

A quasistatic contact problem with friction in magneto-electro-elasticity

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Abstract

We consider a quasistatic contact problem between a magneto-electro-elastic material and a foundation. The contact is modelled with a normal compliance condition and a version of Coulomb's law of dry friction. We derive a variational formulation of the mechanical problem and, under a smallness assumption, we establish an existence theorem of a weak solution. The proof is based on the time-discretization method, the Banach fixed point theorem, arguments of lower semicontinuity, compactness and monotonicity.

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1 Introduction

The magneto-electro-elastic effects result from the coupling between elastic, electric and magnetic fields. This interaction leads to the appearance of a simultaneous induction of both the ferromagnetic moment and the electric polarization by the application of a mechanical stress, and conversely, the presence of an electric or a magnetic field leads to generate a mechanical stress. Moreover, coupling occurs between magnetic and electric subsystems.

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This enables the control of the electric polarization by a magnetic field (direct effect) and the manipulation of the ferromagnetic moment by an electric field (converse effect), see [2, 12, 14, 15].

Works relating to piezoelectromagnetic materials can be found, for example, in [6, 11, 17]. In [6], a static slip dependent frictional contact problem with a normal compliance and a unilateral constraint for a magneto-electro-elastic body was considered and its weak solvability was discussed within the framework of variational-hemivariational inequalities. In [11], uniqueness and reciprocity theorem for thermo-electro-magneto-elastic problem was discussed. In [17], the asymptotic homogenization method was applied in the study of a set of thermo-electro-magneto-elastic problems with rapidly oscillating coefficients, and comparisons with other previously published results were presented.

The novelty in this paper consists in dealing with a quasistatic contact problem for magneto-electro-elastic materials such that the contact is modelled with a normal compliance and a associated version of Coulomb's law of dry friction, which leads to a new and nonstandard mathematical problem. As in [7, 8, 9], our analysis is based on the time-discretization method. By using the backward Euler scheme, we construct a sequence of elliptic quasi-variational inequalities for which at each time step, under a smallness assumption, we prove the existence of a unique solution. Then, after obtaining the necessary estimates, we prove that the limit of a subsequence of the solutions of approximate problems is a solution of the continuous problem.

The rest of this paper is organized as follows. In Section 2 we present the notation and some preliminaries we shall use in our study. Section 3 and 4 are dedicated to describe the mechanical problem and derive its variational formulation. In Section 5 we establish the existence of a weak solution to the problem.

2 NOTATION AND PRELIMINARIES

We introduce some preliminary materials. For further details, we refer the reader to [5, 16]. Here and below, the indices i, j, k, l run between 1 and d ($d=2, 3$). We denote by \mathbb{S}^d the space of second order symmetric tensors on \mathbb{R}^d , and we define the inner products and the corresponding norms on \mathbb{R}^d

and \mathbb{S}^d by

$$w \cdot v = \sum_{i=1}^d w_i v_i, \quad |v| = \sqrt[2]{v \cdot v} \text{ for all } w = (w_i), v = (v_i) \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sum_{1 \leq i, j \leq d} \sigma_{ij} \tau_{ij}, \quad |\sigma| = \sqrt[2]{\sigma \cdot \sigma} \text{ for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ , let $x = (x_i) \in \Omega \cup \Gamma$ be the spatial variable and let ν be the unit outer normal on Γ . We denote by \mathbf{Q}_∞ the space of fourth-order tensor fields defined by

$$\mathbf{Q}_\infty = \{ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega) \},$$

which is a real Banach space with the norm

$$\| \mathcal{E} \|_{\mathbf{Q}_\infty} = \max_{1 \leq i, j, k, l \leq d} \| \mathcal{E}_{ijkl} \|_{L^\infty(\Omega)}.$$

We denote by $H = L^2(\Omega)$ the standard Lebesgue space for scalar functions, $\mathbf{H} = (L^2(\Omega))^d$ the standard Lebesgue space for vector functions, $H^1(\Omega)$ the standard Sobolev space for scalar functions, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$ the standard Sobolev space for vector functions, and we introduce the following spaces

$$\begin{aligned} \mathcal{Q} &= \{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in H \}, \\ \mathcal{H} &= \{ \sigma \in \mathcal{Q} \mid \text{Div} \sigma \in \mathbf{H} \}, \\ \mathcal{W} &= \{ D \in \mathbf{H} \mid \text{div} D \in H \}. \end{aligned}$$

Here and below, $\text{Div} : \mathcal{H} \rightarrow \mathbf{H}$ is the divergence operator for tensor functions

$$\text{Div} \sigma = (\varsigma_i), \quad \varsigma_i = \sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j} \text{ for all } \sigma \in \mathcal{H},$$

$\text{div} : \mathcal{W} \rightarrow H$ denotes the divergence operator for vector functions

$$\text{div} D = \sum_{i=1}^d \frac{\partial D_i}{\partial x_i} \text{ for all } D = (D_i) \in \mathcal{W},$$

$\varepsilon : \mathbf{H}^1(\Omega) \rightarrow \mathcal{Q}$ is the linearized strain tensor

$$\varepsilon(w) = (\varepsilon_{ij}(w)), \quad \varepsilon_{ij}(w) = \frac{1}{2} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \text{ for all } w \in \mathbf{H}^1(\Omega).$$

Note that \mathbf{H} , \mathcal{Q} , $\mathbf{H}^1(\Omega)$, \mathcal{H} and \mathcal{W} are real Hilbert spaces endowed with the following inner products

$$\begin{aligned} (w, v)_{\mathbf{H}} &= \int_{\Omega} w \cdot v dx \text{ for all } w, v \in \mathbf{H}, \\ (\sigma, \tau)_{\mathcal{Q}} &= \int_{\Omega} \sigma \cdot \tau dx \text{ for all } \sigma, \tau \in \mathcal{Q}, \\ (w, v)_{\mathbf{H}^1(\Omega)} &= (w, v)_{\mathbf{H}} + (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}} \text{ for all } w, v \in \mathbf{H}^1(\Omega), \\ (\sigma, \tau)_{\mathcal{H}} &= (\sigma, \tau)_{\mathcal{Q}} + (\text{Div}\sigma, \text{Div}\tau)_{\mathbf{H}} \text{ for all } \sigma, \tau \in \mathcal{H}, \\ (D, B)_{\mathcal{W}} &= (D, B)_{\mathbf{H}} + (\text{div } D, \text{div } B)_H \text{ for all } D, B \in \mathcal{W}. \end{aligned}$$

The associated norms on the spaces \mathbf{H} , \mathcal{Q} , $\mathbf{H}^1(\Omega)$, \mathcal{H} and \mathcal{W} are denoted by $\|\cdot\|_{\mathbf{H}}$, $\|\cdot\|_{\mathcal{Q}}$, $\|\cdot\|_{\mathbf{H}^1(\Omega)}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{W}}$. Let w be a vector field defined on Γ . Then, w_{ν} denotes the normal component of w and w_{τ} denotes the projection of w on the tangent plane of Γ , and they are given by

$$w_{\nu} = w \cdot \nu, \quad w_{\tau} = w - w_{\nu}\nu \text{ on } \Gamma.$$

Similarly, the normal component and the tangential components of a tensor σ are denoted by σ_{ν} and σ_{τ} , and they are given by

$$\sigma_{\nu} = \sigma\nu \cdot \nu, \quad \sigma_{\tau} = \sigma\nu - \sigma_{\nu}\nu \text{ on } \Gamma.$$

