The Internal Logic of the Category of Vector Spaces

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Abstract

In this paper we study the internal logic of the category $\mathcal{V}ect_{\mathbb{k}}$ of vector spaces. The work is inspired by the treatments of Blute–Scott and Hamano on the interpretation of fragments of linear logic in the category of reflexive topological vector spaces. Rather than fixing the logic in advance, we consider the category of vector spaces equipped with a specified collection of functors and aim to construct a logic whose interpretation on this category is complete. We introduce a 2-categorical generalization of Hamano's functorial semantics, analyze the structure of $\mathcal{V}ect_{\mathbb{k}}$ with respect to the given collection of functors, and construct the logic of vector spaces, VSL, by modifying intuitionistic linear logic. The logic proves to be inherently multisorted, reflecting the distinction between finite- and infinite-dimensional spaces. As the main results, we prove soundness of the interpretation of VSL on $\mathcal{V}ect_{\mathbb{k}}$ and completeness of the interpretation of its classical fragment CVSL on the category of finite-dimensional vector spaces over an infinite field.

Contents

1 Introduction

This work grows out of examining the approaches of Blute–Scott [1] and Hamano [2] to modeling linear logic [3] in the category of topological vector spaces $\mathcal{TV}ect_{\Bbbk}$, with a further restriction to the subcategory $\mathcal{RTV}ect_{\Bbbk}$ of reflexive objects. Their choice of model category was motivated by the goal of achieving full completeness for MLL+Mix. As Barr showed [4], in $\mathcal{TV}ect_{\Bbbk}$ the canonical map $\mathrm{inv}_V : V \to V^{**}$ is an isomorphism of vector spaces; the subcategory $\mathcal{RTV}ect_{\Bbbk}$ restricts by definition to objects where inv_V is furthermore a homeomorphism. This gives $\mathcal{RTV}ect_{\Bbbk}$ the structure of a $*$-autonomous category [5], capturing the involutivity of negation in classical linear logic.

Since the hom-sets between vector spaces are big, it is necessary to restrict the class of morphisms that may serve as interpretations of proofs in order to achieve completeness. One of the obvious candidates for such a restriction is some form of naturality requirement. However, the attempt to interpret proofs of sequents as standard natural transformations between interpretations of formulas breaks down already at the level of formulas. For instance, the expression $\lambda f.\mathcal{C}(f, f)$, corresponding to the formula $X \multimap X$, does not yield a well-typed functor in general.

Blute–Scott address this via functorial polymorphism. They interpret formulas as multivariant functors of the form

$$F : (\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^n \to \mathcal{C} \quad (n \in \mathbb{N})$$

and proofs of sequents as dinatural transformations between multivariant functors. The disadvantages of this approach are complex typing and inability to accommodate the cut rule, since dinatural transformations do not compose in general.

Hamano proposed a simpler semantics in which formulas are interpreted as objects and proofs of sequents as $\mathbb{Z}$-invariant morphisms – those commuting with the automorphisms induced by representations of $\mathbb{Z}$ on base objects (interpretations of atomic formulas). This approach works for $\mathcal{RTV}ect_{\Bbbk}$ over an infinite field $\Bbbk$: restricting to automorphisms avoids the need to work with opposite categories, thereby resolving the typing issue, and naturality over countably many automorphisms turns out to be a sufficiently strong condition to recover full completeness. Moreover, $\mathbb{Z}$-invariant morphisms are closed under composition, which allows the cut rule to be interpreted directly.

In this paper we study a question dual in spirit to the one considered by Blute–Scott and Hamano. Instead of searching for a suitable model for a fragment of linear logic, we fix the category $\mathcal{V}ect_{\Bbbk}$ of vector spaces together with a specified collection of functors corresponding to logical connectives, and ask which logic admits a complete interpretation in this setting. We introduce a 2-categorical generalization of Hamano's semantics, based on isonatural transformations, which are natural transformations valued in the maximal groupoid of the domain category. This approach is more suitable for our purposes than either the Blute–Scott or pure Hamano frameworks, as it retains typing convenience while providing greater structural clarity.

The present work is largely methodological and conceptual, laying the groundwork for further development. It introduces the multiplicative and additive fragments of the logic of $\mathcal{V}ect_{\Bbbk}$, which we call VSL, and situates its classical and intuitionistic variants. The exponential fragment involves a discussion of a non-standard comonad structure on the tensor algebra, which is of independent interest; we defer its treatment to a companion paper. As the main result, we establish completeness for the classical fragment, which corresponds to the restriction to finite-dimensional spaces. Full completeness, requiring additional framework development, is left for future work.

We begin by establishing the categorical framework and fixing notation used throughout the paper. The notion of isonatural transformations is introduced, together with an explicitly typed 2-categorical encoding of the sequent calculus, discussing the notions of model and its soundness and (full) completeness. We then give a structural account of the category $\mathcal{V}ect_{\Bbbk}$ of vector spaces equipped with a specified collection of functors serving as interpretations of the logical connectives of VSL, and establish a criterion for the existence of non-zero isonatural transformations on the subcategory $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$ of finite-dimensional spaces. Next, the sequent calculus for VSL is introduced, obtained by modifying a formal factorization of the intuitionistic linear logic sequent calculus, together with its classical and intuitionistic fragments, CVSL and IVSL respectively. We then prove a theorem characterizing the provable sequents of CVSL, parallel to the categorical criterion established for $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$. Finally, these results are combined to establish soundness of the interpretation of VSL in $\mathcal{V}ect_{\Bbbk}$ and completeness of the interpretation of CVSL in $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$. We conclude with a discussion of open questions and directions for future work.

2 Preliminaries

2.1 General setup

We start by introducing the framework used throughout the article. In this section, we work with an abstract monoidal category $(\mathcal{C}, \otimes, I)$, equipped with additional functors of form

$$F : (\mathcal{C}^{\mathrm{op}})^{n_-} \times \mathcal{C}^{n_+} \to \mathcal{C} \quad (n_-, n_+ \in \mathbb{N}),$$

where n_- and n_+ are called negative and positive variance, respectively.

Throughout the paper we adopt the following conventions:

- Objects of a category $\mathcal{C}$ are denoted $\mathrm{obj}(\mathcal{C})$, arrows are denoted $\mathrm{arr}(\mathcal{C})$; we write $X : \mathrm{obj}(\mathcal{C})$ and $f : \mathrm{arr}(\mathcal{C})$ for membership. We adopt the arrow-first perspective, identifying each object $X : \mathrm{obj}(\mathcal{C})$ with its identity arrow id_X. Accordingly, functors will typically be described through their action on morphisms.

- We use $\simeq$ to denote an arbitrary isomorphism, $\cong$ for emphasizing that an isomorphism is natural in its components.

- The internal hom is denoted by $\multimap : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.

- We treat morphism indices with relative freedom to avoid verbosity when the typing is clear from context.

- Throughout the text, we overload notation for logical formulas and their categorical interpretations, avoiding the use of semantic brackets.

- All depicted diagrams are assumed or proved to commute, unless stated otherwise.

Notation 2.1 (Tuples of morphisms)**.** We use bold font to denote tuples of morphisms. When needed, covariant and contravariant cases are explicitly distinguished using the annotation $(-)^{\mathrm{op}}$:

$$\mathbf{f}^{\mathrm{op}} := (f_1^{\mathrm{op}}, \ldots, f_{n_-}^{\mathrm{op}}) : \mathrm{arr}((\mathcal{C}^{\mathrm{op}})^{n_-}), \quad \mathbf{g} := (g_1, \ldots, g_{n_+}) : \mathrm{arr}(\mathcal{C}^{n_+}).$$

The length of a tuple is denoted $\mathrm{len}(\mathbf{f})$.

$\mathbf{f} \cup \mathbf{g}$ denotes the filtered concatenation of $\mathbf{f}$ and $\mathbf{g}$, preserving relative order and removing duplicates:

$$(f_1, \ldots, f_m) \cup (g_1, \ldots, g_n) := (f_{i_1}, \ldots, f_{i_k}, g_{j_1}, \ldots, g_{j_l}),$$

where $i_1 < \ldots < i_k$, $j_1 < \ldots < j_l$, and no component appears more than once in the result.

Notation 2.2 (Lambda-expressed functors)**.** We use tuples of morphisms to express functors in lambda-notation. Any functor F is identified, up to η-equivalence, with the form

$$\lambda \mathbf{f}^{\mathrm{op}} \mathbf{g}. F(\mathbf{f}, \mathbf{g}) : (\mathcal{C}^{\mathrm{op}})^{\mathrm{len}(\mathbf{f})} \times \mathcal{C}^{\mathrm{len}(\mathbf{g})} \to \mathcal{C},$$

where all morphisms occurring in the tuples $\mathbf{f}^{\mathrm{op}}$ and $\mathbf{g}$ are pairwise distinct. For convenience, we omit the $(-)^{\mathrm{op}}$ annotation within the functorial expression, retaining it only in the binder.

For natural transformations between lambda-expressed functors, variables are matched in order of occurrence. Explicitly, the naturality condition for the transformation

$$\alpha : \lambda \mathbf{f}_1^{\mathrm{op}} \mathbf{g}_1. F(\mathbf{f}_1, \mathbf{g}_1) \to \lambda \mathbf{f}_2^{\mathrm{op}} \mathbf{g}_2. G(\mathbf{f}_2, \mathbf{g}_2)$$

is given by the family of diagrams parametrized by tuples $\mathbf{f}^{\mathrm{op}}$ and $\mathbf{g}$ with compatible shapes:

$$\begin{array}{ccccc}
\mathbf{X}' & \mathbf{X} & F(\mathbf{X}', \mathbf{X}) & \xrightarrow{\alpha_{\mathbf{X}', \mathbf{X}}} & G(\mathbf{X}', \mathbf{X}) \\
{\scriptstyle \mathbf{f}}\uparrow & {\scriptstyle \mathbf{g}}\downarrow & {\scriptstyle F(\mathbf{f}, \mathbf{g})}\downarrow & & \downarrow{\scriptstyle G(\mathbf{f}, \mathbf{g})} \\
\mathbf{Y}' & \mathbf{Y} & F(\mathbf{Y}', \mathbf{Y}) & \xrightarrow[\alpha_{\mathbf{Y}', \mathbf{Y}}]{} & G(\mathbf{Y}', \mathbf{Y})
\end{array}\ .$$

Example 2.3. The functor $\multimap \circ (\otimes^{\mathrm{op}} \times \mathrm{id})$, whose type is given by the diagram

$$\begin{array}{ccc}
(\mathcal{C}^{\mathrm{op}})^2 \times \mathcal{C} & \xrightarrow{\otimes^{\mathrm{op}} \times \mathrm{id}} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \\
& {\scriptstyle \multimap \circ (\otimes^{\mathrm{op}} \times \mathrm{id})}\searrow & \downarrow{\scriptstyle \multimap} \\
& & \mathcal{C}
\end{array}$$

is written in lambda notation as

$$\lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}} g. (f_1 \otimes f_2) \multimap g.$$

The natural isomorphism for the tensor-hom adjunction is then expressed as

$$\mathrm{cur} : (\lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}} g. (f_1 \otimes f_2) \multimap g) \to (\lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}} g. f_1 \multimap (f_2 \multimap g)).$$

Remark 2.4. The above notation inherently prevents variance mismatches. For example, without additional specifications (see Rem. 2.9 below), we are not allowed to write the expression

$$\lambda fg.f \otimes (f \multimap g),$$

since f appears both covariantly and contravariantly. Doing so violates functoriality, since identities would not be preserved.

