CONSTRUCTING TREED RINGS WITH PRESCRIBED PROPERTIES

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Abstract. Let $f : R \to S$ be a ring homomorphism and J be an ideal of S. Then the subring $R \bowtie^f J := \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}$ of $R \times S$ is called the amalgamation of R with S along J with respect to f. In this paper, we characterize when the amalgamation $R \bowtie^f J$ is a treed ring. As an application, we construct examples of treed rings with desired features. In particular, we present examples related to treed rings which are not Prüfer and, in contrast to the domain case, Prüfer rings (with zero-divisors) that are not treed.

1. Introduction

Throughout, let R and S be two commutative rings with unity, let J be a non-zero proper ideal of S and $f : R \to S$ be a ring homomorphism. D'Anna, Finocchiaro, and Fontana in [7] and [8] have introduced the following subring

$$R \bowtie^f J := \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}$$

of $R \times S$, called the *amalgamated algebra* (or *amalgamation*) of R with S along J with respect to f. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [10]). Moreover, several classical constructions such as Nagata's idealization (cf. [14, page 2]), the $R+XS[X]$ and the $R+XS[\![X]\!]$ constructions can be studied as particular cases of this new construction (see [7, Example 2.5 and Remark 2.8]). Amalgamation, in turn, can be realized as a pullback. The construction has proved its worth providing numerous examples and counterexamples in commutative algebra ([9], [3], [4], [13], [5], and etc.).

The study of *treed* rings has attracted attention, with important contributions from Dobbs and other researchers (e.g. [11], [12]). Recall that a ring R is said to be treed if, no maximal ideal of R contains incomparable prime ideals. Note that Prüfer domains are treed. While the transfer of Prüfer-like (i.e., Prüfer, Gaussian or arithmetical) properties under amalgamated constructions has been investigated in [13] and [5], the behavior of the treed property itself under such constructions has not yet been studied. In this paper we pursue previous works and investigate treed condition on amalgamations.

The outline of the paper is as follows. In Section 2 we clarify our notations and recall some necessary facts that we will use later. In Section 3 we prove our main theorem, which provides a characterization of treed property on the amalgamations. Also, we characterize when $R \bowtie^f J$ is a total ring of quotients in the special case

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that $J \subseteq \mathrm{Nil}(S)$. Our attempt results in examples related to treed rings with desired features. These examples are presented in Section 4. In particular, we give examples of the rings that are treed, but not Prüfer. We also construct a Prüfer ring (with zero-divisors) that is not treed, showing that Prüfer no longer implies treed in the presence of zero-divisors.

2. Preliminaries

We begin by fixing some notation used throughout the paper. For a commutative ring A, we denote the set of nilpotent elements, zero-divisors, and units of A by $\mathrm{Nil}(A)$, $\mathrm{Z}(A)$, and $\mathrm{U}(A)$, respectively. The set of prime ideals, minimal prime ideals, and maximal ideals of A, will be denoted by $\mathrm{Spec}(A)$, $\mathrm{Min}(A)$ and $\mathrm{Max}(A)$, respectively. For an ideal I of A, we denote by $\mathrm{V}(I)$ the set of prime ideals of A containing I. If T is a multiplicatively closed subset of A, then $T^{-1}A$ denotes the localization of A with respect to T. The total quotient ring of A will be denoted by $\mathrm{Tot}(A)$. In the sequel, we will use the following remark without explicit reference.

Remark 2.1. ([8, Proposition 2.6]) For $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{q} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$, set

$$\mathfrak{p}'^{f} := \mathfrak{p} \bowtie^{f} J := \{(p, f(p) + j) \mid p \in \mathfrak{p}, j \in J\},$$

$$\overline{\mathfrak{q}}^{f} := \{(r, f(r) + j) \mid r \in R, j \in J, f(r) + j \in \mathfrak{q}\}.$$

Then, the following statements hold.

