GENERALIZED THERMO-VISCO-ELASTICITY FLEXIBLE STRUCTURES: GLOBAL EXISTENCE AND NEW SCENARIO FOR ENERGY DECAY

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Abstract. The main idea of this work is to create a new model that combines the properties of porous-elastic materials and standard linear solids that depend in its model on Hooke's law. In addition to the two previous mechanical systems, we have added a thermal effect of second sound which is modeled by a system that link the heat equation with the heat flow field equation. The model was introduced on the basis of evolution equations and basic equations for flexible-porous materials and a standard solids property whose model is based on Hooke's law. It's about a standard linear model of viscoelasticity for system of flexible structure materials with voids coupled to heat waves that propagate at a finite velocity according to Cattaneo's law for thermal conduction. We show a global existence of the weak solutions and we then interest by the decay rate in time of solutions where the exponential stability has been proved. In this study, a systematization, important from a practical point of view, was carried out, devoted to completed studies for thermo-visco-elasticity on flexible structures.

1. Introduction and relevance of the topic

The development of modern technology is inextricably linked to the emergence of new materials with properties that distinguish them from all previously known. An example of this is the flexible visco-thermal materials, where three important physical properties are combined. In this work, we derived a new model that was not previously worked on. This derivation is based on three basic models that have been previously studied separately. As a result of this, new materials appear with higher physicomechanical properties compared to those that were available for unconnected materials. The appearance of such materials is widely used in new technology, but this can only be achieved through a more comprehensive study of the qualitative properties of already known materials than before. In the 50s and 70s of the last century, when the linear theory of viscoelasticity was intensively developed, the representation of relaxation in the form of time functions to be determined experimentally predominated in concrete experiments and at present, it becomes widely used to represent these kernel by using a large number of decreasing exponential functions that allow describing the decay of energy function related to the evolution systems. Recently, damped systems has been actively studied both quantitatively and qualitatively, which is associated with the evolutional equations (systems). The theory of elastic solids with voids was introduced by Nunziato and Cowin [13] and Iesan in [9, 10] added the thermal effect so that this theory

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became stronger and more real, which attracted the attention of many researchers in different fields of engineering, such as the petroleum industry, material science, biology, ...

In the present paper, we consider a novel system that models the behavior of viscoelastic materials of porous and flexible structure in the presence of the thermal effect of the second sound type in the following coupled system

$$\begin{cases} \alpha u_{ttt} + u_{tt} - a^2\beta\Delta u_t - \alpha(a^2+\lambda+m)\nabla \operatorname{div} u_t \\ \quad -a^2\Delta u + \gamma\nabla\theta - (a^2+\lambda+m)\nabla \operatorname{div} u + m\nabla\phi = 0 & \text{in } \Omega\times\mathbb{R}^+ \\ J\phi_{tt} - b^2\Delta\phi + 2m\phi - m\alpha \operatorname{div} u_t - m \operatorname{div} u - d\theta = 0 & \text{in } \Omega\times\mathbb{R}^+ \\ c\theta_t + \kappa \operatorname{div} q + \gamma\alpha \operatorname{div} u_{tt} + \gamma \operatorname{div} u_t + d\phi_t = 0 & \text{in } \Omega\times\mathbb{R}^+ \\ \tau_0 q_t + \kappa\nabla\theta + q = 0 & \text{in } \Omega\times\mathbb{R}^+, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\Gamma = \partial\Omega$, the boundary conditions is given by

$$u(x,t) = u_t(x,t) = \phi(x,t) = \theta(x,t) = 0, \text{ on } \Gamma\times\mathbb{R}^+, \tag{1.2}$$

and the related initial conditions are

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x) \\ u_{tt}(x,0) = u_2(x), \phi(x,0) = \phi_0(x) \\ \phi_t(x,0) = \phi_1(x), \theta(x,0) = \theta_0(x) \\ q(x,0) = q_0(x), \text{ in } \overline{\Omega}. \end{cases} \tag{1.3}$$

The functions

$$u, q : \Omega\times\mathbb{R}^+ \mapsto \mathbb{R}^n, \tag{1.4}$$

$$\phi, \theta : \Omega\times\mathbb{R}^+ \mapsto \mathbb{R}, \tag{1.5}$$

represent respectively, the vibration of flexible structures, the difference of the volume fraction, the difference of temperature between the actual state and a reference temperature and the heat flux. Here, we note that γ, κ, m, d are the coupling constants and $\alpha, a, J, b, \beta, \lambda, \tau_0, c$ are supposed positive.

The essence of problem solved in this article is as follows: To find and use the relationships between three various mechanisms, which allows taking into account the physical properties of problem, which were revealed in the study of this new model in terms of the global existence of solution as well as knowledge of the asymptotic behavior of this system. The latter corresponds to the fact that this system, by studying the behavior of solutions, turns out to have preserved the advantages of porous-elastic systems whose stability depends on the presence of at least two dissipations in relation to the first that is at the level of the mechanical system and the second that is at the level of thermal effect.

It is important firstly to give a general overview of existing models in the literature to illustrate a general view of the derivation for our model. The authors in [5, 8], considered the differential equations of low amplitude acoustic waves in elastic materials with voids, this material was supposed homogeneous and isotropic and in the absence of the external forces, the authors proposed the following system

$$\begin{cases} \rho u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\phi = 0 \\ \rho\kappa\phi_{tt} - \alpha\Delta\phi + \tau\phi_t + \xi\phi + \beta \operatorname{div} u = 0, \end{cases} \tag{1.6}$$

where u is the displacement field, ϕ is the difference of the volume fraction and $\alpha, \beta, \rho, \mu, \lambda, \tau$ and ξ are positive constitutive coefficients. Quintanilla in [16] showed that the porous viscosity was not strong enough to achieve exponential stability for the solutions of system. This results for the decay of energy which alludes to one-dimensional porous elastic materials opened a field to several investigations (see [2, 11, 12, 16, 17, 18]). Gorain in [6] studied the dynamics of linear vibrations of elastic (flexible) structures that are generally formulated according to Hooke's law, these vibrations are governed by the following linear differential equation

$$\alpha u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0, \tag{1.7}$$

where u is the deflection of flexible structures. More precisely, if the stress σ is proportional to ε deformation by the relation $\sigma = E\varepsilon$ with E being the Young's modulus of elastic structure then the vibrations are modeled by the equation

$$u_{tt} = a^2 \Delta u, \tag{1.8}$$

where $a > 0$, under the assumption that the stress and their time derivative are related according to the following equation

$$\sigma + \alpha \sigma_t = E(\varepsilon + \beta \varepsilon_t), \tag{1.9}$$

where the constants α, β are very small and satisfy $0 < \alpha < \beta$. In this direction, M. S. Alves et *al.* in [1], by coupling equation (1.7) to a heat equation according to the Fourier law of thermal conduction, the authors showed the exponential stability. Other related important work has recently been developed in [7], where an exponential stability was proved.

