Reviewer's report on

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and **A.T. Zhussupova's** paper

Height-two concept lattices: graph decomposition and Horn axiomatizablity

CONTENT:

1. A terminological comment

The standard terminology in lattice theory is to speak of the *height* of an element of a finite lattice L, and the *length* of L. The length of L is the height of its top element 1_L. The most influential authors in the field, including Garrett Birkhoff and George Grätzer, consistently follow this usage.

In this report I adhere to the standard terminology, even when referring to statements in the paper under review. While the authors repeatedly speak of ”height-two (concept) lattices”, including in the title of their paper, I consistently use ”length-two” or ”of length 2” when quoting or discussing these passages.

2. About me, and how to read this report

I admit that I have spent only a limited amount of time on the paper under review. Furthermore, I am not an expert on concept lattices.[1]

[1]Some years ago I looked into Rudolf Wille's ”Restructuring ...” paper, cited as [1] by the authors, and now I have read two or three pages of the Ganter–Wille book [2]. But I have never studied any other paper or book on this topic; concept lattices do not match my mathematical taste or skills.

These two facts create a possibility (I hope only a very unlikely one) that some parts of the present report might be incorrect. In particular, it may happen that the authors believe that my counterexamples fail. In that case, I respectfully ask that ***any additional referee be given access to this report as well***, so that they can evaluate *both the authors' claims and my analysis*. This will ensure that the mathematical issues are assessed fairly and transparently.

Fortunately, there are hundreds of papers on the subject and therefore at least dozens of easily accessible real experts. Indeed, the following table shows how many results are found by a simple search at arXiv.org or MathSciNet (the electronic version of Mathematical Reviews) at the time of writing.[2]

Search text	Search in	arXiv count	MR count	MR>2015 count
"concept lattice"	title	15	264	107
"concept lattice"	all fields	86	693	333
"formal context"	title	14	72	40
"formal context"	all fields	88	290	162

Finally, let me suggest that any future version of the paper, if there is one, should be directed to real experts, not to me.

3. My opinion and recommendation

The authors address two problems that may appear appealing at first glance. Unfortunately, it becomes clear rather soon that the first problem, and consequently the second problem (which depends on the authors' solution of the first), are probably not sufficiently deep for the Siberian Electronic Mathematical Reports (SEMR).

A further difficulty is that the authors have overlooked the fact that a formal context, and its visualization as a bipartite graph, can be more complex than the special cases they consider. This explains why the main result of the paper, Theorem 3.4(1)(2), although correct in the particular NFRNFC case[3], is false in its present general form.

[2] The quotation marks in the search text are needed to exclude results where, for example, "lattice" does not immediately follow "concept".

[3] To be defined soon.

In fact, the paper contains three theorems. As shown by the counterexamples in Section 4, two of them, Theorems 3.1 and 3.4, are false. I do not know whether the third, Theorem 4.1, is true, but its proof is insufficient, since it relies on Theorem 3.4.

In my view, parts (3) and (4) of Theorem 3.4 should not be included; they appear to be only technical additions and are less elegant than parts (1) and (2). I would also not call Theorem 4.1 a "main" result. Its proof in the paper is short, but incorrect,[4] and therefore it cannot serve as a main result. In summary, there is only one candidate for a main result, but it is neither correct in its present form nor sufficiently deep.

Another issue is that the novelty of the paper is smaller than it appears at first sight; this point will be discussed later.

Some of the problems mentioned above could perhaps be corrected[5], but even if they were, I still do not expect the resulting paper to reach the depth required for SEMR. Therefore, with regret,

I cannot recommend this paper for publication

in Siberian Electronic Mathematical Reports. However, an improved version of the paper might have a chance at journals that maintain lower standards.

