed to meet the requirement $\mathcal{R}_{ezz'}$, i. e., $\Phi_{z'}$ c-reduces $\widehat{\rho}$ to d_e. Then $\Phi_{z'}(\bar{b})(k+1)\downarrow$, where $\bar{b} = (b, b, \ldots)$ is a name for y. Suppose for a minute that $\widehat{\rho}$ is constructed in such a way that $\widehat{\rho}(x, y) < 2^{-\varphi_{z'}(\bar{b})(k+1)+1}$. Then $x \in C_{\widehat{\rho}}^{z',\bar{b},k}$. Applying Lemma 3.3 again, we see that $d_e(x, y) < 2^{-k}$. Together with $d_e(x, \lambda) < 2^{-k}$ it implies $d_e(y, \lambda) < 2^{-k+1}$, which is a contradiction to the choice of k, and $\mathcal{R}_{ezz'}$ is satisfied. In a nutshell, we forced x to be "split" between λ and y, meaning that x is too close to both λ and y in the metric d_e, by observing that it is close to these points in metrics ρ and $\widehat{\rho}$, respectively, and using the assumption that ρ and $\widehat{\rho}$ are c-reduced to d_e.

Thus, in order to meet $\mathcal{R}_{ezz'}$, we need to make sure that $\widehat{\rho}(x,y) < 2^{-\varphi_{z'}(\bar{b})(k+1)+1}$. We achieve this by deforming a neighbourhood of x in (X,ρ), making x closer to y and preserving the metric structure of the space outside that neighbourhood. Since λ is a limit point, we can associate each requirement $\mathcal{R}_n$, $n = \langle e, z, z' \rangle$, with its own special point y_n, proceed as the $\mathcal{R}_n$-strategy tells us and choose a second point x_n close to λ that will also be associated specifically with $\mathcal{R}_n$. The fact that the space is deformed only locally within a small neighbourhood of x_n permits us to meet each requirement $\mathcal{R}_n$ in its own distinct area, preventing different requirements from meddling with each other. As a result, the construction will be injury-free. The rest of the paper is devoted to the proof of Theorem 4.1. In the next section we describe the analytical part of the construction. Firstly, we outline how an elementary deformation reducing the distance between two given points looks like, and then we show how these deformations are combined with each other, giving us the resulting metric $\widehat{\rho}$. In Section 6 we effectivize these results and present an effective construction of $\widehat{\rho}$.

5. Analytical part of the construction of $\widehat{\rho}$

Throughout this section we assume that $\mathbf{X} = (X, \tau, W, \nu)$ is an arbitrary Polish space and $\rho \in M(\mathbf{X})$ is a metric on $\mathbf{X}$. The first subsection is devoted to the description of an "elementary deformation" of ρ that yields a new metric ρ_1. The main goal of this subsection is to prove that $\rho_1 \in M(\mathbf{X})$, that is, ρ_1 induces topology τ and is complete. In the second subsection we build a countable collection of metrics ρ_i, $i > 0$, such that each consecutive member of this collection is obtained from the previous one via an elementary deformation, and define the metric $\widehat{\rho}$ as the pointwise limit of metrics ρ_i. The main goal of the second subsection is to prove that $\widehat{\rho} \in M(\mathbf{X})$.

5.1. **Elementary deformation.** Suppose that points $x, y \in X$ and real numbers $r, h > 0$ are given such that $\rho(x,y) > r$. We construct a metric ρ_1 on $\mathbf{X}$ such that $\rho_1(x,y) = h$ and $\rho_1(z,v) = \rho(z,v)$ for $z, v \notin B_\rho(x,r)$. In order to do it, we embed the space (X, ρ, W) into the Banach space ℓ^∞ via the well-known construction of a Fréchet embedding (originally published in [21], see [22] for a background). Recall that ℓ^∞ is the space of bounded sequences of real numbers endowed with the norm $\|(x_n)_{n\in\omega}\| = \sup_{n\in\omega}|x_n|$. Fréchet embedding F is given as follows. For $x \in X$, let $F(x) \in \ell^\infty$ be given by

$$(F(x))_i = \rho(x, w_i) - \rho(w_i, w_0).$$

We abbreviate $(F(x))_i$ as $x_{(i)}$. It is not hard to check [22] that $F\colon (X,\rho) \to \ell^\infty$ is well-defined and is an isometry.

Now we are ready to introduce our deformation. Define a map $\Gamma\colon B_\rho(x,r) \to \ell^\infty$ by the rule

$$\Gamma(z) = \left(\tfrac{r-\rho(x,z)}{r}\, h\right) \frown \left(F(y) - \tfrac{\rho(x,z)}{r}(F(y) - F(z))\right),$$

where $\alpha \frown \beta$ is the concatenation of a real number α and an infinite sequence $\beta = (\beta_0, \beta_1, \ldots) \in \mathbf{R}^\omega$ of real numbers:

$$\alpha \frown \beta = (\alpha, \beta_0, \beta_1, \ldots).$$

We can extend Γ to a mapping $\gamma\colon X \to \ell^\infty$ by putting $\gamma(z) = 0 \frown F(z)$ for $z \notin B_\rho(x,r)$. It is not hard to see that the mapping γ is well-defined, i. e., $\gamma(x)$ is a bounded sequence of real numbers for each $x \in X$. It is also easy to see that the mapping γ is injective.

Let us obtain explicit formulas for the value $\|\gamma(z)-\gamma(v)\|$, $z, v \in X$. Suppose first that $\rho(x,z) < r$, $\rho(x,v) \geqslant r$. We have

$$\begin{aligned}\|\gamma(z)-\gamma(v)\| &= \|\Gamma(z) - (0 \frown F(v))\| \\ &= \max\Big(\big|(\Gamma(z))_0 - 0\big|,\ \sup_k \big|(\Gamma(z))_{k+1} - (F(v))_k\big|\Big) \\ &= \max\Big(\tfrac{r-\rho(x,z)}{r}\, h,\ \sup_k \big|y_{(k)} - \tfrac{\rho(x,z)}{r}(y_{(k)} - z_{(k)}) - v_{(k)}\big|\Big).\end{aligned}$$

Calculate the expression E_k under the sup:

$$\begin{aligned}E_k &= \big|y_{(k)} - \tfrac{\rho(x,z)}{r}(y_{(k)} - z_{(k)}) - v_{(k)}\big| \\ &= \Big|\rho(y,w_k) - \tfrac{\rho(x,z)}{r}\big(\rho(y,w_k) - \rho(z,w_k)\big) - \rho(v,w_k)\Big| \\ &= \Big|\tfrac{r-\rho(x,z)}{r}\rho(y,w_k) + \tfrac{\rho(x,z)}{r}\rho(z,w_k) - \tfrac{r-\rho(x,z)}{r}\rho(v,w_k) - \tfrac{\rho(x,z)}{r}\rho(v,w_k)\Big| \\ &\leqslant \tfrac{r-\rho(x,z)}{r}\big|\rho(y,w_k) - \rho(v,w_k)\big| + \tfrac{\rho(x,z)}{r}\big|\rho(z,w_k) - \rho(v,w_k)\big| \\ &\leqslant \tfrac{r-\rho(x,z)}{r}\rho(y,v) + \tfrac{\rho(x,z)}{r}\rho(z,v).\end{aligned}$$

