REFEREE'S REPORT on the paper
"*Global existence and decay estimates of energy of solutions for a class of $p(x)$-Laplacian heat equations with logarithmic nonlinearity*"
by Sarra Toualbia, Abderahmane Zarai, and Salah Boulaaras.

The authors study the initial boundary value problem for the parabolic $p(x)$-Laplacian with logarithmic source:

$$u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = |u|^{p(x)-2}u \ln |u|, \quad x \in \Omega,\ t > 0,$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$
$$u|_{t=0} = u_0 \quad \text{in} \quad \Omega.$$

Here Ω s a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $p = p(x)$ is a continuous function on $\overline{\Omega}$ satisfying

$$p_- \le p(x) \le p_+, \quad x \in \overline{\Omega}.$$

The authors state three theorems. In Theorem 1 they claim the existence of a "strong" solution on some time interval for all initial data from $W_0^{1,p(x)}(\Omega)$. In Theorem 2 the authors claim the local existence of a weak solution for all initial data from $W_0^{1,p(x)}(\Omega)$. Theorem 3 states that there is a global weak solution for all initial data from a special set $W^+ = \{u : J(u) \le d,\ I(u) > 0\}$ where the functional

$$I(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|u|^{p(x)}\ln|u|}{p(x)}\,dx$$

and

$$J(u) = I(u) + \int_\Omega \frac{|u|^{p(x)}}{p(x)^2}\,dx.$$

I can not recommend this paper for publication in SEMR. Below I give the reasons why.

First, the review of literature on the subject is clearly far from satisfactory. I would start from describing basic properties of equations with constant exponent and sign-definite term on the right-hand side (source or absorption, since the nonlinearity of the present paper combines both) as well as mentioning papers on parabolic equations with variable nonlinearity.

But that is not as serious as the fact that the authors do not introduce the concept of solution. Which is necessary anyway, but twice so in the situation of variable exponent where definition of Sobolev spaces and proofs of existence have their peculiarities. So even when the authors write that "The spaces $W^{1,p(\cdot)}(\Omega)$ and ... are defined in the same way as the usual Sobolev space..." they already miss the fact that there are different definitions of these spaces which may lead to different results (say one can define the space of functions with gradients (in the Sobolev sense) integrable to the power $p(x)$, or take the closure of smooth functions in this space, which may not coincide with the large space). Moreover, the authors do not impose any restrictions on the values of p_- and p_+ (at least $p_- \geq 1$).

Without proper definition of solution (and relevant functional spaces) it is hard to evaluate further contents of this paper.

In the present state of this paper the necessary improvements would amount to rewriting it anew. I recommend the authors to have a close look at the papers

Yu. A. Alkhutov, V. V. Zhikov, Existence and uniqueness theorems for solutions of parabolic equations with a variable nonlinearity exponent, Sb. Math., 205:3 (2014), 307–318;

Yu. A. Alkhutov, V. V. Zhikov, Hölder continuity of solutions of parabolic equations with variable nonlinearity exponent, J. Math. Sci. (N. Y.), 179:3 (2011), 347–389;

Yu. A. Alkhutov, V. V. Zhikov, Existence theorems for solutions of parabolic equations with variable order of nonlinearity, Proc. Steklov Inst. Math., 270 (2010), 15–26;

L. Diening, P. Nägele & M. Ruzicka (2012) Monotone operator theory for unsteady problems in variable exponent spaces, Complex Variables and Elliptic Equations, 57:11, 1209-1231, DOI: 10.1080/17476933.2011.557157

Erhardt, A.H. Compact embedding for p(x, t)-Sobolev spaces and existence theory to parabolic equations with p(x, t)-growth. Rev Mat Complut 30, 35–61 (2017). https://doi.org/10.1007/s13163-016-0211-4

Also a number of interesting works of Antontsev, Shmarev are available on parabolic equations with variable nonlinearity.

CONCLUSION: I can not recommend this paper for publication in *Siberian Electronic Mathematical Reports.* However, the direction of study the authors pursue is interesting and results of this type (cast in proper form!) can present certain interest to the research community.

REFEREE's RECOMMENDATION: Reject.