GLOBAL EXISTENCE AND DECAY ESTIMATES OF ENERGY OF SOLUTIONS FOR A CLASS OF $p(x)-$LAPLACIAN HEAT EQUATIONS WITH LOGARITHMIC NONLINEARITY

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ABSTRACT. The main goal of this work is to study the initial boundary value problem of a quasilinear parabolic equation with logarithmic nonlinearity. By using the logarithmic Sobolev inequality and potential wells method, we establish the some results about the existence of global weak solutions. Second, we get the sufficient conditions for the large time decay of global weak solutions under some conditions.

1. INTRODUCTION

In this paper, we consider the following initial boundary value problem with logarithmic source:

$$\begin{cases} u_t - div\left(|\nabla u|^{p(x)-2}\nabla u\right) = |u|^{p(x)-2} u \ln|u|, & x\in\Omega, t>0, \\ u(x,t)=0, & x\in\partial\Omega, t>0, \\ u(x,0)=u_0, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p(x)$ is a continuous function on $\overline{\Omega}$ such that

$$p_- \leq p(x) \leq p_+, \tag{1.2}$$

where

$$p_- = ess\inf_{x\in\Omega} p(x) \quad \text{and} \quad p_+ = ess\sup_{x\in\Omega} p(x).$$

When $p(x)=p$, existence and blow-up results have been established by Cong Nhan Le and Xuan Truong Le [13]. The authors showed that when $p>2$ solutions blow up at finite time and they obtained sufficient conditions on the existence of global weak solutions.

M. Kbiri Alaoui et al [16] considered the equation

$$u_t - div\left(|\nabla u|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2} u, \quad x\in\Omega, t>0,$$

they showed that any solution with nontrivial initial datum blows up in finite time whenever $\int_\Omega u_0^2 dx > 0$ and

$$\int_\Omega \left(\frac{1}{p(x)}|u_0|^{p(x)} - \frac{1}{m(x)}|\nabla u_0|^{m(x)}\right)dx \geq 0.$$

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For instance, Hua Wang and Yijun He in [19] considered the case where

$$\begin{cases} u_t - \Delta u = |u|^{p(x)}, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0, & x \in \Omega, t > 0 \end{cases} \tag{1.3}$$

and proved that under the condition $1 < p_- \le p_+ \le \frac{n+2}{n-2}$ and certain initial data, the solution of problem (1.3) blow up in finite time for a positive initial energy.

With such literature concerning polynomial nonlinear terms, logarithmic nonlinearity also received great interest of both physicists and mathematicians. The logarithmic nonlinearity was introduced in the nonrelativistic wave equation describing spinning particles moving in an external electromagnetic field and in the relativistic wave equation for spinless particles[1]. Moreover this type of nonlinearity is also met in many branches of physics such as inflationary cosmology [2], nuclear physics [8], optics [9] and geophysics [10]. With all those specific underlying meaning in physics, the global-in-time well-posedness of solution to the problem of evolution equation with such logarithmic type nonlinearity captures lots of attention.

In this paper, we consider problem (1.1) with the presence of nonlinear diffusion $\Delta_{p(x)} = div\left(|\nabla u|^{m(x)-2} \nabla u\right)$ and logarithmic nonlinearity $|u|^{p(x)-2} u \ln|u|$ which extends problem in [13]. Our goal is to exploit potential well method for problem (1.1) in order to obtain global existence and decay estimate of solutions. More precisely we show that the norm $\|u(t)\|^2$ decays algebraically instead of exponential decay compare to the case of $p = 2$ studied in [3].

It is necessary to note that the presence of the logarithmic nonlinearity causes some difficulties in deploying the potential well method. In order to handle this situation we need the following logarithmic Sobolev inequality which was introduced in [15].

Lemma 1. *Let $p > 1, \mu > 0,$ and $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$. Then we have*

$$p \int_{\mathbb{R}^n} |u(x)|^p \log\left(\frac{|u(x)|}{\|u(x)\|_{L^p(\mathbb{R}^n)}}\right) dx + \frac{n}{p} \log\left(\frac{p\mu e}{n\mathcal{L}_p}\right) \int_{\mathbb{R}^n} |u(x)|^p \, dx$$
$$\le \mu \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx,$$

where

$$\mathcal{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)}\right]^{\frac{p}{n}}.$$

Remark 1. *If $u \in W^{1,p(x)}(\Omega)$ and $\Omega = \Omega_+ \cup \Omega_-$ where*

$$\Omega_+ = \{x \in \Omega \quad |u| \ge 1\}, \Omega_- = \{x \in \Omega \quad |u| < 1\},$$

then, by defining $u(x) = 0$ for $x \in (\mathbb{R}^n) \setminus \{\Omega\}$, it holds

$$\begin{aligned} & p_+ \int_{\Omega_+} |u(x)|^{p_+} \log\left(\frac{|u(x)|}{\|u(x)\|_{L^{p_+}(\Omega_+)}}\right) dx + \frac{n}{p_+} \log\left(\frac{p_+\mu e}{n\mathcal{L}_{p_+}}\right) \int_{\Omega_+} |u(x)|^{p_+} \, dx \\ & \le \mu \int_{\Omega_+} |\nabla u(x)|^{p_+} \, dx. \\ & p_- \int_{\Omega_-} |u(x)|^{p_-} \log\left(\frac{|u(x)|}{\|u(x)\|_{L^{p_-}(\Omega_-)}}\right) dx + \frac{n}{p_-} \log\left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right) \int_{\Omega_-} |u(x)|^{p_-} \, dx \\ & \le \mu \int_{\Omega_-} |\nabla u(x)|^{p_-} \, dx. \end{aligned} \tag{1.4}$$

For any real number $\mu > 0$, and

$$\mathcal{L}_{p_+} = \tfrac{p_+}{n}\left(\tfrac{p_+-1}{e}\right)^{p_+-1}\pi^{-\frac{p_+}{2}}\left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p_+-1}{p_+}+1\right)}\right]^{\frac{p_+}{n}}.$$
$$\mathcal{L}_{p_-} = \tfrac{p_-}{n}\left(\tfrac{p_--1}{e}\right)^{p_--1}\pi^{-\frac{p_-}{2}}\left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p_--1}{p_-}+1\right)}\right]^{\frac{p_-}{n}}. \tag{1.5}$$

2. Preliminaries

2.1. **Functional framework.** We give some well-know results about the Lebesgue and Sobolev spaces with variable exponents (see [4]).

