PHYSICALLY UNREALIZABLE FLAT IMMERSIONS OF A DISK

KONSTANTIN V. STOROZHUK

ABSTRACT. Let B be a two-dimensional surface smoothly diffeomorphic to a disk that admits an isometric immersion $f : B \to \mathbb{R}^2$ into the Euclidean plane. We show that such an immersion cannot always be realized physically—that is, as a sheet of paper assembled from several pieces and laid flat (possibly in multiple layers) on a table. Formally, this impossibility is equivalent to the non-existence of an embedding into $\mathbb{R}^3$ obtained by augmenting the image in $\mathbb{R}^2$ with a third coordinate. In other words, there exists no lift $\widetilde{f} : B \to \mathbb{R}^2 \times \mathbb{R}$ whose composition with the projection onto $\mathbb{R}^2$ coincides with the original immersion f.

A smooth closed curve Γ in the plane is defined as the image of the circle S^1 under a smooth regular mapping $\gamma : S^1 \to \mathbb{R}^2$ that admits only finitely many self-intersections, all of which are transversal and of multiplicity 2.

Question: How can we determine whether Γ bounds an immersed disk?

We orient the curve Γ by specifying a direction of traversal, which is encoded by the tangent velocity field γ'.

First necessary condition. For Γ to bound an immersed disk, the total rotation of the tangent vector around the origin must equal $\pm 2\pi$.

Second necessary condition. Consider the bounded regions into which Γ divides the plane. Choose a point in each such region and compute the winding number of Γ with respect to that point. These winding numbers must all be of the same sign (zeros are allowed). As illustrated in Fig. 1, the winding number of a region corresponds to the number of layers of the immersed disk covering it.

It can be shown that these two conditions together are also sufficient for the curve to bound an immersed disk. We omit the proof.

Interestingly, such a disk may not be uniquely determined by the curve. The simplest example is the right-hand curve in Fig. 1; in [1], this is referred to as the Milnor example. Readers who wish to work through this example independently should disregard the right-hand side of Fig. 1.

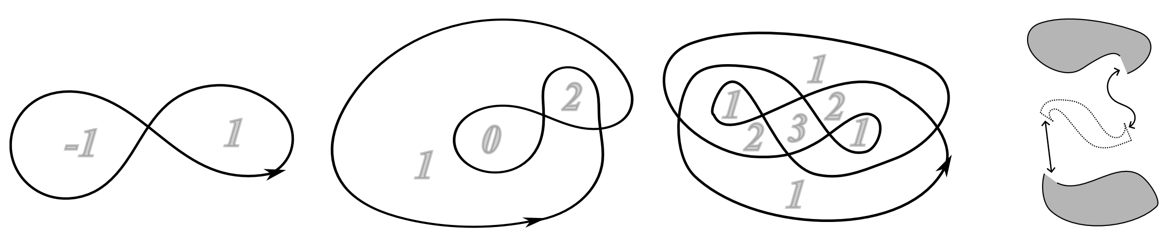

FIGURE 1. Curves bounding the region in 0, 1, and 2 ways

It is straightforward to assemble Milnor's example by gluing together three pieces of paper such that they can be placed flat on a table without "bumps," though not in a single layer. We shall refer to such disks as "physically flat," or simply "physical."

Date: March 15, 2026.

Surprisingly, not every flat immersed region admits a physical realization. The simplest "non-physical" example appears to involve the so-called Bennequin curve, which bounds a flat immersed region in five distinct ways. Two of these are illustrated in Fig. 2: three amoeba-shaped regions "grow" from sides a, c, e of the central hexagon (the first method) or from sides b, d, f (the second method). We shall demonstrate below that neither of these disks is "physical." We omit the remaining three methods: having spent several hours discovering them ourselves, we prefer not to deprive the reader of that pleasure. We note only that these three methods do yield "physical" disks.

The impatient reader may consult the algorithm proposed in [1], which enables one to enumerate all flat immersed disks bounded by a given curve.

