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AN AUXILIARY RESULT FOR THE COMPLEX MONGE – AMPÈRA EQUATION ON POSITIVE CURRENTS OF HIGHER BIDEGREE

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Abstract: We study the induced Monge — Ampère equation on a positive current of bidegree (l, l) in a complex manifold. An auxiliary result is obtained which is crucial in the proof of the a priori inequality for Monge — Ampère operator.

Keywords: Monge – Ampèra equation, positive current, differention form, complex manifold, primitive form, definite quadratic forms, differention operators on current, existence theorems for Monge – Ampèra operator on closed current, currents of higher bidegree.

1 Introduction

Let M be a complex manifold and T a positive current in M. If u and f are smooth differential forms on M we say that

$$(\overline{\partial}\partial u)^k = f \text{ on } T \text{ if } (\overline{\partial}\partial u)^k \wedge T = f \wedge T.$$

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Initially the Monge — Ampère operator is thus only defined on smooth forms but it can later be extended (in different ways) to forms that are in a sense defined only on T.

The question we study in this paper is whether the Monge — Ampère equation can be solved on T, and, if so, what kinds of estimates one can find for the solution.

Solvability of Monge — Ampère equations in the case $k = 1$ is classical (see [1], [2], [3], [4], [7], [8]).

Similarily we may also consider smooth (l, l) currents that are strictly positive in a subdomain D of M and vanish outside of D, which means that we study our equation in D.

The article is organixed in the followind way. In 2 we discuss the linear algebra of forms on a current, in particular a notion of primitive forms (in the sense of Lefschetz) that is used in the a priori inequality.

Let V be a complex vector space of dimension n. A (q, q) form v is strongly positive [5] if it belongs to the cone generated by forms

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_q \wedge \overline{\alpha}_q$$

where $\alpha_j \in \Lambda^{1,0}(V^*)$.

A form $u \in \Lambda^{p,p}(V^*)$ is positive [5] iff $u \wedge v$ is positive for every strongly positive form v bidegree (q, q), with $q + p = n$. On a complex manifold M a differential form $u \in C^\infty_{p,p}(M)$ is strongly positive (resp. positive) if for every $z \in M$, $u(z)$ is strongly positive (resp. positive) as an element of $\Lambda^{p,p}(T^*M)$.

The space $\mathcal{D}'_{(r,s)}(M)$ of currents of bidimension (r, s) on M is by definition the dual of the space $\mathcal{D}_{(r,s)}(M)$ of testforms on M of bidegree (r, s), with respect to the usual inductive limit topology on the space of testforms [5].

A current T of bidimension (p, p) is positive (strongly positive) [5] if $\langle T, u\rangle \geq 0$ for all test forms $u \in \mathcal{D}_{(p,p)}(M)$ that are strongly positive (positive).

Let $c_q = (-1)^{q(q+1)/2} i^q = (-i)^{q^2}$. In local coordinates a current T of bidegree (q, q) can be expressed as

$$T = c_q \sum_{|I|=|J|=q} T_{I,J} d\bar{z}_J \wedge dz_I.$$

The positivity of T then implies that the distributions $T_{I,J}$ are measures and that $\overline{T_{I,J}} = T_{J,I}$. We will therefore further write the action T on a form u on integral form

$$\int u \wedge T.$$

If T is positive it also follows that quantity

$$||T|| = \sum |T_{I,J}|$$

is dominated by a constant times the trace

$$\sum_{|I|=q} T_{I,I}.$$

If ω is a smooth $(1,1)$ positive form we define the trace measure σ_T of a positive current of bidimension (p,p) with respect to ω by

$$\sigma_T := T \wedge \omega_p$$

where $\omega_p = \omega^p/p!$. In particular if $\omega = i\partial\overline{\partial}|z|^2$ the trace measure is the trace defined above multiplied by ω_n. We next define the L^2- norm of forms on a current. Let T is strongly positive and g be a form of bidegree $(0,q)$ in $\mathbb{C}^n$. Then define the L^2-norm of g over T by

$$||g||^2 = \int c_q g \wedge \overline{g} \wedge T \wedge \omega_{p-q}$$

if T is of bidimension (p,p). In particular, if g is a function the norm is exactly the L^2-norm with respect to the trace measure of T. Let $L^2_{(r,s)}(T)$ denote the completion of the space of smooth forms with respect to the norms defined above [5].

The operators ∂ and $\overline{\partial}$ act on currents. The current T is closed if $dT = 0$ [5].