The classical model of Fourier-law heat propagation is transformed into equations using the temperature θ and the heat flux vector q by

$$\begin{cases} \theta_t + \beta \operatorname{div} q = 0 \\ q + \kappa \nabla \theta = 0, \end{cases} \tag{1.10}$$

with positive constants β, κ. This model is based on the physical paradox of the infinite propagation speed of the (thermal) signals, unlike, another model introduced by the law of Cattaneo [3], which eliminates the Fourier paradox, this model is very important for some applications (see [17]), the second sound heat is given by the replacement of the equation $(1.10)_2$ by

$$\tau q_t + q + \kappa \nabla \theta. \tag{1.11}$$

For a comparative study of two laws, please see [4, 14]. The heat flux vector is now considered as other function to be determined from the differential equation where the positive parameter τ is the relaxation time describing the delay of the response of the thermal flux to a temperature gradient.

The purpose of this paper is to develop a thermoviscoelastic model for flexible structure materials by showing a global existence result for the weak solution corresponding to the problem. We proved the asymptotic behavior of the obtained solution and we concluding by some comments on the importance of this results.

Remark 1.1. *Throughout this paper*

- c_0 *is a generic constant, it will change from line to line, which depends in an increasing way on the indicated quantities.*
- *Under a (mathematically) natural hypothesis*

$$(\beta - \alpha) > 0, \tag{1.12}$$

the positivity of the energy is ensured.

2. The existence result via semigroup approach

Here, we prove the existence and uniqueness results for (1.1)-(1.3). All that follows in this section is based on the classic Lumer-Phillips theorem inspired from [15].

Lemma 2.1. [15] *Let $\mathcal{A}$ be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $s(t)$ if and only if*

(1) *$\mathcal{A}$ is dissipative.*
(2) $\Re(I-\mathcal{A})=\mathcal{H}$.

We will use the following standard $L^2(\Omega)$ space, the scalar product and norm are denoted by

$$\langle f,g\rangle_{L^2(\Omega)}=\int_\Omega f\overline{g}dx,\|f\|^2_{L^2(\Omega)}=\int_\Omega |f|^2dx. \tag{2.1}$$

Introducing the vectors

$$z(t)=(u,v,w,\phi,\varphi,\theta,q),\quad z_0=(u_0,v_0,w_0,\phi_0,\varphi_0,\theta_0,q_0), \tag{2.2}$$

such that $u_t=v,\quad v_t=w,\quad \phi_t=\varphi$.
We introduce the energy space

$$\mathcal{H}=H^1_0(\Omega)\times H^1_0(\Omega)\times L^2(\Omega)\times H^1_0(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega). \tag{2.3}$$

equipped with the inner product

$$\begin{aligned}
\langle z,\tilde{z}\rangle_{\mathcal{H}} &= \left\langle \alpha w+v,\alpha\overline{\tilde{w}}+\overline{\tilde{v}}\right\rangle_{L^2(\Omega)}+a^2\left\langle \alpha\nabla v+\nabla u,\alpha\nabla\overline{\tilde{v}}+\nabla\overline{\tilde{u}}\right\rangle_{L^2(\Omega)}\\
&+ (a^2+\lambda+m)\left\langle \alpha\operatorname{div} v+\operatorname{div} u,\alpha\operatorname{div}\overline{\tilde{v}}+\operatorname{div}\overline{\tilde{u}}\right\rangle_{L^2(\Omega)}\\
&+ a^2\alpha(\beta-\alpha)\left\langle\nabla v,\nabla\overline{\tilde{v}}\right\rangle_{L^2(\Omega)}+c\left\langle\theta,\overline{\tilde{\theta}}\right\rangle_{L^2(\Omega)}\\
&+ b^2\left\langle\nabla\phi,\nabla\overline{\tilde{\phi}}\right\rangle_{L^2(\Omega)}+m\left\langle\phi,\overline{\tilde{\phi}}\right\rangle_{L^2(\Omega)}+\tau_0\left\langle q,\overline{\tilde{q}}\right\rangle_{L^2(\Omega)}\\
&+ J\left\langle\varphi,\overline{\tilde{\varphi}}\right\rangle_{L^2(\Omega)}+m\left\langle\operatorname{div}(\alpha v+u)-\phi,\operatorname{div}(\alpha\overline{\tilde{v}}+\overline{\tilde{u}})-\overline{\tilde{\phi}}\right\rangle_{L^2(\Omega)},
\end{aligned}$$

and the norm

$$\begin{aligned}
\|z\|^2_{\mathcal{H}} &= \|\alpha w+v\|^2_{L^2(\Omega)}+a^2\|\alpha\nabla v+\nabla u\|^2_{L^2(\Omega)}\\
&+ (a^2+\lambda)\|\alpha\operatorname{div} v+\operatorname{div} u\|^2_{L^2(\Omega)}+a^2\alpha(\beta-\alpha)\|\nabla v\|^2_{L^2(\Omega)}\\
&+ c\|\theta\|^2_{L^2(\Omega)}+\tau_0\|q\|^2_{L^2(\Omega)}+J\|\varphi\|_{L^2(\Omega)}+m\|\phi\|_{L^2(\Omega)}\\
&+ m\|\operatorname{div}(\alpha v+u)-\phi\|_{L^2(\Omega)}+b^2\|\nabla\phi\|_{L^2(\Omega)},
\end{aligned}$$

we write problem (1.1)-(1.3) as the ODE in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}z(t)=\mathcal{A}z(t)\\ z(0)=z_0. \end{cases} \tag{2.4}$$

The linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}z = \begin{pmatrix} v \\ w \\ \frac{1}{\alpha}\left[a^2\beta\Delta v - w + (a^2+\lambda+m)\nabla \operatorname{div}(\alpha v + u) + a^2\Delta u - \gamma\nabla\theta - m\nabla\phi\right] \\ \varphi \\ \frac{1}{J}\left[b^2\Delta\phi - 2m\phi + m\operatorname{div}(\alpha v + u) + d\theta\right] \\ \frac{1}{c}\left[-\gamma\alpha\operatorname{div} w - \kappa\operatorname{div} q - \gamma\operatorname{div} v - d\varphi\right] \\ \frac{1}{\tau_0}\left[-\kappa\nabla\theta - q\right] \end{pmatrix} \quad (2.5)$$

with domain $\mathcal{D}(\mathcal{A})$ given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} z \in \mathcal{H} : w, \varphi, \theta \in H_0^1(\Omega), q \in W, \quad \phi \in H_0^1(\Omega) \cap H^2(\Omega), \\ -a^2\beta v - a^2 u \in H_0^1(\Omega) \cap H^2(\Omega). \end{array} \right\}$$

where

$$W = \{v \in L^2(\Omega) | \operatorname{div} v \in L^2(\Omega)\}. \quad (2.6)$$

Theorem 2.2. *The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup*

$$s(t) = e^{t\mathcal{A}} : \mathcal{H} \longrightarrow \mathcal{H}. \quad (2.7)$$

Proof. The proof is based on a direct application of Lemma 2.1. □

We introduce the following result.

