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THE COMPLEX MONGE – AMPÈRA EQUATION ON POSITIVE CURRENTS OF HIGHER BIDEGREE

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Abstract: We study the induced Monge — Ampère equation on a positive current in a complex manifold. We show that the L^2-estimates are satisfied for Monge — Ampère equation on a positive closed current of bidegree (l, l) on a pseudoconvex domain in $\mathbb{C}^n$.

Keywords: Monge – Ampèra equation, positive current, differention form, complex manifold, primitive form, definite quadratic forms, differention operators on current, existence theorems for Monge – Ampèra operator on closed current, currents of higher bidegree.

1 Introduction

Let M be a complex manifold and T a positive current in M. If u and f are smooth differential forms on M we say that

$$(\bar{\partial}\partial u)^k = f \text{ on } T \text{ if } (\bar{\partial}\partial u)^k \wedge T = f \wedge T.$$

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Initially the Monge — Ampère operator is thus only defined on smooth forms but it can later be extended (in different ways) to forms that are in a sense defined only on T.

The question we study in this paper is whether the Monge — Ampère equation can be solved on T, and, if so, what kinds of estimates one can find for the solution.

Solvability of Monge — Ampère equations in the case $k = 1$ is classical (see [1], [2], [3], [4], [7], [8]).

Similarily we may also consider smooth (l, l) currents that are strictly positive in a subdomain D of M and vanish outside of D, which means that we study our equation in D.

Let's consider currents of bidegree (l, l). We have then had to assume that T is closed (if the dimension of the ambient space is greater than 2), and we obtain results similar to the manifold case, except that we use weak Monge — Ampère operator for $(nk - pk, qk)$-forms and strong Monge — Ampère operator for (pk, qk)-forms if T is of bidimension (nk, nk) (see Theorems 3, 4 and 5).

Our results for (pk, qk)-forms deal with forms that lie in the L^2-closure of forms with compact support and follows from the result for $(nk - pk, qk)$-forms by a version of Serre duality.

Definition 1. *Let $f \in \Lambda^{p,q}(\mathbb{C}^{n+l})$. The norm of f on T is defined by*

$$|f|^2_{\omega,T}\sigma_T = c_{q+p} f \wedge \overline{\hat{f}} \wedge \omega_{n-q-p} \wedge T, \tag{1}$$

where

$$\hat{f} = f_0 \wedge \omega^p + \sum_{k=1}^{p} \widehat{f_k} \wedge \omega^{p-k},$$

and $f_k \in \Lambda_T^{k,q-p+k}$ are primitive forms,

$$\widehat{f_k} = -\tau^k - \sum_{r=1}^{k-1} (-1)^{k+r} \sigma_r^k + (-1)^k \sigma_0^k.$$

Definition 5.1 [5] is special case of Definition 1 by $p = 0$ and $l = 1$.

Recall that the norm of f in $\mathbb{C}^{n+l}$ measured in the ω-metric is defined by

$$|f|^2_{\omega}\omega_{n+l} = c_{q+p} f \wedge \overline{\hat{f}} \wedge \omega_{n-q-p+l},$$

so

$$(n+1)\dots(n+l)|f|^2_{\omega,\omega_l} = (n-q-p+1)\dots(n-q-p+l)|f|^2_{\omega}$$

in case $T = \omega_l$. In general, since $T \leq tr(T)\omega_l$, we get

$$|f|^2_{\omega,T} \leq \frac{(n-q-p+1)\dots(n-q-p+l)}{l!}|f|^2_{\omega}.$$

Finally, by polarization we obtain a scalar product such that

$$(f, f)_{\omega,T} = |f|^2_{\omega,T}$$

and we shall in the sequel not mention the dependence on ω and T (see [5] by $p = 0$ and $l = 1$).

Proposition 1. *Any $f \in \Lambda_T^{n-p,q}$ can be written $f = \tilde{f} \wedge \omega_{q-p}$ on T for a unique $\tilde{f} \in \Lambda_T^{n-q,p}$. Moreover,*

$$|f|_{\omega,T} = |\tilde{f}|_{\omega,T}.$$

Proposition 5.3 [5] is special case of Proposition 1 by $p = 0$ and $l = 1$.

The next proposition is related to the Lefscherz isomorphism in $\mathbb{C}^{n+l}$ and will be importance when we approximate general currents by smooth forms.

Proposition 2. *Let $T \in \Lambda^{l,l}(\mathbb{C}^{n+l})$ be strictly positive, and let $F \in \Lambda^{n-p+l,q+l}(\mathbb{C}^{n+l}), 0 \leq p \leq q \leq n$. Then there is a unique form $\tilde{F} \in \Lambda^{n-q,p}(\mathbb{C}^{n+l})$ such that*

$$F = \tilde{F} \wedge \omega_{q-p} \wedge T.$$

In particular F can be written

$$F = f \wedge T$$

with $f \in \Lambda^{n-p,q}(\mathbb{C}^{n+l})$.

Proposition 5.4 [5] is special case of Proposition 2 by $p = 0$ and $l = 1$. The proof be analogous to the proof of Proposition 5.4 and is given in [7] by $l = 1$ (see Proposition 2).

On a manifold one can prove that an arbitrary form can be written uniquely

$$f = \sum f_j \wedge \omega_j$$

with f_j primitive (and the degrees are restricted so that $f_j \wedge \omega_j$ is not automatically zero). For primitive k-forms it also holds that the quadratic forms

$$[\gamma, \gamma] = c_k \gamma \wedge \overline{\gamma} \wedge \omega_{n-k}$$

are defined (negative or positive depending on the bidegree) and one can define the norm of f by

$$|f|^2 = \sum c_j [f_j, f_j]$$

for suitably chosen constants c_j (see [5]).

