S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 144–145 (2023)
https://doi.org/10.33048/semi.2023.20.???

УДК 517.97
MSC 35J86,74K20

NON-CONVEX PROBLEM FOR A KIRCHHOFF–LOVE PLATE WITH A CRACK PENETRATING BY A WEDGE-SHAPED OBSTACLE

V.A. KOVTUNENKO AND **N.P. LAZAREV**

Communicated by P.P. Petrov

Abstract: A novel class of non-convex contact problems for a cracked Kirchhoff–Love plate is considered. The variational model is formulated under nonpenetration conditions on the crack faces, which are allowed to come into contact with each other and with a wedge-shaped obstacle. The variational formulation involves non-convexity in the set of admissible displacements. The existence of a non-unique solution to the non-convex problem is established by purchasing the weak closedness of the admissible set. Furthermore, within this approach, four possible configurations of crack faces in the equilibrium state of the plate are examined. For these special cases, the minimization problem is stated over convex sets of admissible functions. Accordingly, optimality and boundary conditions are obtained under the assumption of regularity for the unique solutions of the corresponding variational problems. The modeling is actual for fracture of biological materials like cortical bones.

Keywords: variational inequality, crack, non-penetration conditions, obstacle, non-convex feasible set.

Kovtunenko V.A., Lazarev N.P. Non-convex problem for a Kirchhoff–Love plate.

The work is supported by the Ministry of science and higher education of the Russian Federation, grant number FSRG-2026-0009.

Received april, 16, 2026, Published December, 31, 2026.

1 Introduction

Nonlinear contact problems in the mechanics of deformable solids merit significant interest among researchers, focusing on both the qualitative properties of solutions and their implications for engineering [1, 2, 3, 4]. Given the wide application of thin-walled structures, such as plates and shells, across various technological processes, numerous contact problems associated with these structures have been extensively studied in the scientific literature [5, 6, 7]. The modeling by cracked plates contacting obstacles is motivated by description of fracture due to cutting indentation with a hard, sharp wedge [8]. It is of significance for surgical procedures involving bone tissues and bio-ceramics implants; see [9, 10, 11]. In particular, the interactions along the frontal surfaces of plates in contact with non-deformable obstacles have been examined in references [12, 13], while the contact by their lateral cylindrical surfaces has been analyzed in [14, 15]. The case of contact interaction between plates and inclined undeformable obstacles we refer to [17, 18]. For numerical solution, a semi-smooth Newton method was formulated within the framework of a generalized gradient and developed for the Timoshenko plate in [16].

A number of prior works are related to contact problems for bodies of different dimensions, which also include studies of plates contacting thin elastic beams [19, 20, 21]. Applying the method of fictitious domains, these problems in the case of the contact with nondeformable rigid obstacles are qualitatively connected with crack problems under non-penetration conditions of inequality type [14, 22, 23]. The study of crack problems in elastic plates often involves Signorini-type interface conditions to consistently describe the physical non-penetration between opposite crack surfaces. For example, see contact modeling of homogeneous plates [24], inhomogeneous plates with delaminated inclusions [25, 26], and composite bodies with delaminated junctions [27, 28, 29]. In addition to non-penetration constraints, problems allowing for finite penetration [30], as well as describing nonlinear indentation [31] are also of research interest. It is worth noting that some variational models of solids mechanics have non-convex structure of sets of admissible functions [32, 33].

In this paper, we continue the research started in [34] for the Timoshenko plates. Here we adopt the approach of the mentioned paper to investigate a crack in a Kirchhoff–Love plate interacting with a wedge-shaped obstacle. The geometric singularity of the contact interactions under consideration leads to non-convexity of the corresponding set of admissible displacements. Employing the Weierstrass theorem, the existence of a solution to this non-convex problem is established. Subsequently, for special four cases of known plate and obstacle configurations in the equilibrium state, the admissible sets become convex. In these particular instances, optimality conditions are derived, assuming additional regularity of the unique solutions for the corresponding variational problems.

2 Formulation of a variational problem

Let us describe a spacial configuration of a Kirchhoff–Love plate containing a throughout crack. We assume that the plate has the uniform thickness $2h$ and its midplane coincides with the plane Ox_1x_2 in the reference state. We introduce a bounded simply connected domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary Γ such that a straight line segment

$$\gamma = \{(x_1, x_2)\,|\, x_1 = q, \quad a \le x_2 \le b\}$$

lies strictly inside Ω, where $a < b$ and q are some real numbers. Assume that Γ consists of two curves: Γ_0 and Γ_1 such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. In addition, in order to satisfy the conditions of Korn's inequality, we require that Ω can be partitioned into two simply connected Lipschitz subdomains $\hat{\Omega}_1$ and $\hat{\Omega}_2$ by extending the curve γ so that $\mathrm{meas}(\partial\hat{\Omega}_i \cap \Gamma_0) > 0$, $i = 1, 2$, and $\gamma \subset \partial\hat{\Omega}_1 \cap \partial\hat{\Omega}_2$, $\overline{\Omega} = \overline{\hat{\Omega}_1} \cup \overline{\hat{\Omega}_2}$.

We can parametrize the line segment γ by $t \in I$, $I = [a, b]$ (in the sequel we will remark generalization to curvilinear lines). The normal vector $\boldsymbol{\nu} = (\nu_1, \nu_2)(t)$ pointed from negative to positive direction distinguishes two opposite crack faces γ^+ and γ^-. We denote the jump of functions across the crack

$$[\![\boldsymbol{v}]\!] := \boldsymbol{v}^+ - \boldsymbol{v}^-, \quad \boldsymbol{v}^\pm := \boldsymbol{v}|_{\gamma^\pm},$$

for some vector functions $\boldsymbol{v}$ defined in $\Omega_\gamma = \Omega \backslash \overline{\gamma}$.

Denote by $\boldsymbol{\chi} = \boldsymbol{\chi}(\boldsymbol{x}) = (\boldsymbol{W}, w)$ the displacement vector of the mid-plane points $\boldsymbol{x} \in \Omega_\gamma$, by $\boldsymbol{W} = (w_1, w_2)$ the displacements in the plane $\{x_1, x_2\}$, and by w the displacements along the axis z. The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{W})$, $\sigma_{ij} = \sigma_{ij}(\boldsymbol{W})$, respectively:

$$\varepsilon_{ij}(\boldsymbol{W}) = \frac{1}{2}(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}), \quad \sigma_{ij}(\boldsymbol{W}) = a_{ijkl}\varepsilon_{kl}(\boldsymbol{W}), \quad i, j = 1, 2,$$

where $\{a_{ijkl}\}$ is the given elasticity tensor, assumed to be symmetric and positive definite:

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega_\gamma),$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2, \quad \forall\xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const > 0.$$

A summation convention over repeated indices is used here and in the sequel. Next we denote the bending moments by formulas

$$m_{ij}(w) = -d_{ijkl}w_{,kl}\,, \quad i, j = 1, 2, \quad (w_{,kl} = \frac{\partial^2 w}{\partial x_k \partial x_l})$$

where the tensor $\{d_{ijkl}\}$ has the same symmetry and positive definiteness properties as the tensor $\{a_{ijkl}\}$. Let $B(\cdot\,,\cdot)$ be a bilinear form defined by

$$B(\boldsymbol{\chi}, \bar{\boldsymbol{\chi}}) = \int\limits_{\Omega_\gamma} \{\sigma_{ij}(\boldsymbol{W})\,\varepsilon_{ij}(\bar{\boldsymbol{W}}) - m_{ij}(w)\bar{w}_{,ij}\,\}d\boldsymbol{x},$$

for functions $\boldsymbol{\chi} = (\boldsymbol{W}, w)$, $\bar{\boldsymbol{\chi}} = (\bar{\boldsymbol{W}}, \bar{w})$. The potential energy functional of the plate has the following representation [12]:

$$\Pi(\boldsymbol{\chi}) = \frac{1}{2}B(\boldsymbol{\chi}, \boldsymbol{\chi}) - \int_{\Omega_\gamma} \boldsymbol{F} \cdot \boldsymbol{\chi} d\boldsymbol{x},$$

with the vector $\boldsymbol{F} = (f_1, f_2, f_3) \in L_2(\Omega)^3$ describing external forces.

Introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \left\{ v \in H^1(\Omega_\gamma) \;\middle|\; v = 0 \text{ on } \Gamma_0 \right\},$$
$$H^{2,0}(\Omega_\gamma) = \left\{ v \in H^2(\Omega_\gamma) \;\middle|\; v = \tfrac{\partial v}{\partial \boldsymbol{n}} = 0 \text{ on } \Gamma_0 \right\},$$
$$H(\Omega_\gamma) = H^{1,0}(\Omega_\gamma)^2 \times H^{2,0}(\Omega_\gamma),$$

where $\boldsymbol{n} = (n_1, n_2)$ is a unit normal to the external boundary Γ. Hence, for the displacement function $\boldsymbol{\chi} = (\boldsymbol{W}, w)$ belonging to $H(\Omega_\gamma)$ we have the following clamping condition at the outer boundary of the plate

$$\boldsymbol{W} = \boldsymbol{0}, \quad w = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_0. \tag{1}$$

Note that the coercivity inequality

$$B(\boldsymbol{\chi}, \boldsymbol{\chi}) \geq c\|\boldsymbol{\chi}\|^2 \quad \forall\, \boldsymbol{\chi} \in H(\Omega_\gamma), \quad (\|\boldsymbol{\chi}\| = \|\boldsymbol{\chi}\|_{H(\Omega_\gamma)})$$

with a constant $c > 0$ independent of $\boldsymbol{\chi}$ holds for the bilinear form $B(\cdot, \cdot)$; see [12]. In order to define an obstacle, we introduce two families of generatrices. We assume that for each fixed $t \in I$, $L(t)$ is a straight line given by the following relations recalling the normal $\boldsymbol{\nu}$

$$L(t) = \{\boldsymbol{x} = (x_1, x_2) \,|\, (x_1, x_2) = \boldsymbol{\gamma}(t) + a\boldsymbol{\nu}(t), \quad a \in \mathbb{R}\},$$

where $\boldsymbol{\gamma}(t) = \big(\gamma_1(t), \gamma_2(t)\big)$ is some point of the parametrized curve γ. We introduce a curvilinear wedge $\mathcal{O}$ of the obstacle lying over the crack $\gamma \times [-h, h]$. To construct opposite boundaries of $\mathcal{O}$, we define the generatrix for the positive surface:

$$(\boldsymbol{x} - \boldsymbol{\gamma}(t)) \cdot \boldsymbol{\nu}(t) = k^+(t)(z - h) \quad \text{for } t \in I, \quad \boldsymbol{x} \in L(t), \quad (\boldsymbol{x} - \boldsymbol{\gamma}(t)) \cdot \boldsymbol{\nu}(t) \geq 0, \tag{2}$$

where $0 < k^+ < \infty$, and by the generatrix for the negative surface, respectively:

$$(\boldsymbol{x} - \boldsymbol{\gamma}(t)) \cdot \boldsymbol{\nu}(t) = k^-(t)(z - h) \quad \text{for } t \in I, \quad \boldsymbol{x} \in L(t), \quad (\boldsymbol{x} - \boldsymbol{\gamma}(t)) \cdot \boldsymbol{\nu}(t) \leq 0, \tag{3}$$

where $-\infty < k^- < 0$, see Figure 1.

Remark 1. *According to the proposed reasoning, we can generalize the approach of the obstacle construction for sufficiently smooth simple curve $\gamma \subset \Omega$. It is obvious that generatrices for both negative and positive surfaces can be constructed with respect to the normal $\boldsymbol{\nu}$ to γ and taking into account the equations (2), (3). As an example, we can provide a crack curve by an arc*

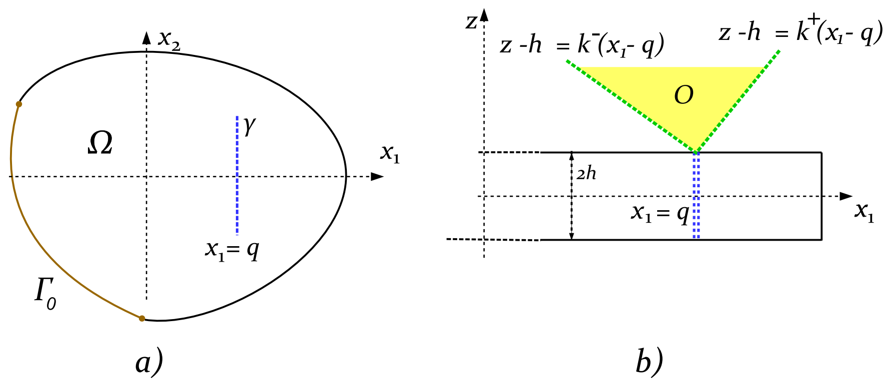

FIG. 1. a) Domain with a cut. b) Obstacle generatrices for a rectilinear γ.

of a circle, for this case obstacle faces can be specified by two different conical surfaces. Note that similar approaches of describing obstacles or cracks were previously used in the framework of plates in [16, 35].

Since the obstacle touches the crack in the reference state, it is necessary to impose non-penetration conditions describing possible contact of crack edges with the obstacle. Following the approach applied for a Timoshenko plate [16, 34], we will specify Signorini-type boundary conditions on γ to exclude penetration of plate points into the rigid obstacle, and the mutual penetration between opposite crack faces. Taking into account the reference configuration of γ and $\mathcal{O}$, we distinguish four cases of possible interactions between crack faces and obstacle. In so doing, we suppose that there are four measurable parts γ_i of the curve γ, $i = 1, 2, 3, 4$, such that $\gamma_i \cap \gamma_j = \emptyset$, $i \neq j$, where certain geometries are realized. Assume that on γ_1 the plate does not contact with the obstacle, so we state a non-penetration condition for crack [12] and two conditions for deflections w^+, w^- on both sides of γ_1:

$$w^+ \leq 0, \quad w^- \leq 0, \quad [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h |[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \quad \text{on} \quad \gamma_1. \tag{4}$$

As the second case of possible interactions of crack faces with the obstacle surfaces we consider the geometry when the positive crack edge may come into contact with the obstacle, and other (negative) crack edge does not contact with the obstacle. Moreover, opposite crack faces may also interact with each other subject to the non-penetration condition:

$$w^+ \geq 0, \quad w^- \leq 0, \quad \boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq k^+ w^+,$$
$$[\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h |[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \quad \text{on} \quad \gamma_2. \tag{5}$$

Note that the third inequality in (5)

$$\boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq k^+ w^+$$

was used in [36] for a nonlinear model describing contact of a plate with an inclined obstacle. The third case is analogous to the previous. Namely, a part of the negative crack edge may come into contact with the obstacle, and other (positive) crack edge does not interact with the obstacle. It is obvious that, by changing the direction of the normal vector $\boldsymbol{\nu}$, third case can be interpreted similar to the previous one:

$$w^- \geq 0, \quad w^+ \leq 0, \quad \boldsymbol{W}^- \cdot \boldsymbol{\nu} - h\frac{\partial w^-}{\partial \boldsymbol{\nu}} \leq k^- w^-, \\ [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \quad \text{on} \quad \gamma_3. \tag{6}$$

Now we describe the most involved case where both crack edges may contact with the obstacle as well as with each other. In so doing, we impose nonlinear constraints given by the following set of inequalities

$$w^+ \geq 0, \quad w^- \geq 0, \quad [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]|, \\ \boldsymbol{W}^- \cdot \boldsymbol{\nu} - h\frac{\partial w^-}{\partial \boldsymbol{\nu}} \leq k^- w^-, \quad \boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq k^+ w^+ \quad \text{on} \quad \gamma_4. \tag{7}$$

We observe that, the fulfillment of the last two inequalities in (7) yields non-penetration for crack points lying on the top plate surface. Indeed, in view of $w^+ \geq 0$, $w^- \geq 0$, $k^+ > 0$, $k^- < 0$, we have

$$\boldsymbol{W}^- \cdot \boldsymbol{\nu} - h\frac{\partial w^-}{\partial \boldsymbol{\nu}} \leq 0, \quad \boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq 0 \quad \text{on} \quad \gamma_4,$$

and, as a consequence,

$$[\![\boldsymbol{W} \cdot \boldsymbol{\nu} - h\frac{\partial w}{\partial \boldsymbol{\nu}}]\!] \geq 0 \quad \text{on} \quad \gamma_4.$$

The last inequality specifies nonpenetration of points lying on crack faces with coordinates $(\boldsymbol{x}, h)$, $\boldsymbol{x} \in \gamma_4$, i.e. on the frontal plate surface. Therefore, it suffices to require instead of the relation $[\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]|$ on γ_4, the following inequality

$$[\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] + h[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!] \geq 0 \quad \text{on} \quad \gamma_4,$$

that holds for points lying on the bottom plate surface, where $z = -h$. This allow us to reduce (7) as follows

$$w^+ \geq 0, \quad w^- \geq 0, \quad \boldsymbol{W}^- \cdot \boldsymbol{\nu} - h\frac{\partial w^-}{\partial \boldsymbol{\nu}} \leq k^- w^-, \\ \boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq k^+ w^+, \quad [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq -h[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!] \quad \text{on} \quad \gamma_4. \tag{8}$$