- $\mathrm{Spec}(R \bowtie^{f} J) = \{\mathfrak{p}'^{f} \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \cup \{\overline{\mathfrak{q}}^{f} \mid \mathfrak{q} \in \mathrm{Spec}(S) \setminus V(J)\}$.
- $\mathrm{Max}(R \bowtie^{f} J) = \{\mathfrak{p}'^{f} \mid \mathfrak{p} \in \mathrm{Max}(R)\} \cup \{\overline{\mathfrak{q}}^{f} \mid \mathfrak{q} \in \mathrm{Max}(S) \setminus V(J)\}$.

We call prime ideals of the form $\mathfrak{p}'^{f}$ as *type 1* prime ideals, and prime ideals of the form $\overline{\mathfrak{q}}^{f}$ as *type 2* prime ideals. Note that $R \bowtie^{f} J$ does not have any type 2 prime ideals if and only if $J \subseteq \mathrm{Nil}(S)$.

We repeatedly use (often without reference) the following lemma ([2, Lemma 2.2 and Lemma 2.3]).

Lemma 2.2. *Let* $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(R)$ *and* $\mathfrak{q}, \mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$. *Then*

1. $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ *if and only if* $\mathfrak{p}_1'^{f} \subseteq \mathfrak{p}_2'^{f}$.
2. $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ *if and only if* $\overline{\mathfrak{q}_1}^{f} \subseteq \overline{\mathfrak{q}_2}^{f}$.
3. $\overline{\mathfrak{q}}^{f} \subseteq \mathfrak{p}'^{f}$ *if and only if* $f^{-1}(\mathfrak{q} + J) \subseteq \mathfrak{p}$.
4. $\mathfrak{p}'^{f} \nsubseteq \overline{\mathfrak{q}}^{f}$.

Remark 2.3. [5, Section 2] examines the behavior of the set of zero-divisors of amalgamated algebras as follows: Let $Z_1 = \{(r, f(r) + j) \mid r \in \mathrm{Z}(R)\}$ and $Z_2 = \{(r, f(r) + j) \mid j'(f(r) + j) = 0 \text{ for some } j' \in J \setminus \{0\}\}$. Then $Z_2 \subseteq \mathrm{Z}(R \bowtie^{f} J) \subseteq Z_1 \cup Z_2$. The amalgamated ring $R \bowtie^{f} J$ is said to have the *condition* $(\star)$ if the equality $\mathrm{Z}(R \bowtie^{f} J) = Z_1 \cup Z_2$ holds.

3. Transfer of the treed condition in the amalgamations

In this section, we characterize when $R \bowtie^{f} J$ is a treed ring (Theorem 3.2). As consequences of this result, we derive several corollaries that specialize the characterization to important particular cases, including the amalgamated duplication and the trivial extension. Also, we investigate when $R \bowtie^{f} J$ is a total ring of quotients. Let us first introduce the concept of *treed* condition for an arbitrary subset of $\mathrm{Spec}(R)$.

Definition 3.1. *Let $X \subseteq \mathrm{Spec}(R)$. We say that X is* treed *if no maximal ideal of R belonging to X contains two incomparable prime ideals. We simply say R is treed if $\mathrm{Spec}(R)$ is treed.*

Note that R is treed in the usual sense, if it is treed in the sense of above definition. The following theorem is one of the main results of this paper.

Theorem 3.2. *$R \bowtie^f J$ is treed if and only if the following conditions hold:*

(1) R and $\mathrm{Spec}(S) \setminus V(J)$ are treed,
(2) For any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$, and any $\mathfrak{m} \in \mathrm{Max}(R)$ containing $f^{-1}(\mathfrak{p}+J)$ and $f^{-1}(\mathfrak{q}+J)$, we have $f^{-1}(\mathfrak{p}+J) \in \mathrm{Min}(R)$, and $\mathfrak{p}$ and $\mathfrak{q}$ are comparable.