Proposition 3. *Let $f \in \Lambda_T^{q,p}$. Then there are uniquely determined primitive forms $f_0 \in \Lambda_T^{q-p,0}$, $f_1 \in \Lambda_T^{q-p+1,1}, \ldots, f_p \in \Lambda_T^{q,p}$ such that*

$$f = \sum_{k=0}^{p} f_k \wedge \omega^{p-k}. \tag{2}$$

Proposition 5.6 [5] is special case of Proposition 3 by $p = 1$. The proof be analogous to the proof of Proposition 5.6 and is given in [7] by $l = 1$ (see Proposition 3).

Similarily we have a primitive decomposition of (p, q)-forms.

The next proposition says that we do get a norm on forms on T.

Proposition 4. *Assume* $\gamma \in \Lambda^{p,q}(\mathbb{C}^{n+l})$ *and* $|\gamma|^2_{\omega,T} = 0$*. Then* $\gamma \wedge T = 0$*.*

Proof. Choose a basis $e_1, \ldots, e_{n+l}$ for the space $(1,0)$-forms in $\mathbb{C}^{n+l}$ that diagonalizes both ω and T, so that

$$\omega = \sum dV_j$$

and

$$T = \sum \lambda_J dV_J$$

$(dV_j = ie_j \wedge \overline{e}_j)$ and $dV_J = \bigwedge_J dV_j)$. Then $tr(T) = \sum \lambda_J$ and

$$T \wedge \omega_{n-p-q} = \sum_{|K|=n+l-p-q} \lambda_K dV_K,$$

if we put

$$\lambda_K = \Sigma_{\substack{|J|=l \\ J \subset K}} \lambda_J .$$

Writing $\gamma = \sum_{k=0}^{p} \gamma_k \wedge \omega^{p-k}$,

$$\gamma_k = \sum \gamma^k_{JK} e_J \wedge \overline{e}_K \in \Lambda_T^{k,q-p+k}$$

are primitive forms, and partition γ_k into a sum $\tau^k + \sigma^k$ depending on whether J belongs to K (τ-part) or not

$$\gamma_k = \sum_{j_1 \in K} e_{j_1} \ldots \sum_{j_k \in K} \gamma^k_{JK} e_{j_k} \wedge \overline{e}_K +$$

$$+ \Big(\sum_{r=1}^{k-1} \sum_{|M|=r} \sum_{j_1 \notin K} e_{j_1} \ldots \sum_{j_{m_1-1} \notin K} e_{j_{m_1-1}} \sum_{j_{m_1} \in K} e_{j_{m_1}} \sum_{j_{m_1+1} \notin K} e_{j_{m_1+1}} \ldots$$

$$\ldots \sum_{j_{m_r-1} \notin K} e_{j_{m_r-1}} \sum_{j_{m_r} \in K} e_{j_{m_r}} \sum_{j_{m_r+1} \notin K} e_{j_{m_r+1}} \ldots \sum_{j_k \notin K} \gamma^k_{JK} e_{j_k} \wedge \overline{e}_K +$$

$$+ \sum_{j_1 \notin K} e_{j_1} \ldots \sum_{j_k \notin K} \gamma^k_{JK} e_{j_k} \wedge \overline{e}_K \Big) = \tau^k + \Big(\sum_{r=1}^{k-1} \sigma^k_r + \sigma^k_0\Big) = \tau^k + \sigma^k .$$

It is easy to verify that $[\sigma^k, \sigma^k] = \sum_{r=0}^{k-1} \sum_{t=0}^{k-1} [\sigma^k_r, \sigma^k_t] = \sum_{r=0}^{k-1} [\sigma^k_r, \sigma^k_r]$ because

$$[\sigma^k_r, \sigma^k_t] \sigma_T = c_{q-p+2k} \sum_{|K|=q-p+k-r} \sum_{|M|=r} \sum_{j_1 \notin K} e_{j_1} \ldots \sum_{j_k \notin K} \sigma^{kr}_{JK} e_{j_k} [M] \wedge dV_{J_M} \wedge \overline{e}_K \wedge$$

$$\wedge \sum_{|L|=q-p+k-t} \sum_{|P|=t} \sum_{s_1 \notin L} \overline{e}_{s_1} \ldots \sum_{s_k \notin L} \overline{\sigma^{kt}_{SL}} \overline{e}_{s_k} [P] \wedge dV_{S_P} \wedge e_L \wedge$$

$$\wedge \sum_{|K|=n-q+p-2k+l} \lambda_K dV_K = 0$$

for $r \neq t$.

We consider

$$[\sigma_r^k,\sigma_r^k]\sigma_T=(-1)^{k+r}\sum_{|K|=q-p+k-r}\sum_{|M|=r}\sum_{|P|=r}\sum_{J_M\cap S_P=\emptyset}\sigma_{JK}^{kr}\times$$

$$\times\overline{\sigma^{kr}_{j_1,\dots,j_{m_1-1},j_{m_1+1},\dots,s_{p_1},\dots,s_{p_r},\dots,j_{m_r-1},j_{m_r+1},\dots,j_k)K}}\sum_{|L|=n-q+p-2k+l}\lambda_L dV_{L\cup J\cup S_P\cup K},$$

$1\le r\le k-1$, $1\le k\le p$,

$$[\sigma_0^k,\sigma_0^k]\sigma_T=(-1)^k\sum|\gamma_{JK}^k|^2\sum_{|L|=n-q+p-2k+l}\lambda_L dV_{L\cup J\cup K}.$$

From here

$$[\sigma_r^k,\sigma_r^k]=(-1)^{k+r}\sum_{|K|=q-p+k-r}\sum_{|M|=r}\sum_{|P|=r}\sum_{J_M\cap L_P=\emptyset}\sigma_{JK}^{kr}\times$$

$$\times\overline{\sigma^{kr}_{(j_1,\dots,j_{m_1-1},j_{m_1+1},\dots,l_{p_1},\dots,l_{p_r},\dots,j_{m_r-1},j_{m_r+1},\dots,j_k)K}}\lambda_{(J\cup L_P\cup K)^c},$$

$$[\sigma_0^k,\sigma_0^k]=(-1)^k\sum|\gamma_{JK}^k|^2\lambda_{(J\cup K)^c},$$

$1\le k\le p$.