Following the free-boundary approach, parts γ_i, $i = 1, 2, 3, 4$, are unknown and may be empty in general. In view of this circumstance, we can assume that at least one of the four conditions (4), (5), (6), (8) holds true. Namely, we require that the following relations are valid almost everywhere on γ:

$$w^+ \leq 0, \quad w^- \leq 0, \quad [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \quad \text{or} \tag{9}$$

$$w^+ \geq 0, \quad w^- \leq 0, \quad \boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq k^+ w^+, \\ [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \quad \text{or} \tag{10}$$

$$w^- \geq 0, \quad w^+ \leq 0, \quad \boldsymbol{W}^- \cdot \boldsymbol{\nu} - h\frac{\partial w^-}{\partial \boldsymbol{\nu}} \leq k^- w^-, \\ [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \quad \text{or} \tag{11}$$

$$w^+ \geq 0, \quad w^- \geq 0, \quad \boldsymbol{W}^- \cdot \boldsymbol{\nu} - h\frac{\partial w}{\partial \boldsymbol{\nu}} \leq k^- w^-, \\ \boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \geq k^+ w^+, \quad [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq -h[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]. \tag{12}$$

This means that for each point of the set $\gamma \backslash B$ (where B has zero measure), at least one of the four set of relations (9)–(12) is true. By this, the non-penetration condition $[\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]|$ for crack holds on γ for all four sets. As can be noted, values of the deflection function w, taken near the crack curve γ, determine possibility of contact with the obstacle. According to the physical sense, for non-negative deflection values, we should impose inequalities preventing possible penetration into the obstacle. For the other case of non-positive deflections, contact with the obstacle does not happen.

Consider the following set of admissible displacements

$$K = \{\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma) \,| \\ \boldsymbol{\chi} \text{ satisfies at least one of the relations (9)} - \text{(12)}\}.$$

Here we treat the constraints written in K in the following sense. We say that a function $\boldsymbol{\chi}(\boldsymbol{x})$ belongs to K if at least one of relations (9)–(12) is fulfilled at points $\boldsymbol{x}$ on the crack. The set K is evidently non-empty. One can note that K is not convex. Since the restriction describing non-penetration for the crack holds, we have the inclusion $K \subset K_c$, where

$$K_c = \{\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma) \,|\, [\![\boldsymbol{W} \cdot \boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]| \text{ on } \gamma\}$$

is a convex set formulated for a Kirchhoff–Love plate with a crack earlier in [12].

Let us give a variational formulation of the equilibrium problem. It is required to find a function $\boldsymbol{\xi} = (\boldsymbol{U}, u) \in K$, such that

$$\Pi(\boldsymbol{\xi}) = \inf_{\boldsymbol{\chi} \in K} \Pi(\boldsymbol{\chi}). \tag{13}$$

Theorem 1. *The minimization problem* (13) *has a solution.*

Proof. We will prove the existence of a solution to the problem with the help of the Weierstrass theorem [37]. The energy functional Π has the quadratic form obeying coercivity and weak lower semicontinuity on $H(\Omega_\gamma)$. Below we justify that the set K is weakly closed in $H(\Omega_\gamma)$; see details in [34].

Let a sequence of functions $\{\boldsymbol{\chi}_n\} \subset K$, $\boldsymbol{\chi}_n = (\boldsymbol{W}_n, w_n)$, $n \in \mathbb{N}$, be such that $\boldsymbol{\chi}_n \to \boldsymbol{\chi}$ weakly in the space $H(\Omega)$ as $n \to \infty$. By virtue of the embedding theorems, this implies that there exist subsequences $\{\boldsymbol{\chi}_n\}$, $\{\frac{\partial w_n}{\partial \boldsymbol{\nu}}\}$ still denoted in the same way, that converge almost everywhere on the whole crack curve to $\boldsymbol{\chi}$, $\frac{\partial w}{\partial \boldsymbol{\nu}}$, respectively. For an arbitrarily fixed point $\boldsymbol{x} \in \gamma$, we get that one of the conditions (9)–(12) is fulfilled for the limiting function $\boldsymbol{\chi}$. Then the limit function $\boldsymbol{\chi}$ also belongs to K. □

Remark 2. *In order to simplify the mathematical model for a plate contacting a thin rigid obstacle, we can put $k^- = k^+ = 0$ in non-penetration conditions* (9)–(12)*. In this case we have a thin non-deformable obstacle defined by the cylindrical surface*

$$\{(\boldsymbol{x}, z) \mid \boldsymbol{x} \in \gamma, z \in [h, +\infty)\}.$$

It is obvious, that the assertion of the Theorem 1 can be extended for values $k^- = k^+ = 0$ in (9)–(12)*.*

3 Optimality conditions for particular cases of known contact zones

In this section, we assume that the crack interacts with the obstacle according to previous configurations. Namely, we consider the four cases of possible geometries leading to separate problems. Each problem deals with a convex set of possible displacements and, as a consequence, the corresponding problem yields optimality condition in the form of a variational inequality. Assuming regularity of its solution allows us to derive equilibrium equations in the domain Ω_γ, and complementary boundary conditions satisfied on γ.

Suppose that crack γ can be extended to a closed curve Σ so that the domain Ω_γ is divided into two Lipschitz subdomains Ω_1, Ω_2 with boundaries of class $C^{1,1}$ and $\partial\Omega_1 = \Sigma$, $\bar{\Omega}_1 \subset \Omega$, $\partial\Omega_2 = \Sigma \cup \Gamma$. For simplicity, within the current section, we assume that Γ_0 coincides with Γ. Previously defined possible contact configurations fit certain signs of deflection function w on γ^+ and γ^-. Namely, we suppose that four cases under consideration correspond to appropriate non-penetration conditions (4), (5), (6), (8) holding on the whole γ. Then, assuming that $\gamma_i = \gamma$, $i = 1, 2, 3, 4$, we introduce the sets of admissible displacements as follows

$$K_1 = \{\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma) \,|\, \boldsymbol{\chi} \text{ satisfies } (4)\},$$
$$K_2 = \{\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma) \,|\, \boldsymbol{\chi} \text{ satisfies } (5)\},$$
$$K_3 = \{\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma) \,|\, \boldsymbol{\chi} \text{ satisfies } (6)\},$$
$$K_4 = \{\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma) \,|\, \boldsymbol{\chi} \text{ satisfies } (8)\}.$$

One can see that all K_i, $i = 1, 2, 3, 4$, are convex and closed. In the following case study we formulate variational problems over sets K_i, $i = 1, 2, 3, 4$ and obtain optimality conditions under the assumption of the additional regularity of solutions.

3.1. Case of the set K_1. Due to the convexity of the set K_1, the following minimization problem

$$\Pi(\boldsymbol{\xi}) = \inf_{\boldsymbol{\chi} \in K_1} \Pi(\boldsymbol{\chi}) \tag{14}$$

is equivalent to the variational inequality

$$\boldsymbol{\xi} \in K_1, \quad B(\boldsymbol{\xi}, \boldsymbol{\chi} - \boldsymbol{\xi}) \geq \int\limits_{\Omega_\gamma} \boldsymbol{F} \cdot (\boldsymbol{\chi} - \boldsymbol{\xi}) d\boldsymbol{x} \quad \forall \boldsymbol{\chi} \in K_1. \tag{15}$$

Observe that problem (15) admits unique solution $\boldsymbol{\xi} \in K_1$. Suppose that the solution $\xi = (\boldsymbol{U}, u) \in K_1$ is sufficiently regular. Testing (15) with functions $\boldsymbol{\chi} = \boldsymbol{\xi} \pm \tilde{\boldsymbol{\chi}}$, $\tilde{\boldsymbol{\chi}} \in C_0^\infty(\Omega_\gamma)^3$, we obtain the equilibrium equations

$$-\sigma_{ij,j}(\boldsymbol{U}) = f_i \quad - m_{ij,ij}(u) = f_3 \quad \text{in} \quad \Omega_\gamma. \tag{16}$$