The Karaganda paper [8] is slightly better, since the assumption $n > 2$ immediately above Theorem 2 there makes the counterexamples in this report irrelevant. However, it is still not difficult to refute[6] Theorem 2 of [8]. It is possible that the shortcomings of [8], together with the authors' intention to write a corrigendum (in fact, a substantially extended paper), may justify choosing the Karaganda journal. (For clarity, this remark does not concern the standards of the Karaganda journal; these standards are not known to me.)

4. Four counterexamples refuting Theorems 3.1 and 3.4

4.1. **First counterexample.** To see a trivial counterexample, let $|G| = |M| = 1$ and let $I = G \times M$. Take $\mathbb{K} := (G, M, I)$. Then the corresponding context graph $\boldsymbol{G}_{\mathbb{K}}$ is the two-element digraph with one edge, so it is a complete bipartite graph. Thus, with $n = 1$, the graph

[4] The proof of Theorem 4.1 relies on Theorem 3.4, which is incorrect. I have not investigated whether Theorem 4.1 can be repaired or whether such an effort would be worthwhile.

[5] Proposition 6.1 and Section 9 of this report may offer some ideas.

[6] As will become clear later, based on Figure 1, one can always spoil a formal context by adding full rows and columns without changing the concept lattice; the associated graph is then spoiled as well.

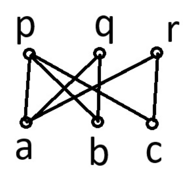
$\boldsymbol{G}_{\mathbb{K}}$ satisfies condition (2) of Theorem 3.4. However, $L(\mathbb{K})$ is the one-element lattice, so it has length 0, and condition (1) of Theorem 3.4 fails. Therefore, even the smallest formal context refutes Theorem 3.4.

To eliminate this counterexample, some modifications of the theorem are clearly necessary.

Interestingly, it seems to me (although I have not verified this) that the presence of the word "subdirect" in Theorem 3.4 excludes the possibility that $L(\mathbb{K})$ has length 1.

4.2. **Second counterexample.** The second counterexample is slightly more involved. Consider the context $\mathbb{K} = (G, M, I)$, where $G = \{a, b, c\}$, $M = \{p, q, r\}$, and $I = \{(x, y)$: there is a $\times$ entry at the intersection of the x-labeled row and the y-labeled column in the table below$\}$.

	p	q	r
a	×	×	×
b	×	×	
c	×		×



The corresponding graph $\boldsymbol{G}_{\mathbb{K}}$ is also shown above. Its vertex set is $\{a, b, c, p, q, r\}$, and its edge set is

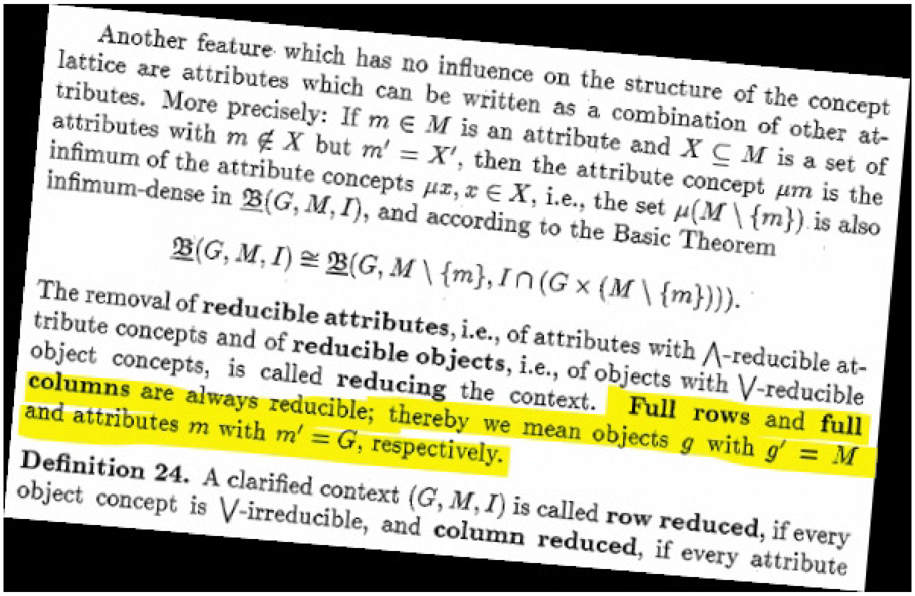
$$\{(a, p), (a, q), (a, r), (b, p), (b, q), (c, p), (c, r)\}.$$