In the sequel we will frequently speak of *forms on* T. This always means an element in the completion of the space of smooth form in the ambient space with respect to L^2-norms we have defined on T [5].

2 Linear algebra and L^2-spaces on currents of bidegree (l,l)

For the Monge — Ampère problem on currents of bidegree (l,l) let us give a more detailed discussion of the linear algebra of forms on a current. We need this to develop a version of the Kähler identities on a current, that will in what follows be used in the proof of the Kodaira — Nakano — Hörmander a priori estimate.

We begin with discussing forms and currents at a fixed point, i. e. we consider T a non-negative element in $\Lambda^{l,l}(\mathbb{C}^{n+l})$ and forms $f \in \Lambda^{*,*}(\mathbb{C}^{n+l})$. The space of (p,q)-forms on T, $\Lambda_T^{p,q}$ is determined [5] as the space of all $f \in \Lambda^{p,q}(\mathbb{C}^{n+l})$, modulo the subspace of forms such that

$$f \wedge T = 0.$$

In order to avoid too heavy notation, we use the same letter to denote an element in $\Lambda^{p,q}(\mathbb{C}^{n+l})$ and the corresponding element in $\Lambda_T^{p,q}$.

On a manifold the space of $(0,q)$-forms is the exterior algebra of the space of $(0,1)$-forms, but that this need not be the case in our setting ([5], p. 397).

To define norms on Λ_T, we need an auxiliary $(1,1)$-form $\omega > 0$, that will determine the metric on T. Let $\omega_k = \omega^k/k!$ [5].

Let σ_T be the trace of T with respect to ω considered as a form of maximal degree, i.e.

$$\sigma_T = T \wedge \omega_n.$$

This means that $\sigma_T = tr(T)\omega_{n+1}$, where $tr(T)$ is the trace considered as number [5].

It can be proved (see e.g. [6], p. 170) that a k-form f on a n-dimensional manifold is primitive if and only if $k \leq n$ and

$$f \wedge \omega_{n-k+1} = 0.$$

This condition has sense on Λ_T.

Definition 1. [5] *A k-form f is primitive on T if $k \leq n$ and*

$$f \wedge \omega_{n-k+1} \wedge T = 0.$$

The next proposition is important in the proof of the a priori inequality for the Monge — Ampère operator.

Let $e_1, \dots, e_{n+1}$ be a basis for the space of $(1,0)$-forms in $\mathbb{C}^{n+l}$. Write

$$\gamma = \sum \gamma_{JK} e_J \wedge \overline{e}_K$$

and partition γ into a sum $\tau + \sigma$ depending on whether J belongs to K (the τ-part) or not

$$\begin{aligned}
\gamma = \sum_{j_1 \in K} e_{j_1} \dots \sum_{j_p \in K} \gamma_{JK} e_{j_p} \wedge \overline{e}_K + \\
+ \Big(\sum_{r=1}^{p-1} \sum_{|M|=r} \sum_{j_1 \notin K} e_{j_1} \dots \sum_{j_{m_1-1} \notin K} e_{j_{m_1-1}} \sum_{j_{m_1} \in K} e_{j_{m_1}} \sum_{j_{m_1+1} \notin K} e_{j_{m_1+1}} \dots \\
\dots \sum_{j_{m_r-1} \notin K} e_{j_{m_r-1}} \sum_{j_{m_r} \in K} e_{j_{m_r}} \sum_{j_{m_r+1} \notin K} e_{j_{m_r+1}} \dots \sum_{j_p \notin K} \gamma_{JK} e_{j_p} \wedge \overline{e}_K + \\
+ \sum_{j_1 \notin K} e_{j_1} \dots \sum_{j_p \notin K} \gamma_{JK} e_{j_p} \wedge \overline{e}_K\Big) = \tau + \Big(\sum_{r=1}^{p-1} \sigma_r + \sigma_0\Big) = \tau + \sigma.
\end{aligned}$$

In the case $p = 1$ this partition given in [5].

Proposition 1. *The quadratic form defined by*

$$[\gamma, \gamma]\sigma_T = c_{q+p} \gamma \wedge \overline{\gamma} \wedge \omega_{n-q-p} \wedge T, \tag{1}$$

decompose on positive definite $[\sigma_r, \sigma_r]\sigma_T$ if $(-1)^{p+r} = -1$ and negative definite $[\tau, \tau]\sigma_T$, $[\sigma_r, \sigma_r]\sigma_T$ if $(-1)^{p+r} = 1$, $1 \leq r \leq p-1$ on the space of primitive forms in $\Lambda_T^{p,q}$. (In the case $p = 0$ form $[\tau, \tau]\sigma_T$ is positive definite, under $p = 2k+1$ form $[\sigma_0, \sigma_0]\sigma_T$ is negative and under $p = 2k$ is positive definite.)