Let $\tilde{\gamma} : \mathbf{H}^1(\Omega) \rightarrow (L^2(\Gamma))^d$ be the trace map. We recall that $\tilde{\gamma}$ is a compact operator, i.e., for any bounded sequence $\{v_n\}$ in $\mathbf{H}^1(\Omega)$, there is a subsequence of $\{v_n\}$ which is convergent in $(L^2(\Gamma))^d$. For every element $v \in \mathbf{H}^1(\Omega)$ we use the notation v to denote the trace $\tilde{\gamma}(v)$ of v on Γ . For a regular (say C^1) tensor field σ the following Green formula holds

$$(\sigma, \varepsilon(w))_{\mathcal{Q}} + (\text{Div}\sigma, w)_{\mathbf{H}} = \int_{\Gamma} \sigma\nu \cdot w da \text{ for all } w \in \mathbf{H}^1(\Omega),$$

where da is the surface measure element. Also, when $D \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(D, \nabla\phi)_{\mathbf{H}} + (\text{div } D, \phi)_H = \int_{\Gamma} D \cdot \nu\phi da \text{ for all } \phi \in H^1(\Omega).$$

For every real Banach space $(X, \|\cdot\|_X)$, we denote by $C([0, T]; X)$ the space of continuous functions from $[0, T]$ to X , with norm

$$\|v\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|v(t)\|_X.$$

Also, we use the standard notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$, $p \in [1, \infty]$ and $k \geq 1$.

3 Problem statement

The physical setting is as follows. A deformable body occupies the reference configuration $\Omega \subset \mathbb{R}^{d=2,3}$ which is a bounded open set with a Lipschitz boundary Γ . The body is assumed to have a magneto-electro-elastic behaviour and the process is quasistatic in the time interval $[0, T]$. We consider three partitions of Γ into three open disjoint parts $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma_1^{el} \cup \Gamma_2^{el} \cup \Gamma_3 = \Gamma_1^{ma} \cup \Gamma_2^{ma} \cup \Gamma_3$ such that $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma_1^{el}) > 0$ and $\text{meas}(\Gamma_1^{ma}) > 0$. We assume that body forces of density f_0 , electric charges of density q_0 and electric current of density p_0 act in Ω . Homogeneous Dirichlet conditions of displacement, electrical potential and magnetic potential fields are considered on Γ_1 , Γ_1^{el} and Γ_1^{ma} , respectively. We assume that surface tractions of density f_2 , surface free electric charges of density q_2 and a surface magnetic induction of density p_2 are prescribed on Γ_2 , Γ_2^{el} and Γ_2^{ma} , respectively. The body is supposed to be in contact over Γ_3 with a foundation and, moreover, both a normal compliance and a version of Coulomb's law of dry friction are included. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 1 *Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \rightarrow \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}$, an electric displacement field $D : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a magnetic potential field $\psi : \Omega \times [0, T] \rightarrow \mathbb{R}$ and a magnetic induction field $B : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ such*

that

$$\sigma = \mathcal{A}\varepsilon(u) + \mathcal{E}^*\nabla\varphi + \mathcal{M}^*\nabla\psi \text{ in } \Omega \times (0, T), \quad (1)$$

$$D = \mathcal{E}\varepsilon(u) - \mathcal{C}\nabla\varphi - \mathcal{S}\nabla\psi \text{ in } \Omega \times (0, T), \quad (2)$$

$$B = \mathcal{M}\varepsilon(u) - \mathcal{S}\nabla\varphi - \mathcal{Z}\nabla\psi \text{ in } \Omega \times (0, T), \quad (3)$$

$$\text{Div}\sigma + f_0 = 0 \text{ in } \Omega \times (0, T), \quad (4)$$

$$\text{div}D - q_0 = 0 \text{ in } \Omega \times (0, T), \quad (5)$$

$$\text{div}B - p_0 = 0 \text{ in } \Omega \times (0, T), \quad (6)$$

$$u = 0 \text{ on } \Gamma_1 \times (0, T), \quad (7)$$

$$\sigma\nu = f_2 \text{ on } \Gamma_2 \times (0, T), \quad (8)$$

$$\varphi = 0 \text{ on } \Gamma_1^{el} \times (0, T), \quad (9)$$

$$D \cdot \nu = q_2 \text{ on } \Gamma_2^{el} \times (0, T), \quad (10)$$

$$\psi = 0 \text{ on } \Gamma_1^{ma} \times (0, T), \quad (11)$$

$$B \cdot \nu = p_2 \text{ on } \Gamma_2^{ma} \times (0, T), \quad (12)$$

$$D \cdot \nu = 0 \text{ on } \Gamma_3 \times (0, T), \quad (13)$$

$$B \cdot \nu = 0 \text{ on } \Gamma_3 \times (0, T), \quad (14)$$

$$-\sigma_\nu = p_\nu (u_\nu - g) \text{ on } \Gamma_3 \times (0, T), \quad (15)$$

$$\begin{cases} |\sigma_\tau| \leq p_\tau (u_\nu - g), \\ |\sigma_\tau| < p_\tau (u_\nu - g) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = p_\tau (u_\nu - g) \Rightarrow \exists \lambda \geq 0 \\ \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases} \text{ on } \Gamma_3 \times (0, T), \quad (16)$$

$$(u(0), \varphi(0), \psi(0)) = (u_0, \varphi_0, \psi_0) \text{ in } \Omega. \quad (17)$$

Equations (1)-(3) represent the magneto-electro-elastic constitutive law in which $\varepsilon(u)$ denotes the linearized strain tensor, $-\nabla\varphi$ is the electric field, $-\nabla\psi$ represents the magnetic field, \mathcal{A} stands for the elasticity operator, \mathcal{E} denotes the piezoelectric operator, \mathcal{E}^* is the transposed of \mathcal{E} , \mathcal{M} denotes the piezo-magnetic operator, \mathcal{M}^* represents the transposed of \mathcal{M} , \mathcal{C} stands for the electric permittivity operator, \mathcal{Z} is the magnetic permeability operator and \mathcal{S} denotes the electromagnetic coupling operator. Equations (4)-(6) represent the balance equations for the stress, electric displacement and magnetic induction fields, respectively. Equations (7)-(8) are the displacement-traction

boundary conditions where ν stands for the unit outward normal vector to Γ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (9)-(10) characterize the electric boundary conditions over $\Gamma_1^{el} \cup \Gamma_2^{el}$, whereas (11)-(12) represent the magnetic boundary conditions over $\Gamma_1^{ma} \cup \Gamma_2^{ma}$. In (13)-(14), the foundation is assumed to be non-conductive such that the normal components of the electrical displacement and the magnetic induction fields vanish over Γ_3 . The relation (15) is a general expression of the normal compliance condition, where g represents the gap in the reference configuration between Γ_3 and the foundation. The function p_ν is a nonnegative prescribed one which vanishes for negative arguments so that when $u_\nu - g < 0$ there is no contact and the normal pressure vanishes; and when the contact takes place then $u_\nu - g \geq 0$ is a measure of the penetration of the surface asperities into those of the foundation. A possible choice of the function p_ν is

$$p_\nu(r) = \begin{cases} 0 & \text{for all } r < 0, \\ c_\nu r & \text{for all } r \geq 0 \end{cases}$$

where c_ν is the surface stiffness coefficient. We note that an early attempt to study the quasistatic contact problem with the normal compliance model was done in [1, 10]. The relations (16) represent a version of Coulomb's law of dry friction where p_τ is a prescribed nonnegative function, the so-called friction bound. Given p_ν , we may choose the friction bound function p_τ such as

$$p_\tau(r) = \mu p_\nu(r),$$

where $\mu \geq 0$ is the coefficient of friction, for more details, (see, e.g., [16]). Here and below the dot above a variable represents its derivative with respect to the time variable. Finally, (17) is the initial condition.