Definition 2.5 (Constant and diagonal functors)**.** For a category $\mathcal{J}$, the *diagonal functor* is defined as

$$\triangle^{(\mathcal{J})} : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}, \quad \triangle^{(\mathcal{J})} := \lambda A \xrightarrow{f} B.(\triangle_A^{(\mathcal{J})} \xrightarrow{\alpha^{(f)}} \triangle_B^{(\mathcal{J})}),$$

where

$$\triangle_X^{(\mathcal{J})} := \lambda f.\mathrm{id}_X : \mathcal{J} \to \mathcal{C}$$

is the *constant functor* at $X : \mathrm{obj}(\mathcal{C})$, and $\alpha^{(f)}$ is the natural transformation corresponding to a morphism f. When $\mathcal{J}$ is small and discrete with object set J, we instead write

$$\triangle^{(J)} := \lambda f.(f)_{j \in J} : \mathcal{C} \to \mathcal{C}^J.$$

Definition 2.6 (Maximal groupoid)**.** The *maximal groupoid* $\mathcal{GC}$ of a category $\mathcal{C}$ is the subcategory of $\mathcal{C}$ consisting of all its isomorphisms. We denote its *inclusion into the base category* as

$$\mathrm{inc}_{\mathcal{C}} : \mathcal{GC} \to \mathcal{C}.$$

Remark 2.7. The maximal groupoid $\mathcal{GC}$ and its opposite $\mathcal{GC}^{\mathrm{op}}$ are isomorphic by taking the inverse:

$$\lambda f.f^{-1} : \mathcal{GC} \xrightarrow{\simeq} \mathcal{GC}^{\mathrm{op}}.$$

This allows us to work with restrictions of functors to $\mathcal{GC}$ as purely covariant: the functor

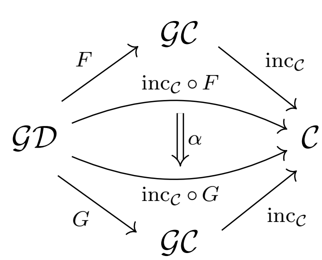
$$\lambda \mathbf{f}^{\mathrm{op}}\mathbf{g}.F(\mathbf{f}, \mathbf{g}) : (\mathcal{C}^{\mathrm{op}})^{n_-} \times \mathcal{C}^{n_+} \to \mathcal{C}$$

restricts to

$$\lambda \mathbf{fg}.F(\mathbf{f}^{-1}, \mathbf{g}) : \mathcal{GC}^{n_- + n_+} \to \mathcal{GC}.$$

Note that the codomain may be taken to be $\mathcal{GC}$, since functors preserve isomorphisms.

Definition 2.8 (Isonatural transformations)**.** Let $F, G : \mathcal{GD} \to \mathcal{GC}$ be functors. A natural transformation α of the form



is called an *isonatural transformation.* Its naturality conditions are evaluated on isomorphisms only, while composing with $\mathrm{inc}_{\mathcal{C}}$ allows all arrows of $\mathcal{C}$ to appear as components.

In our notation for isonatural transformations, we write

$$\alpha : F \to G \quad \text{or} \quad \alpha : \lambda \mathbf{f}.F(\mathbf{f}) \to \lambda \mathbf{f}.G(\mathbf{f}),$$

omitting compositions with inclusion in the short form and implicitly evaluating on isomorphisms in the expanded one. Further on, we will work with transformations with $\mathcal{D} = \mathcal{C}^n$, which we refer to as isonatural transformations on $\mathcal{C}$.

Remark 2.9. By typing, vertical compositions of isonatural transformations are freely available, while horizontal compositions are not in general. The isonatural setting allows us to consider diagonalizations of functors that would not be well-defined on $\mathcal{C}$. For instance, the diagonalization $\lambda f.f \multimap f$ of the internal hom can only be defined on $\mathcal{GC}$, where it takes the form

$$\lambda f.(\lambda \varphi.f \circ \varphi \circ f^{-1}) \quad = \quad \mathcal{GC} \xrightarrow{\triangle^{(2)}} \mathcal{GC}^2 \xrightarrow{\multimap} \mathcal{GC}.$$

At the same time, most of the functors of interest will be well-defined on the original category $\mathcal{C}$, and we will embed them into the isonatural framework by restricting to $\mathcal{GC}$.

Definition 2.10 (Iterated monoidal product)**.** The *iterated monoidal product* is defined inductively:

$$\bigotimes_n := \mathcal{C}^n \xrightarrow{\mathrm{id}_\mathcal{C} \times \bigotimes_{n-1}} \mathcal{C}^2 \xrightarrow{\otimes} \mathcal{C},$$

with base cases

- $\bigotimes_0 := \lambda f.\mathrm{id}_I : 1 \to \mathcal{C}$,
- $\bigotimes_1 := \mathrm{id}_\mathcal{C}$,
- $\bigotimes_2 := \otimes$.

Remark 2.11. Note that well-definedness of the iterated monoidal product follows from Mac Lane's coherence theorem. The definition applies to any monoidal product on $\mathcal{C}$; in our case, it will be both the tensor product $\otimes$ and the direct sum $\oplus$ on $\mathcal{V}ect_\Bbbk$.

Definition 2.12 (Functor applications)**.** Let $(G_k : \mathcal{E}_k \to \mathcal{D}_k)_{k=1}^n$ and $F : \prod_{k=1}^n \mathcal{D}_k \to \mathcal{C}$ be functors. The *application* $F(G_1, \ldots, G_n)$ is defined as

$$F(G_1, \ldots, G_n) := \prod_{k=1}^n \mathcal{E}_k \xrightarrow{\prod_{k=1}^n G_k} \prod_{k=1}^n \mathcal{D}_k \xrightarrow{F} \mathcal{C}.$$

In particular, the monoid product of functors $(G_k : \mathcal{D}_k \to \mathcal{C})_{k=1}^n$ is defined as

$$\bigotimes_{k=1}^n G_k := \prod_{k=1}^n \mathcal{D}_k \xrightarrow{\prod_{k=1}^n G_k} \mathcal{C}^n \xrightarrow{\bigotimes_n} \mathcal{C}.$$

We will refer to partially instantiated functors as *partial applications.* These can be made explicit via constant functors, as in the following example for a bifunctor F and $X : \mathrm{obj}(\mathcal{C})$:

$$F(X, -) \quad := \quad \mathcal{C} \xrightarrow{\simeq} 1 \times \mathcal{C} \xrightarrow{\triangle_X \times \mathrm{id}_\mathcal{C}} \mathcal{C} \times \mathcal{C} \xrightarrow{F} \mathcal{C}.$$

The corresponding lambda-expression is $\lambda f.F(\mathrm{id}_X, f)$.

Notation 2.13. We use a shorthand $\dashv\vdash$ for binary sequents with interderivable formulas:

$$A \dashv\vdash B \quad :\Leftrightarrow \quad A \vdash B \text{ and } B \vdash A.$$

2.2 2-categorical encoding of a sequent calculus

We introduce a 2-categorical encoding of an abstract sequent calculus. The primary motivation is to capture its data entirely within category theory, avoiding appeal to informal notions of interpretation. This approach eliminates the need for auxiliary constructions such as $\mathbb{Z}$-invariant morphisms, as we work directly within the standard 2-categorical framework.

Unlike the standard approach, we work with structural elements of a sequent calculus modulo α-equivalence.

Definition 2.14 (Classes of structural elements of a sequent calculus)**.** The *classes of formulas, sequents and inference rules modulo α-equivalence* are defined as classes of formulas, sequents and inference rules modulo renaming of propositional variables. They are expressed in lambda notation as

$$(A)_\alpha := \lambda \mathbf{X}.A(\mathbf{X});$$

$$(A_1, \ldots, A_n \vdash A)_\alpha := \lambda \bigcup_{k=1}^{n+1} \mathbf{X}_k.(A_1(\mathbf{X}_1), \ldots, A_n(\mathbf{X}_n) \vdash A(\mathbf{X}_{n+1}));$$

$$\left(\frac{A_1, \ldots, A_m \vdash A}{B_1, \ldots, B_n \vdash B}\ R\right)_\alpha := \lambda \bigcup_{k=1}^{m+n+2} \mathbf{X}_k. \left(\frac{A_1(\mathbf{X}_1), \ldots, A_m(\mathbf{X}_m) \vdash A(\mathbf{X}_{m+1})}{B_1(\mathbf{X}_{m+2}), \ldots, B_n(\mathbf{X}_{m+n+1}) \vdash B(\mathbf{X}_{m+n+2})}\ R\right).$$

The inference rule case is written for a single premise rule and extends analogously to multiple premises and *proof trees*. For a given sequent, the α-equivalence class of its proof tree will simply be called a proof of the corresponding sequent class.

We consider a logic built upon a finite collection of connectives $(C_k)_{k=1}^m$ with fixed arities $(n_k)_{k=1}^m$. Nullary connectives among them serve as logical units. The data of formula construction can be encoded by the free Lawvere theory $\mathcal{L}$.

Definition 2.15 (Free Lawvere theory for a logic)**.** The *free Lawvere theory* for a logic upon connectives $(C_k)_{k=1}^m$ is a category $\mathcal{L}$ consisting of the following:

- Objects are finite product powers of the generator Form.
- Morphisms are freely generated by the connectives $(C_k : \mathrm{Form}^{n_k} \to \mathrm{Form})_{k=1}^m$ and the cartesian structure of $\mathcal{L}$ via composition.

The free Lawvere theory allows us to work with explicitly typed formula templates, given by morphisms of $\mathcal{L}$ with codomain Form, rather than individual formulas with fixed variables. The templates are built by iterated application, which is a composition of morphisms of appropriate types:

$$C(C_1, \ldots, C_m) := \prod_{k=1}^m \mathrm{Form}^{n_k} \xrightarrow{\prod_{k=1}^m C_k} \mathrm{Form}^m \xrightarrow{C} \mathrm{Form}\,.$$

Moreover, the free Lawvere theory has projections and diagonals

$$\pi_k : \mathrm{Form}^n \to \mathrm{Form}, \quad \triangle^{(n)} : \mathrm{Form} \to \mathrm{Form}^n$$

built in as counits of the diagonal adjunction and universal maps to the product, respectively. This identifies its morphisms landing in Form with classes of formulas modulo α-equivalence.

To accommodate the sequent calculus, the framework introduced above needs to be 2-categorified. We want classes of logical formulas to be interpreted as functors rather than 1-categorical arrows, and classes of sequent proofs to be encoded by isonatural transformations. There are several possible approaches, and we seek one that captures the essential structure of the interpretation. We define the 2-category $\mathcal{S}yn^{\uparrow}$ of functors and isonatural transformations, built on top of the category $\mathcal{S}yn$, described below. We first consider the case of simple logic, in which the rules of the sequent calculus act uniformly on all formulas, with special cases reserved only for units.

Definition 2.16 (2-category $\mathcal{S}yn^{\uparrow}$ for a simple logic)**.** The *2-category* $\mathcal{S}yn^{\uparrow}$ for a logic upon connectives $(C_k)_{k=1}^m$ with a fixed sequent calculus consists of the following:

- The objects of $\mathcal{S}yn^{\uparrow}$ are the syntactic category $\mathcal{S}yn$ together with all natural powers of its maximal groupoid $\mathcal{GS}yn$. $\mathcal{S}yn$ is described externally: its objects are logical formulas, obtained as images of 1-morphisms of $\mathcal{S}yn^{\uparrow}$, and its morphisms are the components of 2-morphisms of $\mathcal{S}yn^{\uparrow}$, isonatural transformations obtained from the sequent calculus.