Clearly, $\boldsymbol{G}_{\mathbb{K}}$ is not a disjoint union of complete bipartite digraphs.

A straightforward computation yields

$$\begin{aligned} L(\mathbb{K}) = \big\{ &(\{a\}, \{p, q, r\}),\ (\{a, b\}, \{p, q\}), \\ &(\{a, c\}, \{p, r\}),\ (\{a, b, c\}, \{p\}) \big\}. \end{aligned} \tag{4.1}$$

(Importantly, $(\emptyset, \{p, q, r\})$ and $(\{a, b, c\}, \emptyset)$ do not belong to the lattice.) Hence $L(\mathbb{K})$ is the four-element Boolean lattice, that is, M_2, which is a lattice of length 2. Thus $\mathbb{K}$ satisfies part (1) of Theorem 3.4. However, part (2) of Theorem 3.4 fails. Therefore, (4.1) is a counterexample that refutes Theorem 3.4.



Another feature which has no influence on the structure of the concept lattice are attributes which can be written as a combination of other attributes. More precisely: If $m \in M$ is an attribute and $X \subseteq M$ is a set of attributes with $m \notin X$ but $m' = X'$, then the attribute concept μm is the infimum of the attribute concepts $\mu x, x \in X$, i.e., the set $\mu(M \setminus \{m\})$ is also infimum-dense in $\mathfrak{B}(G, M, I)$, and according to the Basic Theorem

$$\mathfrak{B}(G, M, I) \cong \mathfrak{B}(G, M \setminus \{m\}, I \cap (G \times (M \setminus \{m\}))).$$

The removal of **reducible attributes**, i.e., of attributes with $\bigwedge$-reducible attribute concepts and of **reducible objects**, i.e., of objects with $\bigvee$-reducible object concepts, is called **reducing** the context. **Full rows** and **full columns** are always reducible; thereby we mean objects g with $g' = M$ and attributes m with $m' = G$, respectively.

Definition 24. A clarified context (G, M, I) is called **row reduced**, if every object concept is $\bigvee$-irreducible, and **column reduced**, if every attribute

FIGURE 1. The bottom of page 24 of Ganter and Wille [2]

Clearly, (4.1) is also a counterexample to Theorem 3.1.

4.3. **Third counterexample.** We have the following.

Claim 4.1. Let $\mathbb{K} = (A, B, I)$ be a subdirect formal context such that the context graph $\boldsymbol{G}_{\mathbb{K}} = (A \cup B, I)$ is a complete bipartite digraph. Then $|L(\mathbb{K})| \leq 2$.

There are infinitely many finite subdirect[7] formal contexts $\mathbb{K}$ satisfying the condition in Claim 4.1. Each of these formal contexts is a counterexample for the $n = 1$ case of Theorem 3.4(1)–(2), because a complete bipartite digraph is the disjoint union of n complete bipartite digraphs with $n = 1$, but $|L(\mathbb{K})| \leq 2$ excludes the possibility that $L(\mathbb{K})$ is isomorphic to the length-two (that is, size-three) lattice M_1.

Proof. Since $\boldsymbol{G}_{\mathbb{K}} = (A \cup B, I)$ is a complete bipartite digraph, all rows and all columns of K (the matrix describing the formal context) are full, that is, they consist only of ones (the $\times$ symbols). In other words, every entry of K is 1.