We have

$$[\tau^k,\sigma^k]=\sum_{r=0}^{k-1}[\tau^k,\sigma_r^k]=0,$$

since

$$[\tau^k,\sigma_r^k]\sigma_T=c_{q-p+2k}\sum_{|K|=q-p}\tau_{JK}^k dV_J\wedge\overline{e}_K\wedge\sum_{|L|=q-p+k-r}\sum_{|P|=r}\sum_{s_1\notin L}\overline{e}_{s_1}\dots$$

$$\dots\sum_{s_k\notin L}\overline{\sigma_{SL}^{kr}}\overline{e}_{s_k}[P]\wedge e_L\wedge dV_{S_P}\wedge\sum_{|L|=n-q+p-2k+l}\lambda_L dV_L=0.$$

And $[\tau^k,\tau^k]\sigma_T$ equals

$$\sum_{|K|=q-p}\sum_{J\cap S=\emptyset}\tau_{JK}^k\overline{\tau_{SK}^k}\sum_{|L|=n-q+p-2k+l}\lambda_L dV_{L\cup J\cup S\cup K}.$$

From here

$$[\tau^k,\tau^k]=\sum_{|K|=q-p}\sum_{J\cap L=\emptyset}\tau_{JK}^k\overline{\tau_{LK}^k}\lambda_{(J\cup L\cup K)^c}.$$

We find

$$c_{q+p}\gamma\wedge\overline{\tilde{\gamma}}\wedge\omega_{n-q-p}\wedge T=\sum_{k=1}^{p}\Big(-\sum_{|K|=q-p}\sum_{J\cap L=\emptyset}\tau_{JK}^k\overline{\tau_{LK}^k}\lambda_{(J\cup L\cup K)^c}\omega_{n+l}-$$

$$-\sum_{r=1}^{k-1}\sum_{|K|=q-p+k-r}\sum_{|M|=r}\sum_{|P|=r}\sum_{J_M\cap L_P=\emptyset}\sigma_{JK}^{kr}\times$$

$$\times\overline{\sigma^{kr}_{(j_1,\dots,j_{m_1-1},j_{m_1+1},\dots,l_{p_1},\dots,l_{p_r},\dots,j_{m_r-1},j_{m_r+1},\dots,j_k)K}}\lambda_{(J\cup L_P\cup K)^c}\omega_{n+l}+$$

$$+\sum |\gamma^k_{JK}|^2\lambda_{(J\cup K)^c}\omega_{n+l}\Big) + \sum_{|K|=q-p} |\gamma^0_K|^2\lambda_{K^c}\omega_{n+l}.$$

If $|\gamma|^2_{\omega,T}=0$, it follows that for each $1\le k\le p$ and for each multiindexes J, K either $\tau^k_{JK}=0$ or $\lambda_S=0$ for all $S\notin J\cup L\cup K$, and for each multiindexes J, K, $1\le r\le k-1$ either $\sigma^{kr}_{JK}=0$ or $\lambda_S=0$ for all $S\notin J\cup L_P\cup K$, for all J, K either $\gamma^k_{JK}=0$ or $\lambda_S=0$ for all $S\notin J\cup K$, and at last for each multiindex K either $\gamma^0_K=0$ or $\lambda_S=0$ for all $S\notin K$. This gives

$$\gamma\wedge T=\sum_{k=1}^{p}\Big(\tau^k\wedge\omega^{p-k}\wedge T+\sum_{r=0}^{k-1}\sigma^k_r\wedge\omega^{p-k}\wedge T\Big)+\gamma_0\wedge\omega^p\wedge T=$$

$$=\sum_{k=1}^{p}\Big(\sum_{|K|=q-p}\tau^k_{JK}\sum_{|L|=p-k+l}\lambda_L dV_{J\cup L}\wedge\overline{e}_K+$$

$$+\sum_{r=1}^{k-1}\sum_{|K|=q-p+k-r}\sum_{|M|=r}\sum_{j_1\notin K}e_{j_1}\dots\sum_{j_k\notin K}\sigma^{kr}_{JK}e_{j_k}[M]\wedge\sum_{|L|=p-k+l}\lambda_L dV_{J_M\cup L}\wedge\overline{e}_K+$$

$$+\sum_{j_1\notin K}e_{j_1}\dots\sum_{j_k\notin K}\gamma^k_{JK}e_{j_k}\wedge\sum_{|L|=p-k+l}\lambda_L dV_L\wedge\overline{e}_K\Big)+$$

$$+\sum_{|K|=q-p}\gamma^0_K\sum_{|L|=p+l}\lambda_L dV_L\wedge\overline{e}_K=0.$$

□

Proposition 5.2 [5] is special case of Proposition 4 by $p=0$ and $l=1$.