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^N$ with $N \geq 2$, the $p(x)$ modular of a measurable function $u : \Omega \to \mathbb{R}^N$ is defined as

$$\zeta_{p(x)} = \int_{\Omega-\Omega_\infty} |u(x)|^{p(x)}\, dx + ess\sup_{x\in\Omega_\infty} |u(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\zeta_{p(.)}(\lambda u) < \theta, \text{ for some } \lambda > 0.$$

The Luxembourg norm on this space is defined as

$$\|u\|_{p(.)} = \inf\left\{\lambda > 0 : \zeta_{p(.)}(\lambda u) \leq 1\right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space (see [4]).

The variable-exponent space $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$. This space is a Banach space with respect to the norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgues spaces in many aspects. For the following assertions, see [5] :

-The Hölder inequality holds.

-They are reflexive if and only $1 < p_- < p_+ < \infty$.

-Continuous functions are dense if $p_+ < \infty$.

-If Ω has finite measure and p, q are variable exponents so that $p(x) \leq q(x)$ almost every where in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ holds.

-The space $W^{1,p(\cdot)}(\Omega)$ and $W^{1,p'(\cdot)}(\Omega)$ are defined in the same way as the usual Sobolev spaces where $p'(x)$ is the function such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Lemma 2. *Let ϱ be a positive number. Then the following inequality holds*

$$\log s \leq \frac{e^{-1}}{\varrho} s^{\varrho}, \textit{ for all } s \in [1, +\infty].$$

Lemma 3. [11]*(a) For any function $u \in W_0^{1,p}(\Omega)$, we have the inequality*

$$\|u\|_q \leq B_{q,p} \|\nabla u\|_p ,$$

for all $q \in [1, \infty)$ if $n \leq p$, and $1 \leq q \leq \frac{np}{n-p}$ if $n > p$. Then best constant depends $B_{q,p}$ only on Ω, n, p and q.

We will denote the constant $B_{p,p}$ by B_p.

(b) Let $2 \leq p < q < p^*$. For any $u \in W_0^{1,p}(\Omega)$ we have

$$\|u\|_q \leq C \|\nabla u\|_p^{\alpha} \|u\|_p^{1-\alpha} ,$$

where C is a positive constant and

$$\alpha = \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{p}\right)^{-1}.$$

Remark 2. *It follows from Lemma 2 that*

$$s^p \log s \leq \frac{e^{-1}}{\varrho} s^{p+\varrho}, \text{ for all } \varrho > 0 \text{ and } s \in [1, +\infty).$$

Now, we passe to the definition of the strong solutions of problem (1.1).

Theorem 1. *For all $u_0 \in W_0^{1,p(x)}(\Omega)$, there exists $T > 0$ such that problem (1.1) has a strong solution u on $(0, T]$ satisfying*

$$u \in C_w\left([0, T], W_0^{1,p(x)}(\Omega)\right) \cap C\left([0, T], L^{p(x)}(\Omega)\right) \cap W^{1,2}\left(0, T, L^2(\Omega)\right).$$

2.2. **Potential well.** Considering the functionals J and I defined on $X_0 = W_0^{1,p(x)}(\Omega) \setminus \{0\}$ as follows

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \ln |u| \, dx + \int_\Omega \frac{1}{p(x)^2} |u|^{p(x)} dx. \tag{2.1}$$

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \ln |u| \, dx. \tag{2.2}$$

The functions I and J are continuous (they are defined as in [13] with some modifications). Moreover, we have

$$J(u) = I(u) + \int_\Omega \frac{1}{p(x)^2} |u|^{p(x)} dx. \tag{2.3}$$

Let $u \in X_0$ and consider the real function $j : \lambda \to J(\lambda u)$ for $\lambda > 0$, defined by

$$\begin{aligned} j(\lambda) &= \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |\nabla u|^{p(x)} dx - \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} \ln |u| \, dx \\ &\quad + \int_\Omega \frac{\lambda^{p(x)}}{p(x)^2} |u|^{p(x)} dx - \ln \lambda \int_\Omega \frac{\lambda^{p(x)}}{p(x)} |u|^{p(x)} dx. \end{aligned} \tag{2.4}$$

The following Lemma shows that $j(\lambda)$ has a unique positive critical point $\lambda^* = \lambda^*(u)$ see [14].

Lemma 4. *Let $u \in X_0$. Then it holds*

(1) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$,

(2) there is a $\lambda^* = \lambda^*(u) > 0$ *such that* $j^{'}(\lambda^*) = 0$,

(3) $j(\lambda)$ *is increasing on* $(0,\lambda^*)$, *decreasing on* $(\lambda^*,+\infty)$ *and attains the maximum at* λ^*,

(4) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and* $I(\lambda^* u) = 0$.