Either directly or by consulting Figure 1, we see that we can repeatedly omit full rows and full columns from K (equivalently, omit objects and attributes from the formal context) without changing the concept lattice. Eventually a 1×1 matrix remains, that is, the formal context has exactly one object and one attribute.

[7]One could also add the adjective *non-reduced*.

Let $\mathbb{K}' = (A', B', I')$ denote this reduced context; here $|A'| = |B'| = 1$. For a set X, let $\mathcal{P}(X)$ denote the powerset lattice of X. Since

$$L(\mathbb{K}) \cong L(\mathbb{K}') \cong \mathcal{L}_{\mathbb{K}}(A') \subseteq \mathcal{P}(A'),$$

we obtain $|L(\mathbb{K})| \leq |\mathcal{P}(A')| \leq 2$. □

Note that the argument above could also be concluded as follows: for the 1×1 formal context, the concept lattice is the singleton lattice, which cannot be isomorphic to the three-element lattice M_1.

4.4. **Fourth counterexample.** See Claim 9.1, to be presented later.

5. Why does the main result seem too easy?

Clearly, $\boldsymbol{G}_{\mathbb{K}}$ is nothing more than another visualization of $\mathbb{K}$. That is, $\boldsymbol{G}_{\mathbb{K}}$ and $\mathbb{K}$ represent the same mathematical structure in two different languages.

In the table of contents of Ganter and Wille [2], the content of Chapter 6 is listed as follows:

6. Properties of concept lattices
   - 6.1 Distributivity
   - 6.2 Semimodularity and Modularity
   - 6.3 Semidistributivity and Local Distributivity
   - 6.4 Dimension
   - 6.5 Hint and References

Therefore, the reader will quickly suspect that if much more complicated lattice properties have already been studied in terms of formal contexts (or equivalently, bipartite graphs), then a paper devoted to the much simpler property of "having length 2" cannot be deep and is perhaps not worth reading.

6. Why is the main result too easy?

Since the proofs in the paper look technical and long, while Theorem 3.4(1)(2) appears easy, many readers will try to find their own proofs rather than reading the authors' arguments. Such readers will probably discover that the proof is indeed quite simple.

The authors' "disjoint unions of finitely many complete bipartite digraphs" correspond to staircase block-matrices of ones, which I will call a $\boxed{\text{staircase matrix}}$ for short. Instead of giving a precise formal definition, I illustrate the idea with two examples:

$$K_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{6.1}$$

Here K_1 has exactly three blocks of ones (the red, green, and magenta blocks), while K_2 has two blocks (a magenta block and a red block). These matrices correspond to subdirect formal contexts $\mathbb{K}_1$ and $\mathbb{K}_2$: the ones represent the $\times$ symbols (the TRUE entries), and the zeros represent the empty entries (the FALSE entries).

We also need one more concept. Let us call a formal context $\mathbb{K} = (G, M, I)$ an NFRNFC context[8] if there exists no $g \in G$ with $\{g\} \times M \subseteq I$, and there exists no $m \in M$ with $G \times \{m\} \subseteq I$. We are now ready to state the following.

Proposition 6.1. *For an integer $n \geq 2$ and a finite NFRNFC subdirect formal context $\mathbb{K}$, the following two conditions are equivalent:*

(α) *The graph $G_{\mathbb{K}}$ associated with $\mathbb{K}$ is the disjoint union of n finite complete bipartite graphs.*

(β) $L(\mathbb{K}) \cong M_n$.

Note that every reduced formal context (see Figure 1) is NFRNFC, but not conversely. Thus Proposition 6.1 could have been formulated with "reduced subdirect" instead of "NFRNFC subdirect", but the formulation above is stronger.