4 Assumptions and variational formulation

In order to obtain a variational formulation of the mechanical problem (1)-(17), we introduce the space V defined by

$$V = \{w \in \mathbf{H}^1(\Omega) \mid w = 0 \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds

$$C_K \|w\|_{\mathbf{H}^1(\Omega)} \leq \|\varepsilon(w)\|_{\mathcal{Q}} \text{ for all } w \in V, \quad (18)$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1 . A proof of Korn's inequality can be found, for instance, in [13, page 79]. Over the space V , we consider the inner product and its associated norm

$$(w, v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}}, \quad \|w\|_V = \|\varepsilon(w)\|_{\mathcal{Q}} \text{ for all } w, v \in V. \quad (19)$$

It follows from (18) that $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ and $\|\cdot\|_V$ are equivalent norms on V . Moreover, by the Sobolev trace theorem, there exists a positive constant c_0 depending only on the domain Ω , Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \text{ for all } v \in V. \quad (20)$$

We introduce the functional spaces

$$W_{el} = \{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \Gamma_1^{el}\},$$

$$W_{ma} = \{\vartheta \in H^1(\Omega) \mid \vartheta = 0 \text{ on } \Gamma_1^{ma}\},$$

which are closed subspaces of $H^1(\Omega)$. Since $meas(\Gamma_1^{el}) > 0$ and $meas(\Gamma_1^{ma}) > 0$, the following Friedrichs–Poincaré's inequalities hold

$$C_1^{el} \|\phi\|_{H^1(\Omega)} \leq \|\nabla \phi\|_{\mathbf{H}} \text{ for all } \phi \in W_{el},$$

$$C_1^{ma} \|\vartheta\|_{H^1(\Omega)} \leq \|\nabla \vartheta\|_{\mathbf{H}} \text{ for all } \vartheta \in W_{ma},$$

here $C_1^{el} > 0$ is a positive constant depending only on Ω and Γ_1^{el} , whereas $C_1^{ma} > 0$ is a positive constant depending only on Ω and Γ_1^{ma} . Over the spaces W_{el} and W_{ma} , we consider the inner products given by

$$(\varphi, \phi)_{W_{el}} = (\nabla \varphi, \nabla \phi)_{\mathbf{H}} \text{ for all } \varphi, \phi \in W_{el}, \quad (21)$$

$$(\psi, \vartheta)_{W_{ma}} = (\nabla \psi, \nabla \vartheta)_{\mathbf{H}} \text{ for all } \psi, \vartheta \in W_{ma}, \quad (22)$$

where the associated norms are denoted by $\|\cdot\|_{W_{el}}$ and $\|\cdot\|_{W_{ma}}$. Next, we use the functional space

$$W = W_{el} \times W_{ma}.$$

W is a real Hilbert space endowed with the inner product given by

$$((\varphi, \psi), (\phi, \vartheta))_W = (\nabla \varphi, \nabla \phi)_{\mathbf{H}} + (\nabla \psi, \nabla \vartheta)_{\mathbf{H}} \text{ for all } (\varphi, \psi), (\phi, \vartheta) \in W.$$

We consider the following assumptions. We assume that $\mathcal{A} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \mathcal{A}\varepsilon \cdot \varepsilon \geq m_{\mathcal{A}} |\varepsilon|^2 \text{ a.e. } x \in \Omega, \text{ for all } \varepsilon \in \mathbb{S}^d; \\ \text{(ii) } \mathcal{A} \in \mathbf{Q}_{\infty}. \end{array} \right. \quad (23)$$

The electric permittivity operator $\mathcal{C} = (\mathcal{C}_{ij}) : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } (\mathcal{C}w)_i = \sum_{1 \leq j \leq d} \mathcal{C}_{ij} w_j \\ \text{a.e. } x \in \Omega, \text{ for all } w = (w_j) \in \mathbb{R}^d; \\ \text{(ii) } \mathcal{C}_{ij} = \mathcal{C}_{ji} \in L^{\infty}(\Omega). \end{array} \right. \quad (24)$$

The operator $\mathcal{E} = (\mathcal{E}_{ijk}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } (\mathcal{E}\varepsilon)_i = \sum_{1 \leq j, k \leq d} \mathcal{E}_{ijk} \varepsilon_{jk} \\ \text{a.e. } x \in \Omega, \text{ for all } \varepsilon \in \mathbb{S}^d; \\ \text{(ii) } \mathcal{E}_{ijk} = \mathcal{E}_{ikj} \in L^{\infty}(\Omega). \end{array} \right. \quad (25)$$

The operator $\mathcal{E}^* : \Omega \times \mathbb{R}^d \rightarrow \mathbb{S}^d$ is defined by

$$\varepsilon \cdot \mathcal{E}^* w = \mathcal{E}\varepsilon \cdot w \text{ for all } w \in \mathbb{R}^d, \text{ for all } \varepsilon \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \quad (26)$$

The magnetic permeability operator $\mathcal{Z} = (\mathcal{Z}_{ij}) : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } (\mathcal{Z}w)_i = \sum_{1 \leq j \leq d} \mathcal{Z}_{ij} w_j \\ \text{a.e. } x \in \Omega, \text{ for all } w = (w_j)_{1 \leq j \leq d} \in \mathbb{R}^d; \\ \text{(ii) } \mathcal{Z}_{ij} = \mathcal{Z}_{ji} \in L^{\infty}(\Omega). \end{array} \right. \quad (27)$$

The operator $\mathcal{M} = (\mathcal{M}_{ijk}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } (\mathcal{M}\varepsilon)_i = \sum_{1 \leq j, k \leq d} \mathcal{M}_{ijk} \varepsilon_{jk} \\ \text{a.e. } x \in \Omega, \text{ for all } \varepsilon \in \mathbb{S}^d; \\ \text{(ii) } \mathcal{M}_{ijk} = \mathcal{M}_{ikj} \in L^{\infty}(\Omega). \end{array} \right. \quad (28)$$

The operator $\mathcal{M}^* : \Omega \times \mathbb{R}^d \rightarrow \mathbb{S}^d$ is defined by

$$\varepsilon \cdot \mathcal{M}^* w = \mathcal{M} \varepsilon \cdot w \text{ a.e. } x \in \Omega, \text{ for all } w \in \mathbb{R}^d, \text{ for all } \varepsilon \in \mathbb{S}^d. \quad (29)$$

The operator $\mathcal{S} = (\mathcal{S}_{ij}) : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i) } (\mathcal{S}w)_i = \sum_{1 \leq j \leq d} \mathcal{S}_{ij} w_j \\ \text{a.e. } x \in \Omega, \text{ for all } w = (w_j) \in \mathbb{R}^d; \\ \text{(ii) } \mathcal{S}_{ij} = \mathcal{S}_{ji} \in L^\infty(\Omega). \end{array} \right. \quad (30)$$

We assume that there exists $m > 0$ such that

$$\left\{ \begin{array}{l} \mathcal{C}w_1 \cdot w_1 + 2\mathcal{S}w_1 \cdot w_2 + \mathcal{Z}w_2 \cdot w_2 \\ \geq m(|w_1|^2 + |w_2|^2) \text{ a.e. } x \in \Omega \text{ for all } w_1, w_2 \in \mathbb{R}^d. \end{array} \right. \quad (31)$$