The proof is given in [7] by $l = 1$ (see Proposition 2).

Proposition 5.7 [5] is special case of Proposition 1 by $p = 1$ and $l = 1$. The proof of Lemma 10.1 [5] can be replaced by the proof of Lemma 1 (see [9], [10], [12]).

Lemma 1. *The determinant*

$$\Delta_n = \begin{vmatrix} (1-\lambda_1)^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_{n-1} & \lambda_1\lambda_n \\ \lambda_2\lambda_1 & (1-\lambda_2)^2 & \dots & \lambda_2\lambda_{n-1} & \lambda_2\lambda_n \\ \dots & \dots & \ddots & \dots & \dots \\ \lambda_{n-1}\lambda_1 & \lambda_{n-1}\lambda_2 & \dots & (1-\lambda_{n-1})^2 & \lambda_{n-1}\lambda_n \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & \dots & \lambda_n\lambda_{n-1} & (1-\lambda_n)^2 \end{vmatrix} \geq 0, \qquad (2)$$

here $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, *and* $\Delta_n > 0$ *for* $n \geq 3$ *and* $\Delta_1 = \Delta_2 = 0$.

Доказательство. The determinant Δ_n is a special case of problem 406 of [11], and its value is a symmetric polynomial:

$$\Delta_n = \Big(1 + \Sigma_{i=1}^n \frac{\lambda_i^2}{1-2\lambda_i}\Big)\Pi_{i=1}^n(1-2\lambda_i). \qquad (3)$$

The proof of formula (3) (as well as the more general one [11]) is carried out by the method of mathematical induction on n. I express my deep gratitude to Mikhail Yuryevich Tyaglov for the provided link.

We can assume that $\lambda_i \neq 0$, since due to the symmetry of the polynomial at $\lambda_i = 0$ Δ_n becomes Δ_{n-1}.

It is easy to check that when $n = 1$, $\Delta_1 = [1 + \frac{1}{1-2\lambda}](1-2\lambda) = 0$.

When $n = 2$, $\Delta_2 = [1 + \Sigma_{i=1}^2 \frac{\lambda_i^2}{1-2\lambda_i}]\Pi_{i=1}^2(1-2\lambda_i) = 0$.

From formula (3), the positivity of Δ_n for $0 < \lambda_i \leq \frac{1}{2}$, $i = \overline{1,n}$ is obvious (moreover, only one λ_i can be equal to $\frac{1}{2}$ due to condition $\sum_{i=1}^n \lambda_i = 1$) for $n \geq 3$ ($\Delta_1 = \Delta_2 = 0$).

Let there be $\lambda_i > \frac{1}{2}$. There can only be one such number due to the condition of the lemma, and without loss of generality we can assume that $\lambda_1 > \frac{1}{2}$ due to the invariance of the determinant with respect to permutations of the variables λ_i. Then $\sum_{i=2}^n \lambda_i < \frac{1}{2}$, and the positivity of Δ_n is proven as follows:

$$\Delta_n = \Big((1-2\lambda_1) + \lambda_1^2 + \Sigma_{i=2}^n \frac{\lambda_i^2(1-2\lambda_1)}{1-2\lambda_i}\Big)\Pi_{i=2}^n(1-2\lambda_i) =$$

$$= \Big((1-\lambda_1)^2 + \Sigma_{i=2}^n \frac{\lambda_i^2(1-2\lambda_1)}{1-2\lambda_i}\Big)\Pi_{i=2}^n(1-2\lambda_i) =$$

$$= \Big((\lambda_2 + \ldots + \lambda_n)^2 + \Sigma_{i=2}^n \frac{\lambda_i^2(1-2\lambda_1)}{1-2\lambda_i}\Big)\Pi_{i=2}^n(1-2\lambda_i) =$$

$$= \Big(2\Sigma_{2\leq s<r\leq n}\lambda_s\lambda_r + 2\Sigma_{i=2}^n \frac{\lambda_i^2(1-\lambda_1-\lambda_i)}{1-2\lambda_i}\Big)\Pi_{i=2}^n(1-2\lambda_i) > 0,$$

since $0 < \lambda_i < \frac{1}{2}$, $i = \overline{2,n}$ and $\sum_{i=1}^n \lambda_i = 1$.