Choose a sequence of special points w_{k_n} converging to v. Then for E_{k_n} it holds

$$\begin{aligned}E_{k_n} &= \Big|\tfrac{r-\rho(x,z)}{r}\rho(y,w_{k_n}) + \tfrac{\rho(x,z)}{r}\rho(z,w_{k_n}) - \tfrac{r-\rho(x,z)}{r}\rho(v,w_{k_n}) - \tfrac{\rho(x,z)}{r}\rho(v,w_{k_n})\Big| \\ &\to \tfrac{r-\rho(x,z)}{r}\rho(y,v) + \tfrac{\rho(x,z)}{r}\rho(z,v),\end{aligned}$$

consequently, $\sup_k E_k = \frac{r-\rho(x,z)}{r}\rho(y,v) + \frac{\rho(x,z)}{r}\rho(z,v)$ and

$$\|\gamma(z)-\gamma(v)\| = \max\big(\tfrac{r-\rho(x,z)}{r}\, h,\ \tfrac{r-\rho(x,z)}{r}\rho(y,v) + \tfrac{\rho(x,z)}{r}\rho(z,v)\big).$$

Suppose now that $\rho(x,z), \rho(x,v) < r$. We assume that $\rho(x,v) \leqslant \rho(x,z)$, the other case being symmetric.

$$\begin{aligned}\|\gamma(z)-\gamma(v)\| = \|\Gamma(z)-\Gamma(v)\| = \max\Big(&\big|\tfrac{r-\rho(x,z)}{r} - \tfrac{r-\rho(x,v)}{r}\big|\, h, \\ &\sup_k \big|y_{(k)} - \tfrac{\rho(x,z)}{r}(y_{(k)} - z_{(k)}) - y_{(k)} + \tfrac{\rho(x,v)}{r}(y_{(k)} - v_{(k)})\big|\Big).\end{aligned}$$

The value E'_k under the supremum is

$$\begin{aligned}E'_k &= \Big|y_{(k)} - \tfrac{\rho(x,z)}{r}(y_{(k)} - z_{(k)}) - y_{(k)} + \tfrac{\rho(x,v)}{r}(y_{(k)} - v_{(k)})\Big| \\ &= \Big|\tfrac{\rho(x,z)}{r}\rho(z,w_k) - \tfrac{\rho(x,v)}{r}\rho(v,w_k) + \tfrac{\rho(x,v)-\rho(x,z)}{r}\rho(y,w_k)\Big| \\ &= \Big|\tfrac{\rho(x,v)}{r}\rho(z,w_k) - \tfrac{\rho(x,v)}{r}\rho(v,w_k) + \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(z,w_k) - \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(y,w_k)\Big| \\ &\leqslant \tfrac{\rho(x,v)}{r}\big|\rho(z,w_k) - \rho(v,w_k)\big| + \tfrac{\rho(x,z)-\rho(x,v)}{r}\big|\rho(z,w_k) - \rho(y,w_k)\big| \\ &\leqslant \tfrac{\rho(x,v)}{r}\rho(z,v) + \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(z,y).\end{aligned}$$

Choosing $w_{k_n} \to z$, we see that

$$\begin{aligned}E'_{k_n} = \Big|&\tfrac{\rho(x,v)}{r}\rho(z,w_{k_n}) - \tfrac{\rho(x,v)}{r}\rho(v,w_{k_n}) + \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(z,w_{k_n}) \\ &- \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(y,w_{k_n})\Big| \to \tfrac{\rho(x,v)}{r}\rho(z,v) + \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(z,y),\end{aligned}$$

so $\sup_k E'_k = \frac{\rho(x,v)}{r}\rho(z,v) + \frac{\rho(x,z)-\rho(x,v)}{r}\rho(z,y)$ and

$$\|\gamma(z)-\gamma(v)\| = \max\big(\tfrac{\rho(x,z)-\rho(x,v)}{r}\,h,\ \tfrac{\rho(x,v)}{r}\rho(z,v) + \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(z,y)\big).$$

If $\rho(x,z), \rho(x,v) \geqslant r$, then $\|\gamma(z)-\gamma(v)\| = \|F(z)-F(v)\| = \rho(z,v)$.

Concluding all of the above, the distance $\|\gamma(z)-\gamma(v)\|$ can be calculated as follows:

$$\|\gamma(z)-\gamma(v)\| = \begin{cases} \max\big(\tfrac{r-\rho(x,z)}{r}\,h,\ \tfrac{\rho(x,z)}{r}\rho(z,v) + \tfrac{r-\rho(x,z)}{r}\rho(y,v)\big), & (5.1)\\ \qquad \text{if } \rho(x,z) < r \text{ and } \rho(x,v) \geqslant r; \\ \max\big(\tfrac{r-\rho(x,v)}{r}\,h,\ \tfrac{\rho(x,v)}{r}\rho(z,v) + \tfrac{r-\rho(x,v)}{r}\rho(y,z)\big), & (5.2)\\ \qquad \text{if } \rho(x,v) < r \text{ and } \rho(x,z) \geqslant r; \\ \max\big(\tfrac{\rho(x,z)-\rho(x,v)}{r}\,h,\ \tfrac{\rho(x,v)}{r}\rho(z,v) + \tfrac{\rho(x,z)-\rho(x,v)}{r}\rho(z,y)\big), & (5.3)\\ \qquad \text{if } \rho(x,z), \rho(x,v) < r \text{ and } \rho(x,v) \leqslant \rho(x,z); \\ \max\big(\tfrac{\rho(x,v)-\rho(x,z)}{r}\,h,\ \tfrac{\rho(x,z)}{r}\rho(z,v) + \tfrac{\rho(x,v)-\rho(x,z)}{r}\rho(v,y)\big), & (5.4)\\ \qquad \text{if } \rho(x,z), \rho(x,v) < r \text{ and } \rho(x,z) < \rho(x,v); \\ \rho(z,v), \ \text{if } \rho(x,z), \rho(x,v) \geqslant r. & (5.5) \end{cases}$$

Since γ is injective, $\rho_1(z,v) = \|\gamma(z)-\gamma(v)\|$ is a correctly defined metric on X. This is the desired deformed metric. We are ready to prove the main result of this subsection.

Proposition 5.1. *ρ_1 induces topology τ and is complete.*

Proof. In order to show that ρ_1 induces τ, we need to check that

$$\rho_1(z_n, z) \underset{n\to\infty}{\longrightarrow} 0 \text{ if and only if } \rho(z_n, z) \underset{n\to\infty}{\longrightarrow} 0$$

for every sequence $(z_n)_{n\in\omega}$ and point z in X. Fix a sequence $z_0, z_1, \ldots$ and a point $z \in X$.

For the "if" direction, suppose that $\rho(z_n, z) \to 0$. We need to show that $\rho_1(z_n, z) \to 0$. Consider the following three possibilities.