- The 1-morphisms of $\mathcal{S}yn^{\uparrow}$ are defined as follows. First, as an auxiliary step, we introduce the binary comma connective to the original logic:
$$, : \mathrm{Form}^2 \to \mathrm{Form}\,.$$
Then we build a free Lawvere theory for the updated logic and 2-categorify it by assigning a functor on $\mathcal{GS}yn$ to each morphism of $\mathcal{L}$:
$$C_k : \mathrm{Form}^{n_k} \to \mathrm{Form} \quad \mapsto \quad C_k : \mathcal{GS}yn^{n_k} \to \mathcal{GS}yn.$$
This extends to classes of logical formulas:
$$(A)_\alpha = \lambda \mathbf{X}.A(\mathbf{X}) \quad \mapsto \quad \lambda \mathbf{f}.A(\mathbf{f}) : \mathcal{GS}yn^{\mathrm{len}(\mathbf{f})} \to \mathcal{GS}yn.$$
Finally, we add the inclusion $\mathrm{inc}_{\mathcal{S}yn} : \mathcal{GS}yn \to \mathcal{S}yn$ to the obtained collection of 1-morphisms.

- 2-morphisms encode the data of sequent calculus proofs. A proof of a sequent class $(A_1, \ldots, A_n \vdash A)_\alpha$ is mapped to an isonatural transformation of type
$$A_1\,, \cdots, A_n \to A \quad := \quad \lambda \bigcup_{k=1}^{n+1} \mathbf{f}_k.(A_1(\mathbf{f}_1), \cdots, A_n(\mathbf{f}_n)) \to \lambda \bigcup_{k=1}^{n+1} \mathbf{f}_k.A(\mathbf{f}_{n+1}).$$
Precompositions with diagonal functors and compositions with inclusions are omitted in the brief notation. The basic generating cases for 2-morphisms are obtained by mapping classes of inference rules to maps of isonatural transformations (see Rem. 2.18 below):
$$\left(\frac{A_1, \ldots, A_m \vdash A}{B_1, \ldots, B_n \vdash B}\ R\right)_\alpha \quad \mapsto \quad \frac{\xi : A_1\,, \cdots, A_m \to A}{\phi_{\mathrm{R}}(\xi) : B_1\,, \cdots, B_n \to B},$$

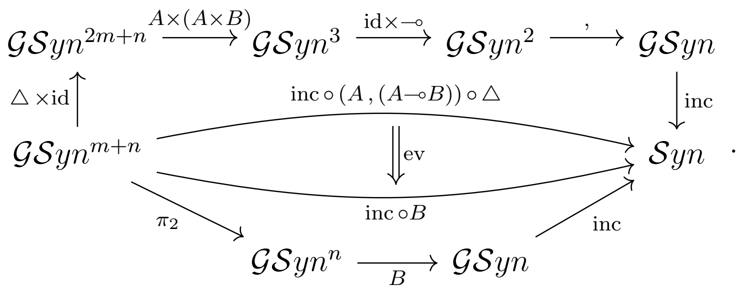
where shared variables are tracked across the whole rule and respected by ϕ_R. The equational theory on 2-morphisms is in accordance with the equational theory on proofs of the sequent calculus.

Example 2.17. The following correspondence in linear logic

$$\frac{\overline{A \vdash A} \quad \overline{B \vdash B}}{A, A \multimap B \vdash B} \multimap_L \quad \mapsto \quad \frac{\overline{\mathrm{id}_A : A \to A} \quad \overline{\mathrm{id}_B : B \to B}}{\mathrm{ev}(\mathrm{id}_A, \mathrm{id}_B) : A\,,(A \multimap B) \to B}$$

yields an isonatural transformation, whose type is given explicitly by

$$\lambda \mathbf{fg}.A(\mathbf{f})\,,(A(\mathbf{f}) \multimap B(\mathbf{g})) \to \lambda \mathbf{fg}.B(\mathbf{g});$$



The definition of $\mathcal{S}yn^{\uparrow}$ involves several design choices worth commenting on.

Remark 2.18.

- We use functors on $\mathcal{GS}yn$ rather than on $\mathcal{S}yn$ in order to remain within the framework of natural transformations while imposing the fewest possible restrictions.

- If one wants to categorify a sequent calculus with multivariant connectives under the correspondence $\mathcal{F}orm \mapsto \mathcal{S}yn$, they must use less restrictive transformations between functors, such as dinatural transformations. In general, this breaks the 2-categorical structure, as composition of such transformations is no longer freely available, and consequently the cut rule cannot be incorporated.

- In the present approach, the sequent calculus must include identity axioms and the cut rule, corresponding to the presence of identity transformations and vertical compositions.

- The comma bifunctor is required to formally model the corresponding syntactic element of the sequent calculus. It need not be monoidal in general, though it is so for most logics of interest. For instance, in standard linear logic it is isomorphic to the multiplicative conjunction $\otimes$; a counterexample is non-associative linear logic.

- We do not introduce a general typing for ϕ_R, as it is treated at the meta level in this work. It suffices to specify the interpretation for each concrete rule of the given logic.

What we lack so far in comparison with Hamano's approach is the ability to handle multisorted logic. This is significant for our purposes: finite- and infinite-dimensional spaces admit different collections of isonatural transformations, and additional transformations arise between functors whose domains come from the cartesian closure of $\{\mathcal{GF}d\mathcal{V}ect_{\Bbbk}, \mathcal{GI}nf\mathcal{V}ect_{\Bbbk}\}$. We therefore extend Definition 2.16 as follows.

Definition 2.19 (2-category $\mathcal{S}yn^{\uparrow}$ for a multisorted logic)**.** Given a *multisorted logic* upon connectives $(C_k)_{k=1}^{m}$ with a fixed sequent calculus, we extend Definition 2.16 by allowing categories from the cartesian closure of $\{\mathcal{GS}yn_t \mid t \in T\}$ to appear as categorical domains of isonatural transformations. Here $\mathcal{S}yn_t$ denotes the full subcategory of $\mathcal{S}yn$ consisting of formulas of type t, and we assume that formulas of different types have distinct names, so α-equivalence extends automatically. Explicitly, the correspondence for the class of an n-ary formula F is:

$$\left(F : \prod_{k=1}^{n} \mathrm{Form}_{t_k} \to \mathrm{Form}\right)_{\alpha} \quad \mapsto \quad F : \prod_{k=1}^{n} \mathcal{GS}yn_{t_k} \to \mathcal{GS}yn.$$

We also add the inclusions $\mathcal{GS}yn_t \to \mathcal{GS}yn$ to the collection of 1-morphisms of $\mathcal{S}yn^{\uparrow}$.

Remark 2.20. The provided extension corresponds to 2-categorifying the framework from the beginning of this subsection using a multisorted Lawvere theory $\mathcal{L}_T$ in place of $\mathcal{L}$. We restrict to the case where each connective accepts arguments of any type.

The last point to address is the notion of a model of logic, together with soundness and (full) completeness. We begin by recalling the original formulations of Hamano's results.

Proposition (Soundness theorem of (I)MLL)**.** For any (I)MLL proof π of $\Gamma \vdash \Delta$, $\pi^{\bullet}$ is a G-map.

Theorem (Completeness theorem for a binary simple sequent)**.** If a binary sequent $A \vdash B$ is not derivable, then $\mathrm{Hom}_{\mathbb{Z}}(A^{\bullet}, B^{\bullet}) = 0$.

Theorem (Full completeness theorem for a binary sequent)**.** The associated binary space for $M \vdash N$ has a basis consisting of the interpretations of the distinct (modulo reductions for normalization) proofs of $M \vdash N$.

Here interpretations are denoted by $(-)^{\bullet}$, G-map is a shorthand for G-invariant map, and $\mathrm{Hom}_{\mathbb{Z}}$ denotes the space of $\mathbb{Z}$-invariant morphisms. More precise forms of the completeness and full completeness theorems are given in the subsequent sections of Hamano's article. In particular, full completeness is stated as the full faithfulness of a certain functor between auxiliary categories.

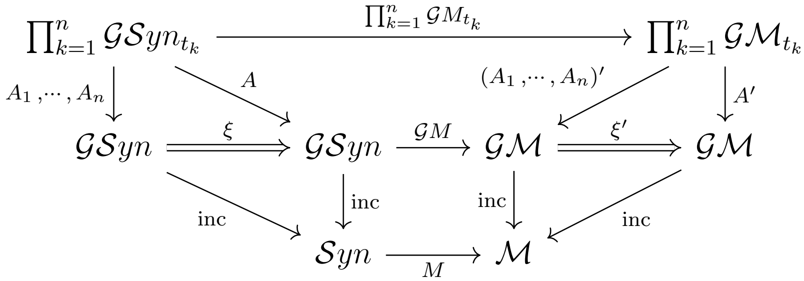
From the categorical perspective, a model of a logic is a functor

$$M : \mathcal{S}yn \to \mathcal{M},$$

where $\mathcal{M}$ is the model category. Soundness of the interpretation can then be stated as the extensibility of M to the 2-functor

$$M^{\uparrow} : \mathcal{S}yn^{\uparrow} \to \mathcal{M}^{\uparrow},$$

which we require to be strict in this work, where $\mathcal{M}^{\uparrow}$ is the 2-category built on top of $\mathcal{M}$ in the same manner as $\mathcal{S}yn^{\uparrow}$. The mental model for the multisorted case is given by the diagram



Here $\mathcal{G}M$ denotes the restriction of M to groupoids, ξ is the categorification of a proof of a sequent class $(A_1, \ldots, A_n \vdash A)_\alpha$, and ξ' is its interpretation in $\mathcal{M}^\uparrow$.

Completeness and full completeness are more subtle. Conceptually, they correspond to the fullness and full faithfulness of $M^\uparrow$ on 2-cells representing proofs, respectively, though the precise formulation requires separate analysis. For instance, to match Hamano's notion, one may replace the standard sequent calculus with its linearized version, in which proofs admit formal scalar multiplication by elements of the field.

In this paper we use more concrete notions of soundness and completeness, following modifications of Hamano's formulations, made precise in Section 4.

3 Category of Vector Spaces

3.1 Structural account of $\mathcal{V}ect_\Bbbk$

We denote the category of vector spaces over a field $\Bbbk$ as $\mathcal{V}ect_\Bbbk$, with full subcategories $\mathcal{F}d\mathcal{V}ect_\Bbbk$ and $\mathcal{I}nf\mathcal{V}ect_\Bbbk$ of finite- and infinite-dimensional spaces respectively. No assumptions on the field are made so far. The goal of this subsection is to describe the functorial structure of these categories required for the sequent calculus interpretation.

We consider $\mathcal{V}ect_\Bbbk$ as a category equipped with the following *collection of functors*, closed under finite products and composition:

- Structural functors:
  - Identity $\mathrm{id}_{\mathcal{V}ect_\Bbbk} = \lambda f.f : \mathcal{V}ect_\Bbbk \to \mathcal{V}ect_\Bbbk$,
  - Diagonals $\triangle^{(n)} := \lambda f.(f)_{k=1}^n : \mathcal{V}ect_\Bbbk \to \mathcal{V}ect_\Bbbk^n \quad (n \in \mathbb{N})$,
  - Projections $\pi_k = \lambda \mathbf{f}.f_k : \mathcal{V}ect_\Bbbk^n \to \mathcal{V}ect_\Bbbk \quad (1 \le k \le n)$.
- Constant functors at the field and the zero vector space:

$$\triangle_0 = \lambda f.\mathrm{id}_0 : 1 \to \mathcal{V}ect_\Bbbk, \quad \triangle_\Bbbk = \lambda f.\mathrm{id}_\Bbbk : 1 \to \mathcal{V}ect_\Bbbk.$$

- Monoidal product $\otimes : \mathcal{V}ect_\Bbbk^2 \to \mathcal{V}ect_\Bbbk$ – the tensor product of vector spaces;
- Biproduct $\oplus : \mathcal{V}ect_\Bbbk^2 \to \mathcal{V}ect_\Bbbk$ – the direct sum of vector spaces;
- Internal hom $\multimap : \mathcal{V}ect_\Bbbk^{\mathrm{op}} \times \mathcal{V}ect_\Bbbk \to \mathcal{V}ect_\Bbbk$ – the vector space of linear maps.