Proof. For $u \in X_0$, by the definition of j and by some calculation, we obtain

$$\begin{aligned} \frac{d}{d\lambda}j(\lambda) &= \int_\Omega \lambda^{p(x)-1}\left|\nabla u\right|^{p(x)} dx - \int_\Omega \lambda^{p(x)-1}\left|u\right|^{p(x)} \ln|u|\, dx \\ &\quad - \ln\lambda \int_\Omega \lambda^{p(x)-1}\left|u\right|^{p(x)} dx. \end{aligned} \tag{2.5}$$

$$\frac{d}{dt}j(\lambda^*) = 0$$

which implies that

$$\ln\lambda^* = \frac{\int_\Omega {\lambda^*}^{p(x)-1}\left[\left|\nabla u\right|^{p(x)} dx - |u|^{p(x)}\ln|u|\right] dx}{\int_\Omega \lambda^{p(x)-1}\left|u\right|^{p(x)} dx}.$$

$$\lambda^* = \exp\frac{\int_\Omega \lambda^{*p(x)-1}\left[\left|\nabla u\right|^{p(x)} dx - |u|^{p(x)}\ln|u|\right] dx}{\int_\Omega \lambda^{*p(x)-1}\left|u\right|^{p(x)} dx}. \tag{2.6}$$

The last property (1.4), follows from the fact that

$$\begin{aligned} I(u) &= \lambda\left[\int_\Omega \frac{\lambda^{p(x)-1}}{p(x)}\left|\nabla u\right|^{p(x)} dx - \int_\Omega \frac{\lambda^{p(x)-1}}{p(x)}\left|u\right|^{p(x)}\ln|u|\, dx - \ln\lambda\int_\Omega \frac{\lambda^{p(x)-1}}{p(x)}\left|u\right|^{p(x)} dx\right] \\ &= \lambda j^{'}(\lambda). \end{aligned}$$

The proof of the lemma is complete. □

Next we denote

$$\begin{aligned} R_1 &= \left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right)^{\frac{n}{p_-^2}}, \\ R_2 &= \left(\frac{p_+\mu e}{n\mathcal{L}_{p_+}}\right)^{\frac{n}{p_-p_+}}, \end{aligned}$$

where $\mathcal{L}_{p_-}, \mathcal{L}_{p_+}$ are defined as in Remark 2.

Lemma 5. *(1) if* $I(u) > 0$ *then*

$$\begin{cases} 0 < \|u\|_{L^{p_-}(\Omega_-)} < R_1. \\ 0 < \|u\|_{L^{p_+}(\Omega_+)} < R_2. \end{cases}$$

(2) if $I(u) < 0$ *then*

$$\begin{cases} \|u\|_{L^{p_-}(\Omega_-)} > R_1. \\ \|u\|_{L^{p_+}(\Omega_+)} > R_2. \end{cases}$$

(3) if $I(u) = 0$ *then*

$$\begin{cases} \|u\|_{L^{p_-}(\Omega_-)} \geq R_1. \\ \|u\|_{L^{p_+}(\Omega_+)} \geq R_2. \end{cases}$$

Proof. From $\mu > 0$, it follows from the logarithm inequality (1.4) that

$$\begin{aligned}
I(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \ln |u|\, dx \\
&\geq \frac{1}{p_-^2}\left[\mu \int_{\Omega_-} |\nabla u|^{p_-}\, dx - p_- \int_{\Omega_-} |u|^{p_-} \ln \frac{|u|}{\|u\|_{L^{p_-}(\Omega_+)}} dx\right] \\
&\quad - \left[\frac{1}{p_-}\int_{\Omega_-} |u|^{p_-} \ln \|u\|_{L^{p_-}(\Omega_-)}\, dx + \frac{1}{p_+}\int_{\Omega_+} |u|^{p_+} \ln \|u\|_{L^{p_+}(\Omega_+)}\, dx\right] \\
&\quad + \frac{1}{p_-p_+}\left[\mu \int_{\Omega_+} |\nabla u|^{p_+}\, dx - p_+ \int_{\Omega_+} |u|^{p_+} \ln \frac{|u|}{\|u\|_{L^{p_+}(\Omega_+)}} dx\right] \\
&\quad + \frac{1}{p_+}\int_{\Omega_+}\left[|\nabla u|^{p_-}\, dx - \frac{\mu}{p_-}|\nabla u|^{p_+}\right] dx + \int_{\Omega_-}\left[\frac{1}{p_+}|\nabla u|^{p_+} - \frac{\mu}{p_-^2}|\nabla u|^{p_-}\right] dx.
\end{aligned} \tag{2.7}$$

We choose $\mu \leq \min\left\{p_- \frac{\|\nabla u\|^{p_-}_{L^{p_-}(\Omega_+)}}{\|\nabla u\|^{p_+}_{L^{p_+}(\Omega_+)}}, p_-^2 \frac{\|\nabla u\|^{p_+}_{L^{p_+}(\Omega -)}}{\|\nabla u\|^{p_-}_{L^{p_-}(\Omega_-)}}\right\}$, so we get

$$\begin{aligned}
I(u) &\geq \frac{1}{p_-^2}\left[\mu \int_{\Omega_-} |\nabla u|^{p_-}\, dx - p_- \int_{\Omega_-} |u|^{p_-} \ln \frac{|u|}{\|u\|_{L^{p_-}(\Omega_-)}} dx\right] \\
&\quad + \frac{1}{p_-p_+}\left[\mu \int_{\Omega_+} |\nabla u|^{p_+}\, dx - p_+ \int_{\Omega_+} |u|^{p_+} \ln \frac{|u|}{\|u\|_{L^{p_+}(\Omega_+)}} dx\right] \\
&\geq \frac{n}{p^3{}_-} \ln\left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right)\int_{\Omega_-} |u|^{p_-}\, dx + \frac{n}{p^2{}_+p_-} \ln\left(\frac{p_+\mu e}{n\mathcal{L}_{p+}}\right)\int_{\Omega_+} |u|^{p_+}\, dx \\
&\quad - \left[\frac{1}{p_-}\int_{\Omega_-} |u|^{p_-} \ln \|u\|_{L^{p_-}(\Omega_-)}\, dx + \frac{1}{p_+}\int_{\Omega_+} |u|^{p_+} \ln \|u\|_{L^{p_+}(\Omega_+)}\, dx\right].
\end{aligned} \tag{2.8}$$