Case 1. $\rho(x,z) < r$. Let $B = B_\rho(z, r - \rho(x,z))$, then $B \subseteq B_\rho(x,r)$. There is an $N \geqslant 0$ such that the Nth tail $(z_n)_{n\geqslant N}$ of our sequence is contained in B. Divide $(z_n)_{n\geqslant N}$ into two subsequences $(z_i)_{i\in I}$ and $(z_j)_{j\in J}$ such that $\rho(x,z_i) \leqslant \rho(x,z)$ for $i \in I$ and $\rho(x,z_j) > \rho(x,z)$ for $j \in J$ (throughout the proof, when dividing a sequence into subsequences this way, w. l. o. g. we assume that the sets I and J are both infinite). By (5.3), for $i \in I$,

$$\rho_1(z_i, z) = \max\big(\tfrac{\rho(x,z)-\rho(x,z_i)}{r}\,h,\ \tfrac{\rho(x,z_i)}{r}\rho(z,z_i) + \tfrac{\rho(x,z)-\rho(x,z_i)}{r}\rho(z,y)\big) \underset{i\to\infty}{\longrightarrow} 0$$

since $\rho(z_i, z) \to 0$. Similarly, by (5.4), $\rho_1(z_j, z) \underset{j\to\infty}{\longrightarrow} 0$ for $j \in J$. Since $(z_i)_{i\in I}$ and $(z_j)_{j\in J}$ partition $(z_n)_{n\geqslant N}$, then $\rho_1(z_n, z) \underset{n\to\infty}{\longrightarrow} 0$.

Case 2. $\rho(x,z) > r$. Again there is an open neighbourhood B of z such that $\rho(x,v) > r$ for $v \in B$. There is an $N \geqslant 0$ such that $(z_n)_{n\geqslant N} \subseteq B$. By (5.5), for $n \geqslant N$, $\rho_1(z_n, z) = \rho(z_n, z) \to 0$.

Case 3. $\rho(x,z) = r$. Then $(z_n)_{n\in\omega}$ can be split into subsequences $(z_i)_{i\in I}$ and $(z_j)_{j\in J}$ such that $\rho(x,z_i) < r$ for $i \in I$ and $\rho(x,z_j) \geqslant r$ for $j \in J$. By (5.5),

$\rho_1(z_j, z) = \rho(z_j, z) \to 0$. By (5.2),

$$\rho_1(z_i, z) = \max\bigl(\tfrac{r-\rho(x,z_i)}{r}\, h,\ \tfrac{\rho(x,z_i)}{r}\rho(z, z_i) + \tfrac{r-\rho(x,z_i)}{r}\rho(y, z)\bigr) \to 0$$

since $\rho(x, z_i) \to \rho(x, z) = r$ and $\rho(z, z_i) \to 0$. Thus, $\rho_1(z_n, z) \to 0$.

For the "only if" direction, suppose that $\rho_1(z_n, z) \to 0$. We consider the same possibilities as above.

Case 1. $\rho(x, z) < r$. Then $B = B_{\rho_1}(z, \frac{r-\rho(x,z)}{r}h) \subseteq B_\rho(x, r)$: indeed, if $\rho(x, v) \geqslant r$, then $\rho_1(z, v) \geqslant \frac{r-\rho(x,z)}{r}h$ by (5.1). There is $N \geqslant 0$ such that $(z_n)_{n\geqslant N} \subseteq B$. Divide $(z_n)_{n\geqslant N}$ into two subsequences $(z_i)_{i\in I}$ and $(z_j)_{j\in J}$ such that $\rho(x, z_i) \leqslant \rho(x, z)$ for $i \in I$ and $\rho(x, z_j) > \rho(x, z)$ for $j \in J$. Then the distances $\rho_1(z_i, z)$ and $\rho_1(z_j, z)$ are calculated according to formulas (5.3) and (5.4), respectively. Since $\rho_1(z_n, z) \to 0$, these formulas give us that $\bigl|\frac{\rho(x,z_n)-\rho(x,z)}{r}\bigr|h \to 0$, so $\rho(x, z_n) \to \rho(x, z)$. Thus, if $z = x$, then immediately $\rho(z_n, z) \to 0$. If $z \neq x$, then, by (5.3) and (5.4),

$$\tfrac{\rho(x,z_i)}{r}\rho(z_i, z) \leqslant \tfrac{\rho(x,z_i)}{r}\rho(z_i, z) + \tfrac{\rho(x,z)-\rho(x,z_i)}{r}\rho(z, y) \leqslant \rho_1(z_i, z) \to 0$$

and

$$\tfrac{\rho(x,z)}{r}\rho(z_j, z) \leqslant \tfrac{\rho(x,z)}{r}\rho(z_j, z) + \tfrac{\rho(x,z_j)-\rho(x,z)}{r}\rho(z, y) \leqslant \rho_1(z_j, z) \to 0.$$

The latter implies that $\rho(z_j, z) \to 0$. Consider now the former expression. Since $\rho(x, z_i) \to \rho(x, z)$ and $z \neq x$, there are $\varepsilon > 0$ and $i_0 \in I$ such that $\rho(x, z_i) > \varepsilon$ for all $i \geqslant i_0$. For $i \geqslant i_0$ we have

$$\tfrac{\varepsilon}{r}\rho(z_i, z) < \tfrac{\rho(x,z_i)}{r}\rho(z_i, z) \to 0,$$

which finally means that $\rho(z_i, z) \to 0$. As a result, $(z_n)_{n\in\omega}$ converges to z in ρ.

Case 2. $\rho(x, z) > r$. Before handling this case, let us first show that $B_\rho[x, r]$ is closed in (X, ρ_1). Suppose that $v \notin B_\rho(x, r)$ belongs to the closure of $B_\rho[x, r]$ in (X, ρ_1), i.e., there is a sequence $(v_n)_{n\in\omega} \subseteq B_\rho[x, r]$ converging to v in ρ_1. We show that $\rho(v_n, v) \to 0$, this implies that $v \in B_\rho[x, r]$. As usual, split $(v_n)_{n\in\omega}$ into sequences $(v_i)_{i\in I}$ and $(v_j)_{j\in J}$ such that $\rho(x, v_i) < \rho(x, v)$ for $i \in I$ and $\rho(x, v_j) = \rho(x, v)$ for $j \in J$. Then the distances $\rho_1(v_i, v)$ and $\rho_1(v_j, v)$ are calculated according to formulas (5.1) and (5.5), respectively. By (5.5), $\rho(v_j, v) = \rho_1(v_j, v) \to 0$. By (5.1),

$$\tfrac{r-\rho(x,v_i)}{r}h \leqslant \rho_1(v_i, v) \to 0,$$

that is, $\rho(x, v_i) \to r$. Again by (5.1),

$$\tfrac{\rho(x,v_i)}{r}\rho(v_i, v) \leqslant \tfrac{\rho(x,v_i)}{r}\rho(v_i, v) + \tfrac{r-\rho(x,v_i)}{r}\rho(y, v) \leqslant \rho_1(v_i, v) \to 0,$$

which means that $\rho(v_i, v) \to 0$, so $\rho(v_n, v) \to 0$ and $v \in B_\rho[x, r]$.

Suppose now that $\rho(x, z) > r$. Since $B_\rho[x, r]$ is closed in (X, ρ_1), there is an open neighbourhood U of z in (X, ρ_1) such that $U \cap B_\rho[x, r] = \varnothing$. There exists an $N \geqslant 0$ such that $z_n \in U$ for all $n \geqslant N$. By (5.5), for $n \geqslant N$ we have $\rho(z_n, z) = \rho_1(z_n, z) \to 0$.

Case 3. $\rho(x, z) = r$. Divide $(z_n)_{n\in\omega}$ into subsequences $(z_i)_{i\in I}$ and $(z_j)_{j\in J}$ such that $\rho(x, z_i) < r$ for $i \in I$ and $\rho(x, z_j) \geqslant r$ for $j \in J$. An easy application of the same reasoning as before shows that $\rho(z_n, z) \to 0$. We conclude that ρ and ρ_1 induce the same topology on X.