With the help of Green's formulas holding for Lipschitz domains Ω_1, Ω_2, we have

$$\int\limits_{\Omega_\gamma} \sigma_{ij}(\boldsymbol{U})\, \varepsilon_{ij}(\bar{\boldsymbol{W}}) dx = -\int\limits_{\Omega_\gamma} \sigma_{ij,j}(\boldsymbol{U}) \bar{w}_i \, dx - \\ - \int\limits_{\gamma} \left[\!\left[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U}) \bar{\boldsymbol{W}} \cdot \boldsymbol{\nu} + \sigma_{\boldsymbol{\tau}}(\boldsymbol{U}) \bar{\boldsymbol{W}}_{\boldsymbol{\tau}} \right]\!\right] d\gamma, \tag{17}$$

$$\int\limits_{\Omega_\gamma} m_{ij}(u) \bar{w}_{,ij} \, dx = \int\limits_{\Omega_\gamma} m_{ij,ij}(u) \, \bar{w} \, dx - \int\limits_{\gamma} \left[\!\left[t_{\boldsymbol{\nu}}(u) \bar{w} - m_{\boldsymbol{\nu}}(u) \frac{\partial \bar{w}}{\partial \boldsymbol{\nu}} \right]\!\right] d\gamma, \tag{18}$$

$$\sigma_{\boldsymbol{\nu}}(\boldsymbol{U}) = \sigma_{ij}(\boldsymbol{U}) \nu_i \nu_j, \quad m_{\boldsymbol{\nu}}(u) = -m_{ij} \nu_i \nu_j,$$
$$\sigma_{\boldsymbol{\tau}}(\boldsymbol{U}) = (\sigma^1_{\boldsymbol{\tau}}(\boldsymbol{U}), \sigma^2_{\boldsymbol{\tau}}(\boldsymbol{U})) = (\sigma_{1j}(\boldsymbol{U}) \nu_j, \sigma_{2j}(\boldsymbol{U}) \nu_j) - \sigma_{\boldsymbol{\nu}}(\boldsymbol{U}) \boldsymbol{\nu},$$
$$t_{\boldsymbol{\nu}}(u) = -m_{ij,k} \tau_k \tau_j \nu_i - m_{ij,j} \nu_i, \quad \boldsymbol{\tau} = (-\nu_2, \nu_1),$$
$$\bar{\boldsymbol{W}} \cdot \boldsymbol{\nu} = \bar{w}_i \nu_i, \quad \bar{\boldsymbol{W}}_{\boldsymbol{\tau}} = (\bar{\boldsymbol{W}}^1_{\boldsymbol{\tau}}, \bar{\boldsymbol{W}}^2_{\boldsymbol{\tau}}), \quad \bar{w}_i = (\bar{\boldsymbol{W}} \cdot \boldsymbol{\nu}) \nu_i + \bar{\boldsymbol{W}}^i_{\boldsymbol{\tau}}, \quad i = 1, 2,$$

for $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in H(\Omega_\gamma)$; see [12]. Transforming (15) with the help of (16)–(18), we have

$$\int_\gamma \Big[\!\Big[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\big((\boldsymbol{W}-\boldsymbol{U})\cdot\boldsymbol{\nu}\big) + \sigma_{\boldsymbol{\tau}}(\boldsymbol{U})\big(\boldsymbol{W}_{\boldsymbol{\tau}} - \boldsymbol{U}_{\boldsymbol{\tau}}\big) + \\ + t_{\boldsymbol{\nu}}(u)(w-u) - m_{\boldsymbol{\nu}}(u)\frac{\partial(w-u)}{\partial\boldsymbol{\nu}} \Big]\!\Big] d\gamma \le 0 \quad \text{for all} \quad \boldsymbol{\chi} \in K_1. \tag{19}$$

Since for an arbitrary test function $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_1$ the values $\boldsymbol{W}_{\boldsymbol{\tau}}$ do not affect validity of the non-penetration inequalities (5), we get

$$\sigma^{\pm}_{\boldsymbol{\tau}}(\boldsymbol{U}) = \mathbf{0} \quad \text{on} \quad \gamma^{\pm}. \tag{20}$$

Substituting $\boldsymbol{\chi} = 2\boldsymbol{\xi}$ and $\boldsymbol{\chi} = \mathbf{0}$ in (19), due to (20), we obtain

$$\int_\gamma \Big\{ [\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{U}\cdot\boldsymbol{\nu}]\!] + [\![t_{\boldsymbol{\nu}}(u)u]\!] - [\![m_{\boldsymbol{\nu}}(u)\frac{\partial u}{\partial \boldsymbol{\nu}}]\!] \Big\} d\gamma = 0.$$

Inserting in (19) the test functions $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_1$ with the properties $\boldsymbol{W} = \boldsymbol{U}$, $\frac{\partial w}{\partial \boldsymbol{\nu}} = \frac{\partial u}{\partial \boldsymbol{\nu}}$ on γ, we get

$$I = \int_\gamma t^+_{\boldsymbol{\nu}}(u)(w^+ - u^+) - t^-_{\boldsymbol{\nu}}(u)(w^- - u^-)\Big\} d\gamma \le 0 \tag{21}$$

for all $w^+ \le 0$ and $w^- \le 0$ on γ, $w \in H^2_0(\Omega_\gamma)$. Then taking in (21) test functions $w^+ = u^+ + \tilde{w}^+$, $\tilde{w}^+ \le 0$ and $\tilde{w}^- = u^-$ on γ for $\tilde{w} \in H^2_0(\Omega_\gamma)$, we obtain

$$t^+_{\boldsymbol{\nu}}(u) \ge 0 \quad \text{on} \quad \gamma. \tag{22}$$

By analogous reasoning one can show that

$$t^-_{\boldsymbol{\nu}}(u) \le 0 \quad \text{on} \quad \gamma. \tag{23}$$

Using standard variational arguments for test functions $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_1$ with properties $\boldsymbol{W} \in H^1_0(\Omega)^2$, $w \in H^2_0(\Omega)$, we arrive at

$$[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] = [\![m_{\boldsymbol{\nu}}(u)]\!] = 0 \quad \text{on} \quad \gamma. \tag{24}$$

The last equations allow us to represent (19) as follows

$$\int_\gamma \Big\{ \sigma_{\boldsymbol{\nu}}(\boldsymbol{U})[\![\boldsymbol{W} - \boldsymbol{U}]\!]\boldsymbol{\nu} - m_{\boldsymbol{\nu}}(u)[\![\frac{\partial(w-u)}{\partial\boldsymbol{\nu}}]\!] \Big\} d\gamma \le 0 \quad \text{for all} \quad \boldsymbol{\chi} \in K_1. \tag{25}$$

From (25) we obtain

$$|m_{\boldsymbol{\nu}}(u)| \le -h\sigma_{\boldsymbol{\nu}}(\boldsymbol{U}) \quad \text{on} \quad \gamma. \tag{26}$$

In view of (22)–(24) and the last inequality, we get the equality

$$\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})[\![\boldsymbol{U}\cdot\boldsymbol{\nu}]\!] + t_{\boldsymbol{\nu}}(u)[\![u]\!] - m_{\boldsymbol{\nu}}(u)[\![\frac{\partial u}{\partial\boldsymbol{\nu}}]\!] = 0 \quad \text{on} \quad \gamma. \tag{27}$$

Following the scheme of the proof in [12, 35], we can establish the converse assertion. Namely, assuming that for some regular function $\boldsymbol{\xi} \in K_1$ the equilibrium equations (16) and boundary conditions (20), (22)–(24), (26), (27) are fulfilled, this follows the variational inequality (15).

3.2. Case of the set K_2. Now we consider the problem

$$\Pi(\boldsymbol{\xi}) = \inf_{\boldsymbol{\chi}\in K_2} \Pi(\boldsymbol{\chi}) \tag{28}$$

for the set K_2 subject to the following constraints on γ

$$w^+ \geq 0, \quad w^- \leq 0, \quad (\boldsymbol{W}^+ \cdot \boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}}) \geq k^+ w^+, \quad [\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!] \geq h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]|.$$

As in the previous case the solution is unique. Furthermore, we can derive from the equivalent variational inequality

$$\boldsymbol{\xi} \in K_2, \quad B(\boldsymbol{\xi}, \boldsymbol{\chi} - \boldsymbol{\xi}) \geq \int\limits_{\Omega_\gamma} \boldsymbol{F}\cdot(\boldsymbol{\chi}-\boldsymbol{\xi})\boldsymbol{d}x \quad \forall \boldsymbol{\chi} \in K_2, \tag{29}$$

the equilibrium equations (16), and the inequality

$$\int\limits_\gamma [\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\big((\boldsymbol{W}-\boldsymbol{U})\cdot\boldsymbol{\nu}\big) + \sigma_{\boldsymbol{\tau}}(\boldsymbol{U})\big(\boldsymbol{W}_{\boldsymbol{\tau}} - \boldsymbol{U}_{\boldsymbol{\tau}}\big) +$$

$$+ t_{\boldsymbol{\nu}}(u)(w-u) - m_{\boldsymbol{\nu}}(u)\frac{\partial(w-u)}{\partial\boldsymbol{\nu}}]\!] d\gamma \leq 0 \quad \text{for all} \quad \boldsymbol{\chi}\in K_2. \tag{30}$$

Applying the arguments of the previous subsection, one can show that

$$\sigma^{\pm}_{\boldsymbol{\tau}}(\boldsymbol{U}) = \boldsymbol{0} \quad \text{on } \gamma^{\pm}. \tag{31}$$