We also define the dual space functor as

$$* := \lambda f.f^* := \lambda f.f \multimap \mathrm{id}_{\Bbbk} : \mathcal{V}ect_{\Bbbk}^{\mathrm{op}} \to \mathcal{V}ect_{\Bbbk}.$$

The structure of these functors and the natural transformations encoding their interactions are described in the remainder of this subsection. All maps below are defined on simple tensors and extended by linearity; the lambda notation is accordingly abused to range over generators rather than arbitrary elements.

1. $\mathrm{id}_{\mathcal{V}ect_{\Bbbk}}$, $\triangle^{(n)}$, and π_k generate the cartesian structure on finite powers of $\mathcal{V}ect_{\Bbbk}$. In particular, they allow to treat $\mathcal{V}ect_{\Bbbk}^n$ as the domain of n-ary functors, as well as to duplicate and discard variables.

2. $(\mathcal{V}ect_{\Bbbk}, \otimes, \Bbbk)$ is a symmetric monoidal category. The structural isomorphisms are:

$$\alpha_{U,V,W} := \lambda(u \otimes v) \otimes w.u \otimes (v \otimes w) : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W);$$

$$\sigma_{U,V} := \lambda u \otimes v.v \otimes u : U \otimes V \xrightarrow{\cong} V \otimes U;$$

$$\lambda_V := \lambda r \otimes v.r \cdot v : \Bbbk \otimes V \xrightarrow{\cong} V, \qquad \rho_V := \lambda v \otimes r.v \cdot r : V \otimes \Bbbk \xrightarrow{\cong} V.$$

The symmetric monoidal structure is not strict – the coherence isomorphisms α, λ, ρ, and σ are not identities. Morever, the monoidal unit $\Bbbk$ is defined up to unique isomorphism, as any two one-dimensional vector spaces are canonically isomorphic.

The scalar multiple of a linear map is defined using the endomorphisms of $\Bbbk$:

$$r \cdot f := V \xrightarrow{\lambda_V^{-1}} \Bbbk \otimes V \xrightarrow{(\lambda s.r \cdot s) \otimes f} \Bbbk \otimes W \xrightarrow{\lambda_W} W.$$

3. $(\mathcal{V}ect_{\Bbbk}, \otimes, \Bbbk; \multimap, \Bbbk)$ is a symmetric monoidal closed category. The functors $V \otimes -$ and $- \otimes V$, isomorphic via symmetry, have a common right adjoint $V \multimap -$, the partial internal hom:

$$V \otimes - \dashv V \multimap -, \quad - \otimes V \dashv V \multimap -.$$

The unit and counit of the adjunction are:

$$\mathrm{coev}_{U,V} := \eta_V^{(U)} = \lambda v.(\lambda u.u \otimes v) : V \to (U \multimap U \otimes V),$$

$$\mathrm{ev}_{U,V} := \varepsilon_V^{(U)} = \lambda u \otimes \varphi.\varphi(u) : U \otimes (U \multimap V) \to V,$$

called coevaluation and evaluation, respectively.

Since $\mathcal{V}ect_{\Bbbk}$ is enriched over itself, the hom-set isomorphism is also an isomorphism of vector spaces:

$$\mathrm{cur}_{U,V,W} := \lambda(\lambda u \otimes v.w).(\lambda u.(\lambda v.w)) : (U \otimes V) \multimap W \xrightarrow{\cong} U \multimap (V \multimap W).$$

The natural tensor-hom and double dual inclusions

$$\mathrm{com}_{V,W} := \lambda v' \otimes w.(\lambda v.v'(v) \cdot w) : V^* \otimes W \to V \multimap W,$$

$$\mathrm{inv}_V := \lambda v.(\lambda v'.v'(v)) : V \to V^{**}$$

are constructed using the adjunction:

$$\mathrm{com}_{V,W} = \mathrm{cur}^{-1}_{V^*\otimes W,V,W}(V^* \otimes W \otimes V \xrightarrow{V^*\otimes\sigma_{W,V}} V^* \otimes V \otimes W \xrightarrow{(\rho_V\circ \mathrm{ev}_{V,\Bbbk})\otimes W} W),$$

$$\mathrm{inv}_V = \mathrm{cur}_{V,V^*,\Bbbk}(V \otimes (V \multimap \Bbbk) \xrightarrow{\mathrm{ev}_{V,\Bbbk}} \Bbbk).$$

The identity functor is represented by $\Bbbk$, as each linear map $\Bbbk \to V$ is determined by the image of the multiplicative unit $1 \in \Bbbk$:

$$\Bbbk \multimap - \simeq \mathrm{id}_{\mathcal{V}ect_\Bbbk}, \qquad \lambda(\lambda r.r \cdot v).v : (\Bbbk \multimap V) \xrightarrow{\cong} V.$$

4. The additive monoidal structure $(\mathcal{V}ect_\Bbbk, \oplus, 0)$ is given by direct sum and zero vector space. Zero vector space 0 is the unique zero object in $\mathcal{V}ect_\Bbbk$, with annihilating natural isomorphisms

$$a_V : \lambda v \otimes 0.0 : V \otimes 0 \xrightarrow{\cong} 0, \quad a'_V : \lambda 0 \otimes v.0 : 0 \otimes V \xrightarrow{\cong} 0.$$

Partial tensor products are left adjoints, and thus they preserve arbitrary coproducts:

$$\mathrm{dst}_{U,(V_j)_{j\in J}} = \lambda u \otimes \bigoplus_{j\in J} v_j. \bigoplus_{j\in J} u \otimes v_j : U \otimes \bigoplus_{j\in J} V_j \xrightarrow{\cong} \bigoplus_{j\in J} U \otimes V_j,$$

similarly for $\mathrm{dst}_{(U_j)_{j\in J},V}$.

Combined, these make $(\mathcal{V}ect_\Bbbk, \otimes, \Bbbk; \multimap, \Bbbk; \oplus, 0)$ a symmetric closed rig category. The additive structure on morphisms is introduced using biproducts: for $f, g : V \to W$, the sum $f + g$ is defined as

$$f + g := V \xrightarrow{\lambda v.v\oplus v} V \oplus V \xrightarrow{f\oplus g} W \oplus W \xrightarrow{\lambda w_1\oplus w_2.w_1+w_2} W.$$

5. $\mathcal{V}ect_\Bbbk$ is bicomplete – all small limits and colimits exist. While finite products and coproducts coincide, infinite products and coproducts are distinct:

$$\langle \pi_j \rangle_{j\in J} : \bigoplus_{j\in J} V_j \overset{\ncong}{\hookrightarrow} \prod_{j\in J} V_j, \ |J| \geq \aleph_0.$$

Product is the right adjoint and coproduct is the left adjoint to the diagonal functor

$$\bigoplus_J \dashv \triangle^{(J)} \dashv \prod_J, \quad \triangle^{(J)} : \mathcal{V}ect_\Bbbk \to \mathcal{V}ect_\Bbbk^J.$$

The corresponding adjunction isomorphisms take the form

$$U \multimap \left(\prod_{j\in J} V_j\right) \cong \prod_{j\in J}(U \multimap V_j), \qquad \left(\bigoplus_{j\in J} U_j\right) \multimap W \cong \prod_{j\in J}(U_j \multimap W).$$

In particular, the dual of the finite biproduct satisfies

$$\left(\bigoplus_{j=1}^{n} V_j\right)^* \cong \bigoplus_{j=1}^{n} V_j^*.$$

The coproduct inclusions are the units of the left adjoint, and the product projections are the counits of the right adjoint. The direct sum admits both inclusions and projections (the latter not to be confused with the structural projections above):

$$i_k : \lambda\mathbf{f}.f_k \to \lambda\mathbf{f}.\bigoplus_{j\in J} f_j, \quad \pi_k : \lambda\mathbf{f}.\bigoplus_{j\in J} f_j \to \lambda\mathbf{f}.f_k,$$

where $\mathbf{f} : \mathrm{arr}(\mathcal{V}ect_{\Bbbk}^J)$ and $k \in J$.

6. The category $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$ is compact and rigid – the tensor-hom map $\mathrm{com}_{V,W}$ and the double dual map inv_V are natural isomorphisms:

$$\mathrm{com}_{V,W} : V^* \otimes W \xrightarrow{\cong} V \multimap W, \qquad \mathrm{inv}_V : V \xrightarrow{\cong} V^{**}.$$

The standard form of compactness

$$\mathrm{com}'_{V,W} : (V \otimes W)^* \xrightarrow{\cong} V^* \otimes W^*$$

can be obtained via currying:

$$\begin{array}{ccccc} (V\otimes W)^* & = & (V\otimes W)\multimap \mathbf{k} & \xrightarrow{\mathrm{cur}_{V,W,\Bbbk}} & V \multimap (W\multimap \Bbbk) \\ \downarrow{\scriptstyle \mathrm{com}'_{V,W}} & & & & \downarrow{\scriptstyle \mathrm{com}^{-1}_{V,W^*}} \\ V^*\otimes W^* & & = & & (V\multimap \Bbbk)\otimes(W\multimap\Bbbk) \end{array} .$$

In $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$ the field $\Bbbk$ serves as the dualizing object, with evaluation and coevaluation defined as

$$\mathrm{ev}_V := \lambda v' \otimes v.v'(v) : V^* \otimes V \to \Bbbk \quad \text{and} \quad \mathrm{coev}_V := \lambda r.r \cdot \mathrm{com}^{-1}_{V,V}(\mathrm{id}_V) : \Bbbk \to V^* \otimes V.$$

Thus, $(\mathcal{F}d\mathcal{V}ect_{\Bbbk}, \otimes, \Bbbk, \multimap)$ is the classical example of a $*$-autonomous category.

In $\mathcal{I}nf\mathcal{V}ect_{\Bbbk}$ the maps $\mathrm{com}_{V,W}$ and inv_V are proper inclusions for infinite-dimensional V and W, as their codomains are strictly larger than their domains:

$$\dim(V) < \dim(V^*) < \dim(V^{**}), \quad \dim(V^* \otimes W) < \dim(V \multimap W).$$

The dimensions of composite vector spaces are calculated as follows:

$$\dim(V \oplus W) = \dim(V) + \dim(W), \quad \dim(V \otimes W) = \dim(V) \cdot \dim(W);$$

$$\dim(V \multimap W) = \begin{cases} \dim(V) \cdot \dim(W), \ \dim(V) < \aleph_0 \\ \max(|\Bbbk|, \dim(W))^{\dim(V)}, \ \dim(V) \geq \aleph_0 \end{cases} .$$

The dimensions of $V^* \otimes W$ and $V \multimap W$ can coincide even when both V and W are infinite-dimensional, but in such cases $\mathrm{com}_{V,W}$ does not serve as an isomorphism. It can be shown that all isonatural maps from $V^* \otimes W$ to $V \multimap W$ are scalar multiples of $\mathrm{com}_{V,W}$. A related result appears in Hamano's work in a slightly different setting. We do not verify this explicitly, as our completeness results are restricted to $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$.

3.2 Isonatural transformations on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$

Before turning to logic, we prove a result used later in the analysis of categorical semantics: a criterion for the existence of non-zero isonatural transformations between given functors on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$. For convenience, we also refer to these as finite-dimensional isonatural transformations.