As our second counterexample shows, the NFRNFC condition cannot be omitted. Proposition 6.1 fails for $n = 1$. For example, if $\boldsymbol{G}_{\mathbb{K}}$ is

[8]Based on the usual visualization of a formal context by a binary table, NFRNFC is an acronym for "no full row, no full column". This is only a temporary name. Since formal contexts have been studied for more than four decades, beginning with [1], it is unlikely that such a basic property has no established name. As I am not an expert, it is the authors' task to check the literature for the appropriate terminology.

the two-element complete bipartite graph (a single edge between two vertices), then (α) holds but $L(\mathbb{K})$ is a singleton. Furthermore, if $\mathbb{K}$ is a subdirect NFRNFC formal context, then $L(\mathbb{K}) \ncong M_1$ (the three-element chain); see Claim 9.1 for a stronger statement.

Remark 6.2. We know from Wille [1] that for an arbitrary (not necessarily staircase) bitmatrix K, the formal concepts of the corresponding formal context $\mathbb{K}$ are exactly the maximal submatrices[9] consisting entirely of ones, that is, the maximal *one-submatrices*. The order of these maximal one-submatrices is the $\subseteq$ order of their row sets, or equivalently, the $\supseteq$ order of their column sets. Extending Proposition 6.1 to all positive integers n might be possible, see Section 9, but it would probably not be worthwhile for SEMR, since the elegance of the proposition would likely be reduced, while the depth would not significantly increase.

Visual Proof of Proposition 6.1 Let $\mathbb{K} = (A, B, I)$ be a finite NFRNFC subdirect formal context, and let K be its matrix (which describes I, as before).

Assume (α). It follows from Remark 6.2 that K is a staircase bitmatrix with n blocks of ones, that these blocks are maximal one-submatrices, and that the concept lattice is (isomorphic to) M_n. Thus (α) implies (β).

Next assume (β). The NFRNFC property easily implies that

$$\begin{aligned} &(\emptyset, B) \in L(\mathbb{K}) \text{ and } 0_{L(\mathbb{K})} = (\emptyset, B), \\ &(A, \emptyset) \in L(\mathbb{K}) \text{ and } 1_{L(\mathbb{K})} = (A, \emptyset). \end{aligned} \tag{6.2}$$

Consider a maximal one-submatrix U of the bitmatrix K such that U corresponds to neither the bottom nor the top of $L(\mathbb{K})$. By the assumption $L(\mathbb{K}) \cong M_n$, the submatrix U is both an atom and a coatom of $L(\mathbb{K})$.

By (6.2), U has at least one row but fewer rows than K, and similarly for the columns. We work with the concrete bitmatrix K given in (6.3) below; modulo permuting its rows and columns, this matrix reflects the general situation.

[9]As usual, a submatrix need not be contiguous; that is, we do not require that rows or columns that are neighbors in the submatrix be neighbors in the original matrix. It is another matter that examples are often easier to visualize with contiguous submatrices.

$$K = \begin{pmatrix} x & x & z & z & z & x & x & x & x & x & x & x & x & x & x \\ y & y & 1 & 1 & 1 & y & y & \boxed{y} & y & y & y & y & y & y & y \\ y & y & 1 & 1 & 1 & y & y & \boxed{y} & y & y & y & y & y & y & y \\ y & y & 1 & 1 & 1 & y & y & \boxed{y_1} & y & y & y & y & y & y & y \\ y & y & 1 & 1 & 1 & y & y & \boxed{y} & y & y & y & y & y & y & y \\ x & x & z & z & z & x & x & x & x & x & x & x & x & x & x \\ x & x & z & \boxed{z_1} & z & x & x & x & x & x & x & x & x & x & x \\ x & x & z & z & z & x & x & x & x & x & x & x & x & x & x \end{pmatrix} \tag{6.3}$$

In (6.3), the red submatrix is U. Its row set and column set are

$$R_U = \{2, 3, 4, 5\}, \qquad C_U = \{3, 4, 5\}.$$

The symbols x, y, z, y_1, z_1 are placeholders; different occurrences may represent different bits. Additional colors indicate additional submatrices. For example, the submatrix with row set R_U and column set $\{1, 2, \dots, 15\} \setminus C_U$ is blue.