Proof. Choose a basis $e_1, \ldots, e_{n+l}$ for the space $(1,0)$-forms in $\mathbb{C}^{n+l}$ that diagonalizes both ω and T. Let $dV_j = ie_j \wedge \overline{e}_j$ and $dV_J = \bigwedge_J dV_j$. Then

$\omega = \sum dV_j$, $T = \sum \lambda_J dV_J$ and

$$T \wedge \omega_{n-q-p+1} = \sum_{|K|=n+l-q-p+1} \lambda_K dV_K$$

if we let

$$\lambda_J = \sum_{\substack{|S|=l \\ S\subset J}} \lambda_S.$$

Easily to check that $[\sigma,\sigma] = \sum_{r=0}^{p-1}\sum_{t=0}^{p-1}[\sigma_r,\sigma_t] = \sum_{r=0}^{p-1}[\sigma_r,\sigma_r]$ because

$$[\sigma_r,\sigma_t]\sigma_T = c_{q+p}\sum_{|M|=r}\sum_{j_1\notin K} e_{j_1}\dots\sum_{j_{m_1-1}\notin K} e_{j_{m_1-1}}\sum_{j_{m_1}\in K} e_{j_{m_1}}\sum_{j_{m_1+1}\notin K} e_{j_{m_1+1}}\dots$$

$$\dots\sum_{j_{m_r-1}\notin K} e_{j_{m_r-1}}\sum_{j_{m_r}\in K} e_{j_{m_r}}\sum_{j_{m_r+1}\notin K} e_{j_{m_r+1}}\dots\sum_{j_p\notin K}\gamma_{JK}e_{j_p}\wedge\overline{e}_K\wedge\sum_{|P|=t}\sum_{s_1\notin L}\overline{e}_{s_1}\dots$$

$$\dots\sum_{s_{p_1-1}\notin L}\overline{e}_{s_{p_1-1}}\sum_{s_{p_1}\in L}\overline{e}_{s_{p_1}}\sum_{s_{p_1+1}\notin L}\overline{e}_{s_{p_1+1}}\dots\sum_{s_{p_t-1}\notin L}\overline{e}_{s_{p_t-1}}\sum_{s_{p_t}\in L}\overline{e}_{s_{p_t}}\wedge$$

$$\wedge\sum_{s_{p_t+1}\notin L}\overline{e}_{s_{p_t+1}}\dots\sum_{s_p\notin L}\overline{\gamma}_{SL}\overline{e}_{s_p}\wedge e_L\wedge\omega_{n-q-p}\wedge T =$$

$$= c_{q+p}\sum_{|K|=q-r}\sum_{|M|=r}\sum_{j_1\notin K} e_{j_1}\dots\sum_{j_p\notin K}\sigma^r_{JK}e_{j_p}[M]\wedge dV_{J_M}\wedge\overline{e}_K\wedge$$

$$\wedge\sum_{|L|=q-t}\sum_{|P|=t}\sum_{s_1\notin L}\overline{e}_{s_1}\dots\sum_{s_p\notin L}\overline{\sigma^t_{SL}}\overline{e}_{s_p}[P]\wedge dV_{S_P}\wedge e_L\wedge\omega_{n-q-p}\wedge T =$$

$$= c_{q+p}\sum_{|K|=q-r}\sum_{|M|=r}\sum_{j_1\notin K} e_{j_1}\dots\sum_{j_p\notin K}\sigma^r_{JK}e_{j_p}[M]\wedge dV_{J_M}\wedge\overline{e}_K\wedge$$

$$\wedge\sum_{|L|=q-t}\sum_{|P|=t}\sum_{s_1\notin L}\overline{e}_{s_1}\dots\sum_{s_p\notin L}\overline{\sigma^t_{SL}}\overline{e}_{s_p}[P]\wedge dV_{S_P}\wedge e_L\wedge\sum_{|K|=n-q-p+l}\lambda_K dV_K = 0$$

for $r \neq t$. Here

$$\sum_{j_1\notin K} e_{j_1}\dots\sum_{j_p\notin K} e_{j_p}[M] = \sum_{j_1\notin K} e_{j_1}\dots\sum_{j_{m_1-1}\notin K} e_{j_{m_1-1}}\sum_{j_{m_1+1}\notin K} e_{j_{m_1+1}}\dots$$