We assume that the function $p_\alpha : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}^+$, ($\alpha = \nu, \tau$), satisfies

$$\left\{ \begin{array}{l} \text{(i) There exists } L_\alpha > 0 \text{ such that} \\ |p_\alpha(x, r_1) - p_\alpha(x, r_2)| \leq L_\alpha |r_1 - r_2|, \\ \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(ii) } p_\alpha(x, r) = 0 \text{ for all } r \leq 0, \text{ a.e. } x \in \Gamma_3; \\ \text{(iii) The mapping } x \mapsto p_\alpha(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \text{for all } r \in \mathbb{R}. \end{array} \right. \quad (32)$$

Finally, we assume that

$$(i) f_0 \in W^{1,\infty}(0, T; \mathbf{H}), \quad (ii) f_2 \in W^{1,\infty}(0, T; \mathbf{L}^2(\Gamma_2)), \quad (33)$$

$$(i) q_0 \in W^{1,\infty}(0, T; H), \quad (ii) q_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2^{el})), \quad (34)$$

$$(i) p_0 \in W^{1,\infty}(0, T; H), \quad (ii) p_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2^{ma})), \quad (35)$$

$$(u_0, \varphi_0, \psi_0) \in V \times W_{el} \times W_{ma}. \quad (36)$$

Let $f : [0, T] \rightarrow V$, $q : [0, T] \rightarrow W_{el}$ and $p : [0, T] \rightarrow W_{ma}$ be functions defined by

$$(f(t), w)_V = \int_{\Omega} f_0(t) \cdot w dx + \int_{\Gamma_2} f_2(t) \cdot w da \text{ for all } w \in V, \quad (37)$$

$$(q(t), \phi)_{W_{el}} = \int_{\Omega} q_0(t) \phi dx - \int_{\Gamma_2^{el}} q_2(t) \phi da \text{ for all } \phi \in W_{el}, \quad (38)$$

$$(p(t), \vartheta)_{W_{ma}} = \int_{\Omega} p_0(t) \vartheta dx - \int_{\Gamma_2^{ma}} p_2(t) \vartheta da \text{ for all } \vartheta \in W_{ma}. \quad (39)$$

In the sequel, we use the functional $\mathcal{R} : V \times V \rightarrow \mathbb{R}$ defined by

$$\begin{cases} \mathcal{R}(v, w) = \int_{\Gamma_3} p_\nu (v_\nu - g) w_\nu da + \int_{\Gamma_3} p_\tau (v_\tau - g) |w_\tau| da \\ \text{for all } v, w \in V. \end{cases} \quad (40)$$

Assume u , φ , ψ , σ , D and B are smooth functions satisfying (1)-(17) and use integration by parts to obtain the following variational formulation.

Problem 2 Find a displacement field $u : [0, T] \rightarrow V$, an electric potential field $\varphi : [0, T] \rightarrow W_{el}$ and a magnetic potential field $\psi : [0, T] \rightarrow W_{ma}$ such that

$$\begin{cases} (\mathcal{A}\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} + (\mathcal{E}^* \nabla \varphi(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} \\ + (\mathcal{M}^* \nabla \psi(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} + \mathcal{R}(u(t), w) - \mathcal{R}(u(t), \dot{u}(t)) \\ \geq (f(t), w - \dot{u}(t))_V \text{ for all } w \in V, \text{ for a.e. } t \in (0, T), \end{cases} \quad (41)$$

$$\begin{cases} (\mathcal{C} \nabla \varphi(t), \nabla \phi)_{\mathbf{H}} + (\mathcal{Z} \nabla \psi(t), \nabla \vartheta)_{\mathbf{H}} \\ + (\mathcal{S} \nabla \psi(t), \nabla \phi)_{\mathbf{H}} + (\mathcal{S} \nabla \varphi(t), \nabla \vartheta)_{\mathbf{H}} \\ = (\mathcal{E} \varepsilon(u(t)), \nabla \phi)_{\mathbf{H}} + (\mathcal{M} \varepsilon(u(t)), \nabla \vartheta)_{\mathbf{H}} \\ + (q(t), \phi)_{W_{el}} + (p(t), \vartheta)_{W_{ma}} \\ \text{for all } (\phi, \vartheta) \in W, \text{ for all } t \in [0, T], \end{cases} \quad (42)$$

$$(u(0), \varphi(0), \psi(0)) = (u_0, \varphi_0, \psi_0). \quad (43)$$

5 Existence of a weak solution

The following theorem is the main result of this paper.

Theorem 3 *Assume (23)-(36) and, in addition, assume the following additional assumptions*

$$\left\{ \begin{array}{l} (\mathcal{A}\varepsilon(u_0), \varepsilon(w))_{\mathcal{Q}} + (\mathcal{E}^*\nabla\varphi_0, \varepsilon(w))_{\mathcal{Q}} + (\mathcal{M}^*\nabla\psi_0, \varepsilon(w))_{\mathcal{Q}} \\ + \mathcal{R}(u_0, w) \geq (f(0), w)_V \text{ for all } w \in V, \end{array} \right. \quad (44)$$

$$\left\{ \begin{array}{l} (\mathcal{C}\nabla\varphi_0, \nabla\phi)_{\mathbf{H}} + (\mathcal{Z}\nabla\psi_0, \nabla\vartheta)_{\mathbf{H}} + (\mathcal{S}\nabla\psi_0, \nabla\phi)_{\mathbf{H}} \\ + (\mathcal{S}\nabla\varphi_0, \nabla\vartheta)_{\mathbf{H}} = (\mathcal{E}\varepsilon(u_0), \nabla\phi)_{\mathbf{H}} \\ + (\mathcal{M}\varepsilon(u_0), \nabla\vartheta)_{\mathbf{H}} + (q(0), \phi)_{W_{el}} \\ + (p(0), \vartheta)_{W_{ma}} \text{ for all } (\phi, \vartheta) \in W. \end{array} \right. \quad (45)$$

$$L_\tau + L_\nu < \frac{m_A}{c_0^2}. \quad (46)$$

Then, Problem (41)-(43) has a solution which satisfies

$$u \in W^{1,\infty}(0, T; V), \quad (47)$$

$$\varphi \in W^{1,\infty}(0, T; W_{el}), \quad (48)$$

$$\psi \in W^{1,\infty}(0, T; W_{ma}). \quad (49)$$

We will divide the proof of Theorem 3 into several steps.