We claim that

$$\text{all the blue entries are zeros.} \tag{6.4}$$

To see this, assume that one of the blue entries, say y_1, is 1. Expand the one-by-one submatrix $\{y_1\}$ to a maximal one-submatrix V. Then $4 \in R_V$ and $8 \in C_V$. Since $4 \in R_U \cap R_V$, the meet $U \wedge V$ is nonzero in $L(\mathbb{K})$ by (6.2). Because U is an atom, this implies $U \leq V$. But U is also a coatom, so either $V = U$ or $V = 1_{L(\mathbb{K})}$. The fact that $8 \in C_V \setminus C_U$ rules out $V = U$, while $8 \in C_V$ together with (6.2) rules out $V = 1_{L(\mathbb{K})}$. This contradiction proves (6.4).

Since M_n is selfdual, the same argument applied dually yields that

$$\text{all the magenta entries are zeros.} \tag{6.5}$$

Because $\mathbb{K}$ is subdirect, K has no zero row and no zero column. Therefore (6.4) and (6.5) imply that, up to permuting rows and columns, K is a staircase bitmatrix with t blocks for some positive integer t. Hence $\boldsymbol{G}_{\mathbb{K}}$ is the disjoint union of t complete bipartite graphs.

Since $n \geq 2$, the submatrix U is not the only atom of $L(\mathbb{K})$, so $t \geq 2$. By the already proven implication $(\alpha) \Rightarrow (\beta)$, we obtain $L(\mathbb{K}) \cong M_t$. Finally, $M_n \cong L(\mathbb{K}) \cong M_t$ implies $t = n$. Thus (β) implies (α), and the visual proof of Proposition 6.1 is complete. ☐

Is the visual proof above a rigorous proof? Yes and no. It becomes rigorous after adding a few explanatory sentences, for example:

Clearly, we may write k instead of 15 and use $\dots$, $\vdots$, $\ddots$, etc. in the matrix. Furthermore, in the definition of a formal context, there is no fixed order on the set of objects or on the set of attributes. Thus we may reorder these sets so that the red entries form a contiguous submatrix; see also Footnote 9. Alternatively, we may regard the colorful matrix as a visualization only, and we could add row labels $i_1, i_2, \dots$ and column labels $j_1, \dots, j_k$ to make everything completely rigorous.

In any case, the proof above is not a polished version suitable for publication; producing such a version would require more time or more familiarity with the topic. My only purpose here was to present a correct statement needed for my report, in particular for Section 9. A Concept Lattice Expert could surely give a better proof, and such a proof may already exist.

7. Overlapping with an earlier paper, [8]

There is another concern that arises at first sight. According to its title, [8] is devoted to bipartite digraphs with *modular* concept lattices of length 2. The majority of the present paper deals with the same class of lattices, except that modularity is not assumed. However, by Dedekind's Modularity Criterion, it is obvious that the present paper and [8] (restricted to the finite case) *deal with exactly the same lattices.* In fact, the scope of [8] is slightly wider, since it also includes M_ω, so the present paper covers fewer lattices.

There is a substantial overlap between Theorem 2 of [8] and the interesting part of Theorem 3.4 of the present paper. Indeed, parts (3) and (4) of Theorem 3.4 appear to be only technical additions, while the equivalence of $\boxed{\text{(1) and (2)}}$ is already stated in Theorem 2 of [8].

Immediately after the "proof" of Theorem 3.4, the authors write that the equivalence of parts (2) and (3) was proved in Theorem 2 of [8]. Writing $\boxed{\text{(2) and (3)}}$ instead of the correct $\boxed{\text{(1) and (2)}}$ is probably only a typo, but the consequence of this typo is significant: the reader may attribute more merit and novelty to the present paper than it actually deserves.