$$\dots\sum_{j_{m_r-1}\notin K} e_{j_{m_r-1}}\sum_{j_{m_r+1}\notin K} e_{j_{m_r+1}}\dots\sum_{j_p\notin K} e_{j_p},$$

$$dV_{J_M} = dV_{j_{m_1}}\wedge\dots\wedge dV_{j_{m_r}}.$$

We consider

$$[\sigma_r,\sigma_r]\sigma_T = c_{q+p}\sum_{|K|=q-r}\sum_{|M|=r}\sum_{j_1\notin K} e_{j_1}\dots\sum_{j_p\notin K}\sigma^r_{JK}e_{j_p}[M]\wedge dV_{J_M}\wedge\overline{e}_K\wedge$$

$$\wedge\sum_{|L|=q-r}\sum_{|P|=r}\sum_{s_1\notin L}\overline{e}_{s_1}\dots\sum_{s_p\notin L}\overline{\sigma^r_{SL}}\overline{e}_{s_p}[P]\wedge dV_{S_P}\wedge e_L\wedge\omega_{n-q-p}\wedge T =$$

$$= c_{q+p}\sum_{|K|=q-r}\sum_{|M|=r}\sum_{j_1\notin K} e_{j_1}\dots\sum_{j_p\notin K}\sigma^r_{JK}e_{j_p}[M]\wedge dV_{J_M}\wedge\overline{e}_K\wedge$$

$$\wedge \sum_{|L|=q-r} \sum_{|P|=r} \sum_{s_1 \notin L} \overline{e}_{s_1} \dots \sum_{s_p \notin L} \overline{\sigma^r_{SL}} \overline{e}_{s_p}[P] \wedge dV_{S_P} \wedge e_L \wedge \sum_{|K|=n-q-p+l} \lambda_K dV_K =$$

$$= (-1)^{p+r} \sum_{|K|=q-r} \sum_{|M|=r} \sum_{|P|=r} \sum_{j_1 \notin K} ie_{j_1} \wedge \overline{e}_{j_1} \dots \sum_{j_p \notin K} ie_{j_p} \wedge \overline{e}_{j_p}[M] \wedge dV_{J_M} \wedge$$

$$\wedge dV_{S_P} \wedge c_{q-r} \overline{e}_K \wedge e_K \sigma^r_{JK} \overline{\sigma^r_{(j_1,\dots,j_{m_1-1},j_{m_1+1},\dots,s_{p_1},\dots,s_{p_r},\dots,j_{m_r-1},j_{m_r+1},\dots,j_p)K}} \wedge$$

$$\wedge \sum_{|K|=n-q-p+l} \lambda_K dV_K = (-1)^{p+r} \sum_{|K|=q-r} \sum_{|M|=r} \sum_{|P|=r} \sum_{J_M \cap S_P = \emptyset} \sigma^r_{JK} \times$$

$$\times \overline{\sigma^r_{(j_1,\dots,j_{m_1-1},j_{m_1+1},\dots,s_{p_1},\dots,s_{p_r},\dots,j_{m_r-1},j_{m_r+1},\dots,j_p)K}} \sum_{|L|=n-q-p+l} \lambda_L dV_{L \cup J \cup S_P \cup K},$$

$1 \le r \le p-1$,

$$[\sigma_0, \sigma_0] \sigma_T = (-1)^p \sum |\gamma_{JK}|^2 \sum_{|L|=n-q-p+l} \lambda_L dV_{L \cup J \cup K}.$$

Here notation $(j_1, \dots, j_{m_1-1}, j_{m_1+1}, \dots, s_{p_1}, \dots, s_{p_r}, \dots, j_{m_r-1}, j_{m_r+1}, \dots, j_p)$ means that in the index S the expression $S[P] = (s_1, \dots, s_{p_1-1}, s_{p_1+1}, \dots$ $\dots, s_{p_r-1}, s_{p_r+1}, \dots, s_p)$ substituted on $J[M]$.