Lemma 3.1 (Structure of functors on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$)**.** *Let*

$$F : (\mathcal{F}d\mathcal{V}ect_{\Bbbk}^{\mathrm{op}})^{n-} \times \mathcal{F}d\mathcal{V}ect_{\Bbbk}^{n+} \to \mathcal{F}d\mathcal{V}ect_{\Bbbk}$$

be a functor from the collection specified above, restricted to $\mathcal{GF}d\mathcal{V}ect_{\Bbbk}$, with diagonals excluded. Then F is isomorphic to a normal form

$$\mathrm{NF}(F) := \lambda \mathbf{f}^{\mathrm{op}}\mathbf{g}. \bigoplus_{k=1}^{m} \left(\left(\bigotimes_{l=1}^{p_{k-}} f^*_{k_l^-} \right) \otimes \left(\bigotimes_{l=1}^{p_{k+}} g_{k_l^+} \right) \right),$$

where $f_{k_l^-}$ and $g_{k_l^+}$ are elements of $\mathbf{f}$ and $\mathbf{g}$, respectively. For the definition to cover all cases, we also allow:

- *Empty tensor products, equal to $\Bbbk$;*
- *The empty direct sum, equal to 0.*

The normal form is defined up to coherence isomorphism.

Proof. To reach the normal form, we apply the following natural isomorphisms as rewriting rules:

1. Compactness to replace internal homs with tensor products throughout the functor expression:
$$\mathrm{com}^{-1} : \lambda f^{\mathrm{op}} g. f \multimap g \xrightarrow{\simeq} \lambda f^{\mathrm{op}} g. f^* \otimes g.$$
2. The standard form of compactness for $\otimes$ and the adjunction property for $\oplus$ to carry the dualization inside the expressions:
$$\lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}}.(f_1 \otimes f_2)^* \xrightarrow{\simeq} \lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}}. f_1^* \otimes f_2^*, \quad \lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}}.(f_1 \oplus f_2)^* \xrightarrow{\simeq} \lambda f_1^{\mathrm{op}} f_2^{\mathrm{op}}. f_1^* \oplus f_2^*.$$
3. Distributivity of the tensor product over the direct sum:
$$\mathrm{inv}^{-1} : \lambda g. g^{**} \xrightarrow{\simeq} \lambda g. g.$$
4. Distribution of tensor over direct sum:
$$\mathrm{dst} : \lambda g_1 g_2 g_3. g_1 \otimes (g_2 \oplus g_3) \xrightarrow{\simeq} \lambda g_1 g_2 g_3.(g_1 \otimes g_2) \oplus (g_1 \otimes g_3).$$
5. Isomorphisms for constant functors:
   - $\lambda g. g \otimes \mathrm{id}_0 \xrightarrow{\simeq} \lambda g. \mathrm{id}_0$;

- $\lambda g.g \otimes \mathrm{id}_{\Bbbk} \xrightarrow{\simeq} \lambda g.g$;
- $\lambda g.g \oplus \mathrm{id}_0 \xrightarrow{\simeq} \lambda g.g$.

Here we use the fact that in lambda expressions, constant functors at 0 and $\Bbbk$ instantiate the bound variables as id_0 or $\mathrm{id}_{\Bbbk}$, respectively.

We apply these steps until no further transformations are possible, arriving at the required normal form. We do not specify the order of transitions, since we intend for it to be determined by the logic later, and the confluence of the process modulo coherence isomorphisms at the object level is straightforward. $\square$

Corollary 3.2 (Structure of functors on $\mathcal{GF}d\mathcal{V}ect_{\Bbbk}$)**.** *Let*

$$F : \mathcal{GF}d\mathcal{V}ect^n_{\Bbbk} \to \mathcal{GF}d\mathcal{V}ect_{\Bbbk}$$

be a functor from the collection specified above, restricted to $\mathcal{GF}d\mathcal{V}ect_{\Bbbk}$. Then F is isomorphic to a normal form

$$\mathrm{NF}(F) := \lambda \mathbf{f}. \bigoplus_{k=1}^{m} \left(\bigotimes_{l=1}^{p_{k-}} f^*_{k^-_l} \right) \otimes \left(\bigotimes_{l=1}^{p_{k+}} f_{k^+_l} \right),$$

where $f_{k^-_l}$, $f_{k^+_l}$ are the elements of $\mathbf{f}$.

Proof. Follows from the restriction of Lem. 3.1 to $\mathcal{GF}d\mathcal{V}ect_{\Bbbk}$ and formal identification of variables in the lambda expression. All diagonalizations are well-defined in $\mathcal{GF}d\mathcal{V}ect_{\Bbbk}$ and can be expressed, among other ways, by precomposing with a combined diagonal functor. $\square$

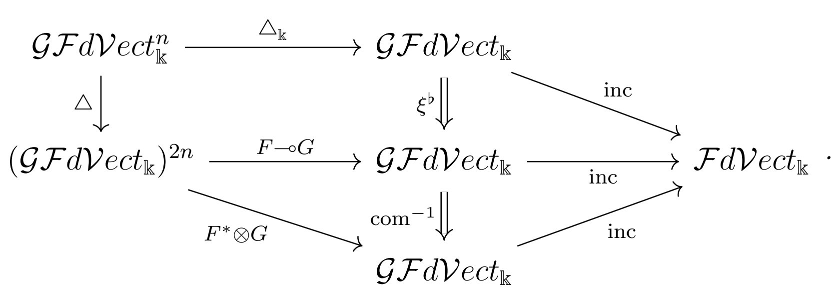
Lemma 3.3 (Structure of isonatural transformations on $\mathcal{GF}d\mathcal{V}ect_{\Bbbk}$)**.** *Every finite-dimensional isonatural transformation $\xi : F \to G$ is equivalently given by a transformation*

$$\mathrm{com}^{-1} \circ \xi^\flat = (\triangle_{\Bbbk} \xrightarrow{\xi^\flat} (F \multimap G) \circ \triangle \xrightarrow{\mathrm{com}^{-1}} (F^* \otimes G) \circ \triangle),$$

where $\triangle$ denotes the functor $\lambda \mathbf{f}.(\mathbf{f}, \mathbf{f})$ for $\mathbf{f} : \mathrm{arr}(\mathcal{GF}d\mathcal{V}ect^n_{\Bbbk})$, $\triangle_{\Bbbk}$ denotes $\lambda \mathbf{f}.\mathrm{id}_{\Bbbk}$, and $\xi^\flat$ is defined as

$$\xi^\flat := \lambda r.r \cdot \xi, \quad \xi^\flat_{\mathbf{V}} := \lambda r.r \cdot \xi_{\mathbf{V}} : \Bbbk \to F(\mathbf{V}) \multimap G(\mathbf{V}).$$

The explicit type is given by the following diagram:



Proof. The naturality of ξ with respect to a tuple $\mathbf{f}$ of isomorphisms is expressed by the commutativity of the diagram

$$\begin{array}{ccccc} \mathbf{V} & \qquad & F(\mathbf{V}) & \xrightarrow{\xi_{\mathbf{V}}} & G(\mathbf{V}) \\ {\scriptstyle \mathbf{f}}\big\downarrow & & {\scriptstyle F\mathbf{f}}\big\downarrow & & \big\downarrow{\scriptstyle G\mathbf{f}} \\ \mathbf{W} & & F(\mathbf{W}) & \xrightarrow[\xi_{\mathbf{W}}]{} & G(\mathbf{W}) \end{array},$$

equivalently reformulated, via the tensor-hom map, as

$$\begin{array}{ccccccc} \mathbf{V} & \qquad & \Bbbk & \xrightarrow{\lambda r.r\cdot\xi_{\mathbf{V}}} & F(\mathbf{V}) \multimap G(\mathbf{V}) & \xrightarrow{\mathrm{com}^{-1}_{F(\mathbf{V}),G(\mathbf{V})}} & (F(\mathbf{V}))^* \otimes G(\mathbf{V}) \\ {\scriptstyle \mathbf{f}}\big\downarrow & & {\scriptstyle \mathrm{id}_{\Bbbk}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda\varphi.G\mathbf{f}\circ\varphi\circ(F\mathbf{f})^{-1}} & & \big\downarrow{\scriptstyle (F\mathbf{f})^*\otimes G\mathbf{f}} \\ \mathbf{W} & & \Bbbk & \xrightarrow[\lambda r.r\cdot\xi_{\mathbf{W}}]{} & F(\mathbf{W}) \multimap G(\mathbf{W}) & \xrightarrow[\mathrm{com}^{-1}_{F(\mathbf{W}),G(\mathbf{W})}]{} & (F(\mathbf{W}))^* \otimes G(\mathbf{W}) \end{array}.$$

The equivalence holds since the internal hom is covariantly defined on $\mathcal{GFdV}ect_{\Bbbk}$ and the tensor-hom map is invertible in the finite-dimensional case. $\square$

Definition 3.4 (Balanced tensor product of morphisms)**.** We call a non-zero tensor product of morphisms in $\mathcal{GV}ect_{\Bbbk}$ *balanced* if each morphism appears exactly once together with its dual. The normal form of a balanced tensor product is

$$\left(\bigotimes_{l=1}^{p} f_l^*\right) \otimes \left(\bigotimes_{l=1}^{p} f_l\right), \quad f_k \neq 0. \tag{3.1}$$

A similar definition applies to functors, obtained as lambda-abstractions of (3.1).

Lemma 3.5. *There exists a nonzero isonatural transformation on $\mathcal{FdV}ect_{\Bbbk}$ over an infinite field $\Bbbk$ from $\triangle_{\Bbbk}$ to a tensor product if and only if this tensor product is balanced.*

Proof. If the tensor product is balanced, the isonatural transformation to its normal form is given by coevaluation and compactness:

$$\lambda\mathbf{f}.\mathrm{id}_{\Bbbk} \xrightarrow{\mathrm{coev}} \lambda\mathbf{f}.\left(\bigotimes_{l=1}^{p} f_l\right)^* \otimes \left(\bigotimes_{l=1}^{p} f_l\right) \xrightarrow{\cong} \lambda\mathbf{f}.\left(\bigotimes_{l=1}^{p} f_l^*\right) \otimes \left(\bigotimes_{l=1}^{p} f_l\right).$$

For the converse, the normal form of an unbalanced tensor product takes the form

$$\lambda g\mathbf{f}.(g^*)^{\otimes n} \otimes g^{\otimes p} \otimes H(\mathbf{f}),$$

where $H(\mathbf{f})$ denotes the remaining factors, $n \neq p$, and g does not appear among the elements of $\mathbf{f}$. Taking scalar multiplications as isomorphisms, the naturality condition takes the form

$$\begin{array}{ccc} \Bbbk & \xrightarrow{\phi_{V,U}} & (V^*)^{\otimes n} \otimes V^{\otimes p} \otimes H(\mathbf{U}) \\ {\scriptstyle \mathrm{id}_{\Bbbk}}\big\downarrow & & \big\downarrow{\scriptstyle \lambda w.w\cdot r^{p-n}\cdot s} \\ \Bbbk & \xrightarrow[\phi_{V,U}]{} & (V^*)^{\otimes n} \otimes V^{\otimes p} \otimes H(\mathbf{U}) \end{array}.$$

Here r and s are elements of $\Bbbk \setminus \{0,1\}$, and s is obtained as a product of powers of scalars chosen arbitrarily on $\mathbf{U}$. The diagram commutes only if $r^{p-n} \cdot s = 1$, and since $\Bbbk$ is infinite, one can always choose r for a given s such that this equality fails. □

Theorem 3.6 (Existence of non-zero finite-dimensional isonatural transformations)**.** *There exists a nonzero isonatural transformation $\xi : F \to G$ on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$ over an infinite field $\Bbbk$ if and only if at least one of the summands in the normal form of $\lambda\mathbf{f}.F^*(\mathbf{f}) \otimes G(\mathbf{f})$ is balanced.*

Proof. By Lem. 3.3 and Cor. 3.2, we can equivalently consider isonatural transformations of form

$$\tilde{\xi} : \triangle_{\Bbbk} \to \mathrm{NF}(\lambda\mathbf{f}.F^*(\mathbf{f}) \otimes G(\mathbf{f})).$$

A map to a direct sum corresponds to a collection of maps to its components. Each component is a tensor product, so at least one must be balanced to yield a nonzero isonatural transformation; conversely, one balanced summand suffices. Since $\Bbbk$ is infinite, new scalar multiple isomorphisms can always be introduced when needed, regardless of the biproduct structure. □

4 Vector Space Logic

4.1 Sequent calculus for VSL

We introduce the sequent calculus for the multiplicative and additive fragments of VSL, the logic of the category of vector spaces. It is obtained by modifying the corresponding fragments of the sequent calculus for intuitionistic linear logic to reflect the structure of $\mathcal{V}ect_{\Bbbk}$. We make the formal identifications

$$1 = \bot, \quad 0 = \top, \quad \& = \oplus.$$

Inference rules are written using 1, 0 and $\oplus$; the notation $(-)^{\perp}$ for negation is retained. The $\oplus$ in this setting is the unified additive connective, which we call *direct sum*, as it combines the rules for both additive conjunction and disjunction. We also drop the rule

$$\overline{\Gamma \vdash 1},$$

since there is no isonatural family of linear maps $\xi_V : V \to \Bbbk$ on $\mathcal{V}ect_{\Bbbk}$. This brings the calculus to the following intermediate form, which inference rules will be referred to as the *main rules*.