Comparing two inequalities obtained from (30) by substitution $\boldsymbol{\chi} = \boldsymbol{0}$, $\boldsymbol{\chi} = 2\boldsymbol{\xi}$, we obtain in view of (31) that

$$\int\limits_\gamma [\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{U}\cdot\boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)u - m_{\boldsymbol{\nu}}(u)\frac{\partial u}{\partial\boldsymbol{\nu}}]\!] d\gamma = 0, \tag{32}$$

$$\int\limits_\gamma [\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{W}\cdot\boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)w - m_{\boldsymbol{\nu}}(u)\frac{\partial w}{\partial\boldsymbol{\nu}}]\!] d\gamma \leq 0 \quad \text{for all} \quad \boldsymbol{\chi}\in K_2. \tag{33}$$

One can observe that the integrand of (33) can be expressed as

$$\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{W}^+\cdot\boldsymbol{\nu} - \sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{W}^-\cdot\boldsymbol{\nu} + t^+_{\boldsymbol{\nu}}(u)w^+ - t^-_{\boldsymbol{\nu}}(u)w^- -$$

$$-m^+_{\boldsymbol{\nu}}(u)\frac{\partial w^+}{\partial\boldsymbol{\nu}} + m^-_{\boldsymbol{\nu}}(u)\frac{\partial w^-}{\partial\boldsymbol{\nu}} = [\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{W}^+\cdot\boldsymbol{\nu} + \sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U})[\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!] +$$

$$+ [\![t^{\nu}(u)]\!]w^+ + t^-_{\boldsymbol{\nu}}(u)[\![w]\!] - [\![m_{\boldsymbol{\nu}}(u)]\!]\frac{\partial w^+}{\partial\boldsymbol{\nu}} - m^-_{\boldsymbol{\nu}}(u)[\![\frac{\partial w}{\partial\boldsymbol{\nu}}]\!].$$

Then we restrict test functions $\boldsymbol{\chi}\in K_2$ in (33) by relations

$$\boldsymbol{W}^+\cdot\boldsymbol{\nu} - h\frac{\partial w^+}{\partial\boldsymbol{\nu}} \geq 0, \quad [\![\boldsymbol{W}]\!] = \boldsymbol{0}, \quad [\![\frac{\partial w}{\partial\boldsymbol{\nu}}]\!] = w = 0 \quad \text{on} \quad \gamma.$$

This provides the following inequality

$$\int\limits_\gamma \Big([\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{W}^+\cdot\boldsymbol{\nu} - [\![m_{\boldsymbol{\nu}}(u)]\!]\frac{\partial w^+}{\partial\boldsymbol{\nu}}\Big)d\gamma \leq 0$$

leading to relations

$$[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] \le 0, \quad [\![m_{\boldsymbol{\nu}}(u)]\!] \le 0, \quad h[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] - [\![m_{\boldsymbol{\nu}}(u)]\!] = 0 \quad \text{on} \quad \gamma. \tag{34}$$

We can substitute in (33) test functions $\boldsymbol{\chi} \in K_2$ satisfying

$$[\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!] \ge h|[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]|, \quad \boldsymbol{W}^+ = \boldsymbol{0}, \quad \frac{\partial w^+}{\partial \boldsymbol{\nu}} = w = 0 \quad \text{on} \quad \gamma,$$

and obtain

$$\int\limits_{\gamma} \Big(\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U})[\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!] - m^-_{\boldsymbol{\nu}}(u)[\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!]\Big) d\gamma \le 0,$$

which yields

$$\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) \le 0, \quad h\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) - m^-_{\boldsymbol{\nu}}(u) \le 0, \quad h\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) + m^-_{\boldsymbol{\nu}}(u) \le 0 \quad \text{on} \quad \gamma. \tag{35}$$

Choosing test functions $\boldsymbol{\chi} \in K_2$ such that

$$\boldsymbol{W}^+\cdot\boldsymbol{\nu} = k^+ w^+, \quad [\![\boldsymbol{W}]\!] = \boldsymbol{0}, \quad w^+ \ge 0, \quad w^- = 0, \quad \frac{\partial w}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \gamma,$$

from (33) we get

$$\int\limits_{\gamma} \Big([\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{W}^+\cdot\boldsymbol{\nu} + t^+_{\boldsymbol{\nu}}(u) w^+\Big) d\gamma \le 0,$$

which means that

$$t^+_{\boldsymbol{\nu}}(u) \le -[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] k^+ \quad \text{on} \quad \gamma. \tag{36}$$

Therefore, by (34) we conclude with

$$t^+_{\boldsymbol{\nu}}(u) \le -k^+ \frac{[\![m_{\boldsymbol{\nu}}(u)]\!]}{h} \quad \text{on} \quad \gamma.$$

It is evident that if test function satisfies $\boldsymbol{W} = \boldsymbol{0}$, $w^+ = 0$, $w^- \le 0$, $\frac{\partial w}{\partial \boldsymbol{\nu}} = 0$ on γ, then we obtain

$$t^-_{\boldsymbol{\nu}}(u) \le 0 \quad \text{on} \quad \gamma. \tag{37}$$

In view of the obtained relations, (32) provides

$$[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{U}^+\cdot\boldsymbol{\nu} - [\![m_{\boldsymbol{\nu}}(u)]\!]\frac{\partial u^+}{\partial \boldsymbol{\nu}} + t^+_{\boldsymbol{\nu}}(u) u^+ = 0 \quad \text{on} \quad \gamma,$$

$$\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U})[\![\boldsymbol{U}\cdot\boldsymbol{\nu}]\!] - m^-_{\boldsymbol{\nu}}(u)[\![\frac{\partial u}{\partial \boldsymbol{\nu}}]\!] = 0, \quad t^-_{\boldsymbol{\nu}}(u) u^- = 0 \quad \text{on} \quad \gamma.$$

Gathering the previous formulas necessitates equilibrium and boundary conditions stated in the following theorem.

Theorem 2. *A sufficiently smooth solution of the variational inequality* (29) *is equivalent to the following boundary value problem: find* $\boldsymbol{\xi} = (\boldsymbol{U}, u) \in K_2$ *such that*

$$-\sigma_{ij,j}(\boldsymbol{U}) = f_i \qquad \textit{in} \quad \Omega_\gamma \ (i = 1, 2), \tag{38}$$

$$-m_{ij,ij}(u) = f_3 \quad \textit{in} \quad \Omega_\gamma, \tag{39}$$

$$[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] \le 0, \quad [\![m_{\boldsymbol{\nu}}(u)]\!] \le 0, \quad h[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] - [\![m_{\boldsymbol{\nu}}(u)]\!] = 0 \quad \textit{on} \quad \gamma,$$

$$\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) \le 0, \quad h\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) - m^-_{\boldsymbol{\nu}}(u) \le 0, \quad h\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) + m^-_{\boldsymbol{\nu}}(u) \le 0 \quad \textit{on} \quad \gamma,$$

$$\sigma_{\boldsymbol{\tau}}^{\pm}(\boldsymbol{U}) = 0, \quad t_{\boldsymbol{\nu}}^{-}(u) \le 0, \quad t_{\boldsymbol{\nu}}^{+}(u) \le -[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]k^{+} \quad on \quad \gamma,$$

$$[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{U}^{+} \cdot \boldsymbol{\nu} - [\![m_{\boldsymbol{\nu}}(u)]\!]\frac{\partial u^{+}}{\partial \boldsymbol{\nu}} + t_{\boldsymbol{\nu}}^{+}(u)u^{+} = 0 \quad on \quad \gamma,$$

$$\sigma_{\boldsymbol{\nu}}^{-}(\boldsymbol{U})[\![\boldsymbol{U} \cdot \boldsymbol{\nu}]\!] - m_{\boldsymbol{\nu}}^{-}(u)[\![\frac{\partial u}{\partial \boldsymbol{\nu}}]\!] = 0, \quad t_{\boldsymbol{\nu}}^{-}(u)u^{-} = 0 \quad on \quad \gamma.$$

Proof. Conversing necessity assertion, it remains to prove sufficiency of the above conditions. We start with multiplying equilibrium equations (38) by $(w_i - u_i)$, $i = 1, 2$, and (39) by $w - u$, where $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_2$. Then summing up and integrating over the domain Ω_γ, we obtain

$$- \int\limits_{\Omega_\gamma} \Big(\sigma_{ij,j}(\boldsymbol{U})(w_i - u_i) + m_{ij,ij}(u)(w - u)\Big) d\boldsymbol{x} =$$

$$= \int\limits_{\Omega_\gamma} \big(f_i(w_i - u_i) + f_3(w - u)\big)\, d\boldsymbol{x}. \quad (40)$$