Proposition 2.3. *The operator $\mathcal{A}$ generates a c_0-semigroup $s(t)$ of contractions on the space $\mathcal{H}$.*

Proof. Using the operator in (2.5) and the vector solution in (2.2), then we have

$$\begin{aligned} \langle \mathcal{A}z, z\rangle_{\mathcal{H}} &= \left\langle a^2\beta\Delta v + (a^2+\lambda+m)\nabla\operatorname{div}(\alpha v + u) + a^2\Delta u, \alpha\overline{w} + \overline{v}\right\rangle_{L^2(\Omega)} \\ &+ a^2\left\langle \alpha\nabla w + \nabla v, \alpha\nabla\overline{v} + \nabla\overline{u}\right\rangle_{L^2(\Omega)} + a^2\alpha(\alpha-\beta)\left\langle\nabla w, \nabla\overline{v}\right\rangle_{L^2(\Omega)} \\ &+ (a^2+\lambda)\left\langle \operatorname{div}(\alpha w + v, \operatorname{div}\alpha\overline{v} + \overline{u})\right\rangle_{L^2(\Omega)} + \left\langle -\kappa\nabla\theta - q, \overline{\theta}\right\rangle_{L^2(\Omega)} \\ &- \left\langle \operatorname{div}(\gamma\alpha w + \gamma v + \kappa q) + d\varphi, \overline{\theta}\right\rangle_{L^2(\Omega)} + b^2\left\langle\nabla\varphi, \nabla\overline{\phi}\right\rangle_{L^2(\Omega)} \\ &+ m\left\langle\varphi, \overline{\phi}\right\rangle_{L^2(\Omega)} + \left\langle b^2\Delta\phi - 2m\phi + m\operatorname{div}(\alpha v + u) + d\theta, \overline{\varphi}\right\rangle_{L^2(\Omega)} \\ &+ m\left\langle\operatorname{div}(\alpha w + v) - \varphi, \operatorname{div}(\alpha\tilde{v} + \tilde{u})\right\rangle_{L^2(\Omega)} - \left\langle\gamma\nabla\theta, \alpha\overline{w} + \overline{v}\right\rangle_{L^2(\Omega)} \\ &+ \left\langle m\nabla\phi - v, \alpha\overline{w} + \overline{v}\right\rangle_{L^2(\Omega)}. \end{aligned} \quad (2.8)$$

Hence, we get

$$\begin{aligned} \Re\langle\mathcal{A}z, z\rangle_{\mathcal{H}} &= -\langle q, \bar{q}\rangle_{L^2(\Omega)} - a^2(\beta-\alpha)\langle\nabla v, \nabla v\rangle_{L^2(\Omega)} \\ &= -\|q\|^2_{L^2(\Omega)} - \|\nabla\|^2_{L^2(\Omega)} \\ &\leq 0. \end{aligned}$$

□

Proposition 2.4. *We have $\Re(\mu I - \mathcal{A}) = \mathcal{H}$, where $\mathcal{A}$ introduced in (2.5).*

Proof. We show that for all $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$, there exists a unique $z \in \mathcal{D}(\mathcal{A})$ such that $(\mu I - \mathcal{A})z = F$, that is

$$\begin{cases} \mu u - v = f_1 \in H_0^1(\Omega) \\ \mu v - w = f_2 \in H_0^1(\Omega) \\ (\alpha\mu + 1)w - a^2\beta\Delta v - (a^2 + \lambda + m)\nabla \operatorname{div}(\alpha v + u) \\ \quad -a^2\Delta u + \gamma\nabla\theta + m\nabla\phi = \alpha f_3 \in L^2(\Omega) \\ \mu\phi - \varphi = f_4 \in H_0^1(\Omega) \\ J\mu\varphi - b^2\Delta\phi + 2m\phi - m \operatorname{div}(\alpha v + u) - d\theta = J f_5 \in L^2(\Omega) \\ c\mu\theta + \gamma\alpha \operatorname{div} w + \kappa \operatorname{div} q + \gamma \operatorname{div} v + d\varphi = c f_6 \in L^2(\Omega) \\ (1 + \tau_0\mu)q + \kappa\nabla\theta = \tau_0 f_7 \in L^2(\Omega). \end{cases} \tag{2.9}$$

Substitute $(2.9)_1$, $(2.9)_2$ and $(2.9)_4$ in $(2.9)_3$,$(2.9)_5$, $(2.9)_6$ and $(2.9)_7$, then we get

$$\begin{cases} \mu^2(\alpha\mu + 1)u - a^2(1 + \mu\beta)\Delta u - (1 + \alpha\mu)(a^2 + \lambda + m)\nabla \operatorname{div} u \\ \quad +\gamma\nabla\theta + m\nabla\phi = \Phi_1 \\ (J\mu^2 + 2m)\phi - b^2\Delta\phi - m(\alpha\mu - 1) \operatorname{div} u - d\theta = \Phi_2 \\ \mu c\theta + (\mu^2\gamma\alpha + \mu\gamma) \operatorname{div} u + \kappa \operatorname{div} q + d\mu\phi = \Phi_3 \\ (1 + \tau_0\mu)q + \kappa\nabla\theta = \Phi_4, \end{cases} \tag{2.10}$$

where

$$\begin{cases} \Phi_1 = \alpha f_3 + (\alpha\mu + 1)(\mu f_1 + f_2) + a^2\beta\Delta f_1 - \alpha(a^2 + \lambda + m)\nabla \operatorname{div} f_1 \\ \Phi_2 = J f_5 + J\mu f_4 - \alpha m \operatorname{div} f_1 \\ \Phi_3 = c f_6 + (\gamma\alpha\mu - \gamma) \operatorname{div} f_1 + \gamma\alpha \operatorname{div} f_2 + d f_4 \\ \Phi_4 = \tau_0 f_7. \end{cases}$$

Substitute $(2.10)_3$ and $(2.10)_4$ then the system (2.10) will be

$$\begin{cases} \mu^2(\alpha\mu + 1)u - a^2(1 + \mu\beta)\Delta u - (1 + \alpha\mu)(a^2 + \lambda + m)\nabla \operatorname{div} u \\ \quad +\gamma\nabla\theta + m\nabla\phi = \Phi_1 \\ (J\mu^2 + 2m)\phi - b^2\Delta\phi - m(\alpha\mu - 1) \operatorname{div} u - d\theta = \Phi_2 \\ \mu c\theta + (\mu^2\gamma\alpha + \mu\gamma) \operatorname{div} u + \frac{\kappa^2}{1+\tau_0\mu}\Delta\theta + d\mu\phi = \Phi_3 + \frac{\kappa}{1+\tau_0\mu} \operatorname{div} \Phi_4. \end{cases} \tag{2.11}$$