For $I(u) > 0$, we get

$$\begin{cases} \|u\|_{L^{p_-}(\Omega_-)} < \left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right)^{\frac{n}{p^2{}_-}} = R_1. \\ \|u\|_{L^{p_+}(\Omega_+)} < \left(\frac{p_+\mu e}{n\mathcal{L}_{p+}}\right)^{\frac{n}{p_+p_-}} = R_2. \end{cases}$$

That's mean

$$\begin{cases} 0 < \|u\|_{L^{p_-}(\Omega_-)} < R_1. \\ 0 < \|u\|_{L^{p_+}(\Omega_+)} < R_2. \end{cases}$$

(2) $I(u) < 0$

$$\begin{cases} \|u\|_{L^{p_-}(\Omega_-)} > \left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right)^{\frac{n}{p^2{}_-}} = R_1. \\ \|u\|_{L^{p_+}(\Omega_+)} > \left(\frac{p_+\mu e}{n\mathcal{L}_{p+}}\right)^{\frac{n}{p_+p_-}} = R_2. \end{cases}$$

that's mean

$$\begin{cases} \|u\|_{L^{p_-}(\Omega_-)} > R_1. \\ \|u\|_{L^{p_+}(\Omega_+)} > R_2. \end{cases}$$

(3) We can argue similarly the proof (2).

The proof of the lemma is complete. □

Let us denote by $\mathcal{N}$ the Nehari manifold

$$\mathcal{N} = \{u \in X_0 : I(u) = 0\}.$$

By virtue of Lemma 4, it is clear that $\mathcal{N}$ is not empty. Moreover, J is coercive on $\mathcal{N}$.

Now as in [13], we define the subsets of $H_0^1(W)$ related to problem (1.1). Set

$$\begin{aligned} W_1 &= \{u \in X_0 : J(u) < d\}, \quad W_2 = \{u \in X_0 : J(u) = d\}, \quad W = W_1 \cup W_2, \\ W_1^+ &= \{u \in W_1 : I(u) > 0\}, \quad W_2^+ = \{u \in W_2 : I(u) > 0\}, \quad W^+ = W_1^+ \cup W_2^+, \\ W_1^- &= \{u \in W_1 : I(u) < d\}, \quad W_2^- = \{u \in W_2 : I(u) < d\}, \quad W^- = W_1^- \cup W_2^-, \end{aligned} \tag{2.9}$$

Remark 3. *Since the equality (2.3), it is not difficult to see that*

$$W_1^+ = \{u \in X_0 : \ 0 < J(u) < d, \ I(u) > 0\}.$$

2.3. **Global Existence and decay Estimates.** In this section, we first establish the local existence and existence of global weak solutions to problem (1.1) and then prove the decay estimate, provided that the initial data come from W_1^+. For proving the last result we need the following lemma due to Martinez [9].

Lemma 6. *(see* [14]*) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and σ is a nonnegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \leq \frac{1}{\omega} f^\sigma(0) f(t). \quad \forall t \geq 0.$$

Then we have
(a) $f(t) \leq f(0)e^{1-\omega t}$, for all $t \geq 0$, whenever $\sigma = 0$,
(b) $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega t}\right)^{\frac{1}{\sigma}}$, for all $t \geq 0$, whenever $\sigma > 0$.

We now begin with local existence of solutions to problem (1.1).

Theorem 2. (*Locale existence*) *Let $u_0 \in X_0$. Then there exists a positive constant T_0 such that the problem (1.1) has a weak solution $u(x,t)$ on $\Omega\times$ $(0, T_0)$. Furthermore, $u(x,t)$ satisfies the energy inequality*

$$\int_0^t \|u_s(s)\|_2^2 \, ds + J(u(t)) \leq J(u_0), \quad \forall t \in [t, T_0). \tag{2.10}$$

Proof. The Faedo–Galerkin's methods is used. In the space $W_0^{1,p(x)}(\Omega)$, we take a basis $\{w_j\}_{j=1}^\infty$ and define the finite dimensional space

$$V_m = span\{w_1, w_2, ...w_m,\}.$$

Let u_{0m} be an element of V_m such that

$$u_{0m} = \sum_{j=1}^m \alpha_{mj} w_j \to \mathrm{u}_0 \text{ strongly in } W_0^{1,p(x)}(\Omega). \tag{2.11}$$

as $m \to +\infty$, We find the approximate solution $u_m(x,t)$ of the problem (1.1) in the form

$$u_m(x,t) = \sum_{j=1}^m \alpha_{mj}(t) w_j(x).$$

where the coefficients $\alpha_{mj}(1 \leq j \leq m)$ satisfy the system of ordinary differential equations

$$\int_\Omega u_{mt} w_i dx - \int_\Omega |\nabla u_m|^{p(x)-2} \nabla u_m \nabla w_i dx = \int_\Omega |u_m|^{p(x)-2} u_m w_i \log |u_m| \, dx \tag{2.12}$$

$i \in \{1, 2.., m\}$, with the initial conditions

$$\alpha_{mj}(0) = a_{m,j}, \quad j \in \{1, 2.., m\} \tag{2.13}$$

The existence of a local solution of system (2.12)–(2.13) is guaranteed by Peano's theorem.