To see that ρ_1 is complete, fix a Cauchy sequence $(z_n)_{n\in\omega}$ in (X, ρ_1) and consider the following two cases.

First, suppose that $\forall n\, \exists k_n \geqslant n\ \rho(z_{k_n}, x) \geqslant r$. Then $\rho_1(z_{k_n}, z_{k_p}) = \rho(z_{k_n}, z_{k_p})$ for all $n, p \in \omega$, so $(z_{k_n})_{n\in\omega}$ is Cauchy in $(X, \rho) \Rightarrow$ it converges in $(X, \rho) \Rightarrow$ it converges in $(X, \rho_1) \Rightarrow (z_n)_{n\in\omega}$ converges in (X, ρ_1).

Suppose now that $\exists N\, \forall n \geqslant N\ \rho(z_n, x) < r$. Since $(z_n)_{n\in\omega}$ is Cauchy in (X, ρ_1), by (5.3) and (5.4) we have

$$\forall \varepsilon > 0\, \exists N_\varepsilon > N\, \forall n, p > N_\varepsilon\ \left|\tfrac{\rho(x,z_n)-\rho(x,z_p)}{r}\right| h \leqslant \rho_1(z_n, z_p) < \varepsilon,$$

which means that the sequence $(\rho(x, z_n))_{n\in\omega}$ is Cauchy in $\mathbf{R}$, thus it has a limit $L \in \mathbf{R}$. If $L = 0$, then $\rho(z_n, x) \to 0$, i.e., $(z_n)_{n\in\omega}$ converges in (X, ρ). Assume that $L > 0$. Fix an $\varepsilon > 0$ and let $\delta = \frac{L\varepsilon}{2r}$. Choose $N'_\varepsilon > N$ such that $\forall n > N'_\varepsilon$ $\rho(x, z_n) > \frac{L}{2}$ and $\forall n, p > N'_\varepsilon$ $\rho_1(z_n, z_p) < \delta$. Take arbitrary $n, p > N'_\varepsilon$. We will assume that $\rho(x, z_p) \leqslant \rho(x, z_n)$, the other case being symmetric. By (5.3),

$$\tfrac{L}{2r}\rho(z_n, z_p) < \tfrac{\rho(x,z_n)}{r}\rho(z_n, z_p) \leqslant \rho_1(z_n, z_p) < \delta,$$

so $\rho(z_n, z_p) < \frac{2r\delta}{L} = \varepsilon$. We have shown that $(z_n)_{n\in\omega}$ is Cauchy in (X, ρ). As above, it implies that $(z_n)_{n\in\omega}$ converges in (X, ρ_1). □

Let $z \in X$ be such that $z \neq y$ and $\rho(x, z) > r$. For each $v \in B_\rho[x, r]$ it holds:

$$\begin{aligned}\rho_1(z, v) &= \max\left(\tfrac{r-\rho(x,v)}{r}\, h,\ \tfrac{\rho(x,v)}{r}\rho(z, v) + \tfrac{r-\rho(x,v)}{r}\rho(y, z)\right)\\ &\geqslant \tfrac{\rho(x,v)}{r}\rho(z, v) + \tfrac{r-\rho(x,v)}{r}\rho(y, z)\\ &\geqslant \min(\rho(z, v), \rho(z, y))\\ &\geqslant \min(\rho(x, z) - r, \rho(z, y)).\end{aligned}$$

Define a function $\Delta\colon X \to \mathbf{R}$ by $\Delta(z) = \min(\rho(x, z) - r, \rho(z, y))$. Then for all $z \notin B_\rho[x, r] \cup \{y\}$ we have $\Delta(z) > 0$ and $B_{\rho_1}(z, \Delta(z)) \cap B_\rho[x, r] = \varnothing$, as shown above. Moreover, the following holds.

Proposition 5.2. *Suppose that $z \notin B_\rho[x, r] \cup \{y\}$. Then the identity map id_X induces an isometry from $B_\rho(z, \Delta(z))$ onto $B_{\rho_1}(z, \Delta(z))$. In particular, $B_\rho(z, \Delta(z)) = B_{\rho_1}(z, \Delta(z))$.*

Proof. Fix $z \notin B_\rho[x, r] \cup \{y\}$ and denote $\Delta = \Delta(z)$. Clearly, $B_\rho(z, \Delta)$ and $B_\rho[x, r]$ are disjoint. By (5.5), $\rho_1(z, v) = \rho(z, v)$ for all $v \in B_\rho(z, \Delta)$. Thus, $B_\rho(z, \Delta) \subseteq B_{\rho_1}(z, \Delta)$, and $\mathrm{id}_X \restriction B_\rho(z, \Delta)$ is an isometry from $B_\rho(z, \Delta)$ into $B_{\rho_1}(z, \Delta)$. To see that this isometry is surjective, suppose that there exists $v \in B_{\rho_1}(z, \Delta) - B_\rho(z, \Delta)$. Then $\rho_1(z, v) \neq \rho(z, v)$, so, by (5.5), v must be contained in $B_\rho(x, r)$, but we have shown that $B_{\rho_1}(z, \Delta) \cap B_\rho[x, r] = \varnothing$. □

The following proposition is trivial.

Proposition 5.3. *Let (Y, ρ_Y) and (Z, ρ_Z) be metric spaces and let $F\colon B_{\rho_Y}(y_0, \varepsilon_0) \to B_{\rho_Z}(F(y_0), \varepsilon_0)$ be a surjective isometry. Then, for every ball $B_{\rho_Y}[y_1, \varepsilon_1]$ formally included in $B_{\rho_Y}(y_0, \varepsilon_0)$, $F \restriction B_{\rho_Y}[y_1, \varepsilon_1]$ is an isometry onto $B_{\rho_Z}[F(y_1), \varepsilon_1]$. The same is true of the open ball $B_{\rho_Y}(y_1, \varepsilon_1)$.*

5.2. **Building a sequence of deformed metrics.** In this subsection we construct a countable sequence of metrics on $\mathbf{X}$, each consecutive member of which is obtained from the previous one via an elementary deformation in a certain special way, as follows. Suppose that $\mathbf{X}$ contains at least one limit point λ. We start with a metric $\rho_0 = \rho$. There exist two sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$, both converging to λ, such that:

(1) $x_n \neq x_m$ and $y_n \neq y_m$ for $n \neq m$,
(2) $x_n \neq y_m$, $x_n \neq \lambda$ and $y_m \neq \lambda$ for all n, m.

For each n there is a real number $\widehat{r}_n > 0$ such that $B_{\rho_0}[x_n, \widehat{r}_n] \cap \{y_k\}_{k\in\omega} = \varnothing$ and $B_{\rho_0}[x_n, \widehat{r}_n]$ are pairwise formally disjoint. It is clear that $\lambda \notin B_{\rho_0}[x_n, \widehat{r}_n]$ for all n and $\widehat{r}_n \to 0$. Fix a sequence $(h_n)_{n\in\omega}$ of positive real numbers converging to 0.

