

New Connections between Hardy Fields and the Definability in O-Minimal Expansions

Abstract

In this article, we study the Hardy field associated with an o-minimal expansion of the real numbers. If the set of analytic germs is dense in the Hardy field, then we can definably analytically separate sets in \mathbb{R}^2 , and we can definably analytically approximate definable continuous unary functions. A similar statement holds for definable smooth functions.

Keywords: O-minimal expansions, Hardy fields, Analytic separation

1 Introduction

Let M be an o-minimal expansion of the real field. We assume the reader to be familiar with the basic concepts of o-minimality as they are presented in [2,4]. In the sequel, “definable” always means “definable in M with parameters from \mathbb{R} ”. Many properties of semialgebraic sets and functions follow for M in the corresponding formulation from the axioms of o-minimal structures. However, apart from Nash functions which are the semialgebraic analytic functions, very little is known about the properties of definable analytic functions, as analyticity is not a first order concept. Most results are only proved for dimension 2 or 3. Let \mathbb{R}_{an} denote the real field expanded by restricted analytic functions. Even the Noetherianity of the ring of definable analytic functions on a definable analytic manifold is only known in dimension 2 for o-minimal expansions of \mathbb{R}_{an} , cf. [11]. In the same paper, using Hironaka’s Desingularization Theorem, cf.[12], the Noetherianity of the ring of \mathbb{R}_{an} -definable analytic functions on an \mathbb{R}_{an} -definable analytic manifold of

dimension 3 was proved. It is not yet known whether Noetherianity also holds for higher dimension in any of these cases.

In this paper we study definable analytic separation of sets in \mathbb{R}^2 . We denote by H_M the Hardy field of germs of definable unary functions at $+\infty$, and by H^ω the Hardy subfield consisting of the germs of H_M corresponding to functions which are analytic in some neighbourhood of $+\infty$. We endow H_M with the topology induced by the ordering on H_M . We say that a function $\varphi : U \rightarrow \mathbb{R}$ separates the sets $A, B \subseteq U$ if $A \subseteq \{\varphi > 0\}$ and $B \subseteq \{\varphi < 0\}$.

We relate the separation property of definable analytic functions to the density of H^ω in H_M by proving the following theorem.

Theorem 1.1. The following are equivalent:

- a) $H^\omega \subseteq H_M$ is dense.
- b) For every open interval I and for all definable continuous functions $f : I \rightarrow \mathbb{R}$ and $\varepsilon : I \rightarrow (0, \infty)$, there is a definable analytic function $g : I \rightarrow \mathbb{R}$ such that

$$|f(t) - g(t)| < \varepsilon(t), \quad t \in I.$$

- c) For every definable open $U \subseteq \mathbb{R}^2$ and all disjoint definable subsets $A, B \subseteq U$ which are closed in U , there is a definable analytic function $\varphi : U \rightarrow \mathbb{R}$ separating A and B .

If M is the semialgebraic structure, then Nash separation of disjoint closed semialgebraic sets is valid for arbitrary dimensions; this is known as Mostowski's Theorem.

Of course, it is also interesting to find examples of o-minimal structures which satisfy item (a) of Theorem 1.1. The o-minimal structures which admit analytic cell decomposition clearly satisfy (a) of Theorem 1.1. Examples of such structures are [3,5], [6], [13,15,23]. Moreover, the Pfaffian closure of an o-minimal structure is again o-minimal, cf. [21], and it preserves analytic cell decomposition, cf.[14].

However, analytic cell decomposition is not always required to obtain the density of H^ω in H_M . There are polynomially bounded o-minimal structures not admitting analytic cell decomposition, cf.[18], for which H^ω is dense in H_M .

Our methods also apply to infinitely differentiable functions. We denote by $C^\infty(U, V)$ the infinitely continuously differentiable functions from U to V ,

and H_M^∞ is defined similarly to H^ω . The notion of C^∞ is not well behaved from the model theoretical point of view, cf.[22].

The proof of Theorem 1.1 leads to the following slightly weaker C^∞ version of Theorem 1.1, if we just replace analytic by C^∞ :

Theorem 1.2. Consider the following statements

- a) $H_M^\infty \subseteq H_M$ is dense.
- b) For every open interval I and for all definable continuous functions $f : I \rightarrow \mathbb{R}$ and $\varepsilon : I \rightarrow (0, \infty)$, there is a definable C^∞ function $g : I \rightarrow \mathbb{R}$ such that

$$|f(t) - g(t)| < \varepsilon(t), \quad t \in I.$$

- c) For every definable open $U \subseteq \mathbb{R}^2$ and all disjoint definable subsets $A, B \subseteq U$ which are closed in U , there is a definable C^∞ function $\varphi : U \rightarrow \mathbb{R}$ separating A and B .