Multiplying the ith equation in (2.12) by $\alpha_{mi}(t)$ and summing over i from 1 to m gives us

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_2^2 + \int_\Omega |\nabla u_m|^{p(x)} dx = \int_\Omega |u_m|^{p(x)} \log |u_m| \, dx \tag{2.14}$$

By virtue of Remark 2, we have

$$\begin{aligned} \int_\Omega |u_m|^{p(x)} \log |u_m| \, dx &\leq \int_{\Omega_-} |u_m|^{p_-} \log |u_m| \, dx + \int_{\Omega_+} |u_m|^{p_+} \log |u_m| \, dx \\ &\leq \int_{\Omega_+} |u_m|^{p_+} \log |u_m| \, dx \\ &\leq \frac{e^{-1}}{\varrho} \|u_m\|_{p_+ + \varrho}^{p_+ + \varrho} \leq C \|\nabla u_m\|_{p_+ + \varrho}^{p_+ + \varrho}. \end{aligned} \tag{2.15}$$

By using Lemma (3) and Yong's inequality, it follows from (2.15) that

$$\begin{aligned} \int_\Omega |u_m|^{p(x)} \log |u_m| \, dx &\leq \|\nabla u_m\|_{p_-}^{\alpha(p_+ + \varrho)} . \|u_m\|_2^{(1-\alpha)(p_+ + \varrho)} \\ &\leq \varepsilon \|\nabla u_m\|_{L^{p_-}(\Omega^+)}^{\alpha(p_+ + \varrho)q} + C_\varepsilon \|u_m\|_2^{(1-\alpha)(p_+ + \varrho)p}, \end{aligned}$$

we choose $q = \frac{p_-}{\alpha(p_+ + \varrho)}$, this mean $p = \frac{p_-}{p_- - \alpha(p_+ + \varrho)}$, we get

$$\int_\Omega |u_m|^{p(x)} \log |u_m| \, dx \leq \varepsilon \|\nabla u_m\|_{L^{p_-}(\Omega^+)}^{p_-} + C_\varepsilon \|u_m\|_2^{2\gamma}, \tag{2.16}$$

here for the second inequality, we choose $0 < \varrho < \frac{2p}{n}$ in order to $\alpha (p_+ + \varrho) < p_+$, where

$$\alpha = \left(\frac{1}{2} - \frac{1}{p - + \varrho} \right) \left(\frac{1}{n} - \frac{1}{p_-} + \frac{1}{2} \right)^{-1} \quad \text{and } \gamma = \frac{p_- (1-\alpha)(p_+ + \varrho)}{2 [p_- - \alpha (p_+ + \varrho)]}$$

combining (2.14) and (2.16) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_m\|_2^2 &= \int_\Omega |u_m|^{p(x)} \log |u_m| \, dx - \int_\Omega |\nabla u_m|^{p(x)} dx \\ &\leq \varepsilon \|\nabla u_m\|_{L^{p_-}(\Omega^+)}^{p_-} + C_\varepsilon \|u_m\|_2^{2\gamma} - \int_{\Omega_+} |\nabla u_m|^{p_-} dx \\ &\leq (\varepsilon - 1) \|\nabla u_m\|_{L^{p_-}(\Omega^+)}^{p_-} + C_\varepsilon \|u_m\|_2^{2\gamma}. \end{aligned}$$

By choosing $\varepsilon \leq 1$, we get
we get

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_2^2 \leq C_\varepsilon \left(\|u_m\|_2^2 \right)^\gamma.$$

By using the integral inequality of Gronwall–Bellman–Bihari type, there exists a constant $T_0 > 0$ such that for all $t \in [0, T_0]$ we have

$$\begin{aligned} \|u_m\|_2^2 &\leq \|u_m(0)\|_2^2 e^{2C_\varepsilon t} \\ &\leq C\tau_0, \quad \forall\, t \in [0, T_0]. \end{aligned} \tag{2.17}$$

We next multiply both sides of (2.12) by $\alpha'_{mi}(t)$, and afterwards integrate with respect to time variable on $[0, t]$, and take the sum over $i \in \{1, 2, ..., m\}$, we get

$$\int_0^t \|u_{mt}(s)\|_2^2\, ds + J(u_m(t)) = J(u_m(0)). \tag{2.18}$$

It follows from (2.11) that there exists a positive constant C such that

$$J(u_m(0)) \leq C. \tag{2.19}$$

By (2.16) and (2.17), we have

$$\begin{aligned} J(u_m(t)) &\geq -\int_\Omega \frac{1}{p(x)} |u_m|^{p(x)} \log|u_m|\, dx + \frac{1}{p_+^2} \int_{\Omega_+} |u_m|^{p_-}\, dx \frac{1}{p_+} \int_{\Omega_+} |\nabla u_m|^{p_-}\, dx \\ &\geq -\varepsilon \|\nabla u_m\|_{L^{p_-}(\Omega_+)}^{p_-} - C_{\tau_0} + \frac{1}{p_+^2} \|u_m\|_{L^{p_-}(\Omega_+)}^{p_-} + \frac{1}{p_+} \|\nabla u_m\|_{L^{p_-}(\Omega_+)}^{p_-} \\ &= \frac{1 - \varepsilon p_+}{p_+} \|\nabla u_m\|_{L^{p_-}(\Omega_+)}^{p_-} + \frac{1}{p_+^2} \|u_m\|_{L^{p_-}(\Omega_+)}^{p_-} - C_{\tau_0}. \end{aligned} \tag{2.20}$$