Step 1. In this step, we present some properties of certain operators used in the proof of Theorem 3. It follows from (20), (32) and (40), that

$$\left\{ \begin{array}{l} \mathcal{R}(w_1, w_3) - \mathcal{R}(w_1, w_4) + \mathcal{R}(w_2, w_4) - \mathcal{R}(w_2, w_3) \\ \leq c_0^2 (L_\tau + L_\nu) \|w_1 - w_2\|_V \|w_3 - w_4\|_V, \end{array} \right. \quad (50)$$

$$\mathcal{R}(w_1, -w_2) - \mathcal{R}(w_1, w_1 - w_2) \leq c_0^2 (L_\tau + L_\nu) \|w_1\|_V^2, \quad (51)$$

$$\mathcal{R}(w_1, w_2) - \mathcal{R}(w_1, w_3) \leq \mathcal{R}(w_1, w_2 - w_3), \quad (52)$$

$$\mathcal{R}(w_1, w_3) - \mathcal{R}(w_2, w_3) \leq c_0^2 (L_\tau + L_\nu) \|w_1 - w_2\|_V \|w_3\|_V, \quad (53)$$

$$|\mathcal{R}(w_1, w_2) - \mathcal{R}(w_1, w_3)| \leq c_0^2 (L_\tau + L_\nu) \|w_1\|_V \|w_2 - w_3\|_V, \quad (54)$$

$$\mathcal{R}(w_1, w_2) \leq c_0 (L_\tau + L_\nu) \|w_1\|_V \|w_2\|_{L^2(\Gamma_3)^d}, \quad (55)$$

for all $w_1, w_2, w_3, w_4 \in V$. As in [6], under the assumptions (24)-(31), we define the linear bounded operators $\mathcal{K} : W \rightarrow W$, $\mathcal{F} : W \rightarrow V$ and $\mathcal{N} : V \rightarrow W$ by

$$\left\{ \begin{array}{l} (\mathcal{K}(\varphi, \psi), (\phi, \vartheta))_W = (\mathcal{C}\nabla\varphi, \nabla\phi)_\mathbf{H} + \\ (\mathcal{Z}\nabla\psi, \nabla\vartheta)_\mathbf{H} + (\mathcal{S}\nabla\varphi, \nabla\vartheta)_\mathbf{H} + (\mathcal{S}\nabla\psi, \nabla\phi)_\mathbf{H} \\ \text{for all } (\varphi, \psi), (\phi, \vartheta) \in W, \end{array} \right. \quad (56)$$

$$\left\{ \begin{array}{l} (\mathcal{F}(\phi, \vartheta), w)_V = (\mathcal{E}^*\nabla\phi, \varepsilon(w))_\mathcal{Q} + (\mathcal{M}^*\nabla\vartheta, \varepsilon(w))_\mathcal{Q} \\ \text{for all } w \in V, \text{ for all } (\phi, \vartheta) \in W, \end{array} \right. \quad (57)$$

$$\left\{ \begin{array}{l} (\mathcal{N}w, (\phi, \vartheta))_W = (\mathcal{E}\varepsilon(w), \nabla\phi)_\mathbf{H} + (\mathcal{M}\varepsilon(w), \nabla\vartheta)_\mathbf{H} \\ \text{for all } w \in V, \text{ for all } (\phi, \vartheta) \in W, \end{array} \right. \quad (58)$$

We note that the operator \mathcal{K} is a bijection. Moreover, the linear operator $\mathcal{G} : V \rightarrow V$ defined by

$$(\mathcal{G}w, v)_V = (\mathcal{A}\varepsilon(w), \varepsilon(v))_\mathcal{Q} + (\mathcal{F}\mathcal{K}^{-1}\mathcal{N}w, v)_V \text{ for all } w, v \in V. \quad (59)$$

satisfies

$$m_{\mathcal{A}} \|w_1 - w_2\|_V^2 \leq (\mathcal{G}w_1 - \mathcal{G}w_2, w_1 - w_2)_V \text{ for all } w_1, w_2 \in V, \quad (60)$$

$$\|\mathcal{G}w_1 - \mathcal{G}w_2\|_V \leq L_{\mathcal{G}} \|w_1 - w_2\|_V \text{ for all } w_1, w_2 \in V \quad (61)$$

with $L_{\mathcal{G}} > 0$. Define the functions $\beta : [0, T] \rightarrow W$ and $\eta : [0, T] \rightarrow V$ by

$$\left\{ \begin{array}{l} (\beta(t), (\phi, \vartheta))_W = (q(t), \phi)_{W_{el}} + (p(t), \vartheta)_{W_{ma}} \\ \text{for all } (\phi, \vartheta) \in W, \text{ for all } t \in [0, T]. \end{array} \right. \quad (62)$$

$$\left\{ \begin{array}{l} (\eta(t), w)_V = (f(t) - \mathcal{F}\mathcal{K}^{-1}\beta(t), w)_V \\ \text{for all } w \in V, \text{ for all } t \in [0, T]. \end{array} \right. \quad (63)$$

Using (33)-(35), (37)-(39) and (62)-(63), we get

$$\beta \in W^{1,\infty}(0, T; W), \quad (64)$$

$$\eta \in W^{1,\infty}(0, T; V). \quad (65)$$

Taking into account (44)-(45), (56)-(58), (62)-(63) and (59), we obtain

$$(\mathcal{G}u_0, w)_V + \mathcal{R}(u_0, w) \geq (\eta(0), w)_V \text{ for all } w \in V. \quad (66)$$

Step 2. For each $m \in \mathbb{N}^*$, we introduce a uniform partition of the time interval $[0, T]$, denoted by $t_i^m = ih_m$, $h_m = \frac{T}{m}$, $i = 0, \dots, m$. For a sequence $\{w_m^i\}_{i=0}^m$, we denote $\delta w_m^{i+1} = \frac{w_m^{i+1} - w_m^i}{h_m}$ and for a continuous function $z \in C([0, T]; X)$ with values in a normed space X , we use the notation $z_i^m = z(t_i^m)$, $i = 0, \dots, m$. Consider the following incremental problems \mathcal{P}_m^{i+1} , $i \in \{0, \dots, m-1\}$.

Problem 4 (\mathcal{P}_m^{i+1}) Find a function $u_m^{i+1} \in V$ such that

$$\begin{cases} (\mathcal{G}u_m^{i+1}, w - \delta u_m^{i+1})_V + \mathcal{R}(u_m^{i+1}, w) - \mathcal{R}(u_m^{i+1}, \delta u_m^{i+1}) \\ \geq (\eta_{i+1}^m, w - \delta u_m^{i+1})_V \text{ for all } w \in V, \end{cases} \quad (67)$$

where u_m^j is the unique solution of Problem \mathcal{P}_m^j , $j = 1, \dots, i$,

$$\eta_{i+1}^m = \eta(t_{i+1}^m), \quad i = 0, \dots, m-1, \quad (68)$$

$$u_m^0 = u_0. \quad (69)$$

Setting $w = \frac{v - u_m^i}{h_m}$ in (67), we deduce that \mathcal{P}_m^{i+1} is formally equivalent to the following problem.

Problem 5 (\mathcal{Q}_m^{i+1}) Find a function $u_m^{i+1} \in V$ such that

$$\begin{cases} (\mathcal{G}u_m^{i+1}, v - u_m^{i+1})_V + \mathcal{R}(u_m^{i+1}, v - u_m^i) \\ -\mathcal{R}(u_m^{i+1}, u_m^{i+1} - u_m^i) \geq (\eta_{i+1}^m, v - u_m^{i+1})_V \text{ for all } v \in V, \end{cases} \quad (70)$$

where $\{\eta_{i+1}^m\}$ and u_m^0 are given by (68)-(69) and u_m^j is the unique solution to Problem \mathcal{P}_m^j , $j = 1, \dots, i$.