Proof. ($\Rightarrow$): Assume that $R \bowtie^f J$ is a treed ring. Then R is treed, since the treed property is stable under factor rings. Next, let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(S)$, and $\mathfrak{n} \in \mathrm{Max}(S) \setminus \mathrm{V}(J)$, such that $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{n}$. Hence $\overline{\mathfrak{p}}^f, \overline{\mathfrak{q}}^f \subseteq \overline{\mathfrak{n}}^f$, hence that $\overline{\mathfrak{p}}^f$ and $\overline{\mathfrak{q}}^f$ are comparable, and therefore so are $\mathfrak{p}$ and $\mathfrak{q}$.
Finally, fix $\mathfrak{p}, \mathfrak{q}, \mathfrak{m}$ as mentioned above. Then, by Lemma 2.2, $\overline{\mathfrak{p}}^f, \overline{\mathfrak{q}}^f \subseteq \mathfrak{m}'^f$. By our hypothesis, prime ideals $\overline{\mathfrak{p}}^f$ and $\overline{\mathfrak{q}}^f$ are comparable, and the same holds for $\mathfrak{p}$ and $\mathfrak{q}$. It remains to show that, for any $\mathfrak{n} \in \mathrm{Spec}(R)$ with $\mathfrak{n} \subseteq \mathfrak{m}$, one has $f^{-1}(\mathfrak{p} + J) \subseteq \mathfrak{n}$. Since $\mathfrak{n}'^f, \overline{\mathfrak{p}}^f \subseteq \mathfrak{m}'^f$, our assumption together with Lemma 2.2(4) implies that $\overline{\mathfrak{p}}^f \subseteq \mathfrak{n}'^f$. Another use of Lemma 2.2 gives the desired result: $f^{-1}(\mathfrak{p} + J) \subseteq \mathfrak{n}$.

($\Leftarrow$): Assume that conditions (1) and (2) hold. We show that any pair of prime ideals $\mathcal{P}, \mathcal{Q} \in \mathrm{Spec}(R \bowtie^f J)$ contained in a maximal ideal $\mathcal{M} \in \mathrm{Max}(R \bowtie^f J)$, are comparable. The argument splits into two cases:
Case 1. $\mathcal{M} = \overline{\mathfrak{m}}^f$ for some $\mathfrak{m} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$. Then, by Lemma 2.2, $\mathcal{P} = \overline{\mathfrak{p}}^f$ and $\mathcal{Q} = \overline{\mathfrak{q}}^f$, where $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$ and $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{m}$. By assumption, $\mathfrak{p}$ and $\mathfrak{q}$ are comparable, and consequently $\overline{\mathfrak{p}}^f$ and $\overline{\mathfrak{q}}^f$ are comparable as well.
Case 2. $\mathcal{M} = \mathfrak{m}'^f$ for some $\mathfrak{m} \in \mathrm{Spec}(R)$. When both $\mathcal{P}$ and $\mathcal{Q}$ are type 1 or type 2 prime ideals, the proof follows easily from our assumption together with Lemma 2.2, so we omit the details since they parallel the above cases. Now let $\mathcal{P} = \mathfrak{p}'^f$ and $\mathcal{Q} = \overline{\mathfrak{q}}^f$, for some $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{q} \in \mathrm{Spec}(S) \setminus \mathrm{V}(J)$. Then, by Lemma 2.2, $\mathfrak{p} \subseteq \mathfrak{m}$ and $f^{-1}(\mathfrak{q} + J) \subseteq \mathfrak{m}$. By assumption, $f^{-1}(\mathfrak{q} + J) \in \mathrm{Min}(R)$, hence $f^{-1}(\mathfrak{q} + J) \subseteq \mathfrak{p}$, which implies $\overline{\mathfrak{q}}^f \subseteq \mathfrak{p}'^f$. □

Recall that if $f := id_R$ is the identity homomorphism on R, and I is an ideal of R, then $R \bowtie I := R \bowtie^{id_R} I$ is called the amalgamated duplication of R along I.