1. Structural rules:

$$\frac{}{A \vdash A}\ \mathit{Identity} \quad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \mathit{Cut} \quad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \mathit{Exchange}$$

2. Units:

$$\frac{}{\vdash 1}\, 1_R \quad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\, 1_L \quad \frac{}{\Gamma, 0 \vdash A}\, 0_L$$

3. Multiplicative conjunction:

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\, \otimes_R \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\, \otimes_L$$

4. Linear implication:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\, \multimap_R \quad \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\, \multimap_L$$

5. Direct sum:

- Disjunction part:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\, \oplus^{(+)}_{R-1} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\, \oplus^{(+)}_{R-2} \quad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\, \oplus^{(+)}_{L}$$

- Conjunction part:

$$\frac{\Gamma, A \vdash C}{\Gamma, A \oplus B \vdash C}\, \oplus^{(\times)}_{L-1} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\, \oplus^{(\times)}_{L-2} \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \oplus B}\, \oplus^{(\times)}_{R}$$

Remark 4.1 (Trivial proofs)**.** We immediately encounter an important structural feature of this calculus. Every sequent of the form $A \vdash B$ is derivable via the *trivial proof*:

$$\frac{\dfrac{\dfrac{}{A \vdash A}\, Id}{A \vdash A \oplus B}\, \oplus^{(+)}_{R-1} \quad \dfrac{\dfrac{}{B \vdash B}\, Id}{A \oplus B \vdash B}\, \oplus^{(\times)}_{L-2}}{A \vdash B}\, Cut \; .$$

The corresponding class of proofs is interpreted on $\mathcal{V}ect_{\Bbbk}$ by the zero isonatural transformation:

$$(A \xrightarrow{i_1} A \oplus B \xrightarrow{\pi_2} B) = 0.$$

Thus, the logic reflects the categorical situation, where there is always a zero map between any pair of vector spaces, while non-zero isonatural maps correspond to non-trivial proofs. For this reason, and since the sequent calculus allows us to distinguish proofs, completeness will be formulated relative to derivability without trivial proofs, giving it meaningful content.

For consistency, we also call the 0_L and the identity $\overline{0 \vdash 0}$ trivial proofs, imposing an implicit partial equational theory on proofs. Informally, a proof is trivial if it yields the zero transformation on $\mathcal{V}ect_{\Bbbk}$.

Next, we introduce a set of *additional rules* aimed at matching the structure of $\mathcal{V}ect_{\Bbbk}$ as closely as possible. To account for the distinction between finite- and infinite-dimensional vector spaces, which corresponds to the distinction between the classical and intuitionistic fragments of VSL, we partition formulas into *classical* and *intuitionistic* types according to the following definition.

Definition 4.2 (Classical and intuitionistic formulas)**.** We inductively annotate each logical formula as either *classical* or *intuitionistic*, reflecting the dimension of the corresponding vector space:

- The units 0 and 1 are classical;
- There are two sorts of propositional variables: classical $X^{(C)}$ and intuitionistic $X^{(I)}$;
- For a binary connective H, the formula $H(A, B)$ is classical if
  - both A and B are classical, or
  - H is $\otimes$ or $\multimap$, and at least one of A, B is 0.

We omit the annotation when classicality is clear from context.

The main rules hold regardless of the classicality of their constituent formulas. For classical formulas, there is an additional rule representing the involutivity of negation:

$$\frac{\Gamma \vdash (A^{(C)})^{\perp\perp}}{\Gamma \vdash A^{(C)}}\ \textit{Rigidity}$$

Similarly, a rule representing compactness applies whenever a linear implication contains at least one classical component: for $(P_1, P_2) \neq (I, I)$,

$$\frac{\Gamma \vdash A^{(P_1)} \multimap B^{(P_2)}}{\Gamma \vdash (A^{(P_1)})^{\perp} \otimes B^{(P_2)}}\ \textit{Compactness}$$

We now collect the above into formal definitions.

Definition 4.3 (VSL)**.** *Vector Space Logic* (VSL) is defined as a logic containing both intuitionistic and classical formulas, whose sequent calculus consists of both the main and additional rules introduced above.

Definition 4.4 (CVSL, IVSL)**.** Classical Vector Space Logic (CVSL) is the fragment of VSL containing only classical formulas. Intuitionistic Vector Space Logic (IVSL) is the fragment of VSL containing only intuitionistic formulas.

4.2 Structure of CVSL

We describe the structure of CVSL with a view to proving the completeness of its interpretation. We begin by explicitly deriving the sequents used in the completeness proof of CVSL.

Lemma 4.5 (Compactness and Rigidity in CVSL)**.** *In CVSL, the sequents*

$$A^{\perp} \otimes B \dashv\vdash A \multimap B \quad \textit{and} \quad A \dashv\vdash A^{\perp\perp}$$

are derivable without the use of trivial proofs.

Proof.

1. The derivability of $A^{\perp} \otimes B \vdash A \multimap B$ and $A \vdash A^{\perp\perp}$ is standard for linear logic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathit{Id} \qquad \dfrac{\dfrac{}{B \vdash B}\,\mathit{Id}}{1, B \vdash B}\,1_L}{A, A^{\perp}, B \vdash B}\,\multimap_L}{A^{\perp}, B, A \vdash B}\,\mathit{Exc}}{A^{\perp} \otimes B, A \vdash B}\,\otimes_L}{A^{\perp} \otimes B \vdash A \multimap B}\,\multimap_R \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathit{Id} \qquad \dfrac{}{1 \vdash 1}\,\mathit{Id}}{A, A^{\perp} \vdash 1}\,\multimap_L}{A \vdash A^{\perp\perp}}\,\multimap_R$$

2. The sequents $A \multimap B \vdash A^{\perp} \otimes B$ and $A^{\perp\perp} \vdash A$ are derivable in CVSL by applying the *Compactness* and *Rigidity* rules, respectively:

$$\dfrac{\dfrac{}{A \multimap B \vdash A \multimap B}\,\mathit{Id}}{A \multimap B \vdash A^{\perp} \otimes B}\,\mathit{Comp} \qquad \dfrac{\dfrac{}{A^{\perp\perp} \vdash A^{\perp\perp}}\,\mathit{Id}}{A^{\perp\perp} \vdash A}\,\mathit{Rig}$$

□

Lemma 4.6 (De Morgan laws in CVSL)**.** *In CVSL, the sequents*

$$A^{\perp} \otimes B^{\perp} \dashv\vdash (A \otimes B)^{\perp} \quad \textit{and} \quad A^{\perp} \oplus B^{\perp} \dashv\vdash (A \oplus B)^{\perp}$$

are derivable without the use of trivial proofs.

Proof.

1. The derivability of $A^{\perp} \otimes B^{\perp} \vdash (A \otimes B)^{\perp}$ is standard in linear logic, while the converse follows from the *Compactness* rule. The proofs partially recall those in Lemma 4.5:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathit{Id} \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,\mathit{Id} \qquad \dfrac{}{1 \vdash 1}\,\mathit{Id}}{B, B^{\perp} \vdash 1}\,\multimap_L}{1, B, B^{\perp} \vdash 1}\,1_L}{A, A^{\perp}, B, B^{\perp} \vdash 1}\,\multimap_L}{A^{\perp}, B^{\perp}, A, B \vdash 1}\,\mathit{Exc}}{\dfrac{A^{\perp} \otimes B^{\perp}, A \otimes B \vdash 1}{A^{\perp} \otimes B^{\perp} \vdash (A \otimes B)^{\perp}}\,\multimap_R}\,\otimes_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathit{Id} \qquad \dfrac{}{B \vdash B}\,\mathit{Id}}{A, B \vdash A \otimes B}\,\otimes_R \qquad \dfrac{}{1 \vdash 1}\,\mathit{Id}}{A, B, (A \otimes B)^{\perp} \vdash 1}\,\multimap_L}{(A \otimes B)^{\perp}, A, B \vdash 1}\,\mathit{Exc}}{\dfrac{(A \otimes B)^{\perp} \vdash A \multimap B^{\perp}}{(A \otimes B)^{\perp} \vdash A^{\perp} \otimes B^{\perp}}\,\mathit{Comp}}\,\multimap_R$$

2. The derivability of $A^\perp \oplus B^\perp \dashv\vdash (A \oplus B)^\perp$ follows from the unification of additive conjunction and disjunction in VSL. It holds for both intuitionistic and classical cases:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,Id \quad \dfrac{}{1 \vdash 1}\,Id}{A^\perp, A \vdash 1}\,\multimap_L}{A^\perp \oplus B^\perp, A \vdash 1}\,\oplus^{\times}_{L-1} \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,Id \quad \dfrac{}{1 \vdash 1}\,Id}{B^\perp, B \vdash 1}\,\multimap_L}{A^\perp \oplus B^\perp, B \vdash 1}\,\oplus^{(\times)}_{L-2}}{A^\perp \oplus B^\perp, A \oplus B \vdash 1}\,\oplus^{(+)}_{L}}{A^\perp \oplus B^\perp \vdash (A \oplus B)^\perp}\,\multimap_R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,Id}{A \vdash A \oplus B}\,\oplus^{(+)}_{R-1} \quad \dfrac{}{1 \vdash 1}\,Id}{A, (A \oplus B)^\perp \vdash 1}\,\multimap_R}{(A \oplus B)^\perp, A \vdash 1}\,Exc}{(A \oplus B)^\perp \vdash A^\perp}\,\multimap_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,Id}{B \vdash A \oplus B}\,\oplus^{(+)}_{R-2} \quad \dfrac{}{1 \vdash 1}\,Id}{B, (A \oplus B)^\perp \vdash 1}\,\multimap_R}{(A \oplus B)^\perp, B \vdash 1}\,Exc}{(A \oplus B)^\perp \vdash B^\perp}\,\multimap_R}{(A \oplus B)^\perp \vdash A^\perp \oplus B^\perp}\,\oplus^{(\times)}_{R}$$

□

Lemma 4.7 (Distributivity in VSL)**.** *In VSL, the pair of sequents*

$$A \otimes (B \oplus C) \dashv\vdash (A \otimes B) \oplus (A \otimes C)$$

is derivable without the use of trivial proofs.