Let $T\ge 0$ be a (l,l)-current in $\mathbb{C}^{n+l}$. Such a current can be written

$$T=i^{l^2}\sum T_{J\overline{K}}dz_J\wedge d\overline{z}_K$$

where the coefficients are measures absolutely continuous with respect to trace measure

$$\sigma_T=T\wedge\omega_n.$$

Let $tr(T)$ be the $(0,0)$-current defined by

$$tr(T)\omega_{n+l}=\sigma_T.$$

Then T can be written

$$T=\tilde{T}tr(T),$$

where $\tilde{T}$ is a form with coefficients defined a.e. with respect to σ_T. Since the coefficients of $\tilde{T}$ make up a semidefinite matrix with trace equal to one, it follows by the Cauchy inequality that

$$T=i^{l^2}\sum\tilde{T}_{J\overline{K}}dz_J\wedge d\overline{z}_K tr(T)$$

where $|\tilde{T}_{J\overline{K}}|\le 1$.

If f is a continuous (p,q)-form in $\mathbb{C}^{n+l}$, we define the L^2-norm of f on T by

$$||f||^2_{\omega,T} = \int |f|^2_{\omega,\tilde{T}}\sigma_T. \tag{3}$$

Equation (3) means that

$$||f||^2_{\omega,T} = c_{p+q}\int f\wedge\overline{\hat{f}}\wedge\omega_{n-p-q}\wedge T$$

since

$$c_{p+q}f\wedge\overline{\hat{f}}\wedge\omega_{n-p-q}\wedge T = c_{p+q}f\wedge\overline{\hat{f}}\wedge\omega_{n-p-q}\wedge\tilde{T}tr(T) = |f|^2_{\omega,\tilde{T}}\sigma_T,$$

and $tr(\tilde{T}) = 1$.

We define the L^2-spaces of (p,q)-forms on T, $L^2_{p,q}(T)$, as a completion of smooth (p,q)-forms with respect L^2-norms. Thus smooth forms are by definition dense in L^2-spaces.

There is a problem that the isomorphism from Proposition 1, or equivalently the $*$-operator, does not in general preserve smoothness. If $f\in L^2_{n-q,n-p}$ there is some $g\in L^2_{p,q}$ suth that $f = *g$ but g is notnecessarily smooth. The next lemma states that we could also have defined $L^2_{n-p,q}(T)$ by taking the closure of forms $g\wedge\omega_{q-p}$, with g smooth and bidegree $(n-q,p)$. In particular, the $*$-operator gives an isomorphism between $L^2_{n-q,p}$ and $L^2_{n-p,q}(T)$.

Lemma 1. *$*D_{(n-q,p)}$ is dense in $L^2_{n-p,q}(T)$.*

Proof. Analogously to the proof of Lemma 5.8 by $p = 0$ and $l = 1$ from [5]. □

Corollary 1. *Let $f\in L^2_{n-p,q}(T)$. Then*

$$||f|| = \sup|\int f\wedge\overline{g}\wedge T|$$

where supremum is taken over $g\in D_{n-q,p}, ||g||_{\omega,T}\leq 1$.

Proof. Analogously to the proof of Corollary 5.9 by $p = 0$ and $l = 1$ from [5]. □

If at last φ is a Borel weight function, we define

$$L^2_{p,q}(T, e^{-\varphi})$$

as the space of those $f\in L^2_{p,q,loc}$ that satisfy

$$||f||^2_{\omega,T,\varphi} = \int |f|^2_{\omega,\tilde{T}}e^{-\varphi}\sigma_T < \infty.$$

In general weight function φ will be strictly plurisubharmonic, and when we don‘t mention metric form ω explicitly it should be understood that we have taken $\omega = i\partial\overline{\partial}\varphi$.

2 Differential operators on T

In this section we define various Monge — Ampère operators. We need to extend the definition of Monge — Ampère operator on smooth forms to get Monge — Ampère operators on $L^2(T)$. First we define the weak extension of Monge — Ampère operator. Assume that T is closed.

Definition 2. *If $u \in L^2_{p,q,loc}(T)$, we say that*

$$(\overline{\partial}\partial_w u)^k = f$$

on T, if $f \in L^2_{(p+1)k,(q+1)k,loc}(T)$ and

$$(\overline{\partial}\partial u)^{k-1} \wedge \overline{\partial}\partial(u \wedge T) = f \wedge T$$

in the sense of currents.

The strong extension of Monge — Ampère operator is defined as follows.

Definition 3. *If $u \in L^2_{p,q,loc}(T)$ and $f \in L^2_{(p+1)k,(q+1)k,loc}(T)$, we say that*

$$(\overline{\partial}\partial_s u)^k = f,$$

if there is a sequence of smooth (p,q)-forms u_n such that

$$u_n \to u \ in \ L^2_{loc}(T)$$

and

$$(\overline{\partial}\partial u_n)^k \to f \ in \ L^2_{loc}(T).$$

Let φ be a Borel measurable weightя function. We then get densely defined operators, $(\overline{\partial}\partial_w u)^k$ and $(\overline{\partial}\partial_s u)^k$ on $L^2_{p,q}(T, e^{-\varphi})$ with domains consisting from all u such that

$$||(\overline{\partial}\partial u)^k||_{T,\varphi} < \infty$$

with $(\overline{\partial}\partial u)^k = (\overline{\partial}\partial_w u)^k$ or $(\overline{\partial}\partial_s u)^k$.

In the existence theorem for Monge — Ampère operator on (p,q)-forms we will need to consider an L^2-variant of Monge — Ampère operator with compact support, defined in the following way. (D is a domain in $\mathbb{C}^{n+l}$.)