Corollary 3.3. *$R \bowtie I$ is a treed ring if and only if so is R, and I is coprime to every $\mathfrak{q} \in \mathrm{Spec}(R) \setminus \mathrm{V}(I)$.*

Proof. ($\Rightarrow$): Assume that $R \bowtie I$ is a treed ring. Then, by Theorem 3.2, R is a treed ring. Now suppose contrarily that, there exists $\mathfrak{q} \in \mathrm{Spec}(R) \setminus \mathrm{V}(I)$ such that I is not coprime to $\mathfrak{q}$. Then, for some $\mathfrak{m} \in \mathrm{Max}(R)$, $I + \mathfrak{q} \subseteq \mathfrak{m}$. By Theorem 3.2, $I + \mathfrak{q}$ must be the *minimal* prime ideal contained in $\mathfrak{m}$. Therefore $I + \mathfrak{q} \subseteq \mathfrak{q}$, absurd.

($\Leftarrow$): Since R is a treed ring, $\mathrm{Spec}(R) \setminus \mathrm{V}(I)$ is also treed. This establishes condition (1) of Theorem 3.2. Also, for any $\mathfrak{p} \in \mathrm{Max}(R)$ and any $\mathfrak{q} \in \mathrm{Spec}(R) \setminus \mathrm{V}(I)$, we have $\mathfrak{q} + I = R \nsubseteq \mathfrak{p}$. Hence, condition (2) of Theorem 3.2 is vacuously true. Therefore $R \bowtie I$ is a treed ring. □

Corollary 3.4. *If I is an ideal of R such that $\mathrm{ht}(I) \geq 1$, then $R \bowtie I$ is not a treed ring.*

Proof. Since $\mathrm{ht}(I) \geq 1$, we may take $\mathfrak{q} \in \mathrm{Min}(R) \setminus \mathrm{V}(I)$ such that I is not coprime to $\mathfrak{q}$. Then, by Corollary 3.3, $R \bowtie I$ is not a treed ring. □

It has been repeatedly observed in several contexts that $R \bowtie^f J$ possesses a property $\mathcal{P}$ if and only if R has the property $\mathcal{P}$ and a suitable *refined version* of the property $\mathcal{P}$ holds on $\mathrm{Spec}(S) \setminus \mathrm{V}(J)$ (see, for instance, [2, Theorem 3.2] and [3, Theorem 3.3], among others). Theorem 3.2 establishes that if $R \bowtie^f J$ is treed, then so are R and $\mathrm{Spec}(S) \setminus \mathrm{V}(J)$. It is therefore natural to ask whether the converse holds; i.e., whether condition (2) in the "only if" direction of Theorem 3.2 is superfluous. We show that this is not the case.

Example 3.5. *Let R be a valuation domain with $J \in \mathrm{Spec}(R)$ of height one. Then $\overline{0}^{id}$ and $0'^{id}$ are two incomparable prime ideals of $R \bowtie J$ contained in J'^{id}. Thus $R \bowtie J$ is not treed. Note that, all conditions in the "only if" direction of Theorem 3.2 hold, except for the requirement in condition (2) that $f^{-1}(0 + J) \in \mathrm{Min}(R)$.*

We now present an example in which all the conditions in the "only if" direction of Theorem 3.2 hold, except for the final one.

Example 3.6. *Let $R := k$, $S' = k[x, y]$, $T = S' \setminus \langle x, y\rangle$ and $S := T^{-1}k[x, y]$ be the localization of $k[x, y]$ at the maximal ideal $\langle x, y\rangle$. Let $J := T^{-1}\langle x, y\rangle$, and $f : R \to S$ be the natural ring homomorphism. Let $\mathfrak{p} := T^{-1}\langle x\rangle$ and $\mathfrak{q} := T^{-1}\langle y\rangle$. We have*