To solve system (2.11) we consider the bilinear form

$$\mathfrak{B} : (H_0^1(\Omega))^3 \times (H_0^1(\Omega))^3 \longrightarrow \mathbb{R}, \tag{2.12}$$

given by

$$\begin{aligned} \mathfrak{B}\left((u, \phi, \theta), (\chi, \psi, \vartheta)\right) &= \mu^2(\alpha\mu + 1) \langle u, \chi\rangle_{L^2(\Omega)} + a^2(1 + \mu\beta) \langle \nabla u, \nabla\chi\rangle_{L^2(\Omega)} \\ &+ (1 + \alpha\mu)(a^2 + \lambda + m) \langle \operatorname{div} u, \operatorname{div} \chi\rangle_{L^2(\Omega)} + \mu c \langle \theta, \vartheta\rangle_{L^2(\Omega)} \\ &+ \gamma \langle \nabla\theta, \chi\rangle_{L^2(\Omega)} + m \langle \nabla\phi, \chi\rangle_{L^2(\Omega)} + (J\mu^2 + 2m) \langle \phi, \psi\rangle_{L^2(\Omega)} \\ &+ b^2 \langle \nabla\phi, \nabla\psi\rangle_{L^2(\Omega)} - m(\alpha\mu - 1) \langle \operatorname{div} u, \psi\rangle_{L^2(\Omega)} \\ &- d \langle \theta, \psi\rangle_{L^2(\Omega)} + (\mu^2\gamma\alpha + \mu\gamma) \langle \operatorname{div} u, \vartheta\rangle_{L^2(\Omega)} \\ &- \frac{\kappa^2}{1 + \tau_0\mu} \langle \nabla\theta, \nabla\vartheta\rangle_{L^2(\Omega)} + d\mu \langle \phi, \vartheta\rangle_{L^2(\Omega)}, \end{aligned}$$

and the linear form

$$\mathfrak{L} : H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}, \tag{2.13}$$

given by

$$\mathfrak{L}(\chi, \psi, \vartheta) = \langle \Phi_1, \chi\rangle_{L^2(\Omega)} + \langle \Phi_2, \psi\rangle_{L^2(\Omega)} + \langle \Phi_3, \vartheta\rangle_{L^2(\Omega)}$$

$$+ \quad \frac{\kappa}{1+\tau_0\mu} \left\langle \operatorname{div} \Phi_4, \vartheta \right\rangle_{L^2(\Omega)}.$$

For $g \in H_0^1(\Omega)$, we have

$$| \left\langle \operatorname{div} f, g \right\rangle_{L^2(\Omega)} | \leq \|f\|_{L^2(\Omega)} \|g\|_{H_0^1(\Omega)}. \tag{2.14}$$

Then, for $f \in L^2(\Omega)$ and $\| \operatorname{div} f\|_{H^{-1}(\Omega)} \leq \|f\|_{L^2(\Omega)}$.
Using the previous inequality, there exist $c_0 > 0$ such that

$$\mathfrak{B}\left((u,\phi,\theta),(u,\phi,\theta)\right) \geq c_0 \left(\|u\|^2_{H_0^1(\Omega)} + \|\phi\|^2_{H_0^1(\Omega)} + \|\theta\|^2_{H_0^1(\Omega)} \right). \tag{2.15}$$

Thus $\mathfrak{B}$ is coercive.
On the other hand, by using the Hölder inequality we have

$$\begin{aligned} \mathfrak{B}\left((u,\phi,\theta),(\chi,\psi,\vartheta)\right) \quad &\leq \quad c_0 \left(\|u\|_{H_0^1(\Omega)} + \|\phi\|_{H_0^1(\Omega)} + \|\theta\|_{H_0^1(\Omega)} \right)^{\frac{1}{2}} \\ &\times \quad \left(\|\chi\|_{H_0^1(\Omega)} + \|\psi\|_{H_0^1(\Omega)} + \|\vartheta\|_{H_0^1(\Omega)} \right)^{\frac{1}{2}}. \end{aligned}$$

Finally, for the linear form, we have

$$\begin{aligned} |\mathfrak{L}(\chi,\psi,\vartheta)| \quad &\leq \quad c_0 \left(\|\chi\|^2_{H_0^1(\Omega)} + \|\nabla\chi\|^2_{H_0^1(\Omega)} \right)^{\frac{1}{2}} \\ &+ \quad c_0 \left(\|\psi\|^2_{H_0^1(\Omega)} + \|\nabla\psi\|^2_{H_0^1(\Omega)} \right)^{\frac{1}{2}} \\ &+ \quad c_0 \left(\|\vartheta\|^2_{H_0^1(\Omega)} + \|\nabla\vartheta\|^2_{H_0^1(\Omega)} \right)^{\frac{1}{2}}. \end{aligned}$$

A classical calculation shows that $\mathfrak{B}$ and $\mathfrak{L}$ satisfy the conditions of Lax-Milgram theorem, consequently, there exists a unique weak solution $(u,\phi,\theta) \in (H_0^1(\Omega))^3$, satisfying

$$\mathfrak{B}\left((u,\phi,\theta),(\chi,\psi,\vartheta)\right) = \mathfrak{L}(\chi,\psi,\vartheta), \quad \forall(\chi,\psi,\vartheta) \in (H_0^1(\Omega))^3. \tag{2.16}$$

In particular, for all $(\chi, 0, 0)$, where $\chi \in \mathcal{D}(\mathcal{A})$ we have that

$$\begin{aligned} &\mu^2(\alpha\mu+1) \left\langle u, \chi \right\rangle_{L^2(\Omega)} - a^2(1+\mu\beta) \left\langle \nabla u, \nabla\chi \right\rangle_{L^2(\Omega)} \\ &-(1+\alpha\mu)(a^2+\lambda+m) \left\langle \operatorname{div} u, \operatorname{div} \chi \right\rangle_{L^2(\Omega)} \\ &+m \left\langle \nabla\phi, \chi \right\rangle_{L^2(\Omega)} + \gamma \left\langle \nabla\theta, \chi \right\rangle_{L^2(\Omega)} = \left\langle \Phi_1, \chi \right\rangle_{L^2(\Omega)}, \end{aligned}$$

that is,

$$\begin{aligned} &\mu^2(\alpha\mu+1)u + a^2(1+\mu\beta)\Delta u + (1+\alpha\mu)(a^2+\lambda+m)\nabla \operatorname{div} u \\ &\quad +\gamma\nabla\theta + m\nabla\phi = \Phi_1 \text{ in } \mathcal{D}'(\mathcal{A}). \end{aligned}$$