Let $r_0 = \widehat{r}_0$. Define a function $\Delta_0\colon X \to \mathbf{R}$ by

$$\Delta_0(z) = \min(\rho_0(x_0, z) - r_0, \rho_0(z, y_0)).$$

Suppose that metrics $\rho_0, \dots, \rho_n$, real numbers $r_0, \dots, r_n$ and functions $\Delta_0, \dots, \Delta_n$ have already been defined. Define a map $\Gamma_{n+1}\colon B_{\rho_n}(x_n, r_n) \to \ell^\infty$ by the rule

$$\Gamma_{n+1}(z) = \left(\tfrac{r_n - \rho_n(x_n,z)}{r_n} h_n\right) ^\frown \left(F_n(y_n) - \tfrac{\rho_n(x_n,z)}{r_n}(F_n(y_n) - F_n(z))\right),$$

where F_n is the Fréchet embedding of (X, ρ_n, W) into ℓ^∞, i. e., F_n is given by

$$(F_n(z))_i = \rho_n(z, w_i) - \rho_n(w_i, w_0).$$

Extend Γ_{n+1} to a mapping $\gamma_{n+1}\colon X \to \ell^\infty$ by putting $\gamma_{n+1}(z) = 0 ^\frown F_n(z)$ for $z \notin B_{\rho_n}(x_n, r_n)$. Let $\rho_{n+1}(z, v) = \|\gamma_{n+1}(z) - \gamma_{n+1}(v)\|$. Choose a real number r_{n+1} such that

$$0 < r_{n+1} < \min(\widehat{r}_{n+1}, \Delta_0(x_{n+1}), \dots, \Delta_n(x_{n+1}))$$

(we will show that such a number exists). Define a function $\Delta_{n+1}\colon X \to \mathbf{R}$ by

$$\Delta_{n+1}(z) = \min(\rho_{n+1}(x_{n+1}, z) - r_{n+1}, \rho_{n+1}(z, y_{n+1})).$$

Proposition 5.4. *For all $n \in \omega$, metric ρ_n, real number r_n and function Δ_n are well-defined and the following conditions are satisfied:*

(1) *ρ_n is a complete metric inducing topology τ on X;*
(2) *$r_n > 0$;*
(3) *$\rho_n(z, v) = \rho_{n-1}(z, v)$ for all $z, v \notin B_{\rho_{n-1}}(x_{n-1}, r_{n-1})$;*
(4) *The identity map id_X induces a series of surjective isometries $B_{\rho_0}[x_n, r_n] \to B_{\rho_1}[x_n, r_n] \to \dots \to B_{\rho_n}[x_n, r_n]$.*

Proof. We argue by induction. It is clear that the statement holds when $n = 0$.

Suppose that the proposition has been proved for $0, \dots, n$. We have to show that it holds for $n + 1$. By (4) and Proposition 5.3, $B_{\rho_i}(x_i, r_i) = B_{\rho_0}(x_i, r_i)$ for $i < n$. By definition of r_i we have $x_n, y_n \notin B_{\rho_0}(x_i, r_i)$, thus

$$\rho_n(x_n, y_n) = \rho_{n-1}(x_n, y_n) = \dots = \rho_0(x_n, y_n) > \widehat{r}_n > r_n.$$

Since $r_n > 0$, by Proposition 5.1, ρ_{n+1} is a complete metric inducing topology τ. By (5.5), $\rho_{n+1}(z, v) = \rho_n(z, v)$ for all $z, v \notin B_{\rho_n}(x_n, r_n)$.

Consider now the point x_{n+1}. For all $i \leqslant n$, since $B_{\rho_i}[x_i, r_i] = B_{\rho_0}[x_i, r_i]$ and $\rho_0(x_{n+1}, x_i) > \widehat{r}_i > r_i$, then $x_{n+1} \notin B_{\rho_i}[x_i, r_i]$, so $\Delta_i(x_{n+1}) > 0$. Thus, $\min(\widehat{r}_{n+1}, \Delta_0(x_{n+1}), \dots, \Delta_n(x_{n+1})) > 0$, and r_{n+1} is well-defined.

By Propositions 5.2 and 5.3, since $r_{n+1} < \Delta_0(x_{n+1})$, id_X induces an isometry from $B_{\rho_0}[x_{n+1}, r_{n+1}]$ onto $B_{\rho_1}[x_{n+1}, r_{n+1}]$. Similarly, for each $0 < i \leqslant n$, id_X induces an isometry from $B_{\rho_i}[x_{n+1}, r_{n+1}]$ onto $B_{\rho_{i+1}}[x_{n+1}, r_{n+1}]$. □

Proposition 5.4 implies that the construction of ρ_{n+1} works inside the ball $B_{\rho_0}(x_n, r_n)$ and does not change the values of $\rho_n(z, v)$ for $z, v \notin B_{\rho_0}(x_n, r_n)$. Since $B_{\rho_0}(x_n, r_n)$ are pairwise disjoint, for each $z, v \in X$ there are at most two numbers $n \in \omega$ such that $\rho_n(z, v) \neq \rho_{n+1}(z, v)$. Let $\widehat{\rho}(z, v) = \lim_n \rho_n(z, v)$ be the pointwise limit of metrics ρ_n. It is easy to check that $\widehat{\rho}$ is a metric on X.

Recall that $x_n \to \lambda$, $y_n \to \lambda$ and $r_n < \widehat{r}_n \to 0$.

Proposition 5.5. *$\widehat{\rho}$ is a complete metric inducing topology τ on X.*

Proof. In order to show that $\widehat{\rho}$ induces τ, we need to show that every open ρ-ball contains an open $\widehat{\rho}$-ball, and vice versa. Let us first consider balls with center $z \neq \lambda$. Fix such a z. There is a number $n_0 \in \omega$ such that $z \notin D_{n_0}$, where

$$D_m = \bigcup_{n \geqslant m} \big(B_{\rho_0}[x_n, r_n] \cup \{y_n\}\big).$$

Notice that by the previous proposition $D_m = \bigcup_{n\geqslant m}\big(B_{\rho_m}[x_n, r_n] \cup \{y_n\}\big)$ for all m. There is an $\varepsilon_0 > 0$ such that $\rho_{n_0}(x_n, z) > r_n + \varepsilon_0$ for all $n \geqslant n_0$. Indeed, if it is not the case, then for each $k > 0$ there exists $m_k \geqslant n_0$ such that $\rho_{n_0}(x_{m_k}, z) \leqslant r_{m_k} + 1/k$. If $m_k \not\to \infty$, there is an $m^* \geqslant n_0$ such that $m^* = m_k$ for infinitely many k. Then it is clear that $\rho_{n_0}(x_{m^*}, z) \leqslant r_{m^*}$, i. e., $z \in B_{\rho_{n_0}}[x_{m^*}, r_{m^*}] \subseteq D_{n_0}$, which is a contradiction. So, $m_k \underset{k\to\infty}{\longrightarrow} \infty$. For any $\delta > 0$ there exists $k \in \omega$ such that $\rho_{n_0}(x_{m_k}, z) \leqslant r_{m_k} + 1/k < \delta/2$ and $\rho_{n_0}(x_{m_k}, \lambda) < \delta/2$. Consequently, $\rho_{n_0}(z, \lambda) < \delta$ for all $\delta > 0$, so $z = \lambda$, which contradicts the choice of z.