1. *R and $\mathrm{Spec}(S) \setminus V(J)$ are treed.*
2. *$\mathrm{Spec}(S) \setminus \mathrm{V}(J) = \{0, \mathfrak{p}, \mathfrak{q}\}$, and $\mathrm{Max}(R) = \{0\}$.*
3. *$f^{-1}(J) = f^{-1}(\mathfrak{p} + J) = f^{-1}(\mathfrak{q} + J) = 0 \in \mathrm{Min}(R)$,* but *$\mathfrak{p}$ and $\mathfrak{q}$ are* not *comparable.*

Note that, by Lemma 2.2, $\overline{\mathfrak{p}}^f, \overline{\mathfrak{q}}^f \subseteq 0'_f \in \mathrm{Max}(R \bowtie^f J)$ are not comparable, and so $R \bowtie^f J$ is not treed.

Recall that if $J \subseteq \mathrm{Nil}(S)$, then $R \bowtie^f J$ has no type 2 prime ideals. Therefore we have the following corollary.

Corollary 3.7. *Let $J \subseteq \mathrm{Nil}(S)$. Then $R \bowtie^f J$ is treed if and only if so is R.*

Let M (respectively, $N = (M_i)_{i=1}^n$) be an R-module (respectively, a family of R-modules). Then $R \ltimes M$ (respectively, $R \ltimes_n N$) denotes the *trivial extension* of R by M (respectively, the n-trivial extension of R by N [1]). It should be noted that both construction are special cases of amalgamation with $J^n = 0$ (For more details see [7], [4]). Hence the next result follows from the above corollary.

Corollary 3.8. *Let M be an R-module and $N = (M_i)_{i=1}^n$ be a family of R-modules. Then the following hold:*

(1) *$R \ltimes_n N$ is treed iff so is R .*
(2) *$R \ltimes M$ is treed iff so is R .*

Let $R \subseteq S$ be an extension of domains and $\mathbf{X} := \{X_1, \ldots, X_n\}$ a finite set of indeterminates over S. If $f : R \hookrightarrow S[\mathbf{X}]$ is the natural embedding and $J := \mathbf{X}S[\mathbf{X}]$, then it is easy to check that $R \bowtie^f J$ is isomorphic to $R + \mathbf{X}S[\mathbf{X}]$ ([7, Example 2.5]).

If $n \geq 2$, then Theorem 3.2 implies that $R + \mathbf{X}S[\mathbf{X}]$ is not treed, simply because $\mathrm{Spec}(S[\mathbf{X}]) \setminus V(J)$ is not treed. On the other hand, if $n = 1$, one can easily check conditions of Theorem 3.2, and observe that $R + \mathbf{X}S[\mathbf{X}]$ is a treed domain, when R and S are treed. Thus we have the following corollary.

Corollary 3.9. *Let $R \subseteq S$ be an extension of treed domains and $\boldsymbol{X} := \{X_1, \dots, X_n\}$ a finite set of indeterminates over S. Then $R + \boldsymbol{X}S[\boldsymbol{X}]$ is a treed domain if and only if $n = 1$.*

[13, Proposition 4.6] and [5, Proposition 3.13] investigate when $R \bowtie^f J$ is a total ring of quotients in certain special cases. In the following theorem, we provide a complete characterization of this property under the assumption that $J \subseteq \mathrm{Nil}(S)$. This result will be used in Section 4 to construct one of our key examples.

Theorem 3.10. *Assume that $J \subseteq \mathrm{Nil}(S)$. Then $R \bowtie^f J$ is a total ring of quotients if and only if any element of $R \setminus (\mathrm{Z}(R) \cup \mathrm{Z}_R(J))$ is a unit of R.*