The condition that σ_r, $r \ge 1$, is primitive (σ_0 is always primitive since

$$\sum_{j_1 \notin K} e_{j_1} \dots \sum_{j_p \notin K} \gamma_{JK} e_{j_p} \wedge \overline{e}_K \wedge \omega_{n-q-p+1} \wedge T =$$

$$= \sum_{j_1 \notin K} e_{j_1} \dots \sum_{j_p \notin K} \gamma_{JK} e_{j_p} \wedge \overline{e}_K \wedge \sum_{|L|=n-q-p+1+l} \lambda_L dV_L = 0)$$

means that

$$\sigma_r \wedge \omega_{n-q-p+1} \wedge T = \sum_{|K|=q-r} \sum_{|M|=r} \sum_{|L|=n-q-p+1+l} \sum_{j_1 \notin K} e_{j_1} \dots$$

$$\dots \sum_{j_p \notin K} \sigma^r_{JK} \lambda_L e_{j_p}[M] \wedge dV_{J_M \cup L} \wedge \overline{e}_K = 0.$$

We have

$$[\sigma_r, \sigma_r] = (-1)^{p+r} \sum_{|K|=q-r} \sum_{|M|=r} \sum_{|P|=r} \sum_{J_M \cap L_P = \emptyset} \sigma^r_{JK} \times$$

$$\times \overline{\sigma^r_{(j_1,\dots,j_{m_1-1},j_{m_1+1},\dots,l_{p_1},\dots,l_{p_r},\dots,j_{m_r-1},j_{m_r+1},\dots,j_p)K}} \lambda_{(J \cup L_P \cup K)^c},$$

$1 \le r \le p-1$,

$$[\sigma_0, \sigma_0] = (-1)^p \sum |\gamma_{JK}|^2 \lambda_{(J \cup K)^c}.$$

Fix $K, j_1, \dots, j_{m_1-1}, j_{m_1+1}, \dots, j_{m_r-1}, j_{m_r+1}, \dots, j_p$ and renumber the remaining indices $1, \dots, N = n + l - q - p + 2r$.

Put

$$\hat{\lambda}_{J_M} = \sum_{|S|=l} \lambda_S - \sum_{\substack{|S|=l \\ S \subset J_M}} \lambda_S = \sum_{|S|=l} \lambda_S - \lambda_{J_M}$$

and

$$\hat{\lambda}_{J_M L_P} = \sum_{|S|=l} \lambda_S - \lambda_{J_M} - \lambda_{L_P}.$$

What we need to prove is then that if

$$\sum \sigma^r_{J_M} \hat{\lambda}_{J_M} = 0$$

then

$$\sum_{J_M \cap L_P = \emptyset} \sigma^r_{J_M} \overline{\sigma^r_{L_P}} \hat{\lambda}_{J_M L_P} \leq 0.$$

This follows from the next lemma

Lemma 2. *Let* $\lambda_I \geq 0$, *and presuppose* $\sum \lambda_I = 1$. *Then*

$$\sum_{J_M \cap L_P = \emptyset} \sigma^r_{J_M} \overline{\sigma^r_{L_P}} \hat{\lambda}_{J_M L_P} \leq |\sum_{J_M} \hat{\lambda}_{J_M}|^2,$$

$1 \leq r \leq p-1$.

Proof. The proof is a repetition of the proof of Lemma 3.

We have

$$[\tau, \sigma] = \sum_{r=0}^{p-1} [\tau, \sigma_r] = 0,$$

since

$$[\tau, \sigma_r]\sigma_T = c_{q+p} \sum_{|K|=q-p} \tau_{JK} dV_J \wedge \overline{e}_K \wedge \sum_{|P|=r} \sum_{s_1 \notin L} \overline{e}_{s_1} \dots$$

$$\dots \sum_{s_{p_1-1} \notin L} \overline{e}_{s_{p_1-1}} \sum_{s_{p_1} \in L} \overline{e}_{s_{p_1}} \sum_{s_{p_1+1} \notin L} \overline{e}_{s_{p_1+1}} \dots$$

$$\dots \sum_{s_{p_r-1} \notin L} \overline{e}_{s_{p_r-1}} \sum_{s_{p_r} \in L} \overline{e}_{s_{p_r}} \sum_{s_{p_r+1} \notin L} \overline{e}_{s_{p_r+1}} \dots \sum_{s_p \notin L} \overline{\gamma}_{SL} \overline{e}_{s_p} \wedge e_L \wedge \omega_{n-q-p} \wedge T =$$

$$= c_{q+p} \sum_{|K|=q-p} \tau_{JK} dV_J \wedge \overline{e}_K \wedge \sum_{|L|=q-r} \sum_{|P|=r} \sum_{s_1 \notin L} \overline{e}_{s_1} \dots$$

$$\dots \sum_{s_p \notin L} \overline{\sigma^r_{SL}} \overline{e}_{s_p}[P] \wedge e_L \wedge dV_{S_P} \wedge \sum_{|L|=n-q-p+l} \lambda_L dV_L = 0.$$