For $n > m \geqslant n_0$ we have $x_n, z \notin B_{\rho_0}(x_m, r_m) = B_{\rho_m}(x_m, r_m)$. Thus,

$$\rho_n(x_n, z) = \rho_{n-1}(x_n, z) = \ldots = \rho_{n_0}(x_n, z) > r_n + \varepsilon_0.$$

Arguing similarly, observe that there exists an $\varepsilon_1 > 0$ such that $\rho_n(y_n, z) = \rho_{n_0}(y_n, z) > \varepsilon_1$ for $n \geqslant n_0$. Setting $\varepsilon = \min(\varepsilon_0, \varepsilon_1)$, we see that $\Delta_n(z) > \varepsilon$ for all $n \geqslant n_0$. By Propositions 5.2 and 5.3, id_X induces a series of surjective isometries

$$B_{\rho_{n_0}}(z, \varepsilon) \to B_{\rho_{n_0+1}}(z, \varepsilon) \to \ldots \to B_{\rho_n}(z, \varepsilon) \to \ldots$$

It follows that $B_{\rho_{n_0}}(z, \varepsilon) \subseteq B_{\widehat{\rho}}(z, \varepsilon)$. On the other hand, suppose that there is $v \in B_{\widehat{\rho}}(z, \varepsilon) - B_{\rho_{n_0}}(z, \varepsilon)$. Then there must be a number $n \geqslant n_0$ such that $\widehat{\rho}(z, v) = \rho_n(z, v)$, thus $v \in B_{\rho_n}(z, \varepsilon) - B_{\rho_{n_0}}(z, \varepsilon)$, which is a contradiction. So, id_X induces an isometry from $B_{\rho_{n_0}}(z, \varepsilon)$ onto $B_{\widehat{\rho}}(z, \varepsilon)$. By Proposition 5.3, $B_{\rho_{n_0}}(z, \varepsilon') = B_{\widehat{\rho}}(z, \varepsilon')$ for all $\varepsilon' \leqslant \varepsilon$. Since ρ_{n_0} and ρ_0 induce the same topology, this implies that every open ρ_0-neighbourhood of z contains a $\widehat{\rho}$-neighbourhood of z, and vice versa.

Consider now the case $z = \lambda$. Fix an $\varepsilon > 0$ and consider the ball $B_{\widehat{\rho}}(\lambda, \varepsilon)$. There exists an $n_0 > 0$ such that for $n \geqslant n_0$ the balls $B_{\rho_0}[x_n, r_n]$ are formally included in $B_{\rho_0}(\lambda, \varepsilon/3)$, $\rho_0(y_n, \lambda) < \varepsilon/3$ and $h_n < \varepsilon/3$. Choose a $\delta \in (0, \varepsilon/3)$ such that $B_{\rho_0}(\lambda, \delta) \cap \big(\bigcup_{k<n_0} B_{\rho_0}[x_k, r_k]\big) = \varnothing$. For each $v \in B_{\rho_0}(\lambda, \delta)$, either

$\widehat{\rho}(v,\lambda)=\rho_0(v,\lambda)<\delta<\varepsilon$, or $v\in B_{\rho_n}(x_n,r_n)$ for some $n\geqslant n_0$. In the second case,

$$\begin{aligned}\widehat{\rho}(v,\lambda)&=\rho_{n+1}(v,\lambda)\\&\leqslant \rho_{n+1}(v,y_n)+\rho_{n+1}(y_n,\lambda)\\&=\rho_{n+1}(v,y_n)+\rho_0(y_n,\lambda)\\&<\rho_{n+1}(v,y_n)+\varepsilon/3\\&=\max\big(\tfrac{r_n-\rho_n(x_n,v)}{r_n}h_n,\tfrac{\rho_n(x_n,v)}{r_n}\rho_n(v,y_n)+\tfrac{r_n-\rho_n(x_n,v)}{r_n}\rho_n(y_n,y_n)\big)+\varepsilon/3\\&<\max(h_n,\rho_n(v,y_n))+\varepsilon/3\\&=\max(h_n,\rho_0(v,y_n))+\varepsilon/3\\&\leqslant\max(h_n,\rho_0(v,x_n)+\rho_0(x_n,\lambda)+\rho_0(\lambda,y_n))+\varepsilon/3\\&\leqslant\max(h_n,r_n+\rho_0(x_n,\lambda)+\rho_0(\lambda,y_n))+\varepsilon/3\\&\leqslant\max(\varepsilon/3,\varepsilon/3+\varepsilon/3)+\varepsilon/3\\&=2\varepsilon/3+\varepsilon/3=\varepsilon.\end{aligned}$$

We have proved that $B_{\rho_0}(\lambda,\delta)\subseteq B_{\widehat{\rho}}(\lambda,\varepsilon)$.

It remains to show that any ball $B_{\rho_0}(\lambda,\varepsilon)$ contains an open $\widehat{\rho}$-ball. Fix a ball $B_{\rho_0}(\lambda,\varepsilon)$. Choose an $n_0>0$ such that the balls $B_{\rho_0}[x_n,r_n]$, $n\geqslant n_0$, are formally included in $B_{\rho_0}(\lambda,\varepsilon)$. Using functions Δ_k, choose a nonzero $\delta<\varepsilon$ such that $B_{\rho_{n_0}}(\lambda,\delta)=B_{\rho_0}(\lambda,\delta)$ and $B_{\rho_0}(\lambda,\delta)\cap\big(\bigcup_{k<n_0}B_{\rho_0}[x_k,r_k]\big)=\varnothing$. For any $z\in B_{\widehat{\rho}}(\lambda,\delta)$ we either have $z\notin B_{\rho_0}(x_n,r_n)$ for all $n\in\omega$, in which case $\rho_0(\lambda,z)=\widehat{\rho}(\lambda,z)<\delta<\varepsilon$, or $z\in B_{\rho_0}(x_{n_1},r_{n_1})$ for some n_1. If $n_1<n_0$, then $\widehat{\rho}(\lambda,z)=\rho_{n_1+1}(\lambda,z)=\rho_{n_0}(\lambda,z)\geqslant\delta$ by the choice of δ. Thus, $n_1\geqslant n_0$, and $z\in B_{\rho_0}(x_{n_1},r_{n_1})\subseteq B_{\rho_0}(\lambda,\varepsilon)$. We conclude that $B_{\widehat{\rho}}(\lambda,\delta)\subseteq B_{\rho_0}(\lambda,\varepsilon)$ and $\widehat{\rho}$ induces the same topology as ρ_0.

Let us show that $\widehat{\rho}$ is complete. Let $(z_i)_{i\in\omega}$ be a Cauchy sequence in $(X,\widehat{\rho})$. Then either $\rho_0(z_i,\lambda)\to 0$, in which case $\widehat{\rho}(z_i,\lambda)\to 0$, or there are an $\varepsilon>0$ and a subsequence $(z_{i_k})_{k\in\omega}$ of $(z_i)_{i\in\omega}$ such that $\rho_0(z_{i_k},\lambda)>\varepsilon$ for all k. There is an $n_0\in\omega$ such that $\widehat{\rho}(z_{i_k},z_{i_l})=\rho_{n_0}(z_{i_k},z_{i_l})$ for all k,l. Since $(z_{i_k})_{k\in\omega}$ is Cauchy in $(X,\widehat{\rho})$, then $(z_{i_k})_{k\in\omega}$ is Cauchy in (X,ρ_{n_0}), so it converges in (X,ρ_{n_0}), thus it converges in $(X,\widehat{\rho})$. As a result, $(z_i)_{i\in\omega}$ converges in $(X,\widehat{\rho})$. □

6. Construction of $\widehat{\rho}$

In this section we finish the proof of Theorem 4.1. Suppose that $\rho\in M(\mathbf{X})$ is a computable metric and λ is a computable limit point in (X,ρ). Fix a computable ρ-name f_λ for the point λ. We show that the construction described in Subsection 5.2 can be effectivized to yield a computable metric $\widehat{\rho}$ satisfying all requirements $\mathcal{R}_{ezz'}$.