Proof. Assume that $J \subseteq \mathrm{Nil}(S)$. Then $(r, f(r) + i) \in \mathrm{U}(R \bowtie^f J)$, if and only if $(r, f(r) + i) \notin \mathfrak{m}'^f$, for any $\mathfrak{m}'^f \in \mathrm{Max}(R \bowtie^f J)$, if and only if, $r \notin \mathfrak{m}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$, if and only if $r \in \mathrm{U}(R)$. This shows that $\mathrm{U}(R \bowtie^f J) = \{(r, f(r)+i) \in R \bowtie^f J \mid r \in \mathrm{U}(R)\}$. In order to complete the proof, it is enough to show that $\mathrm{Z}(R \bowtie^f J) = \{(r, f(r) + i) \in R \bowtie^f J \mid r \in \mathrm{Z}(R) \cup \mathrm{Z}(J)\}$. This we do:

We preserve the notation of Remark 2.3. First assume that $(r, f(r)+i) \in R \bowtie^f J$ is so that $r \in \mathrm{Z}(R) \cup \mathrm{Z}_R(J)$. By [6, Lemma 2.2], when $J \subseteq \mathrm{Nil}(S)$, $R \bowtie^f J$ has the condition ($\star$). Therefore, if $r \in \mathrm{Z}(R)$, then $(r, f(r) + i) \in \mathrm{Z}_1 \subseteq \mathrm{Z}(R \bowtie^f J)$. Otherwise, for some nonzero $j \in J$, we have $f(r)j = 0$. If $i = 0$, then we have the desired result: $(r, f(r))(0, j) = 0$. Assume that $i \neq 0$ and let n be the smallest positive integer such that $i^n = 0$ and $m \leq n - 1$ the largest integer such that $f(r)ji^m \neq 0$. Then $(r, f(r)+i)(0, ji^m) = 0$, and we get $(r, f(r)+i) \in \mathrm{Z}_2 \subseteq \mathrm{Z}(R \bowtie^f J)$.

Conversely, assume that $(r, f(r) + i) \in \mathrm{Z}(R \bowtie^f J) = Z_1 \cup Z_2$. There is nothing to prove if $(r, f(r) + i) \in \mathrm{Z}_1$. Then consider the case $(r, f(r) + i) \in \mathrm{Z}_2$, and pick a nonzero $j \in J$ such that $j(f(r) + i) = 0$. As above, for a suitable positive integer m, we have $ji^m \neq 0$ and $f(r)ji^m = 0$, which implies $r \in \mathrm{Z}_R(J)$. ☐

4. Examples

In this section, we present examples that illustrate the preceding characterizations and highlight how treed, Prüfer, and related properties may behave in concrete cases. Before proceeding, we note that, while the treed property is rare for amalgamated duplication (Corollary 3.4), it is common for trivial extensions (Corollary 3.8). In fact, given any treed ring R and any R-module M, one can construct a new treed ring $R \ltimes M$.

Prüfer domains have several different characterizations, many of which have been extended to the case of rings with zero-divisors. Among them it is commonly accepted to define *Prüfer rings* as the rings in which every non-zero finitely generated regular ideal is projective. It is well known that Prüfer domains are treed. But the next example shows that this is not true for Prüfer *rings*.

Example 4.1. *Let $\mathbb{Z}$ be the ring of integers, and X be an indeterminate over $\mathbb{Z}$. Let $T = \{p(X) \in \mathbb{Z}[X] \mid p(0)$ is odd$\}$, and $R = T^{-1}\mathbb{Z}[X]$. Let $S = \mathbb{Z}/4\mathbb{Z}$ and $J = \langle 2 \rangle \subseteq S$. Define $f\colon R \to S$ to be $f\left(\frac{p(X)}{q(X)}\right) = p(0) \cdot q(0)^{-1} \bmod 4$, where the inverse $q(0)^{-1}$ is taken modulo 4. The following hold:*

(1) R is not a treed ring. In fact, $\langle 2\rangle$ and $\langle X\rangle$ are incomparable prime ideals of $\mathbb{Z}[X]$ contained in the maximal ideal $\langle 2, X\rangle$, and the set of prime ideals of R is in inclusion-preserving correspondence with the set of prime ideals of $\mathbb{Z}[X]$ that do not intersect T.