Definition 4. *Let $u \in L^2_{p,q}(T, e^{\varphi})$. We say that $u \in Dom((\overline{\partial}\partial^0_s u)^k)$ and $(\overline{\partial}\partial^0_s u)^k = f$ if there is a sequence of test forms $u_n \in \mathcal{D}_{p,q}$ with support in D such that*

$$u_n \to u \ в \ L^2_{p,q}(T, e^{\varphi})$$

and

$$(\overline{\partial}\partial u_n)^k \to f \ в \ L^2_{(p+1)k,(q+1)k}(T, e^{\varphi}).$$

We define formal adjoints. If f is a (p,q)-form such that $*f$ is a smooth, we put [5]

$$\vartheta f = \varepsilon_{p,q} * \partial * f$$

where $\varepsilon_{p,q}$ is chosen so that

$$(g, \vartheta f)_{\omega,T} = (\overline{\partial}_w g, f)_{\omega,T} \qquad (4)$$

if f has compact support. If φ is a weight function, we let [5]

$$\vartheta_\varphi = e^\varphi \vartheta e^{-\varphi}$$

so that

$$(g, \vartheta_\varphi f)_{\omega,T,\varphi} = (\overline{\partial}_w g, f)_{\omega,T,\varphi}.$$

We can define weak and strong extensions of ϑ and ϑ_φ similarly we did for $\overline{\partial}$, but the definitions become a bit cumbersome since φ and $*f$ are not necessarily smooth. In the sequel we will have use only for the weak extension ϑ_φ on (p,q)-forms, and in this case we put

$$\vartheta_{\varphi,w} f = v$$

if

$$\varepsilon_{p,q} \partial * e^{-\varphi} f \wedge T = *ve^{-\varphi} \wedge T$$

in the sense of currents.

Proposition 5. *$\overline{\partial}_s^0$ and $\vartheta_{-\varphi,w}$ are adjoint operators on $L^2_{p,q}(T, e^\varphi)$.*

Proof. Analogously to the proof of Proposition 6.4 from [5] by $p = 0$. □

3 A priori estimates for Monge — Ampère operator

The basic technical result in this section is the following generalization of the Kodaira — Nakano — Hörmander identity. In the formulation of the result we use the notation [5]

$$\partial_{-\varphi} = e^{-\varphi} \partial e^\varphi$$

for a twisted $\overline{\partial}$-operator and

$$(\gamma)_{pk+2}$$

for the primitive part of a $(q, pk+2)$-form γ, defined as in Proposition 3 (see Section 1).

Theorem 1. *Let $T \geq 0$ be a (l,l)-current in a domain D in $\mathbb{C}^{nk+l}$, such that $i\partial\overline{\partial}T$ has mesure coefficients. Let ω be a Kähler form in D. Let finally g be a test form of bidegree (pk, qk) with support in D, and suppose $\varphi \in C^2(D)$. Then*

$$\int c_{(p+q)k+1} \partial g \wedge \overline{\widehat{\partial g}} \wedge i\partial\overline{\partial}\varphi \wedge \omega_{(n-p-q)k-2} \wedge Te^\varphi -$$
$$- \int c_{(p+q)k+1} \partial g \wedge \overline{\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge i\partial\overline{\partial}Te^\varphi +$$
$$+ (\widehat{\overline{\partial}\partial g}, \widehat{\overline{\partial}\partial g})_{\omega,T,-\varphi} + \| (\partial_{-\varphi}\partial g)_{pk+2} \|^2_{\omega,T,-\varphi} =$$
$$= 2Re(\vartheta_{-\varphi}\widehat{\overline{\partial}\partial g}, \widehat{\partial g})_{\omega,T,-\varphi} + \| \vartheta_{-\varphi}\partial g \|^2_{\omega,T,-\varphi} . \quad (5)$$

In particular, if $i\partial\overline{\partial}T$ is strongly negative and $i\partial\overline{\partial}\varphi \geq \omega$, we have

$$((n-p-q)k-1)||\partial g||^2 + \| \overline{\partial}\partial g \|^2_{\omega,T,-\varphi} \leq$$
$$\leq 2Re(\vartheta_{-\varphi}\widehat{\overline{\partial}\partial g}, \widehat{\partial g}) + \| \vartheta_{-\varphi}\partial g \|^2 . \quad (6)$$

If moreover $dT = 0$, *then*

$$((n-p-q)k-1)||\partial g||^2 \leq \| \overline{\partial}\partial g \|^2 + \| \vartheta_{-\varphi}\partial g \|^2 . \tag{7}$$

Proof. It is clear that (6) and (7) follow from (5). To prove (5) we follow Bochner — Kodaira method of [6] and compute

$$\partial\overline{\partial}(\partial g \wedge \overline{\widehat{\partial g}} \wedge e^{\varphi}) = (\partial_{-\varphi}\overline{\partial}\partial g \wedge \overline{\widehat{\partial g}} + (-1)^{(p+q)k+1}\overline{\partial}\partial g \wedge \overline{\overline{\partial}\widehat{\partial g}} +$$

$$+(-1)^{(p+q)k+1}\partial_{-\varphi}\partial g \wedge \overline{\partial_{-\varphi}\widehat{\partial g}} + \partial g \wedge \overline{\overline{\partial}\partial_{-\varphi}\widehat{\partial g}})e^{\varphi}.$$

Using the commutation relation

$$\overline{\partial}\partial_{-\varphi}\widehat{\partial g} + \partial_{-\varphi}\overline{\partial}\widehat{\partial g} = \widehat{\partial g} \wedge \overline{\partial}\partial\varphi$$

we rewrite the last term

$$\partial g \wedge \overline{\overline{\partial}\partial_{-\varphi}\widehat{\partial g}} = -\partial g \wedge \overline{\partial_{-\varphi}\overline{\partial}\widehat{\partial g}} + \partial g \wedge \overline{\widehat{\partial g}} \wedge \partial\overline{\partial}\varphi.$$