Stage s of the construction will be called *active* if $s=0$ or some requirement acts at this stage (see definition in the construction). If a requirement acts at stage s, we output a new metric ρ_s and a function $\Delta_s\colon X\to\mathbf{R}$ at this stage. Let f_0,f_1,f_2 be computable functions from the proof of Theorem 3.1. An element $x=w_{f_1(i)}$ will be called *fresh* if $i>j$ for all $w_{f_1(j)}$ seen in the construction so far.

6.1. **Construction.** *Stage 0.* Let $A_0=\varnothing$, $\rho_0=\rho$.
Stage s+1. Let $A_{s+1}=A_s\cup\{w_s\}$. We work with requirement $\mathcal{R}_{ezz'}$, $\langle e,z,z'\rangle=\langle s+1\rangle_0=n$, if it hasn't been satisfied yet. Let $y=w_b$, where $b=f_0(n)$. Compute the distance $d_e(y,\lambda)$ with precision 2^{-s}, using at most $s+1$ machine steps. If this precision permits us to see that $d_e(y,\lambda)>0$, fix some k such that $d_e(y,\lambda)>2^{-k+1}$.

Compute $\Phi_{z,s+1}(f_\lambda)(k+1)$ and $\Phi_{z',s+1}(\bar{b})(k+1)$. If these computations halt, we say that $\mathcal{R}_{ezz'}$ *acts at stage* $s+1$. Choose a fresh element $x = w_{f_1(i)}$ such that $x \in C^{z,f_\lambda,k}_{\rho_0}$ and $\rho_0(x,w) > 0$ for all $w \in A_{s+1}$. Let S_{s+1} be the set of all active stages before $s+1$ and let s' be the latest previous active stage. Choose a rational number r such that

$$0 < r < \min\bigl(q_{f_2(i)}, \min_{t \in S_{s+1}} \Delta_t(x)\bigr)$$

and $B_{\rho_{s'}}(x,r) \cap A_{s+1} = \varnothing$. Define a mapping $\Gamma_{s+1}\colon B_{\rho_{s'}}(x,r) \to \ell^\infty$ by

$$\Gamma_{s+1}(z) = \bigl(\tfrac{r-\rho_{s'}(x,z)}{r}\, h\bigr) \frown \bigl(F_{s'}(y) - \tfrac{\rho_{s'}(x,z)}{r}(F_{s'}(y) - F_{s'}(z))\bigr),$$

where $h < 2^{-\varphi_{z',s+1}(\bar{b})(k+1)+1}$ is a nonzero rational number and $F_{s'}$ is the Fréchet embedding of $(X, \rho_{s'}, W)$ into ℓ^∞. Extend Γ_{s+1} to a mapping $\gamma_{s+1}\colon X \to \ell^\infty$ by putting $\gamma_{s+1}(z) = 0^\frown F_{s'}(z)$ for $z \notin B_{\rho_{s'}}(x,r)$. Let $\rho_{s+1}(z,v) = \|\gamma_{s+1}(z) - \gamma_{s+1}(v)\|$. Let $\Delta_{s+1}(z) = \min(\rho_{s+1}(x,z) - r, \rho_{s+1}(z,y))$.

6.2. **Verification.** Let S be the set of all active stages of the construction. For $s \in S$, let x_s be the x that we choose at stage s, r_s be the r chosen at stage s, and y_s be $y = w_{f_0(n)}$ associated with requirement $\mathcal{R}_{ezz'}$ we are working with at stage s. Since $w_{f_1(n)} \to \lambda$ and $C^{z,f_\lambda,k}_{\rho_0}$ is a neighbourhood of λ, we see that a suitable special point x_s always exists. Arguing as in the proof of Proposition 5.4 (remember also the properties of f_0, f_1 and f_2) and bearing in mind that $x_s \notin A_s$, we see that there exists a suitable $r_s > 0$. Applying Proposition 5.4, we obtain that, for all $s \in S$, ρ_s is a correctly defined complete metric inducing topology τ. Applying Proposition 5.5, we obtain that

$$\widehat{\rho}(z,v) = \lim_{s \in S} \rho_s(z,v)$$

is a complete metric inducing topology τ. Our next step will be to prove that $(\rho_s)_{s \in S}$ is a computable sequence of computable metrics. It is easy to see that S is a computable set: in order to check if $s \in S$, proceed as follows. Figure out whether $d_e(y,\lambda) > 0$ within at most s machine steps, using the computable d_e-name $\Phi_z(f_\lambda)$ for λ. If $d_e(y,\lambda) > 0$, then the construction fixes some k such that $d_e(y,\lambda) > 2^{-k+1}$, and we can check whether $\Phi_{z,s+1}(f_\lambda)(k+1)\downarrow$ and $\Phi_{z',s+1}(\bar{b})(k+1)\downarrow$, where $w_b = y$. If they do, then $s \in S$.

For $n \in \omega$, let s_n be the nth active stage in strictly increasing order.

Lemma 6.1. *There exists a computable function g such that $\rho_{s_n} = d_{g(n)}$ for $n \in \omega$.*

Proof. We will first show that there exists a partial computable function $\psi\colon \omega^8 \to \omega$ such that if $d_{e_0} = d$ is a computable metric on $\mathbf{X}$, $w_a = x$ and $w_b = y$ are special points, $q_i = r$ and $q_j = h$ are rational numbers such that $d(x,y) > r$, and a metric d' is obtained from d via elementary deformation $\Gamma\colon B_d(x,r) \to \ell^\infty$ defined in the usual manner:

$$\Gamma(z) = \bigl(\tfrac{r-d(x,z)}{r}\, h\bigr) \frown \bigl(F(y) - \tfrac{d(x,z)}{r}(F(y) - F(z))\bigr),$$

then for all $l, m, k \in \omega$ it holds $\psi(e_0, a, b, i, j, l, m, k)\downarrow$ and

$$|d'(w_l, w_m) - \psi(e_0, a, b, i, j, l, m, k)| < 2^{-k}.$$

In two words, that ψ exists should be clear from Formulas (5.1)–(5.5). Let us, though, carry out a more detailed explanation. We need to show how to uniformly compute $d'(w_l, w_n)$ with any given precision. Fix the parameters e_0, a, b, i and j

mentioned above. Let $z = w_l, v = w_m$. The distance $d'(z,v)$ can be computed with precision $\varepsilon < r$ as follows. Compute rational ε-approximations D_1 and D_2 of distances $d(x,z)$ and $d(x,v)$, respectively. There are several opportunities:

(1) $D_1 \geqslant r+\varepsilon$ and
   (a) $D_2 \geqslant r+\varepsilon$,
   (b) $D_2 < r-\varepsilon$,
   (c) $r-\varepsilon \leqslant D_2 < r+\varepsilon$,
(2) $D_1 < r-\varepsilon$ with the same subcases,
(3) $r-\varepsilon \leqslant D_1 < r+\varepsilon$ with the same subcases.