We set $v = \mu u - f_1$ and $w = \mu(\mu u - f_1) - f_2$. Then $v, w \in H_0^1(\Omega)$ solve $(2.9)_1$,$(2.9)_2$. Hence, by using (2.17) we have

$$(\alpha\mu+1)w - a^2\beta\Delta v - (a^2+\lambda)\nabla \operatorname{div}(\alpha v + u) - a^2\Delta u + \gamma\nabla\theta + m\nabla\phi = \alpha f_3, \tag{2.17}$$

which solves $(2.10)_1$, since $F \in \mathcal{H}$, and therefore $\alpha f_3 \in L^2(\Omega)$. Moreover, we note that (2.17) implies

$$-a^2\beta v - a^2 u \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.18}$$

for all $(0, \psi, 0)$, where $\phi \in \mathcal{D}(\mathcal{A})$ we have that

$$\begin{aligned} &(J\mu^2 + 2m) \left\langle \phi, \psi \right\rangle_{L^2(\Omega)} + b^2 \left\langle \nabla\phi, \psi \right\rangle_{L^2(\Omega)} \\ &-m(\alpha\mu-1) \left\langle \operatorname{div} u, \psi \right\rangle_{L^2(\Omega)} - d \left\langle \theta, \psi \right\rangle_{L^2(\Omega)} = \left\langle \Phi_2, \psi \right\rangle_{L^2(\Omega)}, \end{aligned}$$

that is,

$$(J\mu^2+2m)\phi - b^2\Delta\phi - m(\alpha\mu-1)\operatorname{div} u - d\theta = \Phi_2, \quad \text{in} \quad \mathcal{D}'(\mathcal{A}). \tag{2.19}$$

We set $\varphi = \mu\phi - f_4$. Then $\varphi \in H_0^1(\Omega)$ solve $(2.9)_4$. Hence, using $(2.11)_2$ we have

$$J\mu\varphi - b^2\Delta\phi + 2m\phi - m\operatorname{div}(\alpha v + u) - d\theta = Jf_5, \tag{2.20}$$

which solves $(2.9)_5$, since $F \in \mathcal{H}$ and therefore $Jf_5 \in L^2(\Omega)$. Moreover, we note that (2.20) implies

$$\phi \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.21}$$

for all $(0,0,\vartheta)$ with $\psi \in \mathcal{D}(\mathcal{A})$

$$\begin{aligned}&\mu c \langle \theta, \vartheta\rangle_{L^2(\Omega)} + (\mu^2\gamma\alpha + \mu\gamma)\langle \operatorname{div} u, \vartheta\rangle_{L^2(\Omega)} + d\mu\langle \phi, \vartheta\rangle_{L^2(\Omega)}\\ &+\frac{\kappa^2}{1+\tau_0\mu}\langle \nabla\theta, \nabla\vartheta\rangle_{L^2(\Omega)} = \langle \Phi_3, \vartheta\rangle_{L^2(\Omega)} + \frac{\kappa}{1+\tau_0\mu}\langle \operatorname{div}\Phi_4, \vartheta\rangle_{L^2(\Omega)}.\end{aligned}$$

Thus

$$\mu c\theta + (\mu^2\gamma\alpha+\mu\gamma)\operatorname{div} u + \frac{\kappa^2}{1+\tau_0\mu}\Delta\theta + d\mu\phi = \Phi_3 + \frac{\kappa}{1+\tau_0\mu}\operatorname{div}\Phi_4 \in \mathcal{D}'(\mathcal{A}).$$

or, for $cf_6 \in L^2(\Omega)$, we havw

$$c\mu\theta + \gamma\alpha\operatorname{div} w + \kappa\operatorname{div} q + \gamma\operatorname{div} v + d\varphi = cf_6, \tag{2.22}$$

which solves $(2.10)_3$ and $(2.10)_4$. Now, from (2.22) we conclude that

$$\theta \in H_0^1(\Omega) \text{ and } \operatorname{div} q \in L^2(\Omega), \tag{2.23}$$

recalling that by $(2.10)_4$, we have $q \in L^2(\Omega)$. Then we have

$$q \in W(\Omega). \tag{2.24}$$

It is easy to show that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. We conclude that the operator A generates a c_0-semigroup of contractions on the space $\mathcal{H}$. □

We can conclude using the previous result the following theorem.

Theorem 2.5. *For any $z_0 \in \mathcal{H}$, there exists a unique solution z of (1.1)-(1.3) satisfying*

$$\begin{cases} u \in C^1(\mathbb{R}^+; H_0^1(\Omega)) \cap C^2(\mathbb{R}^+; L^2(\Omega)) \\ \phi \in C(\mathbb{R}^+; H_0^1(\Omega)) \\ \theta \in C(\mathbb{R}^+; L^2(\Omega)) \\ q \in C(\mathbb{R}^+; L^2(\Omega)). \end{cases} \tag{2.25}$$

However, if $z_0 \in \mathcal{D}(\mathcal{A})$ then

$$\begin{cases} u \in C^2(\mathbb{R}^+; H_0^1(\Omega)) \cap C^3(\mathbb{R}^+; L^2(\Omega)) \\ \phi \in C(\mathbb{R}^+; H_0^1(\Omega)\cap H^2(\Omega)) \cap C^1(\mathbb{R}^+; H_0^1(\Omega)) \\ \theta \in C(\mathbb{R}^+; H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)) \\ q \in C(\mathbb{R}^+; W(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)). \end{cases} \tag{2.26}$$

3. Asymptotic behavior

In this section, we show the exponential stability of problem (1.1)-(1.3).by using the multiplier techniques in the energy space.
We define the total energy functional of system (1.1) by

$$E(t) = \sum_{i=1}^{i=3} E_i(t), \forall t \geq 0. \tag{3.1}$$

The total energy (3.1) is the sum of the potential energy E_1, the kinetic energy E_2 and the energy coming from the heat conduction E_3, where

$$\begin{aligned}
E_1(t) &= \frac{1}{2}\Big((a^2+\lambda+m)\|\operatorname{div}(\alpha u_t+u)\|^2_{L^2(\Omega)} + m\|\operatorname{div}(\alpha u_t+u)-\phi\|^2_{L^2(\Omega)}. \\
&+ m\|\phi\|^2_{L^2(\Omega)} + a^2\|\nabla(\alpha u_t+u)\|^2_{L^2(\Omega)} + b^2\|\nabla\phi\|^2_{L^2(\Omega)}\Big). \\
E_2(t) &= \frac{1}{2}\Big(\|\alpha u_{tt}+u_t\|^2_{L^2(\Omega)} + a^2\alpha(\beta-\alpha)\|\nabla u_t\|^2_{L^2(\Omega)} + J\|\phi_t\|^2_{L^2(\Omega)}\Big) \\
E_3(t) &= \frac{1}{2}\Big(\tau_0\|q\|^2_{L^2(\Omega)} + c\|\theta\|^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Then, we have

Lemma 3.1. *For any solution $(u,\phi,\theta,q) \in \mathcal{H}$ of the problem (1.1)-(1.3), the energy functional (3.1) satisfies*

$$\frac{d}{dt}E(t) = -\|q\|^2_{L^2(\Omega)} - a^2(\beta-\alpha)\|\nabla u_t\|^2_{L^2(\Omega)}. \tag{3.2}$$

Proof. Multiplying the first equations of system (1.1)-(1.3) by $(\alpha u_t+u)_t, \phi, \theta$ and q respectively, by (1.2) and Green formula, the Lemma (3.1) follows. □

Theorem 3.2. *Let $(u,\phi,\theta,q) \in \mathcal{H}$ be solutions of (1.1)-(1.3) given in Theorem 2.2. Then there exist positive constants ϖ_1 and ϖ_2 such that*

$$E(t) \leq \varpi_1 E(0) e^{-\varpi_2 t}, \forall t \geq 0. \tag{3.3}$$

To proof Theorem 3.2, we need to introduce several Lemmas.