I cannot resent a typo. *But I do resent* that the introduction does not even mention [8] or its content. When two papers are as closely related as the present one and [8], the introductory section (which is otherwise quite talkative) should have explained the overlap and the differences, and should have motivated and justified why a new paper is needed after [8]. This is an important factor that editors and reviewers should take into account.

8. Elegance, as a guiding principle

I believe that good theorems are either elegant or useful; these are not independent attributes.

The concept of a lattice and its Hasse diagram are elegant. By Wille [1], a finite lattice X can be captured by the context $\mathbb{K} = (J(X), M(X), \leq)$, and indeed $X \cong L(\mathbb{K})$. But in this translation most of the elegance is lost. Nevertheless, this loss can be tolerated, because concept lattices have practical applications; the FCA (Formal Concept Analysis) community has demonstrated this extensively.

Graphs, and in particular bipartite digraphs, also carry a certain elegance.[10]

However, I do not see any elegance in parts (3) and (4) of Theorem 3.4. In fact, they are merely technical tools used to prove Theorem 3.4. But the proof in the paper is incorrect, and Theorem 3.4 itself is false. Therefore, I dare to say that parts (3) and (4) are not very useful.[11]

This is why I am not impressed by parts (3) and (4). If Theorem 2 in [8] had been correct, then it would have been superfluous[12] to write the present paper merely for the sake of parts (3) and (4) and the trivial and incorrect Horn-sentence approach.

9. Perspectives

Let me begin with the following statement.

Claim 9.1. There exists no subdirect formal context $\mathbb{K}$ such that $L(\mathbb{K})$ is the three-element (that is, length-two) chain.

Proof. Suppose the contrary, and let $\mathbb{K}$ be a subdirect formal context such that $L(\mathbb{K})$ is the three-element chain. By Figure 1, we may assume

[10]It is a matter of taste whether a relation matrix like K or a graph is more visual, but neither is as visual as a Hasse diagram.

[11]I have often observed that a needlessly overcomplicated approach, instead of an elegant one, increases the chance of mathematical errors.

[12]The authors may even benefit from this observation: the failure of [8] provides some justification for writing a new paper, at least for a journal that is not very selective.

that all full rows and full columns have already been removed, that is, $\mathbb{K}$ has the NFRNFC property.

As in the proof of Proposition 6.1, take the formal concept $U \in L(\mathbb{K})$ which is neither the bottom nor the top of the concept lattice; see the red submatrix in (6.3).

If U coincided with the whole matrix, then $\boldsymbol{G}_{\mathbb{K}}$ would be a complete bipartite digraph, and $|L(\mathbb{K})| \leq 2$ by Claim 4.1, a contradiction. Therefore U is not the whole matrix. In terms of (6.3), this means that at least one of the blue entries y (or y_1), the magenta entries z (or z_1), or the green entries x is present.

Furthermore, at least one of the green entries must exist. If not, then the row set of U would be the set of all rows of the matrix, or analogously for the columns. Either alternative contradicts (6.2), because each formal concept (X, Y) is uniquely determined by X and also by Y.

Thus there is a green entry x. By (6.4) and (6.5), all blue and magenta entries are zeros. Since $\mathbb{K}$ is subdirect, K has no zero row and no zero column. Therefore there is a green entry x_1 whose value is 1. This x_1 forms a 1-by-1 one-submatrix. Extend this submatrix to a maximal one-submatrix V. Since x_1 is green while the blue and magenta entries are zeros, the row set R_V of V is distinct from the row set of K, and the column set C_V of V is distinct from the column set of K. Therefore, because R_V and C_V determine V uniquely, V is not (the matrix describing) $(A, \emptyset)$ or $(\emptyset, B)$, which belong to $L(\mathbb{K})$ by (6.2). Thus, using (6.2) again, the submatrix V is neither the bottom nor the top of $L(\mathbb{K})$. Hence, since $L(\mathbb{K})$ is assumed to be the three-element chain, we must have $V = U$.