Applying Green's formulas (17)–(18) allows to express (40) as follows

$$B(\boldsymbol{\xi}, \boldsymbol{\chi} - \boldsymbol{\xi}) + \int\limits_{\gamma} \Big[\!\!\Big[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\big((\boldsymbol{W} - \boldsymbol{U}) \cdot \boldsymbol{\nu}\big) + \sigma_{\boldsymbol{\tau}}(\boldsymbol{U})\big(\boldsymbol{W}_{\boldsymbol{\tau}} - \boldsymbol{U}_{\boldsymbol{\tau}}\big) +$$

$$+ t_{\boldsymbol{\nu}}(u)(w - u) - m_{\boldsymbol{\nu}}(u)\frac{\partial (w - u)}{\partial \boldsymbol{\nu}}\Big]\!\!\Big] d\gamma = \int\limits_{\Omega_\gamma} \boldsymbol{F} \cdot (\boldsymbol{\chi} - \boldsymbol{\xi})\, d\boldsymbol{x}.$$

Bearing in mind (31) we reduce the last equation to

$$B(\boldsymbol{\xi}, \boldsymbol{\chi} - \boldsymbol{\xi}) - \int\limits_{\Omega_\gamma} \boldsymbol{F} \cdot (\boldsymbol{\chi} - \boldsymbol{\xi}) =$$

$$= - \int\limits_{\gamma} \Big[\!\!\Big[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{W} \cdot \boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)w - m_{\boldsymbol{\nu}}(u)\frac{\partial w}{\partial \boldsymbol{\nu}}\Big]\!\!\Big] d\gamma\,, \quad (41)$$

where the integral in the right side of (41) can be represented in the form

$$\int\limits_{\gamma} \Big[\!\!\Big[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{W} \cdot \boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)w - m_{\boldsymbol{\nu}}(u)\frac{\partial w}{\partial \boldsymbol{\nu}}\Big]\!\!\Big] d\gamma \; =$$

$$= \int\limits_{\gamma} \Big(\sigma_{\boldsymbol{\nu}}^{+}(\boldsymbol{U})\boldsymbol{W}^{+} \cdot \boldsymbol{\nu} - \sigma_{\boldsymbol{\nu}}^{-}(\boldsymbol{U})\boldsymbol{W}^{-} \cdot \boldsymbol{\nu} - m_{\boldsymbol{\nu}}^{+}(u)\frac{\partial w^{+}}{\partial \boldsymbol{\nu}} +$$

$$+ m_{\boldsymbol{\nu}}^{-}(u)\frac{\partial w^{-}}{\partial \boldsymbol{\nu}} + t_{\boldsymbol{\nu}}^{+}(u)w^{+} - t_{\boldsymbol{\nu}}^{-}(u)w^{-}\Big) d\gamma. \quad (42)$$

The inequality (36) together with the last equality in (34) admits the following estimation of the integrand in (42)

$$
\begin{aligned}
&[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{W}^{+}\cdot\boldsymbol{\nu}+\sigma_{\boldsymbol{\nu}}^{-}(\boldsymbol{U})[\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!]-[\![m_{\boldsymbol{\nu}}(u)]\!]\frac{\partial w^{+}}{\partial\boldsymbol{\nu}}-\\
&\qquad -m_{\boldsymbol{\nu}}^{-}(u)[\![\frac{\partial w}{\partial\boldsymbol{\nu}}]\!]+t_{\boldsymbol{\nu}}^{+}(u)w^{+}-t_{\boldsymbol{\nu}}^{-}(u)w^{-}\le\\
&\le[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{W}^{+}\cdot\boldsymbol{\nu}+\sigma_{\boldsymbol{\nu}}^{-}(\boldsymbol{U})[\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!]-[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]h\frac{\partial w^{+}}{\partial\boldsymbol{\nu}}-\\
&\qquad -m_{\boldsymbol{\nu}}^{-}(u)[\![\frac{\partial w}{\partial\boldsymbol{\nu}}]\!]-[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]k^{+}w^{+}-t_{\boldsymbol{\nu}}^{-}(u)w^{-}\le\\
&\qquad \le\Big([\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!](\boldsymbol{W}^{+}\cdot\boldsymbol{\nu}-h\frac{\partial w^{+}}{\partial\boldsymbol{\nu}}-k^{+}w^{+})\Big)+\\
&\qquad +\Big(\sigma_{\boldsymbol{\nu}}^{-}(\boldsymbol{U})[\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!]-m_{\boldsymbol{\nu}}^{-}(u)[\![\frac{\partial w}{\partial\boldsymbol{\nu}}]\!]\Big)+(-t_{\boldsymbol{\nu}}^{-}(u)w^{-}).
\end{aligned}
\tag{43}
$$

One can note that each term in the right side of (43) in brackets is nonpositive due to (34), (35), (37) and relations (5) for test functions $\boldsymbol{\chi}=(\boldsymbol{W},w)\in K_2$. Therefore, recalling (41), we have proved the equivalence of both problem formulations. □

3.3. Case of the set K_3. The problem for the set K_3 can be treated as in the previous case (28) for K_2. It suffices to change the selected direction of the normal vector $\boldsymbol{\nu}$, which in turn implies the interchange between the positive and negative faces of γ.

3.4. Case of the set K_4. In this section we study the interaction model, where both crack faces may come into contact with the obstacle as well as with each other. We impose the following inequality type conditions on γ

$$
\begin{aligned}
&w^{+}\ge 0,\quad w^{-}\ge 0,\quad [\![\boldsymbol{W}\cdot\boldsymbol{\nu}]\!]\ge -h[\![\frac{\partial w}{\partial\boldsymbol{\nu}}]\!],\\
&\qquad \boldsymbol{W}^{-}\cdot\boldsymbol{\nu}-h\frac{\partial w^{-}}{\partial\boldsymbol{\nu}}\le k^{-}w^{-},\quad \boldsymbol{W}^{+}\cdot\boldsymbol{\nu}-h\frac{\partial w^{+}}{\partial\boldsymbol{\nu}}\ge k^{+}w^{+},
\end{aligned}
$$

together with the clamping boundary conditions on Γ. Let us assume that the variational inequality

$$
\boldsymbol{\xi}\in K_4,\quad B(\boldsymbol{\xi},\boldsymbol{\chi}-\boldsymbol{\xi})\ge\int\limits_{\Omega_\gamma}\boldsymbol{F}\cdot(\boldsymbol{\chi}-\boldsymbol{\xi})d\boldsymbol{x}\quad\forall\,\boldsymbol{\chi}\in K_4 \tag{44}
$$

holds for smooth solution $\boldsymbol{\xi}=(\boldsymbol{U},u)\in K_4$. As in the previous sections we can show that variational inequality (44) is equivalent to the following integral inequality over γ

$$
\begin{aligned}
&\int\limits_{\gamma}\Big[\!\!\Big[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\big((\boldsymbol{W}-\boldsymbol{U})\cdot\boldsymbol{\nu}\big)+\sigma_{\boldsymbol{\tau}}(\boldsymbol{U})\big(\boldsymbol{W}_{\boldsymbol{\tau}}-\boldsymbol{U}_{\boldsymbol{\tau}}\big)+\\
&+t_{\boldsymbol{\nu}}(u)(w-u)-m_{\boldsymbol{\nu}}(u)\frac{\partial(w-u)}{\partial\boldsymbol{\nu}}\Big]\!\!\Big]d\gamma\le 0\quad\text{for all}\quad\boldsymbol{\chi}\in K_4.
\end{aligned}
\tag{45}
$$

Varying $\boldsymbol{\chi}$ in (45), one can prove that

$$\sigma_{\boldsymbol{\tau}}^{\pm}(\boldsymbol{U}) = 0 \quad \text{on } \gamma^{\pm},$$

as well as

$$\int\limits_{\gamma} \left[\!\left[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{U}\cdot\boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)u - m_{\boldsymbol{\nu}}(u)\frac{\partial u}{\partial \boldsymbol{\nu}} \right]\!\right] d\gamma = 0,$$

$$\int\limits_{\gamma} \left[\!\left[\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{W}\cdot\boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)w - m_{\boldsymbol{\nu}}(u)\frac{\partial w}{\partial \boldsymbol{\nu}} \right]\!\right] d\gamma \le 0 \quad \text{for all} \quad \boldsymbol{\chi} \in K_4. \tag{46}$$

At first, we consider (46) along with test functions $\boldsymbol{\chi} \in K_4$ satisfying

$$\boldsymbol{W}^+\cdot\boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} = 0, \quad [\![\boldsymbol{W}]\!] = \boldsymbol{0}, \quad w^+ = 0, \quad w^- = 0, \quad [\![\frac{\partial w}{\partial \boldsymbol{\nu}}]\!] = 0 \quad \text{on} \quad \gamma.$$

Since the term $\boldsymbol{W}^+\cdot\boldsymbol{\nu} = h\frac{\partial w^+}{\partial \boldsymbol{\nu}}$ can take negative and positive values, we obtain

$$\int\limits_{\gamma} \Big(h[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!]\boldsymbol{W}^+\cdot\boldsymbol{\nu} - [\![m_{\boldsymbol{\nu}}(u)]\!]\boldsymbol{W}^+\cdot\boldsymbol{\nu} \Big) d\gamma \le 0,$$

that is

$$h[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] - [\![m_{\boldsymbol{\nu}}(u)]\!] = 0 \quad \text{on} \quad \gamma. \tag{47}$$