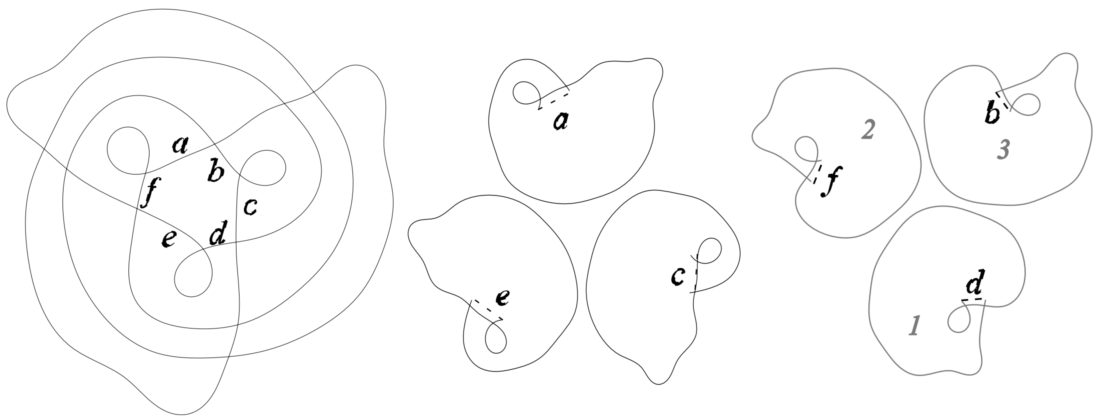

FIGURE 2. The Bennequin curve bounds an immersed disk in five different ways (two "non-physical" configurations are shown)

Let us glue together one of the two surfaces shown in Fig. 2—for example, the one on the right. We denote this surface by Ω_{123}.

When attempting to lay it flat on a table, we encounter an unexpected difficulty: one of the parts 1, 2, or 3 presses against another and prevents the surface from lying flat. To ensure that the surface can lie flat on the table, we must introduce a cut, as shown, for example, in Fig. 4 (marked in red on the right).

To rigorously justify the necessity of such a cut, we introduce the following notation. Let R denote the central hexagon to which regions 1, 2, and 3 of the surface Ω_{123} are attached. Each region 1, 2, and 3, when glued to R, is physically inserted into R along its wider edge. We denote the corresponding preimages of R under the projections by R_1, R_2, and R_3.

First, consider the subset Ω_{ij}, consisting of the central hexagon with only two of the three parts $(1, 2, 3)$ from Fig. 2 attached. It suffices to consider Ω_{13}.

This configuration can still be laid "physically flat," but doing so requires placing R_1 and R_3 on opposite sides of the hexagon R. In other words, if region Ω_i is superimposed on R from above, then region Ω_j must necessarily be placed below R.

Indeed, suppose we wish to arrange the figure Ω_{13} such that the order of the "floors" from bottom to top is R, R_3, R_1. We shall demonstrate that this arrangement is impossible without making a cut.

Consider an arc drawn within region 1, starting at point A on the boundary of R and terminating at point $B \in R_1 \subset 1$, where the projection of this arc lies entirely within the projection of region 3 (see Fig. 3). According to the prescribed floor ordering, the starting point $A \in R$ lies below region 3, while the endpoint $B \in R_1$ lies above region 3. Thus, if the height varies continuously along the curve, there must exist a point on this

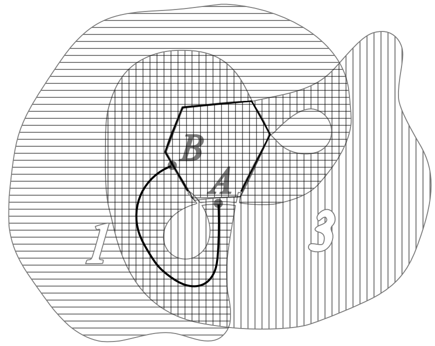

FIGURE 3. The subset Ω_{13}, assembled from regions 1 and 3.

arc that also belongs to region 3. Consequently, it is impossible to interleave regions 1 and 3 in the order R, R_3, R_1 without introducing a cut. The impossibility of other floor orderings in which R does not occupy the middle position can be established by a similar argument.

The reason why the figure Ω_{123} is "non-physical" now becomes clear. Indeed, if this figure were to be laid flat, then two of the three regions 1, 2, and 3 would necessarily lie on the same side of R, which leads to a contradiction.