Lemma 6 *Problem \mathcal{P}_m^{i+1} , $0 \leq i \leq m - 1$, has a unique solution.*

Proof. Define the functional $\Theta_w : V \rightarrow \mathbb{R}$ by

$$\Theta_w(v) = \mathcal{R}(w, v - u_m^i) \text{ for all } v \in V. \quad (71)$$

Consider the following problem. Let $w \in V$, find $z_w \in V$ such that

$$\begin{cases} (\mathcal{G}z_w, v - z_w)_V + \Theta_w(v) - \Theta_w(z_w) \\ \geq (\eta_{i+1}^m, v - z_w)_V \text{ for all } v \in V. \end{cases} \quad (72)$$

We combine (40), (54) and (71) to see that Θ_w is a convex continuous function. From (61) and (60), \mathcal{G} is a strongly monotone and Lipschitz continuous operator. Thus, it follows from a standard result on elliptic variational inequalities of the second kind (cf. [5, p. 60]) that Problem (72) has a unique solution $z_w \in V$. Let $w_1, w_2 \in V$, using (71)-(72), we get

$$\begin{aligned} (\mathcal{G}z_{w_1} - \mathcal{G}z_{w_2}, z_{w_1} - z_{w_2})_V &\leq \mathcal{R}(w_1, z_{w_2} - u_m^i) - \mathcal{R}(w_1, z_{w_1} - u_m^i) \\ &\quad + \mathcal{R}(w_2, z_{w_1} - u_m^i) - \mathcal{R}(w_2, z_{w_2} - u_m^i), \end{aligned}$$

which, together with (50) and (60), implies that

$$m_{\mathcal{A}} \|z_{w_1} - z_{w_2}\|_V^2 \leq c_0^2 (L_\tau + L_\nu) \|w_1 - w_2\|_V \|z_{w_1} - z_{w_2}\|_V. \quad (73)$$

Define the operator $\Psi : V \rightarrow V$ by

$$\Psi(w) = z_w \text{ for all } w \in V,$$

and using (73), we have

$$\|\Psi w_2 - \Psi w_1\|_V \leq \frac{c_0^2 (L_\tau + L_\nu)}{m_{\mathcal{A}}} \|w_1 - w_2\|_V.$$

Thus, under the smallness assumption (46), Ψ is a contraction on the Hilbert space V . Therefore, Ψ has a unique fixed point $w^* \in V$. We have now all the ingredients to prove Lemma 6. Let w^* be the unique fixed point of

Ψ and let z_{w^*} be the unique solution to Problem (72) for $w = w^*$. Then, taking $u_m^{i+1} = z_{w^*}$, we deduce that u_m^{i+1} is a solution to Problem (70) which is formally equivalent to Problem (67). The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Ψ and the uniqueness of the solution of the problem (72). ■

In the rest of this paper, the same letter c will be used to denote different positive constants which do not depend on $m \in \mathbb{N}^*$ nor on $t \in (0, T)$. We have the following result.

Lemma 7 *There exists $c > 0$ such that, for all $m \in \mathbb{N}^*$ and $0 \leq i \leq m - 1$,*

$$\|u_m^{i+1}\|_V \leq c. \quad (74)$$

$$\|\delta u_m^{i+1}\|_V \leq c. \quad (75)$$

Proof. Using (70) with $v = 0_V$, we obtain

$$(\mathcal{G}u_m^{i+1}, u_m^{i+1})_V \leq \mathcal{R}(u_m^{i+1}, -u_m^i) - \mathcal{R}(u_m^{i+1}, u_m^{i+1} - u_m^i) + (\eta_{i+1}^m, u_m^{i+1})_V,$$

and using (60) and (51), we get

$$m_{\mathcal{A}} \|u_m^{i+1}\|_V^2 \leq c_0^2 (L_\tau + L_\nu) \|u_m^{i+1}\|_V^2 + \|\eta_{i+1}^m\|_V \|u_m^{i+1}\|_V.$$

By virtue of (46) and (65), the last inequality gives (74). Setting $v = u_m^0$ in (70) for $i=0$, and $w = u_m^1 - u_m^0$ in (66), adding the two inequalities, we obtain

$$\begin{aligned} (\mathcal{G}u_m^1 - \mathcal{G}u_m^0, u_m^1 - u_m^0)_V &\leq \mathcal{R}(u_m^0, u_m^1 - u_m^0) - \mathcal{R}(u_m^1, u_m^1 - u_m^0) \\ &\quad + (\eta_1^m - \eta_0^m, u_m^1 - u_m^0)_V. \end{aligned}$$

We then use (60) and (53), to see that

$$m_{\mathcal{A}} \|u_m^1 - u_m^0\|_V^2 \leq c_0^2 (L_\tau + L_\nu) \|u_m^1 - u_m^0\|_V^2 + \|\eta_1^m - \eta_0^m\|_V \|u_m^1 - u_m^0\|_V,$$

and thanks to (46) and (65), we get

$$\begin{aligned} \left\| \frac{u_m^1 - u_m^0}{h_m} \right\|_V &\leq c \left\| \frac{\eta_1^m - \eta_0^m}{h_m} \right\|_V \\ &\leq c \|\dot{\eta}\|_{L^\infty(0, T; V)}. \end{aligned}$$

Thus, we have

$$\|\delta u_m^1\|_V \leq c. \quad (76)$$

Taking $w = 0_V$ in problem \mathcal{P}_m^{i+1} , and $w = \frac{u_m^{i+1} - u_m^{i-1}}{h_m}$ in problem \mathcal{P}_m^i , adding the two inequalities, we obtain

$$\begin{cases} (\mathcal{G}u_m^{i+1} - \mathcal{G}u_m^i, \delta u_m^{i+1})_V \leq \left(\mathcal{R}(u_m^i, \frac{u_m^{i+1} - u_m^{i-1}}{h_m}) - \mathcal{R}(u_m^i, \frac{u_m^i - u_m^{i-1}}{h_m}) \right) \\ -\mathcal{R}(u_m^{i+1}, \frac{u_m^{i+1} - u_m^i}{h_m}) + (\eta_{i+1}^m - \eta_i^m, \delta u_m^{i+1})_V. \end{cases}$$

So, from (60), (53) and (52), it follows that

$$m_{\mathcal{A}} \|\delta u_m^{i+1}\|_V^2 \leq c_0^2 (L_\tau + L_\nu) \|\delta u_m^{i+1}\|_V^2 + \left\| \frac{\eta_{i+1}^m - \eta_i^m}{h_m} \right\|_V \|\delta u_m^{i+1}\|_V.$$

Therefore, we have

$$\|\delta u_m^{i+1}\|_V^2 \leq c \|\dot{\eta}\|_{L^\infty(0,T;V)} \quad \text{for } 0 \leq i \leq m-1,$$

which, with (76), gives (75). \blacksquare

Step 3. For each $m \in \mathbb{N}^*$, let u_m^j be the unique solution to Problem \mathcal{P}_m^j , $j = 1, \dots, m$. We introduce the following functions $u_m : [0, T] \rightarrow V$, $\tilde{u}_m : [0, T] \rightarrow V$ and $\eta_m : [0, T] \rightarrow V$ defined, respectively, by

$$u_m(0) = u_0, \quad u_m(t) = u_m^i + (t - t_i^m) \delta u_m^{i+1}, \quad (77)$$

$$\tilde{u}_m(0) = u_0, \quad \tilde{u}_m(t) = u_m^{i+1}, \quad (78)$$

$$\eta_m(0) = \eta(0), \quad \eta_m(t) = \eta_{i+1}^m \quad (79)$$

for all $t \in (t_i^m, t_{i+1}^m]$, $0 \leq i \leq m-1$. Here $\{\eta_{i+1}^m\}$ and u_m^0 are given by (68)-(69). From (77), the function u_m has a derivative function which is given by

$$\dot{u}_m(t) = \delta u_m^{i+1} \quad \text{for all } t \in (t_i^m, t_{i+1}^m), \quad 0 \leq i \leq m-1. \quad (80)$$

We have the following estimate results.