Multiply the intire identity by

$$ic_{(p+q)k+1}\omega_{(n-p-q)k-2} \wedge T$$

and integrate. Because $d\omega = 0$, the result is

$$c_{(p+q)k+1}\int \partial g \wedge \overline{\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge i\partial\overline{\partial}Te^{\varphi} =$$

$$= c_{(p+q)k+1}\int \partial g \wedge \overline{\widehat{\partial g}} \wedge i\partial\overline{\partial}\varphi \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi} +$$

$$+(-1)^{(p+q)k+1}ic_{(p+q)k+1}\int \overline{\partial}\partial g \wedge \overline{\overline{\partial}\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi} +$$

$$+(-1)^{(p+q)k+1}ic_{(p+q)k+1}\int \partial_{-\varphi}\partial g \wedge \overline{\partial_{-\varphi}\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi} +$$

$$+c_{(p+q)k+1}\int (i\partial_{-\varphi}\overline{\partial}\partial g \wedge \overline{\widehat{\partial g}} + \partial g \wedge \overline{i\partial_{-\varphi}\overline{\partial}\widehat{\partial g}}) \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi}. \tag{8}$$

Recall $i(-1)^{(p+q)k+1}c_{(p+q)k+1} = c_{(p+q)k+2}$ so the second term on the right-hand side equals

$$c_{(p+q)k+2}\int \overline{\partial}\partial g \wedge \overline{\overline{\partial}\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi} = \| \overline{\partial}\partial g \|^2 .$$

The third term is

$$-c_{(p+q)k+2}\int \partial_{-\varphi}\partial g \wedge \overline{\partial_{-\varphi}\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi}.$$

Decompose the $(pk+2, qk)$-form $\partial_{-\varphi}\widehat{\partial g}$ by Proposition 3

$$\partial_{-\varphi}\widehat{\partial g} = (\partial_{-\varphi}\widehat{\partial g})_0 \wedge \omega^{pk+2} + (\partial_{-\varphi}\widehat{\partial g})_1 \wedge \omega^{pk+1} + \ldots + (\partial_{-\varphi}\widehat{\partial g})_{pk+2},$$

where

$$\omega_{(n-p-q)k-1} \wedge (\partial_{-\varphi}\widehat{\partial g})_{pk+2} = 0.$$

Recall

$$*\partial g = \overline{c}_{(p+q)k+1}\widehat{\partial g} \wedge \omega_{(n-p-q)k-1}$$

so

$$\partial_{-\varphi} * \partial g = \overline{c}_{(p+q)k+1} \partial_{-\varphi} \widehat{\partial g} \wedge \omega_{(n-p-q)k-1} = \overline{c}_{(p+q)k+1} ((\partial_{-\varphi} \widehat{\partial g})_0 \wedge \omega^{pk+2} + \\ + (\partial_{-\varphi} \widehat{\partial g})_1 \wedge \omega^{pk+1} + \ldots + (\partial_{-\varphi} \widehat{\partial g})_{pk+1} \wedge \omega) \wedge \omega_{(n-p-q)k-1}.$$

Hence

$$i\vartheta_{-\varphi} \partial g = ((\partial_{-\varphi} \widehat{\partial g})_0 \wedge \omega^{pk+1} + \widehat{(\partial_{-\varphi} \widehat{\partial g})_1 \wedge} \omega^{pk} + \ldots + (\partial_{-\varphi} \widehat{\partial g})_{pk+1}).$$

Analogously

$$i\vartheta_{-\varphi} \widehat{\partial g} = ((\partial_{-\varphi} \partial g)_0 \wedge \omega^{pk+1} + \widehat{(\partial_{-\varphi} \partial g)_1 \wedge} \omega^{pk} + \ldots + (\partial_{-\varphi} \partial g)_{pk+1}).$$

We thus see that the third term equals

$$-c_{(p+q)k+2} \int (\partial_{-\varphi} \partial g)_{pk+2} \wedge \overline{(\partial_{-\varphi} \widehat{\partial g})}_{pk+2} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi} - \\ -c_{(p+q)k} \int \widehat{\vartheta_{-\varphi} \widehat{\partial g}} \wedge \overline{\widehat{\vartheta_{-\varphi} \partial g}} \wedge \omega_{(n-p-q)k} \wedge Te^{\varphi} = \\ = c_{(p+q)k+2} \int (\partial_{-\varphi} \partial g)_{pk+2} \wedge \overline{\widehat{(\partial_{-\varphi} \partial g)}_{pk+2}} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi} - \\ - \| \vartheta_{-\varphi} \partial g \|^2 = \| (\partial_{-\varphi} \partial g)_{pk+2} \|^2 - \| \vartheta_{-\varphi} \partial g \|^2$$

since $c_{(p+q)k+2} = c_{(p+q)k}$.

The last term on the right-hand side of (8) can at last be written

$$2Rec_{(p+q)k+1} \int i\partial_{-\varphi} \overline{\partial} \partial g \wedge \overline{\widehat{\partial g}} \wedge \omega_{(n-p-q)k-2} \wedge Te^{\varphi},$$

which equals

$$-2Re(\vartheta_{-\varphi} \widehat{\overline{\partial} \partial g}, \widehat{\partial g})$$

by the definition of the formal adjoint. $\square$

Theorem 1 has a counterpart for forms of bidegree $(nk-pk, qk)$.