Now, we substitute into (46) test functions $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_4$ having the properties $\boldsymbol{W}^- = \boldsymbol{0}$, $w^- = \frac{\partial w^-}{\partial \boldsymbol{\nu}} = 0$ on γ^- and

$$\boldsymbol{W}^+\cdot\boldsymbol{\nu} \pm h\frac{\partial w^+}{\partial \boldsymbol{\nu}} = 0, \quad \boldsymbol{W}^+\cdot\boldsymbol{\nu} \ge 0, \quad w^+ = 0$$

on γ^+. This leads to inequality

$$\int\limits_{\gamma} \Big(h\sigma_{\boldsymbol{\nu}}^+(\boldsymbol{U})\boldsymbol{W}^+\cdot\boldsymbol{\nu} \pm m_{\boldsymbol{\nu}}^+(u)\boldsymbol{W}^+\cdot\boldsymbol{\nu} \Big) d\gamma \le 0,$$

and therefore,

$$h\sigma_{\boldsymbol{\nu}}^+(\boldsymbol{U}) \pm m_{\boldsymbol{\nu}}^+(u) \le 0 \quad \text{on} \quad \gamma. \tag{48}$$

Then, substitution in (46) of the functions $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_4$ with properties

$$\boldsymbol{W}^- = \boldsymbol{0}, \quad w^- = \frac{\partial w^-}{\partial \boldsymbol{\nu}} = 0, \quad \boldsymbol{W}^+\cdot\boldsymbol{\nu} = -h\frac{\partial w^+}{\partial \boldsymbol{\nu}} \ge 0, \quad 2\boldsymbol{W}^+\cdot\boldsymbol{\nu} = k^+w^+,$$

yields

$$\int\limits_{\gamma} \Big(hk^+\sigma_{\boldsymbol{\nu}}^+(\boldsymbol{U})\boldsymbol{W}^+\cdot\boldsymbol{\nu} + k^+m_{\boldsymbol{\nu}}^+(u)\boldsymbol{W}^+\cdot\boldsymbol{\nu} + 2t_{\boldsymbol{\nu}}^+(u)\boldsymbol{W}^+\cdot\boldsymbol{\nu}h \Big) d\gamma \le 0.$$

From there we conclude that the following upper bound takes place

$$hk^+\sigma_{\boldsymbol{\nu}}^+(\boldsymbol{U}) + k^+m_{\boldsymbol{\nu}}^+(u) + 2t_{\boldsymbol{\nu}}^+(u)h \le 0 \quad \text{on} \quad \gamma. \tag{49}$$

By analogous arguments it is possible to extract relations

$$h\sigma_{\boldsymbol{\nu}}^-(\boldsymbol{U}) \pm m_{\boldsymbol{\nu}}^-(u) \le 0, \quad hk^-\sigma_{\boldsymbol{\nu}}^-(\boldsymbol{U}) + k^-m_{\boldsymbol{\nu}}^-(u) + 2t_{\boldsymbol{\nu}}^-(u)h \ge 0 \quad \text{on} \quad \gamma \tag{50}$$

from (46) for test functions $\boldsymbol{\chi} = (\boldsymbol{W}, w) \in K_4$ having zero values on γ^+. For further analysis, we represent the integrand in (46) as follows

$$\frac{1}{2}\Big(\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) - \frac{m^+_{\boldsymbol{\nu}}(u)}{h}\Big)(\boldsymbol{W}^+\cdot\boldsymbol{\nu} + h\frac{\partial w^+}{\partial \boldsymbol{\nu}}) - \frac{1}{2}\Big(\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) - \frac{m^-_{\boldsymbol{\nu}}(u)}{h}\Big)(\boldsymbol{W}^-\cdot\boldsymbol{\nu} + h\frac{\partial w^-}{\partial \boldsymbol{\nu}}) +$$
$$+\frac{1}{2}\Big(\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) + \frac{m^+_{\boldsymbol{\nu}}(u)}{h}\Big)(\boldsymbol{W}^+\cdot\boldsymbol{\nu} - h\frac{\partial w^+}{\partial \boldsymbol{\nu}} - k^+w^+) -$$
$$-\frac{1}{2}\Big(\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) + \frac{m^-_{\boldsymbol{\nu}}(u)}{h}\Big)(\boldsymbol{W}^-\cdot\boldsymbol{\nu} - h\frac{\partial w^-}{\partial \boldsymbol{\nu}} - k^-w^-) +$$
$$+\Big(\frac{k^+}{2}\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) + \frac{m^+_{\boldsymbol{\nu}}(u)k^+}{2h} + t^+_{\boldsymbol{\nu}}(u)\Big)w^+ -$$
$$-\Big(\frac{k^-}{2}\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) + \frac{m^-_{\boldsymbol{\nu}}(u)k^-}{2h} + t^-_{\boldsymbol{\nu}}(u)\Big)w^-. \tag{51}$$

One can observe that the sum of the first and the second terms of (3.4) is nonpositive. Indeed, due to the non-penetration condition for the crack and equality (47) it holds

$$\frac{1}{2}\Big(\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) - \frac{m^+_{\boldsymbol{\nu}}(u)}{h}\Big)(\boldsymbol{W}^+\cdot\boldsymbol{\nu} + h\frac{\partial w^+}{\partial \boldsymbol{\nu}}) - \frac{1}{2}\Big(\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) - \frac{m^-_{\boldsymbol{\nu}}(u)}{h}\Big)(\boldsymbol{W}^-\cdot\boldsymbol{\nu} + h\frac{\partial w^-}{\partial \boldsymbol{\nu}}) =$$
$$= \frac{1}{2}\Big(\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) - \frac{m^+_{\boldsymbol{\nu}}(u)}{h}\Big)[\![\boldsymbol{W}\cdot\boldsymbol{\nu} + h\frac{\partial w}{\partial \boldsymbol{\nu}}]\!] \le 0 \quad \text{on} \quad \gamma.$$

Other terms of (3.4) are also nonpositive in view of the relations (48)–(50) and the non-penetration condition for the obstacle.

Finally, we note that the obtained optimality conditions on γ are sufficient to give a boundary value formulation for (44). In so doing, a smooth function $\boldsymbol{\xi} = (\boldsymbol{U}, u)$ of the set K_4 satisfying

$$\begin{gathered}
-\sigma_{ij,j}(\boldsymbol{U}) = f_i \qquad \text{in} \quad \Omega_\gamma \ (i = 1, 2), \\
-m_{ij,ij}(u) = f_3 \quad \text{in} \quad \Omega_\gamma, \\
\sigma^{\pm}_{\boldsymbol{\tau}}(\boldsymbol{U}) = 0 \quad \text{on } \gamma^{\pm}, \\
h[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})]\!] - [\![m_{\boldsymbol{\nu}}(u)]\!] = 0 \quad \text{on} \quad \gamma, \\
h\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) - m^-_{\boldsymbol{\nu}}(u) \le 0, \quad h\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) + m^-_{\boldsymbol{\nu}}(u) \le 0 \quad \text{on} \quad \gamma, \\
h\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) - m^+_{\boldsymbol{\nu}}(u) \le 0, \quad h\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) + m^+_{\boldsymbol{\nu}}(u) \le 0 \quad \text{on} \quad \gamma, \\
hk^-\sigma^-_{\boldsymbol{\nu}}(\boldsymbol{U}) + k^-m^-_{\boldsymbol{\nu}}(u) + 2t^-_{\boldsymbol{\nu}}(u)h \ge 0 \quad \text{on} \quad \gamma, \\
hk^+\sigma^+_{\boldsymbol{\nu}}(\boldsymbol{U}) + k^+m^+_{\boldsymbol{\nu}}(u) + 2t^+_{\boldsymbol{\nu}}(u)h \le 0 \quad \text{on} \quad \gamma, \\
[\![\sigma_{\boldsymbol{\nu}}(\boldsymbol{U})\boldsymbol{U}\cdot\boldsymbol{\nu} + t_{\boldsymbol{\nu}}(u)u - m_{\boldsymbol{\nu}}(u)\tfrac{\partial u}{\partial \boldsymbol{\nu}}]\!] = 0 \quad \text{on} \quad \gamma
\end{gathered}$$

provides solution to the variational problem (44). This finishes the case study.