Lemma 3.3. *Let $(u,\phi,\theta,q) \in \mathcal{H}$ solution of the problem (1.1)-(1.3). Then, the functional*

$$\Theta_1(t) := \sum_{i=1}^{i=4} \Lambda_i(t), \quad \forall t \geq 0, \tag{3.4}$$

where the functionals Λ_i given by

$$\begin{cases}
\Lambda_1(t) := \langle \alpha u_t+u, \alpha u_{tt}+u_t\rangle_{L^2(\Omega)} \\
\Lambda_2(t) := -\frac{\gamma\tau_0}{\kappa}\langle \alpha u_t+u, q\rangle_{L^2(\Omega)} \\
\Lambda_3(t) := J\langle \phi, \phi_t\rangle_{L^2(\Omega)} \\
\Lambda_4(t) := -\frac{d\tau_0}{\kappa}\langle \phi, \nabla^{-1}q\rangle_{L^2(\Omega)},
\end{cases} \tag{3.5}$$

satisfies for all $t \geq 0$ the following inequality

$$\frac{d}{dt}\Theta_1(t) \quad \leq \quad -E_1(t) + c_0\left(\|\alpha u_t+u\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)} + \|\phi_t\|^2_{L^2(\Omega)}\right)$$

$$+ \quad c_0(\epsilon)\|q\|^2_{L^2(\Omega)}. \tag{3.6}$$

Proof. Take the derivative of Λ_1, Λ_2 use the equations of system (1.1) and Green's formula

$$\begin{aligned}\frac{d}{dt}\Lambda_1(t) &= -(a^2+\lambda+m)\|\operatorname{div}(\alpha u_t + u)\|^2_{L^2(\Omega)} - a^2\|\nabla(\alpha u_t + u)\|^2_{L^2(\Omega)} \\ &- a^2(\beta-\alpha)\left\langle \nabla(\alpha u_t+u), \nabla u_t\right\rangle_{L^2(\Omega)} + \|\alpha u_{tt}+u_t\|^2_{L^2(\Omega)} \\ &- m\left\langle \alpha u_t + u, \nabla\phi\right\rangle_{L^2(\Omega)} + \gamma\left\langle \theta, \alpha\nabla u_t + \nabla u\right\rangle_{L^2(\Omega)}.\end{aligned} \tag{3.7}$$

$$\begin{aligned}\frac{d}{dt}\Lambda_2(t) &= -\frac{\gamma\tau_0}{\kappa}\left\langle \alpha u_{tt}+u_t, q\right\rangle_{L^2(\Omega)} - \gamma\left\langle \theta, \alpha\nabla u_t + \nabla u\right\rangle_{L^2(\Omega)} \\ &+ \frac{\gamma}{\kappa}\left\langle \alpha u_t + u, q\right\rangle_{L^2(\Omega)}.\end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8), using Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\left[\Lambda_1(t)+\Lambda_2(t)\right] &\leq -\frac{a^2}{2}\|\alpha\nabla u_t + \nabla u\|^2_{L^2(\Omega)} - (a^2+\lambda+m)\|\alpha \operatorname{div} u_t + \operatorname{div} u\|^2_{L^2(\Omega)} \\ &+ c_0\|\alpha u_{tt}+u_t\|^2_{L^2(\Omega)} + c_0\|\nabla u_t\|^2_{L^2(\Omega)} \\ &- m\left\langle \alpha u_t + u, \nabla\phi\right\rangle_{L^2(\Omega)} + c_0(\epsilon)\|q\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.9}$$

Take the derivative of Λ_3, Λ_4 use the equations of system (1.1) and Green's formula

$$\begin{aligned}\frac{d}{dt}\Lambda_3(t) &= -b^2\|\nabla\phi\|_{L^2(\Omega)} + J\|\nabla\phi\|_{L^2(\Omega)} - 2m\|\phi_t\|_{L^2(\Omega)} \\ &+ \left\langle m\alpha \operatorname{div} u_t + m \operatorname{div} u + d\theta, \phi\right\rangle_{L^2(\Omega)}.\end{aligned} \tag{3.10}$$

$$\frac{d}{dt}\Lambda_4(t) = -\frac{d\tau_0}{\kappa}\left\langle \phi_t, \nabla^{-1}q\right\rangle_{L^2(\Omega)} + \left\langle \frac{d}{\kappa}\nabla^{-1}q - d\theta, \phi\right\rangle_{L^2(\Omega)}. \tag{3.11}$$

Adding (3.10) and (3.11), using Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\left[\Lambda_3(t)+\Lambda_4(t)\right] &\leq -b^2\|\nabla\phi\|^2_{L^2(\Omega)} - \frac{3m}{2}\|\phi\|^2_{L^2(\Omega)} + c_0\|\phi_t\|^2_{L^2(\Omega)} \\ &+ c_0(\epsilon)\|q\|^2_{L^2(\Omega)} + m\left\langle \alpha \operatorname{div} u_t + \operatorname{div} u, \phi\right\rangle_{L^2(\Omega)}.\end{aligned} \tag{3.12}$$

Summing up inequalities (3.12) and (3.9), then we get (3.6), so the proof is completed. □

Let Φ be the solution of the problem

$$\begin{cases} \nabla\Phi = \theta, \text{ in } \Omega, \\ \\ \Phi = 0, \text{ on } \partial\Omega. \end{cases} \tag{3.13}$$

Since $\theta \in C^1(\mathbb{R}^+; L^2(\Omega))$ then $\Phi \in C^1(\mathbb{R}^+; H^1_0(\Omega))$ and

$$\|\Phi\|_{H^1_0(\Omega)} \leq c_0\|\theta\|_{L^2(\Omega)}. \tag{3.14}$$

Lemma 3.4. *The functional*

$$\Theta_2(t) := \left\langle \alpha u_{tt} + u_t, \Phi\right\rangle_{L^2(\Omega)}. \tag{3.15}$$

satisfies, along the solution of the problem (1.1)-(1.3) the estimate

$$\frac{d}{dt}\Theta_2(t) \leq -\frac{\gamma}{2c}\|\alpha u_{tt}+u_t\|^2_{L^2(\Omega)} + \epsilon_1\|\nabla(\alpha u_t + u)\|^2_{L^2(\Omega)}$$

$$\begin{aligned} &+ \quad c_0(\epsilon)\|\theta\|^2_{L^2(\Omega)} + c_0\|q\|^2_{L^2(\Omega)} + \epsilon_1\|\phi\|^2_{L^2(\Omega)} \\ &+ \quad c_0(\epsilon)\|\nabla u_t\|^2_{L^2(\Omega)} + c_0\|\phi_t\|^2_{L^2(\Omega)} + \epsilon_1\|\alpha \operatorname{div} u_t + \operatorname{div} u\|^2_{L^2(\Omega)}, \end{aligned}$$

for $\epsilon_1 > 0$.