(2) $0 \neq J \subseteq \mathrm{Nil}(S)$.

(3) Let $r = p/q \notin \mathrm{U}(R)$. Then $p(0)$ is even, and so $f(r) \in \{0, 2\} \subseteq S$. Thus $f(r)$ annihilates the nonzero element $2 \in J$.

By Theorem 3.2, $R \bowtie^f J$ is not treed, whereas by Theorem 3.10, $R \bowtie^f J$ is a total ring of quotients, and hence a Prüfer ring.

Remark 4.2. [5, Proposition 3.13] *proves that if $R \bowtie^f J$ is a total ring of quotients, then so is R, under assumption that $f(\mathrm{Reg}(R)) \subseteq \mathrm{Reg}(S)$. The above example, among other things, shows that the assumption is not superfluous.*

It is well known that treed domains are not necessarily Prüfer. In the following, using our results in Section 3, we construct an example of a treed ring that is not Prüfer. To this end, we appeal to [5, Corollary 3.12] which says: assuming that $A \subseteq B$ is an extension of domains and K is the quotient field of A, $A \ltimes B$ is a Prüfer ring if and only if A is a Prüfer domain and $K \subseteq B$.

Example 4.3. *Assume that $R \subseteq S$ is an extension of domains such that R is a Prüfer domain (hence treed) and $K \nsubseteq S$, where K is the quotient field of R. Then, by Corollary 3.8, $R \ltimes S$ is a treed ring, while, by* [5, Corollary 3.12], *it is not a Prüfer ring.*

To construct a concrete example, let $R := \mathbb{Z}$ and so $\mathbb{Q}$ be the quotient field of R. Let $S := \mathbb{Z}\left[\frac{1}{2}\right] := \left\{\frac{a}{2^n} \mid a \in \mathbb{Z},\, n \geq 0\right\}$. Hence $\mathbb{Z} \subsetneq \mathbb{Z}\left[\frac{1}{2}\right] \subsetneq \mathbb{Q}$, hence that $\mathbb{Z} \ltimes \mathbb{Z}\left[\frac{1}{2}\right]$ is a treed ring which is not Prüfer.

Arithmetical rings (locally chain rings) are obvious examples of treed rings. Although Example 4.3 provides treed rings that are not arithmetical, but it is worth to use Corollary 3.3 to construct a bunch of such rings, in an easy way:

Example 4.4. *Let R be a chain ring and I be the minimal non-zero prime ideal of R. Then, by Theorem 3.2, $R \bowtie I$ is a treed ring, while it is not arithmetical, by* [5, Corollary 5.3].

A natural question studied in ring theory, is the following ascent/descent question: If $R \subseteq T$ is an integral extension of domains and if one end is a Prüfer (resp, going down) domain, when is the other end also a Prüfer (resp, going down) domain?

Prüfer domains and going down domains are treed domains. Thus the same question maybe raised for treed rings as well. We now provide an example of integral extensions $R \subseteq R \bowtie I \subseteq R \bowtie J$ such that the sides are treed, but the middle is not.

Example 4.5. *Let k be a field and X an indeterminate over k. Let $R := k[X] \times k$, $I = \langle X\rangle \times 0$, $J = k[X] \times 0$, and $\mathfrak{p} = 0 \times k$. Note that* $\mathrm{Spec}(R) = \{0 \times k,\ \{(X - a)\}_{a\in k} \times k,\ k[X] \times 0\}$*. We have $R \subseteq R \bowtie I \subseteq R \bowtie J$. Note that, by* [8, Lemma 3.6], *$R \bowtie J$ is integral over R.*

One easily observes that R is treed. By Corollary 3.3, $R \bowtie J$ is also treed since J is both maximal and minimal ideal of R, and so is coprime to any other prime ideal of R. But $R \bowtie I$ is not treed since $I + \mathfrak{p} = \langle X\rangle \times k$ is a proper ideal of R.

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