4 Discussion

The new nonlinear mathematical model describing equilibrium of the Kirchhoff–Love plate is proposed as the minimization problem (13). The plate has a crack touching by a wedge-shaped obstacle. For consistency, we impose inequality constraints providing non-penetration of opposite crack faces and conditions of possible frictionless contact between the crack faces and the obstacle edges. The main difficulty of the problem (13) is that the corresponding set of admissible displacements is not convex. By identifying four cases with a pre-defined configuration between the crack and the obstacle, particular cases of the problem with convex sets K_i, $i = 1, 2, 3, 4$, for admissible functions are studied in Section 2. Convexity of these single sets allows us to obtain complementary optimality conditions and boundary value formulations for the four variational problems.

The rigorous mathematical results obtained for this model can be used as from viewpoint of applications, as well as from the point of view of non-convex optimization theory. In particular, further studies would be considerable with respect to numerical simulations, uniqueness issues, optimal control, inverse problems, etc.

References

[1] G. Fichera, Boundary Value Problems of Elasticity with Unilateral Constraints. In *Handbook der Physik VIa/2. Festkörpermechanik*; Flügge, S., Truesdell, C. Eds.; Springer-Verlag: Berlin, Heidelberg, New York, 1972; pp. 391–424.

[2] G. Dal Maso, G. Paderni, *Variational inequalities for the biharmonic operator with variable obstacles*, Annali di Matematica Pura ed Applicata, **153** (1988), 203–227.

[3] E. Hasrati, A. Jain, *Exact analysis of bending of a heterogeneous plate with irregularly-shaped inclusions*, Int. J. Solids Struct., **339**, (2026), 113828.

[4] E. Öner, P.T. Artar, *A comprehensive parametric study of contact behavior in functionally graded bilayer systems*, Appl. Math. Model., **157**, (2026), 116926.

[5] P.G. Ciarlet, P. Piersanti, *An obstacle problem for Koiter's shells*, Math. Mech. Solids., **24**:10 (2019), 3061–3079.

[6] A.G. Grimaldi, *Regularity results for solutions to a class of obstacle problems*, Nonlinear Anal. Real World Appl., **62** (2021), 103377.

[7] P. Piersanti, *Asymptotic analysis of linearly elastic flexural shells subjected to an obstacle in absence of friction*, J. Nonlinear Sci., **33** (2023), 58.

[8] M. Terzano, A. Spagnoli, P. Ståhle, *A fracture mechanics model to study indentation cutting*, Fatigue Fract. Eng. Mater. Struct., **41**:4 (2018), 821–830.

[9] K. Alam, G. Kerckhofs, A.V. Mitrofanov, S. Lomov, M. Wevers, V.V. Silberschmidt, *On-line analysis of cracking in cortical bone under wedge penetration*, Proc. Inst. Mech. Eng. H: J. Eng. Med., **226**:9 (2012), 709–717.

[10] K. Hoffseth, C. Randall, S. Chandrasekar, P. Hansma, H.T.Y. Yang, *Analyzing the effect of hydration on the wedge indentation fracture behavior of cortical bone*, J. Mech. Behav. Biomed. Mater., **69** (2017), 318–326.

[11] S. Kasiri, G. Reilly, D. Taylor, *Wedge indentation fracture of cortical bone: experimental data and predictions*, J. Biomech. Eng., **132**:8, (2010), 081009.

[12] A. Khludnev, V. Kovtunenko, *Analysis of Cracks in Solids*, WIT-Press: Southampton, 2000.

[13] L.A. Caffarelli, A. Friedman, *The obstacle problem for the biharmonic operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **6** (1979), 151–184.

[14] N.P. Lazarev, H. Itou, N.V. Neustroeva, *Fictitious domain method for an equilibrium problem of the Timoshenko-type plate with a crack crossing the external boundary at zero angle*, Jpn. J. Ind. Appl. Math., **33** (2016), 63–80.

[15] N.P. Lazarev, G.M. Semenova, A.S. Nikulin, *Equilibrium problem for a Kirchhoff–Love plate contacting with an inclined and lateral obstacles*, Mathematical notes of NEFU, **31**:2 (2024), 15–31.

[16] V.A. Kovtunenko, N.P. Lazarev, *Variational inequality for a Timoshenko plate contacting at the boundary with an inclined obstacle*, Philos. Transact. A Math. Phys. Eng. Sci., **382** (2024), 20230298.

[17] N. Lazarev, A. Singh, *Contact problem for an inhomogeneous Timoshenko plate interacting with an inclined obstacle*, Z. Angew. Math. Mech., **105**:5 (2025), e70097.

[18] N. Lazarev, S. Verkhovtsev, *Equilibrium problem for a Kirchhoff–Love plate contacting by a thin rigid inclusion with an inclined obstacle*, Comput. Math. Model., **36** (2025), 399–413.

[19] E.M. Rudoi, A.M. Khludnev, *Unilateral contact of a plate with a thin elastic obstacle*, J. Appl. Ind. Math., **4** (2010), 389–398.

[20] A.I. Furtsev, *Energy derivative for a construction consisting of a Timoshenko plate and a thin beam*, Mathematical notes of NEFU, **28** (2021), 68–87.

[21] T.S. Popova, *A contact problem for a viscoelastic plate and an elastic beam*, J. Appl. Industr. Math., **10** (2016), 404–416.

[22] N.A. Nikolaeva, *Method of fictitious domains for Signorini's problem in Kirchhoff–Love theory of plates*, J. Math. Sci., **221**, (2017) 872–882.

[23] V.D. Stepanov, A.M. Khludnev, *The Method of fictitious domains in the Signorini problem*, Siberian Math. J., **44** (2003), 1061–1074.

[24] N.P. Lazarev, E.M. Rudoy, *Shape sensitivity analysis of Timoshenko's plate with a crack under the nonpenetration condition*, Z. Angew. Math. Mech., **94** (2014), 730–739.

[25] A.M. Khludnev, *On bending an elastic plate with a delaminated thin rigid inclusion*, J. Appl. Indust. Math., **5** (2011), 582–594.

[26] A. Khludnev, *On equilibrium problem for T-shape elastic structure*, Axioms, **14** (2025), 49.

[27] N.P. Lazarev, V.A. Kovtunenko, *Asymptotic analysis of the problem of equilibrium of an inhomogeneous body with hinged rigid inclusions of various widths*, J. Appl. Mech. Tech. Phy., **64** (2023), 911–920.

[28] A.M. Khludnev, T.S. Popova, *Junction problem for rigid and semi-rigid inclusions in elastic bodies*, Arch. Appl. Mech., **86** (2016), 1565–1577.

[29] T.S. Popova, *Mathematical modeling of a T-shaped junction of thin anisotropic inclusions in an elastic body in the presence of delamination*, J. Appl. Mech. Tech. Phy. **66** (2025), 560–574.

[30] M. Sofonea, A. Matei, *History-dependent quasi-variational inequalities arising in contact mechanics*, Eur. J. Appl. Math., **22** (2011), 471–491.

[31] V.A. Kovtunenko, *Quasi-variational inequality for the nonlinear indentation problem: a power-law hardening model*, Philos. Trans. R. Soc. A ., **380** (2022), 20210362.

[32] A.M. Bagirov, L. Jin, N. Karmitsa, A. Al Nuaimat, N. Sultanova, *Subgradient method for nonconvex nonsmooth optimization*, J. Optim. Theory Appl., **157**:2 (2013), 416–435.

[33] N.P. Lazarev, V.A. Kovtunenko, *Signorini-type problems over non-convex sets for composite bodies contacting by sharp edges of rigid inclusions*, Mathematics, **10** (2022), 250.

[34] N.P. Lazarev, V. A. Kovtunenko, *Contact problem for a cracked Timoshenko plate with a wedge-shaped obstacle*, J. Comput. Appl. Math., **473** (2026), 116884 .

[35] V.A. Kovtunenko, A.N. Leont'ev, A.M. Khludnev, *An equilibrium problem of a plate with an oblique cut*, J. Appl. Mech. Tech. Phys., **39** (1998), 302–311.

[36] N.P. Lazarev, G.M. Semenova, E.D. Fedotov, *An equilibrium problem for a Kirchhoff–Love plate, contacting an obstacle by top and bottom edges*, Lobachevskii J. Math. **44**:2 (2023), 614–619.

[37] C. Baiocchi, A. Capello, *Variational and Quasivariational Inequalities: Application to free boundary problems*, Wiley, New York, 1984.

Victor Anatol'evich Kovtunenko
Lavrentyev Institute of Hydrodynamics, Siberian Branch of the Russian Academy of Sciences,
pr. Akad. Lavrentieva, 15,
630090, Novosibirsk, Russia,
Department of Mathematics and Scientific Computing, University of Graz, Heinrichstr. 36,
8010 Graz, NAWI Graz, Austria,
Email address: victor.kovtunenko@uni-graz.at

Nyurgun Petrovich Lazarev
Scientific Research Institute of Mathematics, North-Eastern Federal University,
Kulakovsky str., 46,
677000, Yakutsk, Russia
Email address: nyurgunlazarev@yandex.ru