The condition that τ is primitive means that

$$\tau \wedge \omega_{n-q-p+1} \wedge T = \sum_{|K|=q-p} \sum_{|L|=n-q-p+l+1} \tau_{JK} \lambda_L dV_{J \cup L} \wedge \overline{e}_K = 0,$$

and $[\tau, \tau]\sigma_T$ equals

$$c_{q-p} \sum_{|K|=q-p} \sum_{|L|=q-p} \tau_{JK} \overline{\tau}_{SL} \overline{e}_K \wedge dV_J \wedge e_L \wedge dV_S \wedge \sum_{|L|=n-q-p+l} \lambda_L dV_L =$$

$$= \sum_{|K|=q-p} \sum_{J \cap S = \emptyset} \tau_{JK} \overline{\tau}_{SK} \sum_{|L|=n-q-p+l} \lambda_L dV_{L \cup J \cup S \cup K}.$$

From here

$$[\tau,\tau]=\sum_{|K|=q-p}\sum_{J\cap L=\emptyset}\tau_{JK}\overline{\tau}_{LK}\lambda_{(J\cup L\cup K)^c}.$$

Fix K and renumber the remaining indices $1,\dots,N$. Put

$$\hat{\lambda}_J=\sum_{|S|=l}\lambda_S-\sum_{\substack{|S|=l\\ S\subset J}}\lambda_S=\sum_{|S|=l}\lambda_S-\lambda_J$$

and

$$\hat{\lambda}_{JL}=\sum_{|S|=l}\lambda_S-\lambda_J-\lambda_L.$$

What we need to prove is then that if

$$\sum\tau_J\hat{\lambda}_J=0$$

then

$$\sum_{J\cap L=\emptyset}\tau_J\overline{\tau}_L\hat{\lambda}_{JL}\leq 0.$$

This follows from the next lemma

Lemma 3. *Let* $\lambda_I\geq 0$ *and presuppose* $\sum\lambda_I=1$. *Then*

$$\sum_{J\cap L=\emptyset}\tau_J\overline{\tau}_L\hat{\lambda}_{JL}\leq|\sum\tau_J\hat{\lambda}_J|^2.$$

Proof. Note that

$$\hat{\lambda}_J=\sum_{|I|=l}\lambda_I-\sum_{\substack{|S|=1\\ S\subset J}}\lambda_S=1-\lambda_J$$

and

$$\hat{\lambda}_{JL}=1-\lambda_J-\lambda_L,$$

so

$$\hat{\lambda}_J\hat{\lambda}_L=\hat{\lambda}_{JL}+\lambda_J\lambda_L.$$

Expanding the square in the right hand side the inequality becomes

$$\sum|\tau_J|^2(2\lambda_J-1)-\sum_{\substack{J\cap L\neq\emptyset\\ J\neq L}}\tau_J\overline{\tau}_L(1-\lambda_J-\lambda_L)\leq|\sum\tau_J\lambda_J|^2.$$

Because both sides are quadratic forms in τ with real coefficients it is sufficient to check the inequality for τ real. We suppose all τ_J pairwise different. The general case follows by continuity. For τ fixed we assume

$$F(\lambda)=[\sum\tau_J\lambda_J\sum_{L\neq J}\tau_L\lambda_L+\sum(\tau_J)^2(1-\lambda_J)^2]+$$

$$+\sum_{\substack{J\cap L\neq\emptyset\\ J\neq L}}\tau_J\tau_L(1-\lambda_J-\lambda_L)$$

for λ in the simplex $\lambda_I\geq 0$, $\sum\lambda_I=1$.

In the case $p = 1$ the last term is missing since conditions $j \cap l \neq \emptyset$ and $j \neq l$ are impossible simultaneously.

The proof of the inequality

$$F(\lambda) = \sum \tau_j \lambda_j \sum_{l \neq j} \tau_l \lambda_l + \sum (\tau_j)^2 (1 - \lambda_j)^2 \geq 0$$

comes down to the proof of the inequality (see [9], [10],[12], and also Lemma 1)

$$\Delta_n = \begin{vmatrix} (1-\lambda_1)^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_{n-1} & \lambda_1\lambda_n \\ \lambda_2\lambda_1 & (1-\lambda_2)^2 & \dots & \lambda_2\lambda_{n-1} & \lambda_2\lambda_n \\ \dots & \dots & \ddots & \dots & \dots \\ \lambda_{n-1}\lambda_1 & \lambda_{n-1}\lambda_2 & \dots & (1-\lambda_{n-1})^2 & \lambda_{n-1}\lambda_n \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & \dots & \lambda_n\lambda_{n-1} & (1-\lambda_n)^2 \end{vmatrix} \geq 0,$$

here $n = C_N^1 = N$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$.