Proof. The derivation is standard for the additive conjunction of linear logic, hence it suffices to use only the disjunction rules for $\oplus$ in the proof trees:

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,Id \quad \dfrac{}{B \vdash B}\,Id}{A, B \vdash A \otimes B}\,\otimes_R}{A, B \vdash (A \otimes B) \oplus (A \otimes C)}\,\oplus^{(+)}_{R-1} \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,Id \quad \dfrac{}{C \vdash C}\,Id}{A, C \vdash A \otimes C}\,\otimes_R}{A, C \vdash (A \otimes B) \oplus (A \otimes C)}\,\oplus^{(+)}_{R-2}}{A, B \oplus C \vdash (A \otimes B) \oplus (A \otimes C)}\,\oplus^{(+)}_{L}}{A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)}\,\otimes_L$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,Id \quad \dfrac{\dfrac{}{B \vdash B}\,Id}{B \vdash B \oplus C}\,\oplus^{(+)}_{R-1}}{A, B \vdash A \otimes (B \oplus C)}\,\otimes_R}{A \otimes B \vdash A \otimes (B \oplus C)}\,\otimes_L \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,Id \quad \dfrac{\dfrac{}{C \vdash C}\,Id}{C \vdash B \oplus C}\,\oplus^{(+)}_{R-2}}{A, C \vdash A \otimes (B \oplus C)}\,\otimes_R}{A \otimes C \vdash A \otimes (B \oplus C)}\,\otimes_L}{(A \otimes B) \oplus (A \otimes C) \vdash A \otimes (B \oplus C)}\,\oplus^{(+)}_{L}$$

□

Remark 4.8. For a coproduct $+$ on an abstract monoidal category, we only get the direction $(A \otimes B) + (A \otimes C) \to A \otimes (B + C)$ for free. The invertibility of this arrow is an independent requirement. This happens because the rule $\oplus^{(+)}_L$ is stronger than the universal property of coproduct, as it also includes the context.

Lemma 4.9 (Equivalent form of classical sequents)**.** *Every sequent in CVSL is equivalent to a normal form*

$$\vdash \bigoplus_{l=1}^{m}\left(\left(\bigotimes_{k=1}^{p_{l,-}} X_{k_l^-}^{\perp}\right)\otimes\left(\bigotimes_{k=1}^{p_{l,+}} X_{k_l^+}\right)\right),$$

where $X_{k_l^-}, X_{k_l^+}$ are propositional variables. For the definition to cover all cases, we also allow:

- *Empty tensor products, equal to* 1*;*
- *The empty direct sum, equal to* 0*.*

Proof. Given the rules $\otimes_L$ and $\multimap_R$, a sequent $A_1,\ldots,A_n \vdash A$ is equivalent to the form

$$\vdash A_1 \otimes \cdots \otimes A_n \multimap A.$$

By applying compactness and the De Morgan laws for $\otimes$, we obtain

$$(A_1 \otimes \cdots \otimes A_n) \multimap A \dashv\vdash (A_1 \otimes \cdots \otimes A_n)^{\perp} \otimes A \dashv\vdash A_1^{\perp} \otimes \cdots \otimes A_n^{\perp} \otimes A.$$

Next, linear implication is unfolded, negation is pushed inside formulas, and double duals are eliminated:

- If $F = B^{\perp}$, then $F^{\perp} \dashv\vdash B$;
- If $F = B \otimes C$, then $F^{\perp} \dashv\vdash B^{\perp} \otimes C^{\perp}$;
- If $F = B \oplus C$, then $F^{\perp} \dashv\vdash B^{\perp} \oplus C^{\perp}$;
- If $F = B \multimap C$, then $F^{\perp} \dashv\vdash (B^{\perp} \otimes C)^{\perp} \dashv\vdash B \otimes C^{\perp}$.

Then, the brackets are expanded using distributivity:

$$B \otimes (C \oplus D) \dashv\vdash (B \otimes C) \oplus (B \otimes D),$$

similarly with B on the right. We arrive at the direct sum of tensor products. Finally, the units are handled as follows:

- $0^{\perp} \dashv\vdash 0$, $1^{\perp} \dashv\vdash 1$;
- If $F = A \otimes 1$ or $F = 1 \otimes A$, then $F \dashv\vdash A$;
- If $F = A \otimes 0$ or $F = 0 \otimes A$, then $F \dashv\vdash 0$;
- If $F = A \oplus 0$ or $F = 0 \oplus A$, then $F \dashv\vdash A$.

This results in a sequent of the required normal form, whose derivability is equivalent to that of the initial sequent. ∎

Definition 4.10 (Balanced tensor product of atomic formulas)**.** A tensor product of atomic formulas is called *balanced* if it does not include 0 and each propositional variable appears exactly once together with its negation. The normal form of a balanced tensor product from Lem. 4.9 is

$$\left(\bigotimes_{l=1}^{p} X_l^{\perp}\right) \otimes \left(\bigotimes_{l=1}^{p} X_l\right).$$

This form is interderivable with any other balanced tensor product by applying the $\otimes$, 1 and *Exchange* rules.

Lemma 4.11 (Derivability of tensor product)**.** *Tensor product of atomic formulas is derivable without the use of trivial proofs if and only if it is balanced.*

Proof. Let the tensor product be balanced. Then its normal form is derivable via the following proof tree (with empty tensor product being a trivial case):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X_1 \vdash X_1}\,Id}{\vdash X_1 \multimap X_1}\,\multimap_R}{\vdash X_1^{\perp} \otimes X_1}\,Comp \quad \cdots \quad \dfrac{\dfrac{\dfrac{}{X_p \vdash X_p}\,Id}{\vdash X_p \multimap X_p}\,\multimap_R}{\vdash X_p^{\perp} \otimes X_p}\,Comp}{\vdash \left(\bigotimes_{l=1}^{p} X_l^{\perp}\right) \otimes \left(\bigotimes_{l=1}^{p} X_l\right)}\,\otimes_R,\ \text{Exchange}$$

Conversely, let the tensor product be unbalanced. Its normal form is either equal to 0, or it contains at least one atom without a dual of the opposite variance. In the first case, $\vdash 0$ cannot be derived non-trivially; in the second case, any derivation leaves a subtree of the form

$$\dfrac{\vdash (X^{\perp})^{\otimes m} \otimes X^{\otimes n}}{\cdots}$$

with $n \neq m$, that cannot be completed using the rules for $\otimes$ and $\multimap$. □

Theorem 4.12 (Derivability in CVSL)**.** *In CVSL, a sequent is derivable without the use of trivial proofs if and only if at least one of the summands in its normal form is balanced.*

Proof. Consider a sequent $\Gamma \vdash A$ with normal form

$$\mathrm{NF}(\Gamma \vdash A) \quad = \quad \vdash F_1 \oplus \cdots \oplus F_m.$$

Let F_k be balanced. Then the normal form can be derived using the $\oplus_R^{(+)}$ rule and Lem. 4.11:

$$\dfrac{\vdash F_k}{\vdash F_1 \oplus \cdots \oplus F_m}\,\oplus_R^{(+)}$$

Conversely, let all tensor products inside the normal form be unbalanced. It follows from the rules of VSL that a direct sum is derivable if and only if at least one of its summands is derivable. By Lem. 4.11, none of the summands are derivable; hence the sequent is not derivable. □

Remark 4.13 (Simultaneous reduction to normal form)**.** Consider a sequent $A_1, \ldots, A_n \vdash A$ and the corresponding isonatural transformation $A_1 \otimes \cdots \otimes A_n \to A$. The two can be reduced to their normal forms simultaneously, with the sequent class at each step interpreted by the corresponding isonatural transformation. This follows from the parallel structure of the normalization procedures on the logical and categorical sides, and can be verified formally by induction.

4.3 Interpretation of VSL on $\mathcal{V}ect_{\mathbb{k}}$

Following Section 2.2, we introduce the 2-category $\mathcal{S}yn^{\uparrow}$ for VSL. The generator objects of the base category $\mathcal{S}yn$ consist of the units 0 and 1, together with two countable sets of propositional variables annotated as classical or intuitionistic. We define the subcategories $\mathcal{S}yn_{\mathrm{Cl}}$ and $\mathcal{S}yn_{\mathrm{Int}}$ as the full subcategories on objects of classical and intuitionistic type, respectively. This allows isonatural transformations with categorical domains from the cartesian closure of $\{\mathcal{G}\mathcal{S}yn_{\mathrm{Cl}}, \mathcal{G}\mathcal{S}yn_{\mathrm{Int}}\}$.

The 1-morphisms of $\mathcal{S}yn^{\uparrow}$ are generated by the constant functors $\triangle_0$, $\triangle_1$, the bifunctors $\otimes$, $\multimap$, $\oplus$, and the comma. The 2-morphisms are generated by the categorifications of the inference rule classes of the sequent calculus presented below. We do not impose an equational theory on proofs beyond requiring 0 to be the zero object and identifying isonatural transformations corresponding to trivial proofs with the zero map.

The 2-category $\mathcal{V}ect_{\mathbb{k}}^{\uparrow}$ is defined similarly. It is built on top of the category $\mathcal{V}ect_{\mathbb{k}}$ over an infinite field, considered together with the subcategories of finite-dimensional ($\mathcal{F}d\mathcal{V}ect_{\mathbb{k}}$) and infinite-dimensional ($\mathcal{I}nf\mathcal{V}ect_{\mathbb{k}}$) vector spaces. The 1-morphisms of $\mathcal{V}ect_{\mathbb{k}}^{\uparrow}$ are generated by the functors $\triangle_0$, $\triangle_{\mathbb{k}}$, $\otimes$, $\multimap$, $\oplus$. In contrast with $\mathcal{S}yn^{\uparrow}$, the 2-morphisms are controlled directly by the algebraic structure of the theory of vector spaces.

Since our aim is to recover the logic from the category of vector spaces, we present the categorification of the sequent calculus directly on $\mathcal{V}ect_{\mathbb{k}}^{\uparrow}$. The corresponding 2-morphisms of $\mathcal{S}yn^{\uparrow}$ are defined analogously, with the only formal differences being that the comma bifunctor replaces $\otimes$ to connect formulas in sequents, and that the multiplicative unit is 1 rather than $\mathbb{k}$. We write the categorification of a sequent class $(\Gamma \vdash A)_\alpha$ with $\Gamma = A_1, \ldots, A_n$ in the form $\xi : \Gamma \to A$, explicitly given by the expression

$$\xi : \lambda \bigcup_{k=1}^{n+1} \mathbf{f}_k.(A_1(\mathbf{f}_1) \otimes \cdots \otimes A_n(\mathbf{f}_n)) \to \lambda \bigcup_{k=1}^{n+1} \mathbf{f}_k.A(\mathbf{f}_{n+1}),$$

where sharing of variables is controlled by the logic: lambda-abstraction is first applied to the sequent with fixed propositional variables, after which the bound variables are replaced by morphisms and units by identities on the corresponding objects. Some isonatural transformations will be indexed by their (co)domain functors; for instance, $\sigma_{A,B} : A \otimes B \to B \otimes A$ denotes the family of symmetry isomorphisms

$$\sigma_{A,B} : \lambda \mathbf{f} \cup \mathbf{g}.A(\mathbf{f}) \otimes B(\mathbf{g}) \xrightarrow{\cong} B(\mathbf{g}) \otimes A(\mathbf{f}).$$

1. Structural rules:

$\dfrac{}{A \vdash A}\ \mathit{Identity}$	$\dfrac{}{\mathrm{id}_A : A \to A}$
$\dfrac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \mathit{Cut}$	$\dfrac{\xi : \Gamma \to A \quad \eta : A \otimes \Delta \to B}{\eta \circ (\xi \otimes \mathrm{id}_\Delta) : \Gamma \otimes \Delta \to B}$
$\dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \mathit{Exchange}$	$\dfrac{\xi : \Gamma \otimes A \otimes B \otimes \Delta \to C}{\xi \circ (\mathrm{id}_\Gamma \otimes \sigma_{B,A} \otimes \mathrm{id}_\Delta) : \Gamma \otimes B \otimes A \otimes \Delta \to C}$