Thus, the regions Ω_{12}, Ω_{13}, and Ω_{23} are "physical," whereas the entire region Ω_{123} is "non-physical." This situation is conceptually analogous to the well-known Borromean rings: while no two rings are linked pairwise, the three together form an inseparable configuration.

FIGURE 4. The disk can be placed on the table only by introducing a cut

Note that the fact that a flatly immersed surface can be "physically non-flat" is well illustrated by the standard example of the Riemann surface associated with the two-sheeted mapping $\sqrt{z}$. A smooth "double annulus" cut from this surface admits a standard immersion into the plane. In Fig. 5, the two sheets of the "double annulus" are shown slightly shifted relative to each other; the transition between the sheets occurs along the branch cut AB.

We invite the reader to verify that the "double annulus" in Fig. 5 bounded by the curve Γ_1 is "non-physical," whereas the double annulus bounded by the curve Γ_2 is "physical."

Upon a closer examination of Fig. 5, one observes that the curve Γ_1 bounds a smooth annulus in not one but two non-isometric ways, neither of which is "physical." This suggests a possible connection between the number of non-isometric immersions and their "physicality."

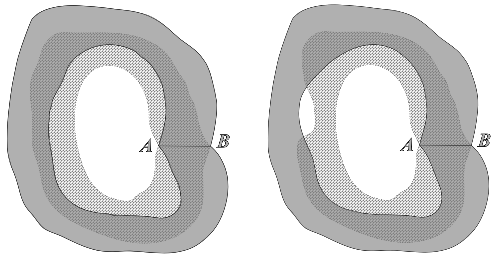

FIGURE 5. Γ_1: boundary of the "non-physical" annuli; Γ_2: boundary of the "physical" annulus.

An annulus, unlike a disk, is not simply connected; however, it remains connected. By relaxing the connectivity requirement, we can construct straightforward examples of non-unique immersibility. The reader is invited to deduce a recurrence formula for the number of ways to glue n concentric circles onto a flat (disconnected) surface and to verify that the resulting sequence $f(n)$ appears in the OEIS (Online Encyclopedia of Integer Sequences) under entry A000085.

In conclusion, we note a visual similarity between the curves depicted in Figures 2–4 and the contours that arise during the eversion of a two-dimensional sphere. Closely related are questions concerning immersions of the projective plane into three-dimensional space; see, for example, [2] and [3]. The visible contours of the surfaces folds in Figure 6 resemble the Milnor and Bennequin curves. This resemblance appears to be more than coincidental; see [4].

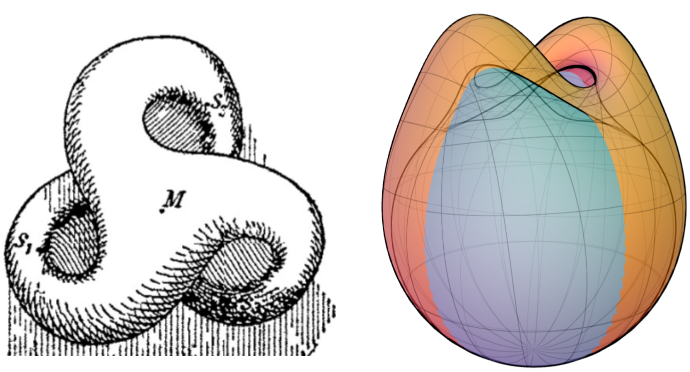

FIGURE 6. Figures from [2] and [3]

References

[1] V. Poenaru, Extension des Immersions en Codimension 1 (d'apres Samuel Blank), Séminaire N. Bourbaki, Volume 1967/68, Expose 342., p.473-505.

[2] W. Boy, Über die Abbildung der projektiven Ebene auf eine im Endlichen geschlossene singularitätenfreie Fläche Kgl. Ges. d. Wiss. Nachrichten, Mathphys. Klasse (Heft 1) (1901), pp. 20-33.

[3] `https://rreusser.github.io/explorations/sphere-eversion/`

[4] K.C. Millet, Generic smooth maps of surfaces, Topology Appl. 18 (1984) pp. 197–215.

SOBOLEV INSTITUTE OF MATHEMATICS. 4 KOPTYUG AV., NOVOSIBIRSK, 630090, RUSSIA
Email address: `stork931@gmail.com`