In case (1a) we are sure that $d(x,z) \geqslant r$ and $d(x,v) \geqslant r$, so $d'(z,v) = d(z,v)$ according to Formula (5.5). Similarly, in case (1b) we are sure that $d(x,z) \geqslant r$ and $d(x,v) < r$, thus $d'(z,v)$ can be computed according to (5.2). Case (1c) is a little trickier as we no longer know whether (5.2) or (5.5) is to be applied. However, since (5.2) and (5.5) give the same result when $d(x,v) = r$, we can use any of these formulas to determine a suitable approximation of $d'(z,v)$. More formally, to compute $d'(z,v)$ up to precision ε, recall that

$$\tilde{d}(z,v) = \max\big(\tfrac{r-d(x,v)}{r}\,h,\ \tfrac{d(x,v)}{r}d(z,v) + \tfrac{r-d(x,v)}{r}d(y,z)\big)$$

is the value of $d'(z,v)$ given by Formula (5.2). Let $N \in \omega$ be such that

$$d(z,v) < N,\ d(y,z) < N \text{ and } h < N.$$

Compute $d(x,v)$ with precision $\frac{\varepsilon r}{8N}$, let D_3 be the corresponding approximate value. If it still holds

$$r - \tfrac{\varepsilon r}{8N} \leqslant D_3 < r + \tfrac{\varepsilon r}{8N},$$

i. e. we are still not sure whether $d(x,v) < r$, then

$$\big|\tfrac{d(x,v)-r}{r}\big|d(z,v) \leqslant \tfrac{\varepsilon}{4},\ \big|\tfrac{d(x,v)-r}{r}\big|d(y,z) \leqslant \tfrac{\varepsilon}{4} \text{ and } \big|\tfrac{d(x,v)-r}{r}\big|h \leqslant \tfrac{\varepsilon}{4}.$$

Now we can compute an approximate value D_4 of $d(z,v)$ with precision $\varepsilon/8$ and check whether $D_4 > \varepsilon + \varepsilon/8$. If not, then $d(z,v) \leqslant \varepsilon + \varepsilon/4$ and

$$\begin{aligned}\tilde{d}(z,v) &= \max\big(\tfrac{r-d(x,v)}{r}\,h,\ \tfrac{d(x,v)}{r}d(z,v) + \tfrac{r-d(x,v)}{r}d(y,z)\big)\\ &= \max\big(\tfrac{r-d(x,v)}{r}\,h,\ \tfrac{d(x,v)-r}{r}d(z,v) + d(z,v) + \tfrac{r-d(x,v)}{r}d(y,z)\big)\\ &\leqslant \max\big(\tfrac{\varepsilon}{4},\ \varepsilon + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4}\big)\\ &< 2\varepsilon.\end{aligned}$$

We see that both $d(z,v) < 2\varepsilon$ and $\tilde{d}(z,v) < 2\varepsilon$, so we can safely output ε as an approximate value of $d'(z,v)$ with precision ε. If $D_4 > \varepsilon + \varepsilon/8$, then $\tilde{d}(z,v) = \frac{d(x,v)}{r}d(z,v) + \frac{r-d(x,v)}{r}d(y,z)$ and

$$\big|\tilde{d}(z,v) - d(z,v)\big| = \big|\tfrac{d(x,v)-r}{r}d(z,v) + \tfrac{r-d(x,v)}{r}d(y,z)\big| \leqslant \tfrac{\varepsilon}{2},$$

so we can output D_4 as an approximate value of $d'(z,v)$ with precision ε.

The other possibilities are treated similarly. Applying the s-m-n theorem to the function ψ, we see that there is a computable function $g'\colon \omega^5 \to \omega$ such that for all e_0, c, a, i, j as above it holds $d' = d_{g'(e_0,c,a,i,j)}$.

Now we can obtain g from the statement of the lemma. Let $g(0)$ be an index of ρ_0. Suppose that $g(0), \dots, g(n)$ have been defined. To compute $g(n+1)$, go to stage s_{n+1}. It is clear that the elements $x = w_a$ and $y = w_b$ appearing at stage s_{n+1} are chosen effectively. Since $\rho_{s_0}, \dots, \rho_{s_n}$ are computable metrics, then

$\Delta_{s_0}(x), \ldots, \Delta_{s_n}(x)$ are computable real numbers. Thus, rational numbers $r = q_i$ and $h = q_j$ are obtained effectively as well. Letting $g(n+1) = g'(g(n), a, b, i, j)$, we see that $\rho_{s_{n+1}} = d_{g(n+1)}$. □

Lemma 6.2. *$\widehat{\rho}$ is a computable metric.*

Proof. We show that, for all $n, m \in \omega$, $\widehat{\rho}(w_n, w_m) = \rho_s(w_n, w_m)$ for the least active stage $s > \max(n, m)$. It implies that $\widehat{\rho}(w_n, w_m)$ is computable uniformly in m, n.

Let s be as above. Then $w_n, w_m \in A_s$. Element x and rational number r are chosen at stage s so that $w_n, w_m \notin B_{\rho_{s'}}(x, r)$, where s' is the previous active stage. By Proposition 5.4, $\rho_s(w_n, w_m) = \rho_{s'}(w_n, w_m)$. Repeating the same argument for all subsequent active stages, we see that this distance will not change from now on, thus $\widehat{\rho}(w_n, w_m) = \rho_s(w_n, w_m)$. □

Lemma 6.3. *Every requirement $\mathcal{R}_{ezz'}$ is satisfied.*

Proof. Suppose that some requirement $\mathcal{R}_{ezz'}$ is not satisfied, i. e., there are a computable metric $d_e \in M(\mathbf{X})$ and Turing functionals Φ_z and $\Phi_{z'}$ reducing ρ and $\widehat{\rho}$ to d_e, respectively. Let $n = \langle e, z, z' \rangle$. At some stage s_0 we find out that $d_e(y, \lambda) > 0$, where $y = w_{f_0(n)} = w_b$, and fix k such that $d_e(y, \lambda) > 2^{-k+1}$. At some stage $s_1 \geqslant s_0$ we find out that $\Phi_{z,s_1}(f_\lambda)(k+1)\downarrow$ and $\Phi_{z',s_1}(\bar{b})(k+1)\downarrow$. At this stage we choose a special point $x \in C_{\rho}^{z,f_\lambda,k}$ and define the metric ρ_{s_1} in such a way that $\rho_{s_1}(x, y) = h < 2^{-\varphi_{z',s_1}(\bar{b})(k+1)+1}$. Propositions 5.2 and 5.4 show that the distance $\rho_{s_1}(x, y)$ remains unchanged after s_1, thus, $\widehat{\rho}(x, y) = \rho_{s_1}(x, y) = h$ and $x \in C_{\widehat{\rho}}^{z',\bar{b},k}$. Applying Lemma 3.3, we see that $d_e(x, \lambda) < 2^{-k}$ and $d_e(x, y) < 2^{-k}$, thus $d_e(y, \lambda) < 2^{-k+1}$, contrary to the choice of k. Thus, either d_e is not a suitable metric on X, or Φ_z does not reduce ρ to d_e, or $\Phi_{z'}$ does not reduce $\widehat{\rho}$ to d_e, which means that $\mathcal{R}_{ezz'}$ is satisfied. □

Proof of Theorem 4.1 is now complete.

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