Lemma 8 *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$,*

$$\|\tilde{u}_m(t)\|_V \leq c \text{ for all } t \in [0, T], \quad (81)$$

$$\|u_m(t)\|_V \leq c \text{ for all } t \in [0, T], \quad (82)$$

$$\|\dot{u}_m(t)\|_V \leq c \text{ a.e. } t \in [0, T], \quad (83)$$

$$\|\tilde{u}_m(t) - u_m(t)\|_V \leq ch_m \text{ for all } t \in [0, T], \quad (84)$$

$$\|\eta_m(t) - \eta(t)\|_V \leq ch_m \text{ for all } t \in [0, T], \quad (85)$$

$$\|u_m(t) - u_m(s)\|_V \leq c|t - s| \text{ for all } t, s \in [0, T], \quad (86)$$

$$\|u_m(t) - u_m(s)\|_{L^2(\Gamma_3)^d} \leq c|t - s| \text{ for all } t, s \in [0, T]. \quad (87)$$

Proof. The proof can be obtained using arguments similar to those used in [7, proof of Lemma 5]. ■

Lemma 9 *There exists a function $u \in W^{1,2}(0, T; V)$ and two subsequences of $\{u_m\}$ and $\{\tilde{u}_m\}$ again denoted by $\{u_m\}$ and $\{\tilde{u}_m\}$, respectively, such that*

$$u_m \rightharpoonup u \text{ weakly in } L^2(0, T; V). \quad (88)$$

$$\dot{u}_m \rightharpoonup \dot{u} \text{ weakly in } L^2(0, T; V). \quad (89)$$

$$u_m \rightarrow u \text{ strongly in } C([0, T]; L^2(\Gamma_3)^d). \quad (90)$$

$$u_m \rightarrow u \text{ strongly in } C([0, T]; V). \quad (91)$$

$$\tilde{u}_m \rightarrow u \text{ strongly in } L^2(0, T; V). \quad (92)$$

Proof. The convergences (88)-(89) follow from (82)-(83) and standard compactness arguments (cf. [3]). Let $E = \{u_m : [0, T] \rightarrow L^2(\Gamma_3)^d, m \in \mathbb{N}^*\}$ be the set of the traces of $\{u_m\}$ on Γ_3 . We use (87) to see that E is equicontinuous. Moreover, since the trace map is a compact operator, it follows from (82) that $E(t) = \{u_m(t), u_m \in E\}$ is relatively compact for all $t \in [0, T]$. Therefore, using a version of the Arzela-Ascoli theorem (cf. [4]) and taking another subsequence if necessary, we obtain (90). We turn now to the proof of (91). To this end, we need to show that the subsequence $\{u_m\}$ obtained in (88)-(90) is a Cauchy sequence in the Banach space $C([0, T]; V)$. We combine (52), (70), (78) and (79) to deduce that

$$\begin{cases} (\mathcal{G}\tilde{u}_m(t), v - \tilde{u}_m(t))_V + \mathcal{R}(\tilde{u}_m(t), v - \tilde{u}_m(t)) \\ \geq (\eta_m(t), v - \tilde{u}_m(t))_V \text{ for all } v \in V, \text{ for all } t \in [0, T]. \end{cases} \quad (93)$$

Let $m, n \in \mathbb{N}^*$ such that $m > n > T$. Taking $(\tilde{u}_m, \eta_m, v) = (\tilde{u}_m, \eta_m, \tilde{u}_n)$ and then $(\tilde{u}_m, \eta_m, v) = (\tilde{u}_n, \eta_n, \tilde{u}_m)$ in (93), and adding the two inequalities, leads us to

$$\begin{cases} (\mathcal{G}\tilde{u}_m(t) - \mathcal{G}\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \leq \mathcal{R}(\tilde{u}_m(t), \tilde{u}_n(t) - \tilde{u}_m(t)) \\ + \mathcal{R}(\tilde{u}_n(t), \tilde{u}_m(t) - \tilde{u}_n(t)) + \\ + (\eta_m(t) - \eta_n(t), \tilde{u}_m(t) - \tilde{u}_n(t))_V \text{ for all } t \in [0, T]. \end{cases}$$

We combine this last inequality with (55), (60), (82) and use the following property

$$ab \leq \frac{a^2}{m_{\mathcal{A}}} + \frac{m_{\mathcal{A}}}{4}b^2 \text{ for all } a, b \in \mathbb{R},$$

to obtain

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 &\leq c \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} + c \|\eta_m(t) - \eta(t)\|_V^2 + \\ &c \|\eta(t) - \eta_n(t)\|_V^2 \text{ for all } t \in [0, T], \end{aligned}$$

and use the inequality

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 &\leq c \|u_m(t) - \tilde{u}_m(t)\|_V^2 + c \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V^2 \\ &+ c \|\tilde{u}_n(t) - u_n(t)\|_V^2 \end{aligned}$$

to deduce that

$$\begin{aligned} \|u_m(t) - u_n(t)\|_V^2 &\leq c \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} + c \|\eta_m(t) - \eta(t)\|_V^2 + \\ &c \|\eta(t) - \eta_n(t)\|_V^2 + c \|u_m(t) - \tilde{u}_m(t)\|_V^2 \\ &+ c \|\tilde{u}_n(t) - u_n(t)\|_V^2. \end{aligned} \quad (94)$$

Taking into account (84) and (20), we find that

$$\begin{aligned} \|\tilde{u}_m(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} &\leq \|\tilde{u}_m(t) - u_m(t)\|_{L^2(\Gamma_3)^d} + \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} \\ &+ \|u_n(t) - \tilde{u}_n(t)\|_{L^2(\Gamma_3)^d} \\ &\leq \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} + ch_m + ch_n. \end{aligned} \quad (95)$$

From (84)-(85) and (94)-(95), we obtain

$$\|u_m(t) - u_n(t)\|_V^2 \leq c \|u_m(t) - u_n(t)\|_{L^2(\Gamma_3)^d} + ch_m + ch_n \text{ for all } t \in [0, T].$$

Therefore, we have

$$\|u_m - u_n\|_{C([0,T];V)}^2 \leq c \|u_m - u_n\|_{C([0,T];L^2(\Gamma_3)^d)} + ch_n,$$

which combined with (90) implies that $\{u_m\}$ is a Cauchy sequence in $C([0, T]; V)$. We now use the convergence (88) to obtain (91). Finally, the convergence (92) is a consequence of (84) and (91). ■

In the rest of this paper u is the function obtained in Lemma 9, $\{u_m\}$, $\{\tilde{u}_m\}$ and $\{f_m\}$ represent appropriate subsequences of $\{u_m\}$, $\{\tilde{u}_m\}$ and $\{f_m\}$ such that the convergences (88)-(92) hold.