Proof. By taking the derivative of Θ_2, using equations of the system (1.1) and by Greens formula, we obtain

$$\begin{aligned} \frac{d}{dt}\Theta_2(t) &= -\frac{\gamma}{c}\|\alpha u_{tt} + u_t\|^2_{L^2(\Omega)} + \gamma\|\nabla\theta\|^2_{L^2(\Omega)} - \frac{\kappa}{c}\langle \alpha u_{tt} + u_t, q\rangle_{L^2(\Omega)} \\ &- (\lambda + m)\langle \operatorname{div}(\alpha u_t + u), \theta\rangle_{L^2(\Omega)} - \frac{d}{c}\left\langle \nabla^{-1}\phi_t, \alpha u_{tt} + u_t\right\rangle_{L^2(\Omega)} \\ &- a^2(\beta - \alpha)\langle \nabla\theta, \nabla u_t\rangle_{L^2(\Omega)} + m\langle \theta, \phi\rangle_{L^2(\Omega)}. \end{aligned} \tag{3.16}$$

Estimate (3.16) follows thanks to Young's and Poincaré's inequalities. □

Next, let Ψ be the solution of the problem

$$\begin{cases} \Delta\Psi = \theta, \text{ in } \Omega, \\ \Psi = 0, \text{ on } \partial\Omega. \end{cases} \tag{3.17}$$

Since $\theta \in C^1(\mathbb{R}^+; L^2(\Omega))$ then $\Psi \in C^1(\mathbb{R}^+; H^2(\Omega) \cap H^1_0(\Omega))$ and

$$\|\Psi\|_{H^2(\Omega)} \leq c_0\|\theta\|_{L^2(\Omega)}. \tag{3.18}$$

Lemma 3.5. *Let* $(u, \phi, \theta, q) \in \mathcal{H}$ *solution of the problem (1.1)-(1.3). Then, the functional*

$$\Theta_3(t) := -\tau_0\langle q, \nabla\Psi\rangle_{L^2(\Omega)}, \tag{3.19}$$

satisfies

$$\begin{aligned} \frac{d}{dt}\Theta_3(t) &\leq -\frac{\kappa}{2}\|\theta\|^2_{L^2(\Omega)} + \epsilon_2\left(\|\alpha u_{tt} + u_t\|^2_{L^2(\Omega)} + \|\phi_t\|^2_{L^2(\Omega)}\right) \\ &+ c_0(\epsilon)\|q\|^2_{L^2(\Omega)}, \end{aligned}$$

for $\epsilon_2 > 0$.

Proof. Taking the derivative of Θ_3, using (1.1) and Greens formula, to get

$$\begin{aligned} \frac{d}{dt}\Theta_3(t) &= -\kappa\|\theta\|^2_{L^2(\Omega)} + \langle \nabla\Psi, q\rangle_{L^2(\Omega)} + \frac{\tau_0\kappa}{c}\left\langle q, \nabla(\Delta^{-1}\operatorname{div} q)\right\rangle_{L^2(\Omega)} \\ &+ \left\langle q, \frac{\tau_0 d}{c}\nabla\Delta^{-1}\phi_t + \frac{\tau_0\gamma}{c}\nabla\Delta^{-1}\operatorname{div}(\alpha u_{tt} + u_t)\right\rangle_{L^2(\Omega)}. \end{aligned} \tag{3.20}$$

Since $\operatorname{div} f \in H^{-1}(\Omega)$ then $\Delta^{-1}(\operatorname{div} f) \in H^1(\Omega)$ with

$$\begin{aligned} \|\nabla(\Delta^{-1}\operatorname{div} f)\|_{L^2(\Omega)} &\leq \|\Delta^{-1}\operatorname{div} f\|_{H^1(\Omega)} \leq c_0\|\operatorname{div} f\|_{H^1(\Omega)} \\ &\leq c_0\|f\|_{L^2(\Omega)}. \end{aligned} \tag{3.21}$$

By using (3.21), Young's and Poincaré's inequalities, the estimate (3.20) follows. □

Lemma 3.6. *The functional*

$$\Theta_4(t) := \Lambda_5(t) + \Lambda_6(t), \tag{3.22}$$

with

$$\begin{cases} \Lambda_5(t) = \langle \phi_t, \theta \rangle_{L^2(\Omega)}, \\ \\ \Lambda_6(t) = \frac{\tau_0 b^2}{\kappa J} \langle \operatorname{div} q, \phi \rangle_{L^2(\Omega)}, \end{cases} \tag{3.23}$$

satisfies

$$\begin{aligned} \frac{d}{dt}\Theta_4(t) &\leq -\frac{d}{2c}\|\nabla \phi_t\|^2_{L^2(\Omega)} + \epsilon_3 \left(\|\phi\|^2_{L^2(\Omega)} + \|\alpha \operatorname{div} u_t + \operatorname{div} u\|^2_{L^2(\Omega)}\right) \\ &+ c_0(\epsilon)\|\theta\|^2_{L^2(\Omega)} + c_0 \left(\|\alpha u_{tt} + u_t\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{3.24}$$

for $\epsilon_3 > 0$.

Proof. Direct computations, using (1.1) and Greens formula, we have

$$\begin{aligned} \frac{d}{dt}\Lambda_5(t) &= -\frac{d}{c}\|\nabla \phi_t\|^2_{L^2(\Omega)} + \frac{d}{J}\|\nabla \theta\|^2_{L^2(\Omega)} + \frac{b^2}{J} \langle \Delta \phi, \theta \rangle_{L^2(\Omega)} \\ &- \frac{2m}{J} \langle \theta, \phi \rangle_{L^2(\Omega)} + \frac{m}{J} \langle \theta, \alpha \operatorname{div} u_t + \operatorname{div} u \rangle_{L^2(\Omega)} \\ &+ \frac{\gamma}{c} \langle \phi_t, \alpha \operatorname{div} u_{tt} + \operatorname{div} u_t \rangle_{L^2(\Omega)} - \frac{\kappa}{c} \langle \phi_t, q \rangle_{L^2(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\Lambda_6(t) &= -\frac{b^2}{J} \langle \Delta \theta, \phi \rangle_{L^2(\Omega)} - \frac{b^2}{\kappa J} \langle \operatorname{div} q, \phi \rangle_{L^2(\Omega)} \\ &+ \frac{\tau_0 b^2}{\kappa J} \langle \operatorname{div} q, \phi_t \rangle_{L^2(\Omega)}. \end{aligned}$$