2. Units:

$$\frac{}{\vdash 1}\ 1_R \quad\Big|\quad \frac{}{\mathrm{id}_{\triangle_{\Bbbk}} : \triangle_{\Bbbk} \to \triangle_{\Bbbk}}$$

$$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\ 1_L \quad\Big|\quad \frac{\xi : \Gamma \to A}{\xi \circ \rho_\Gamma : \Gamma \otimes \triangle_{\Bbbk} \to A}$$

$$\frac{}{\Gamma, 0 \vdash A}\ 0_L \quad\Big|\quad \frac{}{a_\Gamma : \Gamma \otimes \triangle_0 \to A}$$

3. Multiplicative conjunction:

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes_R \quad\Big|\quad \frac{\xi : \Gamma \to A \quad \eta : \Delta \to B}{\xi \otimes \eta : \Gamma \otimes \Delta \to A \otimes B}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes_L \quad\Big|\quad \frac{\xi : (\Gamma \otimes A) \otimes B \to C}{\xi \circ \alpha_{\Gamma,A,B} : \Gamma \otimes (A \otimes B) \to C}$$

4. Linear implication:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap_R \quad\Big|\quad \frac{\xi : \Gamma \otimes A \to B}{\mathrm{cur}^{-1}_{\Gamma,A,B}(\xi) : \Gamma \to A \multimap B}$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\ \multimap_L \quad\Big|\quad \frac{\xi : \Gamma \to A \quad \eta : B \otimes \Delta \to C}{\phi(\xi,\eta) : \Gamma \otimes (A \multimap B) \otimes \Delta \to C}$$

Here $\phi(\xi, \eta)$ denotes the composition

$$\eta \circ (\mathrm{ev}_{A,B} \otimes \mathrm{id}_\Delta) \circ (\xi \otimes \mathrm{id}_{A \multimap B} \otimes \mathrm{id}_\Delta) : \Gamma \otimes (A \multimap B) \otimes \Delta \to C.$$

5. Direct sum:

- Disjunction part:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ \oplus^{(+)}_{R-1} \quad\Big|\quad \frac{\xi : \Gamma \to A}{i_1 \circ \xi : \Gamma \to A \oplus B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \oplus^{(+)}_{R-2} \quad\Big|\quad \frac{\xi : \Gamma \to B}{i_2 \circ \xi : \Gamma \to A \oplus B}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus^{(+)}_{L} \quad\Big|\quad \frac{\xi : \Gamma \otimes A \to C \quad \eta : \Gamma \otimes B \to C}{\xi \circ (\mathrm{id}_\Gamma \otimes \pi_1) + \eta \circ (\mathrm{id}_\Gamma \otimes \pi_2) : \Gamma \otimes (A \oplus B) \to C}$$

- Conjunction part:

$$\frac{\Gamma, A \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus^{(\times)}_{L-1} \quad\Big|\quad \frac{\xi : \Gamma \otimes A \to C}{\xi \circ (\mathrm{id}_\Gamma \otimes \pi_1) : \Gamma \otimes (A \oplus B) \to C}$$

$$\frac{\Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\ \oplus^{(\times)}_{L-2} \quad\Big|\quad \frac{\xi : \Gamma \otimes B \to C}{\xi \circ (\mathrm{id}_\Gamma \otimes \pi_2) : \Gamma \otimes (A \oplus B) \to C}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \oplus B}\ \oplus^{(\times)}_{R} \quad\Big|\quad \frac{\xi : \Gamma \to A \quad \eta : \Gamma \to B}{\langle \xi, \eta \rangle : \Gamma \to A \oplus B}$$

Here $\langle \xi, \eta \rangle$ is the universal arrow to biproduct, which can be written explicitly as

$$i_1 \circ \xi + i_2 \circ \eta : \Gamma \to A \oplus B.$$

6. Rigidity and Compactness:

$$\frac{\Gamma \vdash (A^{(C)})^{\perp\perp}}{\Gamma \vdash A^{(C)}}\ \mathit{Rigidity} \qquad \Bigg| \qquad \frac{\xi : \Gamma \to A^{**}}{\mathrm{inv}^{-1} \circ \xi : \Gamma \to A}$$

$$\frac{\Gamma \vdash A^{(P_1)} \multimap B^{(P_2)}}{\Gamma \vdash (A^{(P_1)})^{\perp} \otimes B^{(P_2)}}\ \mathit{Compactness} \qquad \Bigg| \qquad \frac{\xi : \Gamma \to A \multimap B}{\mathrm{com}^{-1} \circ \xi : \Gamma \to A^{*} \otimes B}$$

In these rules the domains of functors are determined by the types of formulas in the sequent calculus. For instance, the functor typing for one of the mixed cases of the compactness rule is

$$\mathcal{GFdVect}_{\Bbbk}^{n} \times \mathcal{GInfVect}_{\Bbbk}^{m} \xrightarrow{A \times B} \mathcal{GFdVect}_{\Bbbk} \times \mathcal{GInfVect}_{\Bbbk} \xrightarrow{\multimap} \mathcal{GVect}_{\Bbbk} \xrightarrow{\mathrm{inc}} \mathcal{Vect}_{\Bbbk}.$$

Following Section 2.2, we define the model as a 2-functor

$$M : \mathcal{Syn} \to \mathcal{Vect}_{\Bbbk},$$

which maps additive and multiplicative units to 0 and $\Bbbk$, respectively, classical variables to distinct finite-dimensional spaces of dimension ≥ 2, and intuitionistic variables to distinct infinite-dimensional vector spaces. With the specified collections of functors and isonatural maps, the model is sound by construction – M extends to the 2-functor

$$M^{\uparrow} : \mathcal{Syn}^{\uparrow} \to \mathcal{Vect}_{\Bbbk}^{\uparrow},$$

mapping generator 1- and 2-morphisms of $\mathcal{Syn}^{\uparrow}$ to the corresponding ones of $\mathcal{Vect}_{\Bbbk}^{\uparrow}$. We formally state soundness in the standard form, analogous to Hamano's.

Proposition 4.14 (Soundness of interpretation of VSL on $\mathcal{Vect}_{\Bbbk}$)**.** *For any VSL proof of $\Gamma \vdash A$, its interpretation is a component of an isonatural transformation $\Gamma \to A$ on $\mathcal{Vect}_{\Bbbk}$.*

Proof. Follows directly from the construction of the model. Each class of inference rules yields a well-defined pair of isonatural transformations on $\mathcal{Syn}^{\uparrow}$ and $\mathcal{Vect}_{\Bbbk}^{\uparrow}$, and $M^{\uparrow}$ maps one to the other. Interpretations of complex proofs are obtained by composing isonatural transformations. The components of the resulting transformations are the interpretations of individual proofs between sequents with fixed variables. □

The completeness of the interpretation of CVSL on finite dimensional spaces over an infinite field corresponds to the fullness of $M^{\uparrow}$, restricted to $\mathcal{FdVect}_{\Bbbk}^{\uparrow}$, on 2-cells. As it was mentioned in Section 4.1, every sequent of VSL can be derived using trivial rules. Hence, the standard completeness theorem would be trivial (as we required 0 to be the zero object on $\mathcal{Syn}$), and we formulate the extended version instead.

Theorem 4.15 (Completeness of interpretation of CVSL on $\mathcal{FdVect}_{\Bbbk}$ over an infinite field)**.** *If a sequent $\Gamma \vdash A$ of CVSL is not derivable without the use of trivial proofs, then there is no non-zero isonatural transformation $\Gamma \to A$ of $\mathcal{FdVect}_{\Bbbk}$ over an infinite field.*

Proof. By Th. 4.12, a sequent $\Gamma \vdash A$ is derivable without the use of trivial proofs if and only if at least one of the summands in its normal form is balanced:

$$\mathrm{NF}(\Gamma \vdash A) \quad = \quad \vdash F_1 \oplus \cdots \oplus F_m,$$

$$F_k = \left(\bigotimes_{l=1}^{p} X_l^{\perp} \right) \otimes \left(\bigotimes_{l=1}^{p} X_l \right).$$

By Th. 3.6, there exists a nonzero isonatural transformation $\xi : \Gamma \to A$ on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$ over an infinite field $\Bbbk$ if and only if at least one of the summands in the normal form of $\lambda \mathbf{f}.\Gamma^*(\mathbf{f}) \otimes A(\mathbf{f})$ is balanced:

$$\mathrm{NF}(\xi : \Gamma \to A) \ := \ \triangle_{\Bbbk} \to \mathrm{NF}(\lambda \mathbf{f}.\Gamma^*(\mathbf{f}) \otimes A(\mathbf{f})) \ = \ \triangle_{\Bbbk} \to \lambda \mathbf{f}.F_1(\mathbf{f}) \oplus \cdots \oplus F_m(\mathbf{f}),$$

$$F_k = \left(\bigotimes_{l=1}^{p} f_l^* \right) \otimes \left(\bigotimes_{l=1}^{p} f_l \right).$$

By Rem. 4.13, the sequent and the isonatural transformation can be reduced to their normal forms simultaneously, with the sequent class at each step interpreted by the corresponding isonatural transformation. The balancedness condition for the existence of a non-trivial proof and a nonzero isonatural transformation thus match, giving completeness.

□

5 Discussion

In the present work, we studied a question dual in spirit to one considered by Blute–Scott and Hamano. Using a 2-categorical generalization of Hamano's semantics, we constructed the multiplicative and additive fragments of the logic VSL from the category $\mathcal{V}ect_{\Bbbk}$ of vector spaces equipped with a specified collection of functors. These fragments turned out to be relatively close to the corresponding fragments of intuitionistic linear logic, arising from them by factorization and minor modification. The logic proved to be inherently multisorted, since vector spaces of different dimensions admit different collections of isonatural transformations. This necessitated introducing the classical (CVSL) and intuitionistic (IVSL) fragments, connected by a Compactness rule. Within the established framework, we carried out a basic analysis of VSL, proving soundness of the interpretation on $\mathcal{V}ect_{\Bbbk}$ and completeness of the interpretation of CVSL on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$.

The present paper only scratched the surface of what can be studied within the established framework. The most immediate direction for future work is a deeper analysis of VSL itself, with possible refinements to its formulation. We expect the interpretation of CVSL on $\mathcal{F}d\mathcal{V}ect_{\Bbbk}$ to be fully complete; however, the required machinery is more involved than in Blute–Scott and Hamano, where the basis of the interpretation of a sequent consisted of at most one component, and the question therefore warrants careful separate treatment. We also expect the tensor algebra functor to admit a non-standard comonad structure, which would allow an exponential fragment to be added to VSL based on a modified version of the exponential fragment of intuitionistic linear logic. The question of completeness for the full VSL appears considerably harder. In particular, it is unclear whether, in the infinite-dimensional case, there exists an isonatural transformation $\lambda f.f^{**} \to \lambda f.f$ corresponding to the sequent $X^{\perp\perp} \vdash X$, which is not derivable in ILL, since the elementary combinatorial methods employed by Hamano no longer apply in this setting.

As further directions for future work, we mention a more detailed study of the framework itself and its application to other categories. The most natural candidates are the category of sets $\mathcal{S}et$ and the category of abelian groups $\mathcal{A}b$, as two of the most well-studied examples. While $\mathcal{S}et$ is expected to yield a logic close to standard ILL, matching other known semantics, the category $\mathcal{A}b$ has a significantly more complex structure due to its nontrivial homology, and thus only partial results are expected to carry over from the vector space case.

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