From (2.18)–(2.20), it follows that

$$\|u_m(t)\|_{L^\infty(0, T_0, W_0^{1,p(x)})} \leq C. \tag{2.21}$$

and

$$\|u_{mt}\|_{L^2(0, T_0, L^2(\Omega))} \leq C. \tag{2.22}$$

Combining a priori estimate (2.21) and (2.22) we see that there exist a function u and a subsequence of $\{u_m\}_{m=1}^\infty$ still denoted by $\{u_m\}_{m=1}^\infty$ such that

$$u_m \to u \quad \text{Weakly}^* \text{ in } L^\infty(0, T_0, W_0^{1,p(x)}(\Omega)). \tag{2.23}$$

$$u_{mt} \to u \quad \text{Weakly in } L^2(0, T_0, L^2(\Omega)). \tag{2.24}$$

$$|\nabla u_m|^{p(x)-2} \nabla u_m \to \chi \quad \text{Weakly in } L^\infty(0, T_0, W_0^{-1,q(x)}(\Omega)). \tag{2.25}$$

Since (2.23) and (2.24), it follows from Aubin–Lions compactness theorem that

$$u_m \to u \quad \text{strongly in } C([0, T_0], L^r(\Omega)), \quad \forall r \in [2, p^*].$$

Clearly, this implies that

$$\varphi(u_m) \log(|u_m|) \to \varphi(u) \log(|u|) \quad \text{a.e } (x, t) \in \Omega \times [0, T_0]. \tag{2.26}$$

On the other hand, by a direct calculation, we have

$$\begin{aligned} \int_\Omega |\rho_m|^{p'}\, dx &= \int_{\Omega_-} |\rho_m|^{p'}\, dx + \int_{\Omega_+} |\rho_m|^{p'}\, dx \\ &\leq e^{-p'} |\Omega| + \left(\frac{p}{q}\right)^{p'} \int_{\Omega_+} |u_m|^q\, dx \\ &\leq e^{-p'} |\Omega| + \left(\frac{p}{q}\right)^{p'} B_p |\nabla u_m|_p^q \leq C_{T_0}. \end{aligned} \tag{2.27}$$

Where $q \in [p, p^*]$, $\rho_m(x,t) = |u_m(x,t)|^{p-1} \log(|u_m|)$.
and B_p is the best constant of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see Lemma 3).

Hence, by Lion's lemma [8, Lemma 1.3, p. 12], it follows from (2.26) and (2.27) that

$$\varphi(u_m)\log(|u_m|) \to \varphi(u)\log(|u|) \text{ weakly}^* \text{ in } L^\infty(0, T_0, W_0^{1,p(x)}(\Omega)). \tag{2.28}$$

Passing to the limit in (2.12) and (2.13) as $m \to +\infty$, by (2.22)-(2.25) and (2.28), we can show that u satisfies the initial condition $u(0) = u_0$ and

$$\int_\Omega u_t(t)\omega dx = \int_\Omega \chi(t)\nabla\omega dx = \int_\Omega \varphi_p(u)\log|u|\,\omega dx \tag{2.29}$$

for all $\omega \in W_0^{1,p(x)}(\Omega)$ and for almost every $t \in [0, T_0]$. Finally, well known arguments of the theory of monotone operators yields

$$\chi = |\nabla u|^{p(x)-2}\nabla u.$$

which implies the function u is a desired solution of problem (1.1).

We now show that the solution u satisfies the energy inequality (2.10). For this, let θ be the nonnegative function which belongs to $C([0, T_0])$. Then, it follows from (2.18) that

$$\int_0^{T_0}\theta(t)dt\int_0^t \|u_{ms}(s)\|_2^2\,ds + \int_0^{T_0} J(u_m(t))\theta(t)dt = \int_0^{T_0} J(u_m(0))\theta(t)dt. \tag{2.30}$$

The right-hand side of (2.30) converges to

$$\int_0^{T_0} J(u(0))\theta(t)dt,$$

as $m \to \infty$. The second term in left-hand side, $\int_0^{T_0} J(u_m(t))\theta(t)dt$ is lower semi-continuous with respect to the weak topology of $W_0^{1,p(x)}(\Omega)$. Hence

$$\int_0^{T_0} J(u(t))\theta(t)dt \le \lim_{m\to+\infty}\inf\int_0^{T_0} J(u_m(t))\theta(t)dt.$$

Therefore, we get

$$\int_0^{T_0}\theta(t)dt\int_0^t \|u_s(s)\|_2^2\,ds + \int_0^{T_0} J(u(t))\theta(t)dt \le \int_0^{T_0} J(u_0))\theta(t)dt.$$

Since θ is arbitrary, we obtain the energy inequality

$$\int_0^t \|u_s(s)\|_2^2\,ds + J(u(t)) \le J(u(0)), \quad t \in [0, T_0].$$

The proof is complete. □

The following theorem is the main result of this section.

Theorem 3. *Let $u_0 \in W^+$. Then the problem (1.1) admits a global weak solution such that*

$$u(t) \in \overline{W^+} \quad \textit{for } 0 \le t < \infty.$$

and satisfying the energy estimate

$$\int_0^t \|u_s(s)\|_2^2\,ds + J(u(t)) \le J(u_0), \quad t \in [0, T_0]. \tag{2.31}$$

Moreover, the solution decays polynomially, namely,

(i) if $J(u0) < M$, then we have

(2.32)
$$\|u(t)\|_2 \leq \|u(0)\|_2 \min\left\{ \left[\frac{p_+}{\frac{1}{4}\left[2+\tau\left(p_+-2\right)\right]t}\right]^{\frac{2}{p_+-2}}, \left[\frac{p_-}{\frac{1}{4}\left[2+\tau\left(p_--2\right)\right]t}\right]^{\frac{2}{p_--2}} \right\}.$$

Where $\zeta_1 = \frac{1}{p_-}\left\{\log\frac{\left(\frac{p_+\mu e}{n\mathcal{L}_{p_+}}\right)^{\frac{n}{p_+^2}}}{J(u_0)}\right\}$, $\zeta_2 = \left\{\frac{1}{p_-}\log\frac{\left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right)^{\frac{n}{p_-^2}}}{J(u_0)}\right\}$,

$\tau = \min\left(\zeta_1, \zeta_2\right)$, *and* $M = \max\left\{\left(\frac{p_+\mu}{n\mathcal{L}_{p_+}}\right)^{\frac{n}{p_+^2}}, \left(\frac{p_-\mu}{n\mathcal{L}_{p_-}}\right)^{\frac{n}{p_-^2}}\right\}$.