But this is impossible: U consists of the red entries, while $x_1 \in V$ is green. This contradiction proves the claim. □

Armed with Claim 9.1 and the counterexamples in Section 4, a natural question arises: why insist on subdirectness?

As I see it, restricting the study to **subdirect** formal contexts causes considerable harm without offering any benefit. Why not work with NFRNFC formal contexts (or, if weaker results are acceptable, with reduced formal contexts) rather than with subdirect ones?

Another idea for avoiding the difficulties is to introduce the concepts of **weakly bipartite digraphs** and **complete weakly bipartite digraphs**. A digraph $G = (V, R)$ is *weakly bipartite* if there exist disjoint subsets A and B of V such that $V = A \cup B$ and $R \subseteq A \times B$. In other words, a digraph is weakly bipartite if it is a bipartite digraph or an edgeless digraph. Replacing $R \subseteq A \times B$ by $R = A \times B$ yields the notion of a *complete weakly bipartite digraph*; such digraphs are exactly the complete bipartite digraphs and the edgeless digraphs.

The first merit of using weakly bipartite digraphs, and of abandoning the adjective subdirect, is the following observation.

Remark 9.2. The three-element chain, that is, M_1, can be represented by an NFRNFC formal context, and also by a disjoint union of finitely many (in fact, two) complete weakly bipartite digraphs.

Sketch of Proof. Use the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and use the four-element digraph with exactly one edge. □

I guess that, motivated by Remark 9.2 and other parts of this review, the authors could do something meaningful and error-free with concept lattices of length at most 2.

10. Some typos

When there was a hope of a positive report, I made some notes. As the authors may benefit from these notes, I list the typos I found below in a random order. Note that I made no effort to make this list complete; in fact, I have not read some parts of the paper. The typo density of the paper is not particularly high, and it could be lower than that of the present report.

⊛ 1. Change "height" to "length" at all occurrences.

⊛ 2. It is not an optimal style that n in Theorem 3.4 is not specified.

⊛ 3. On every even-numbered page, the running head overlaps the page number.

⊛ 4. P4, second line above Definition 2.4: remove "the" in "For the sets A, B". Explanation: in the previous sentence, A and B occur only as bound variables. They function merely as placeholders; they could just as well have been written X and Y. Therefore, A and B have not been fixed previously, so the definite article "the" is not appropriate here.

⊛ 5. In the Abstract: change ”we make remark concerning lattices” to ”we make remarks concerning lattices” or ”we make a remark concerning lattices”.

⊛ 6. P3, definition of a bipartite graph: I suggest using parentheses, that is, $(V_1 \times V_2) \cup (V_2 \times V_1)$ is what we need. Furthermore, why introduce non-directed graphs? Digraphs would be sufficient for the statements.

⊛ 7. P3 L−3: remove the space from $G\ (M)$.

⊛ 8. P4 L3: I see no reason to use $\preceq$, the ”covered by or equal to” relation, to denote the lattice order of $\mathcal{L}(\mathbb{K})$.

⊛ 9. P7: ”it followers that” (occurring twice) is not correct English; the correct phrase is ”it follows that”.

⊛ 10. When a paper defines well-known notions like bipartite graphs, it is strange that the definition of universal Horn sentences and classes is not presented.

⊛ 11. In Theorem 3.4(3), ”forms n-partition” is ungrammatical; the correct form is ”forms an n-partition”. An additional problem is that the concept of an n-partition is not defined in the paper.

⊛ 12. Why are the items in the References section not listed in alphabetical order by author?

11. Finally

The authors are free to use any idea, (counter)example, statement, and proof from this review; I take no credit for them. However, I am not an expert and cannot tell which of them are genuine, which belong to the folklore, or which have already appeared in print. Furthermore, I cannot guarantee that this writing is error-free; probably it is not.