Theorem 2. *With notation and assumptions as in Theorem 1, let f be a test form of bidegree $(nk-pk, qk)$ with support in D such that $*f$ is smooth. Then*

$$\int c_{(n-q+p)k+1} \partial_{\varphi} \overline{*f} \wedge \overline{\widehat{\partial_{\varphi} \overline{*f}}} \wedge i\partial\overline{\partial}\varphi \wedge \omega_{(q-p)k-2} \wedge Te^{-\varphi} - \\ - \int c_{(n-q+p)k+1} \partial_{\varphi} \overline{*f} \wedge \overline{\widehat{\partial_{\varphi} \overline{*f}}} \wedge \omega_{(q-p)k-2} \wedge i\partial\overline{\partial} Te^{-\varphi} + (\overline{\partial}_{\varphi} \partial_{\varphi} \overline{*f}, \overline{\partial}_{\varphi} \widehat{\partial_{\varphi} \overline{*f}}) - \\ - c_{(n-q+p)k+2} \int (\partial\partial_{\varphi} \overline{*f})_{pk+2} \wedge \overline{\widehat{(\partial\partial_{\varphi} \overline{*f})}_{pk+2}} \wedge \omega_{(q-p)k-2} \wedge Te^{-\varphi} = \\ = 2Re(\vartheta \widehat{\overline{\partial}_{\varphi} \partial_{\varphi} \overline{*f}}, \widehat{\partial_{\varphi} \overline{*f}}) + c_{(n-q+p)k} \int \vartheta \widehat{\partial_{\varphi} \overline{*f}} \wedge \overline{\vartheta \widehat{\partial_{\varphi} \overline{*f}}} \wedge \omega_{(q-p)k} \wedge Te^{-\varphi}.$$

In particular, if $i\partial\overline{\partial} T \leq 0$ and $i\partial\overline{\partial}\varphi \geq \omega$, then

$$((q-p)k-1) ||\partial_{\varphi} \overline{*f}||^2 + \| \overline{\partial}_{\varphi} \partial_{\varphi} \overline{*f} \|^2 + \| (\partial\partial_{\varphi} \overline{*f})_{pk+2} \|^2 \leq \\ \leq 2Re(\vartheta \widehat{\overline{\partial}_{\varphi} \partial_{\varphi} \overline{*f}}, \widehat{\partial_{\varphi} \overline{*f}}) + \| \vartheta \partial_{\varphi} \overline{*f} \|^2 .$$

If moreover $dT = 0$, *then*

$$((q-p)k-1)||\partial_\varphi \overline{*f}||^2 \leq \| \overline{\partial}_\varphi \partial_\varphi \overline{*f} \|^2 + \| \vartheta \partial_\varphi \overline{*f} \|^2 .$$

Proof. Analogously to the proof of Theorem 2 from [7] by $l = 1$. We need apply Theorem 1 to $g = \overline{*f}e^{-\varphi}$. ☐

4 Existence theorems for Monge — Ampère equations on closed currents of bidegree (l, l)

We shall consider closed currents defined in all of $\mathbb{C}^n$ since this case shows the main ideas. Then we discuss closed currents in domains in $\mathbb{C}^n$ and on manifolds.

Theorem 3. *Let* $T \geq 0$ *be a closed* (l, l)*-current in* $\mathbb{C}^{nk+l}$, *and let* $\omega = i\partial\overline{\partial}|z|^2$ *be the Kähler form of the Euclidean metric in* $\mathbb{C}^{nk+l}$. *Let* φ *be a plurisubharmonic function in* $\mathbb{C}^{nk+l}$ *satisfying*

$$i\partial\overline{\partial}\varphi \geq \omega.$$

Then, for any $\overline{\partial}_w$*-closed* $(nk-pk, qk)$*-form* f *on* T *with* $(q-p)k-1 \geq 1$ *there exists an* $(n-p-1, q-1)$*-form* u *on* T *such that*

$$(\overline{\partial}\partial_w u)^k = f$$

on T *and*

$$\int |\partial u \wedge (\overline{\partial}\partial u)^{k-1}|^2_{\omega,T}\sigma_T e^{-\varphi} \leq \frac{1}{(q-p)k-1}\int |f|^2_{\omega,T}\sigma_T e^{-\varphi}.$$

The proof of Theorem 3. Analogously to the proof of Theorem 3 from [7] by $l = 1$.

We have the following version of Theorem 3 for pseudoconvex domain in $\mathbb{C}^{nk+l}$ and general Kähler metrics.

Theorem 4. *Let* $T \geq 0$ *be a closed* (l, l)*-form in a pseudoconvex domain in* $\mathbb{C}^{nk+l}$, *and let* ω *be the Kähler form of a smooth Kähler metric in* D. *Let* φ *be plurisubharmonic in* D *and sstisfy*

$$i\partial\overline{\partial}\varphi \geq \omega.$$

Then for any $\overline{\partial}_w$*-closed* $(nk-pk, qk)$*-form* f *on* T, *there is an* $(n-p-1, q-1)$*-form* u *such that*

$$(\overline{\partial}\partial_w u)^k = f$$

on T *and*

$$||\partial u \wedge (\overline{\partial}\partial u)^{k-1}||^2_{\omega,T,\varphi} \leq \frac{1}{(q-p)k-1}||f||^2_{\omega,T,\varphi}.$$

Proof. Analogously to the proof of Theorem 8.5 from [5] by $p = 0$. ☐

Next objective is the counterpart of Theorem 4 for (p, q)-forms. Recall the definition of $(\overline{\partial}\partial^0_s u)^k$ from Sect. 2.