Proof. We consider the following cases

The case of the Initial Data $u_0 \in W_1^+$

Existence of Global Weak Solutions:

Let the sequence $\{w_j\}_{j=1}^{\infty}$, $\{u_{0m}\}_{m=1}^{\infty}$, and $\{u_m\}_{m=1}^{\infty}$ be the same as those stated in the proof of Theorem 2.

We multiply both sides of (2.12) by $\alpha'_{mi}(t)$, take the sum over $i \in \{1, 2..., m\}$, and afterwards integrate with respect to time variable on $[0, t]$. Then we get the equality

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + J(u_m(t)) = J(u_m(0)) \quad 0 \leq t \leq T_m, \tag{2.33}$$

where T_m is the maximal existence time of solution $u_m(x, t)$.

It follows from (2.11), (2.13), (2.33), and the continuity of J that

$$J(u_m(0)) \ \to J(u_0). \tag{2.34}$$

with $J(u_0) < d$ and

$$\int_0^t \|u_{ms}(s)\|_2^2 \, ds + J(u_m(t)) < d, \quad 0 \leq t < T_m, \tag{2.35}$$

for sufficiently large m. We will show that

$$u_m(t) \in W_1^+, \quad \forall t \geq 0, \tag{2.36}$$

and for sufficiently large m. Indeed, assume that (2.36) does not hold and let t_* be the smallest time for which $u_m(t_*) \notin W_1^+$. Then, by the continuity of $u_m(t)$, one has $u_m(t_*) \in \partial W_1^+$. Hence, it follows that

$$J(u_m(t_*)) = d, \tag{2.37}$$

or

$$I(u_m(t_*)) = 0. \tag{2.38}$$

Nevertheless, it is clear that (2.37) could not occur by (2.35) while if (2.38) holds then, by the definition of d, we have

$$J(u_m(t_*)) \geq \inf_{u \in N} J(u) = d,$$

which also contradicts with (2.35). Hence, we possess (2.36). On the other hand, since $u_m(t) \in W_1^+$ and

(2.39)
$$\int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)} \, dx = J(u_m(t)) + \int_\Omega \frac{1}{p(x)} |u_m|^{p(x)} \ln|u| \, dx - \int_\Omega \frac{1}{p(x)^2} |u_m|^{p(x)} \, dx$$

Remark: For $J(u_m(t)) < d$, wet get

$$\begin{cases} I(u_m(t)) < d, \ \int_0^t \|u_{ms}\|_2^2 \, ds < d. \\ \|u_m\|^{p_-}_{L^{p_-}(\Omega_+)} < p_+^2 d, \ \ \|u_m\|^{p_+}_{L^{p_+}(\Omega_-)} < p_+^2 d. \end{cases} \tag{2.40}$$

for sufficiently large m , $t \in [0, T_m)$, and using Yong equality, we have

$$\begin{aligned} \int_\Omega \frac{1}{p(x)} |\nabla u_m|^{p(x)} \, dx &\leq J(u_m(t)) + \frac{1}{p_-} \int_{\Omega_+} |u_m|^{p_+} |u| \, dx \\ &\leq J(u_m(t)) + \frac{\varepsilon}{2p_-} \int_{\Omega_+} |u_m|^{2p_+} \, dx + \frac{1}{2\varepsilon p_-} \int_{\Omega_+} |u|^2 \, dx \\ &\leq J(u_m(0)) + \frac{\varepsilon C}{2p_-} \left(\|u_m\|^{p_-}_{L^{p_-}(\Omega_+)} \right)^{\frac{2p_+}{p_-}} + \frac{1}{2\varepsilon p_-} \|u_m\|^{p_-}_{L^{p_-}(\Omega_+)} \\ &\leq C_d. \end{aligned} \tag{2.41}$$

we deduce that

$$\begin{cases} \|\nabla u_m\|^{p_-}_{L^{p_-}(\Omega_+)} \leq C_d. \\ \|\nabla u_m\|^{p_+}_{L^{p_+}(\Omega_-)} \leq C_d. \end{cases} \tag{2.42}$$

for all $t \in [0, T_m)$. The above estimates allow one to take $T_m = T$ for all m, with any $T > 0$. From (2.40) and (2.42), by a repetition of the arguments in the proof of Theorem 3.2, we see that the problem (1.1) has a weak solution u in the interval $[0, T]$, and moreover the u satisfies (2.31).