Estimate (3.24) follows thanks to Young and Poincaré's inequalities. □

Proof. (Of Theorem 3.2) We define the Lyapunov functional $\mathfrak{Y}$ by

$$\mathfrak{Y}(t) = \mathfrak{N}E(t) + \sum_{i=1}^{i=2} \Theta_i(t) + \mathfrak{N}_1\Theta_3(t) + \Theta_4(t). \tag{3.25}$$

We note that there exist two positive constants σ_1 and σ_2 such that

$$\sigma_1 E(t) \leq \mathfrak{Y}(t) \leq \sigma_2 E(t), \tag{3.26}$$

which means that $\mathfrak{Y}(t) \equiv E(t)$.
Thanks to Young and Poincaré's inequality to get

$$\sum_{i=1}^{i=2} \Theta_i(t) + \mathfrak{N}_1\Theta_3(t) + \Theta_4(t) \leq \sigma_0 E(t). \tag{3.27}$$

Thus

$$|\mathfrak{Y}(t) - NE(t)| = \sum_{i=1}^{i=2} \Theta_i(t) + \mathfrak{N}_1\Theta_3(t) + \Theta_4(t) \leq \sigma_0 E(t). \tag{3.28}$$

Then we can get (3.26) with $\sigma_1 = \mathfrak{N} - \sigma_0$ and $\sigma_2 = \mathfrak{N} + \sigma_0$ by choosing $\sigma_0 > 0$ such that $\mathfrak{N} - \sigma_0 > 0$.
It follows from previous Lemmas 3.3-Lemma 3.6, that for any $t > 0$

$$\begin{aligned} \frac{d}{dt}\mathfrak{Y}(t) &\leq -\left[\frac{a^2}{2} - \epsilon_1\right] \|\nabla \alpha u_t + \nabla u\|^2_{L^2(\Omega)} \\ &- \left[\frac{m}{2} - \epsilon_3 - \epsilon_1\right] \|\phi\|^2_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{a^2+\lambda}{2} - \epsilon_1 - \epsilon_3\right] \|\operatorname{div}(\alpha u_t + u)\|^2_{L^2(\Omega)} \\
&- \frac{b^2}{2}\|\nabla\phi\|^2_{L^2(\Omega)} - \frac{m}{2}\|\alpha \operatorname{div} u_t + \operatorname{div} u - \phi\|^2_{L^2(\Omega)} \\
&- \left[\frac{\gamma}{2c} - c_0 - \mathfrak{N}_1\epsilon_2\right] \|\alpha u_{tt} + u_t\|^2_{L^2(\Omega)} \\
&- \left[\frac{d}{2c} - c_0 - \mathfrak{N}_1\epsilon_2\right] \|\phi_t\|^2_{L^2(\Omega)} \\
&- \left[\mathfrak{N}_1\frac{\kappa}{2} - c_0(\epsilon)\right] \|\theta\|^2_{L^2(\Omega)} \\
&- \left[\mathfrak{N}a^2\alpha(\beta-\alpha) - c_0(\epsilon)\right] \|\nabla u_t\|^2_{L^2(\Omega)} \\
&- \left[\mathfrak{N} - c_0(\epsilon)\left(\mathfrak{N}_1+1\right)\right] \|q\|^2_{L^2(\Omega)}.
\end{aligned}
$$

We can choose, now, $\mathfrak{N}_1$ large enough such that

$$\mathfrak{N}_1\frac{\kappa}{2} - c_0(\epsilon) > 0. \tag{3.29}$$

Next, we pick $\epsilon_1, \epsilon_2, \epsilon_3$ small enough so that

$$
\begin{cases}
\frac{a^2}{2} - \epsilon_1 > 0 \\
\frac{m}{2} - \epsilon_3 - \epsilon_1 > 0 \\
\frac{a^2+\lambda}{2} - \epsilon_1 - \epsilon_3 > 0 \\
\frac{\gamma}{2c} - c_0 - \mathfrak{N}_1\epsilon_2 > 0 \\
\frac{d}{2c} - c_0 - \mathfrak{N}_1\epsilon_2 > 0.
\end{cases} \tag{3.30}
$$

Then we take $\mathfrak{N}$ large enough and the fact that $\beta - \alpha > 0$ so that

$$
\begin{cases}
\mathfrak{N}a^2(\beta-\alpha) - c_0(\epsilon) > 0 \\
\mathfrak{N} - c_0(\epsilon)\left(\mathfrak{N}_1+1\right) > 0.
\end{cases}
$$

Thus, we arrive at

$$
\begin{aligned}
\frac{d}{dt}\mathfrak{Y}(t) \quad &\le \quad -\mathfrak{C}_1\|\nabla\alpha u_t + u\|^2_{L^2(\Omega)} - \mathfrak{C}_2\|\phi\|^2_{L^2(\Omega)} - \frac{b^2}{2}\|\nabla\phi\|^2_{L^2(\Omega)} \\
&- \quad \mathfrak{C}_3\|\operatorname{div}(\alpha u_t + u)\|^2_{L^2(\Omega)} - \mathfrak{C}_4\|\alpha u_{tt} + u_t\|^2_{L^2(\Omega)} - \mathfrak{C}_5\|\phi_t\|^2_{L^2(\Omega)} \\
&- \quad \frac{m}{2}\|\alpha \operatorname{div} u_t - \operatorname{div} u - \phi\|^2_{L^2(\Omega)} - \mathfrak{C}_6\|\theta\|^2_{L^2(\Omega)} - \mathfrak{C}_7\|\nabla u_t\|^2_{L^2(\Omega)} \\
&- \quad \mathfrak{C}_8\|q\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Then we have for some positive constant ϖ_0

$$\frac{d}{dt}\mathfrak{Y}(t) \le -\varpi_0 E(t), \tag{3.31}$$

On the other hand, by using (3.26) we have

$$\mathfrak{Y}(t) \sim E(t). \tag{3.32}$$

Hence, combination of (3.31) and (3.32) leads to

$$\frac{d}{dt}\mathfrak{Y}(t) \leq -\varpi_4 \mathfrak{Y}(t), \forall \varpi_4 > 0. \tag{3.33}$$

A simple integration of (3.33) and owing to (3.26), the proof is completed. □

4. Conclusion

Through a depth research, we established a positive Lyapunov function equivalent to the energy of our problem, which showed that its nature of decaying is exponential, and then imply that is the same for our system. This work opens a new field of study by adding various dissipations to study the converging behavior of solutions. It also shows us that the properties of standard solids do not affect the stability of porous-elastic systems. As for the works that have been mentioned in the introduction, it is interested in studying each model separately. For example, in 2013, the work published by [1], the standard linear solid model with thermal effect of Fourier law was studied.

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