Theorem 5. *Let D be a pseudoconvex domain in $\mathbb{C}^{nk+l}$ and let $T \geq 0$ be a closed (l,l)-currant in D. Let φ be plurisubharmonic in D and presuppose*

$$i\partial\overline{\partial}\varphi \geq \omega,$$

where ω defines a Kähler metric in D. Then for any $((p+1)k,(q+1)k)$-form g on T with $(q+1)k < nk$, such that

$$\overline{\partial}_s^0 g = 0$$

in $L^2(T, e^{\varphi})$ we can solve

$$(\overline{\partial}\partial_s^0 u)^k = g \tag{9}$$

with

$$||\partial u \wedge (\overline{\partial}\partial u)^{k-1}||^2_{\omega,T,-\varphi} \leq \frac{1}{(n-p-q-2)k+1}||g||^2_{\omega,T,-\varphi}.$$

If $(q+1)k = nk$ we can solve (9) with the same estimate provided g satisfies the moment condition

$$\int g \wedge h \wedge T = 0$$

for any $(nk-(p+1)k, 0)$-form h in $L^2(T, e^{-\varphi})$ satisfying $\overline{\partial}_w h = 0$ on T.

In both cases, if g has compact support in $\{\psi \leq 0\}$ where ψ is some plurisubharmonic exhaustion function, the solution can be taken with support in $\{\psi \leq 0\}$.

Proof. Since by Proposition 5 $\overline{\partial}_s^0 = \vartheta^*_{-\varphi,w}$ the equation $(\overline{\partial}\partial_s^0 u)^k = g$ is equivalent to

$$(g, \overline{\vartheta}_{-\varphi,w} v) = (\partial u \wedge (\overline{\partial}\partial_s^0 u)^{k-1}, \vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w} v)$$

for any $v \in Dom\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}$. Observe that Theorem 3 is equivalent to the statement that for each $((p+1)k,(q+1)k-1)$-form f such that $\vartheta_{-\varphi,w} f = 0$, we can solve

$$\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w} v = f$$

with

$$||\overline{\vartheta}_{-\varphi} v||^2_{\omega,T,-\varphi} \leq \frac{1}{(n-q-p-2)k+1}||f||^2_{\omega,T,-\varphi}$$

(cf. the proof of Theorem 2).

Define an antilinear functional on $R(\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w})$ by

$$L_g(\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w} v) = (g, \overline{\vartheta}_{-\varphi,w} v).$$

We state that L_g is well defined, i.e. that $(g, \overline{\vartheta}_{-\varphi,w} v) = 0$ if $\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w} v = 0$.

If $(q+1)k = nk$, this follows from the moment condition since

$$(g, \overline{\vartheta}_{-\varphi,w} v) = \int g \wedge *\overline{\overline{\vartheta}_{-\varphi} v} \wedge T e^{\varphi}$$

and $*\overline{\overline{\vartheta}_{-\varphi} v} e^{\varphi}$ is $\overline{\partial}_w$-closed if $\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w} v = 0$. If $(q+1)k < nk$, it follows since we can solve

$$\overline{\vartheta}_{-\varphi,w} v = \vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}\xi$$

so that

$$(g, \overline{\vartheta}_{-\varphi,w}v) = (\overline{\partial}_s^0 g, \overline{\vartheta}_{-\varphi,w}\xi) = 0.$$

By Theorem 3, we can solve

$$\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v_0 = \vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v$$

with

$$||\overline{\vartheta}_{-\varphi,w}v_0||^2 \leq \frac{1}{(n-q-p-2)k+1}||\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v||^2.$$

Hence

$$|L_g(\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v)|^2 \leq$$
$$\leq \frac{1}{(n-q-p-2)k+1}||g||^2||\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v||^2.$$

By the Riesz representation theorem, we can therefore find a form u such that

$$L_g(\vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v) = (\partial u \wedge (\overline{\partial}\partial_s^0 u)^{k-1}, \vartheta_{-\varphi}\overline{\vartheta}_{-\varphi,w}v)$$

and

$$||\partial u \wedge (\overline{\partial}\partial_s^0 u)^{k-1}||^2 \leq \frac{1}{(n-q-p-2)k+1}||g||^2.$$

Then $(\overline{\partial}\partial_s^0 u)^k = g$, and we have found a solution to our equation.

Further as in the proof of Theorem 8.6 from [5]. □

We give a version Theorems 4 and 5 for some compact Kähler manifolds.

We assume that the line bundle is equipped with a metric, $\varphi = \{\varphi_j\}$, which with respect to a local trivialisation of L is given by weight functions φ_j. The curvature of the metric is then the globally defined form $i\partial\overline{\partial}\varphi_j = c(\varphi)$, and the line bundle is said to be positive if we can choose the metric so that this form is everywhere positive. Norms of L-valued forms and functions on T are defined in the same way as in scalarvalued case with respect to each local trivialisation and we still denote the metric $|| \cdot ||_\varphi$.

Theorem 6. *Let M be a homogenous compact $(nk+l)$-dimensional Kähler manifold, and let L be a positive line bundle over M. Presuppose moreover that $H^1(M, \mathcal{O}) = 0$. Let finally T be a positive closed current of bidegree (l, l) on M. Then, for any L-valued $(nk-pk, qk)$-form f with $(q-p)k > 1$ in L_T^2 satisfying $\overline{\partial}_w f = 0$ we can solve*

$$(\overline{\partial}\partial_w u)^k = f$$

with u in L_T^2. Analogously, for any L^{-1}-valued (pk, qk)-form g with $(n-q-p)k > 1$ in L_T^2 satisfying $\overline{\partial}_s g = 0$ we can solve

$$(\overline{\partial}\partial_s v)^k = g$$

with v in L_T^2.

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