Decay Estimates : CASE $J(u_0) < M$:

By using the logarithmic Sobolev inequality (1.4) we have

$$\begin{aligned} I(u) &\geq \frac{1}{p_+} \int_{\Omega_+} |\nabla u|^{p_-} \, dx + \frac{1}{p_+} \int_{\Omega_-} |\nabla u|^{p_+} \, dx - \frac{1}{p_-} \int_{\Omega_+} |u|^{p_+} \ln |u| \, dx - \frac{1}{p_-} \int_{\Omega_-} |u|^{p_+} \ln |u| \, dx \\ &\geq \frac{1}{p_+} \int_{\Omega_+} \left(|\nabla u|^{p_-} - \frac{\mu}{p_-} |\nabla u|^{p_+} \right) dx + \int_{\Omega_-} \left(\frac{1}{p_+} |\nabla u|^{p_+} - \frac{\mu}{p_-^2} |\nabla u|^{p_-} \right) dx \\ &\quad + \frac{n}{p_- p_+^2} \log \left(\frac{p_+ \mu e}{n \mathcal{L}_{p_+}} \right) \int_{\Omega_+} |u|^{p_+} \, dx + \frac{n}{p_-^3} \log \left(\frac{p_- \mu e}{n \mathcal{L}_{p_-}} \right) \int_{\Omega_+} |u|^{p_-} \, dx \\ &\quad - \frac{1}{p_-} \int_{\Omega_-} |u|^{p_-} \ln \|u\|_{L^{p_-}} \, dx - \frac{1}{p_-} \int_{\Omega_+} |u|^{p_+} \ln \|u\|_{L^{p_+}} \, dx. \end{aligned}$$

By choosing

$$\mu \leq \min \left(p_- \frac{\|\nabla u\|^{p_-}_{L^{p_-}(\Omega_+)}}{\|\nabla u\|^{p_+}_{L^{p_+}(\Omega_+)}}, \frac{p_-^2 \, \|\nabla u\|^{p_+}_{L^{p_+}(\Omega_-)}}{p_+ \, \|\nabla u\|^{p_-}_{L^{p_-}(\Omega_-)}} \right).$$

We get

$$I(u) \geq \frac{1}{p_-} \left\{ \log \frac{\left(\frac{p_+ \mu e}{n \mathcal{L}_{p_+}} \right)^{\frac{n}{p_+^2}}}{J(u_0)} \right\} \int_{\Omega_+} |u|^{p_+} \, dx + \frac{1}{p_-} \left\{ \log \frac{\left(\frac{p_- \mu e}{n \mathcal{L}_{p_-}} \right)^{\frac{n}{p_-^2}}}{J(u_0)} \right\} \int_{\Omega_-} |u|^{p_-} \, dx.$$

We put $\zeta_1 = \frac{1}{p_-}\left\{ \log \frac{\left(\frac{p_+\mu e}{n\mathcal{L}_{p_+}}\right)^{\frac{n}{p_+^2}}}{J(u_0)} \right\}$, $\zeta_2 = \left\{ \frac{1}{p_-} \log \frac{\left(\frac{p_-\mu e}{n\mathcal{L}_{p_-}}\right)^{\frac{n}{p_-^2}}}{J(u_0)} \right\}$, we get

$$\begin{aligned} I(u) &\geq \zeta_1 \|u\|_{L^{p_+}(\Omega_+)}^{p_+} + \zeta_2 \|u\|_{L^{p_-}(\Omega_-)}^{p_-} \\ &\geq \zeta_1 \|u\|_2^{p_+} + \zeta_2 \|u\|_2^{p_-}. \end{aligned} \tag{2.43}$$

On the other hand, from the equation (1.1) we obtain

$$-\int_\Omega u_t u dx = \int_\Omega |\nabla u|^{p(x)} dx - \int_\Omega |u|^{p(x)} \ln |u| \, dx \tag{2.44}$$

We have

$$\begin{aligned} I(u) &\leq \frac{1}{p_-} \int_\Omega \left(|\nabla u|^{p(x)} - |u|^{p(x)} \ln |u| \right) dx \\ &\leq -\frac{1}{2} \int_\Omega u_t u dx. \end{aligned}$$

By integrating last equality in $[0, t]$

$$\begin{aligned} \int_t^T I(u(s)) ds &\leq -\frac{1}{2} \int_t^T \int_\Omega u_s u dx ds = -\int_t^T \frac{d}{dt} \|u(s)\|_2^2 \, ds \\ &= -\|u(T)\|_2^2 + \|u(t)\|_2^2 \\ &\leq \|u(t)\|_2^2. \end{aligned} \tag{2.45}$$

Combining (2.43) and (2.45), it follows that

$$\zeta_1 \|u\|_2^{p_+} + \zeta_2 \|u\|_2^{p_-} \leq \frac{1}{\tau} \|u(t)\|_2^2, \quad \text{where } \tau = \min(\zeta_1, \zeta_2).$$

that's implied

$$\begin{cases} \|u\|_2^{p_+} \leq \frac{1}{\tau} \|u(t)\|_2^2, \\ \|u\|_2^{p_-} \leq \frac{1}{\tau} \|u(t)\|_2^2, \end{cases} \tag{2.46}$$

Let $T \to +\infty$ and apply Lemma 4,
by choosing $\sigma_1 = \frac{p_+-2}{2}, \sigma_2 = \frac{p_--2}{2}, f = \|u\|_2^2$, we get

$$\begin{cases} \|u(t)\|_2^2 \leq \|u(0)\|_2^2 \left\{ \frac{p_+}{\frac{1}{4}[2+\tau(p_+-2)]t} \right\}^{\frac{2}{p_+-2}}. \\ \|u(t)\|_2^2 \leq \|u(0)\|_2^2 \left\{ \frac{p_-}{\frac{1}{4}[2+\tau(p_--2)]t} \right\}^{\frac{2}{p_--2}}. \end{cases}$$

Finally, we get

$$\|u(t)\|_2 \leq \|u(0)\|_2 \min \left\{ \left[\frac{p_+}{\frac{1}{4}[2+\tau(p_+-2)]t} \right]^{\frac{2}{p_+-2}}, \left[\frac{p_-}{\frac{1}{4}[2+\tau(p_--2)]t} \right]^{\frac{2}{p_--2}} \right\}.$$

□

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