Let us move on to proving the inequality

$$F(\lambda) = [\sum \tau_J \lambda_J \sum_{J \cap L = \emptyset} \tau_L \lambda_L + \sum (\tau_J)^2 (1 - \lambda_J)^2] +$$

$$+ [\sum \tau_J \lambda_J \sum_{\substack{J \cap L \neq \emptyset \\ L \neq J}} \tau_L \lambda_L + \sum_{\substack{J \cap L \neq \emptyset \\ J \neq L}} \tau_J \tau_L (1 - \lambda_J - \lambda_L)] \geq 0$$

in the case $p \neq 1$.

Note that the number of terms in λ_J is equal to C_p^l.

Let‘s rename λ_J, $|J| = p$ to $\lambda_1 \dots \lambda_q \ \lambda_{q+1} \dots \lambda_n$, where $n = C_N^p$. Let us assume that for $J \cap L = \emptyset$ we have $\lambda_1 \dots \lambda_q$, where $q = C_{N-2p}^p$, and for $J \cap L \neq \emptyset$ we have $\lambda_{q+1} \dots \lambda_{q+m}$, where $m = n - q = C_N^p - C_{N-2p}^p$.

The proof of inequality $F(\lambda) \geq 0$ for $p \neq 1$ is reduced to the proof of inequality (see [9], [10], [12], and also Lemma 1)

$$\Delta_q = \begin{vmatrix} (1-\lambda_1)^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_{q-1} & \lambda_1\lambda_q \\ \lambda_2\lambda_1 & (1-\lambda_2)^2 & \dots & \lambda_2\lambda_{q-1} & \lambda_2\lambda_q \\ \dots & \dots & \ddots & \dots & \dots \\ \lambda_{q-1}\lambda_1 & \lambda_{q-1}\lambda_2 & \dots & (1-\lambda_{q-1})^2 & \lambda_{q-1}\lambda_q \\ \lambda_q\lambda_1 & \lambda_q\lambda_2 & \dots & \lambda_q\lambda_{q-1} & (1-\lambda_q)^2 \end{vmatrix} \geq 0,$$

and equality

$$\Delta_m = \begin{vmatrix} (1-\lambda_{q+1})^2 & \dots & \lambda_{q+1}\lambda_n + 1 - \lambda_{q+1} - \lambda_n \\ \lambda_{q+2}\lambda_{q+1} + 1 - \lambda_{q+2} - \lambda_{q+1} & \dots & \lambda_{q+2}\lambda_n + 1 - \lambda_{q+2} - \lambda_n \\ \dots & \ddots & \dots \\ \lambda_n\lambda_{q+1} + 1 - \lambda_n - \lambda_{q+1} & \dots & (1-\lambda_n)^2 \end{vmatrix} =$$

$$= \begin{vmatrix} (1-\lambda_{q+1})^2 & (1-\lambda_{q+1})(1-\lambda_{q+2}) & \dots & (1-\lambda_{q+1})(1-\lambda_n) \\ (1-\lambda_{q+2})(1-\lambda_{q+1}) & (1-\lambda_{q+2})^2 & \dots & (1-\lambda_{q+2})(1-\lambda_n) \\ \dots & \dots & \ddots & \dots \\ (1-\lambda_n)(1-\lambda_{q+1}) & (1-\lambda_n)(1-\lambda_{q+2}) & \dots & (1-\lambda_n)^2 \end{vmatrix} =$$

$$= (1-\lambda_{q+1})(1-\lambda_{q+2})\dots(1-\lambda_n) \begin{vmatrix} 1-\lambda_{q+1} & 1-\lambda_{q+2} & \dots & 1-\lambda_n \\ 1-\lambda_{q+1} & 1-\lambda_{q+2} & \dots & 1-\lambda_n \\ \dots & \dots & \ddots & \dots \\ 1-\lambda_{q+1} & 1-\lambda_{q+2} & \dots & 1-\lambda_n \end{vmatrix} = 0,$$

here $q+m=n=C_N^p$. □

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