Lemma 10 *The following convergences hold.*

$$\lim_{m \rightarrow +\infty} \|\mathcal{G}\tilde{u}_m - \mathcal{G}u\|_{L^2(0,T;V)} = 0. \quad (96)$$

$$\lim_{m \rightarrow +\infty} \|\eta_m - \eta\|_{L^2(0,T;V)} = 0. \quad (97)$$

$$\lim_{m \rightarrow +\infty} \int_0^T [\mathcal{R}(\tilde{u}_m(s), \dot{u}_m(s)) - \mathcal{R}(u(s), \dot{u}_m(s))] ds = 0. \quad (98)$$

$$\liminf_{m \rightarrow +\infty} \int_0^T \mathcal{R}(\tilde{u}_m(s), \dot{u}_m(s)) ds \geq \int_0^T \mathcal{R}(u(s), \dot{u}(s)) ds. \quad (99)$$

Proof. Obviously, (61) and (92) gives (96). On the other hand, (97) follows from (85). Using (53), we deduce the following

$$\begin{cases} \left| \int_0^T [\mathcal{R}(\tilde{u}_m(s), \dot{u}_m(s)) - \mathcal{R}(u(s), \dot{u}_m(s))] ds \right| \\ \leq c \|\tilde{u}_m - u\|_{L^2(0,T;V)} \|\dot{u}_m\|_{L^2(0,T;V)}. \end{cases} \quad (100)$$

Therefore, the convergence (98) follows from (83), (92) and (100). Let $\Phi : L^2(0, T; V) \rightarrow \mathbb{R}$ be the functional defined by

$$\Phi(v) = \int_0^T \mathcal{R}(u(s), v(s)) ds \text{ for all } v \in L^2(0, T; V). \quad (101)$$

Using (40), (54) and (101), we conclude that Φ is convex and continuous. Therefore, Φ is a weakly lower semicontinuous function (cf. [3]). Consequently, taking into account (89), we obtain

$$\liminf_{m \rightarrow +\infty} \Phi(\dot{u}_m) \geq \Phi(\dot{u}). \quad (102)$$

On the other hand, one has

$$\int_0^T \mathcal{R}(\tilde{u}_m(s), \dot{u}_m(s)) ds = \int_0^T [\mathcal{R}(\tilde{u}_m(s), \dot{u}_m(s)) ds - \mathcal{R}(u(s), \dot{u}_m(s))] ds + \Phi(\dot{u}_m). \quad (103)$$

We combine now (98) and (102) when passing to the lim inf as $m \rightarrow +\infty$ in (103) to see that (99) holds. ■

Step 4. We have now all the ingredients to prove Theorem 3. Let $t \in (0, T)$, let $r > 0$ such that $t + r \in (0, T)$. For each $w \in V$, we define a function $v \in L^2(0, T; V)$ by

$$v(s) = \begin{cases} w & \text{for all } s \in (t, t + r), \\ \dot{u}(s) & \text{elsewhere.} \end{cases} \quad (104)$$

We now use (67) and (78)-(80) to obtain the following inequality

$$\left\{ \begin{array}{l} \int_0^T (\mathcal{G}\tilde{u}_m(s), v(s) - \dot{u}_m(s))_V ds \\ + \int_0^T \mathcal{R}(\tilde{u}_m(s), v(s)) ds - \int_0^T \mathcal{R}(\tilde{u}_m(s), \dot{u}_m(s)) ds \\ \geq \int_0^T (\eta_m(s), v(s) - \dot{u}_m(s))_V ds. \end{array} \right. \quad (105)$$

Passing to the lim sup as $m \rightarrow +\infty$ in (105), by using (89), (104) and Lemma 10, we get

$$\left\{ \begin{array}{l} \frac{1}{r} \int_t^{t+r} (\mathcal{G}u(s), w - \dot{u}(s))_V ds \\ + \frac{1}{r} \int_t^{t+r} [\mathcal{R}(u(s), w) - \mathcal{R}(u(s), \dot{u}(s))] ds \\ \geq \frac{1}{r} \int_t^{t+r} (\eta(s), w - \dot{u}(s))_V ds \text{ for all } w \in V. \end{array} \right. \quad (106)$$

Letting $r \rightarrow 0$ in (106), we find that

$$\begin{cases} (\mathcal{G}u(t), w - \dot{u}(t))_V + \mathcal{R}(u(t), w) - \mathcal{R}(u(t), \dot{u}(t)) \\ \geq (\eta(t), w - \dot{u}(t))_V \text{ for all } w \in V, \text{ for a.e. } t \in (0, T), \end{cases} \quad (107)$$

Define the functions $\varphi : [0, T] \rightarrow W_{el}$ and $\psi : [0, T] \rightarrow W_{ma}$ by

$$(\varphi(t), \psi(t)) = \mathcal{K}^{-1}\mathcal{N}u(t) + \mathcal{K}^{-1}\beta(t) \text{ for all } t \in [0, T]. \quad (108)$$

Using (57) and (108), we obtain

$$\begin{cases} (\mathcal{F}(\mathcal{K}^{-1}\mathcal{N}u(t)) + \mathcal{F}(\mathcal{K}^{-1}\beta(t)), w)_V = (\mathcal{E}^*\nabla\varphi(t), \varepsilon(w))_{\mathcal{Q}} \\ + (\mathcal{M}^*\nabla\psi(t), \varepsilon(w))_{\mathcal{Q}} \text{ for all } w \in V, \text{ for all } t \in [0, T]. \end{cases} \quad (109)$$

Taking into account (59), (63), (107), (108) and (109), we infer that

$$\begin{cases} (\mathcal{A}\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} \\ + (\mathcal{M}^*\nabla\psi(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{Q}} + \mathcal{R}(u(t), w) - \mathcal{R}(u(t), \dot{u}(t)) \\ \geq (f(t), w - \dot{u}(t))_V \text{ for all } w \in V, \text{ a.e. } t \in [0, T]. \end{cases}$$

It follows from (56), (58), (62) and (108) that

$$\begin{cases} (\mathcal{C}\nabla\varphi(t), \nabla\phi)_{\mathbf{H}} + (\mathcal{Z}\nabla\psi(t), \nabla\vartheta)_{\mathbf{H}} + (\mathcal{S}\nabla\psi(t), \nabla\phi)_{\mathbf{H}} \\ + (\mathcal{S}\nabla\varphi(t), \nabla\vartheta)_{\mathbf{H}} = (\mathcal{E}\varepsilon(u(t)), \nabla\phi)_{\mathbf{H}} \\ + (\mathcal{M}\varepsilon(u(t)), \nabla\vartheta)_{\mathbf{H}} + (q(t), \phi)_{W_{el}} + (p(t), \vartheta)_{W_{ma}} \\ \text{for all } (\phi, \vartheta) \in W, \text{ for all } t \in [0, T]. \end{cases}$$

Using (77) and (91), we deduce that $u(0) = u_0$. Also, it follows from (45), (58), (62) and (108) that

$$\begin{aligned} (\varphi(0), \psi(0)) &= \mathcal{K}^{-1}\mathcal{N}u(0) + \mathcal{K}^{-1}\beta(0) \\ &= \mathcal{K}^{-1}\mathcal{N}u_0 + \mathcal{K}^{-1}\beta(0) \\ &= (\varphi_0, \psi_0). \end{aligned}$$

On the other hand, using (86), we obtain

$$\begin{aligned} \|u(t) - u(s)\|_V &\leq \|u(t) - u_m(t)\|_V + \|u_m(t) - u_m(s)\|_V + \|u_m(s) - u(s)\|_V \\ &\leq \|u(t) - u_m(t)\|_V + c|t - s| + \|u_m(s) - u(s)\|_V \text{ for all } t, s \in [0, T]. \end{aligned}$$

Passing to the limit as $m \rightarrow +\infty$ in the last inequality and using (91), we get

$$\|u(t) - u(s)\|_V \leq c|t - s| \text{ for all } t, s \in [0, T],$$

and using (108), we find that

$$\|\varphi(t) - \varphi(s)\|_{W_{el}} + \|\psi(t) - \psi(s)\|_{W_{ma}} \leq c\|u(t) - u(s)\|_V \text{ for all } t, s \in [0, T].$$

Hence, we conclude that (u, φ, ψ) is a solution to Problem (41)-(43) which satisfies (47)-(49). Finally, we notice that the uniqueness of the solution remains, as far as we know, an open question.

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