Then (a) and (b) are equivalent, and (a) implies (c).

If M is additionally polynomially bounded, then the statements (a), (b) and (c) are equivalent.

If M is an o-minimal expansion of the real exponential field that additionally admits C^∞ cell decomposition, then definable C^∞ separation and approximation are known for any arbitrary dimension, see [9], and see [10] for a sound study of smooth functions in these structures.

This paper is organized as follows. In Section 2 we show that for the o-minimal structures constructed in [18], the analytic germs H^ω are dense in H_M although these structures do not admit analytic cell decomposition. In Section 3 we prove the equivalence (a) \Leftrightarrow (b) of Theorem 1.1, and in Section 4 we finish the proof of Theorem 1.1. In Section 5 we sketch the proof of Theorem 1.2 which is similar to the proof of Theorem 1.1.

2 Examples

Remark 2.1. Alternatively to H_M we can always consider the Hardy field H of germs of definable functions at 0^+ , and analogously, we define H^ω as the corresponding analytic germs. This is provided by considering $f(1/t)$ instead of $f(t)$. Whenever it makes proofs simpler, we use H instead of H_M .

For the whole section, we fix an o-minimal structure M that is constructed in [18]. These structures are constructed analogously to \mathbb{R}_{an} , but in place of rings of analytic functions, certain quasianalytic Denjoy–Carleman classes are used. These classes contain C^∞ functions that are nowhere analytic. In particular, some of these structures possess unary definable functions which are not piecewise analytic, cf. [18, Corollary after Theorem 2]. Moreover, these structures are polynomially bounded, cf. [18, Theorem 5.4]. Therefore, the rings of definable C^∞ functions are quasianalytic, cf. [16]. That is, if U is a definable open connected subset of \mathbb{R}^n , and if $f : U \rightarrow \mathbb{R}$ is a definable C^∞ function, then $f \equiv 0$ if and only if the Taylor series of f at some point $u \in U$ is the zero series. In these structures the following version of the curve-selection is available, cf. [18, Lemma 5.3].

Lemma 2.2.[Curve-Selection Lemma] Let $B \subset \mathbb{R}^n$ be definable in M , and let $0 \in \text{cl}(B)$. Then, for some $\delta > 0$, there exists a definable C^∞ function $g : (-\delta, \delta) \rightarrow \mathbb{R}^n$ such that $g(0) = 0$ and $g(t) \in B$ for all $t \in (0, \delta)$.

We shall prove the following proposition.

Proposition 2.3. The Nash germs at $+\infty$ are dense in H_M

Proof Let $f : (0, \delta_1) \rightarrow \mathbb{R}$ be a definable continuous function. By multiplying a sufficiently high power of t to $f(t)$, we may assume that f extends continuously to 0 by setting $f(0) = 0$. We have to show that for every $m \in \mathbb{N}$ there is a Nash germ $\varphi : (0, \delta_2) \rightarrow \mathbb{R}$ such that $\varphi(t) - f(t)$ is $o(t^m)$ as $t \rightarrow 0$. Fix $m \in \mathbb{N}$.

By the special version of the Curve-Selection Lemma for M , there are definable C^∞ functions $g_1, g_2 : (-\delta_3, \delta_3) \rightarrow \mathbb{R}$ which both vanish at 0, such that

$$(g_1(t), g_2(t)) \in \{(x, f(x)) : 0 < x < \delta_3\}$$

for small $t > 0$. By quasianalyticity, the zero order of g_2 at 0 is some positive integer q . Denote by $P(t)$ the Taylor polynomial of order mq^2 of g_1 at 0. Neither P nor g_1 is locally constant. So, by the Monotonicity Theorem, both P and g_1 have inverse functions for small $t > 0$, say P^{-1} and g_1^{-1} , respectively. The function P^{-1} is actually a Nash function. Note that

$$P(t) \text{ and } g_1(t) \text{ are } \mathcal{O}(t^q) \text{ but not } o(t^q) \text{ as } t \rightarrow 0^+,$$

so both

$$P^{-1}(t) \text{ and } g_1^{-1}(t) \text{ are } \mathcal{O}(t^{1/q}) \text{ but not } o(t^{1/q}) \text{ as } t \rightarrow 0^+. \quad (1)$$

By Taylor's Theorem,

$$g_1(t) - P(t) = o(t^{mq^2}) \quad \text{as } t \rightarrow 0^+,$$

so

$$P^{-1}(g_1(t)) - t = o(t^{mq}) \quad \text{as } t \rightarrow 0^+.$$

Therefore,

$$P^{-1}(t) - g_1^{-1}(t) = o(t^m) \quad \text{as } t \rightarrow 0^+. \quad (2)$$

The function g_2 is C^∞ and therefore Lipschitz continuous with some constant L in some open neighbourhood $(-\delta_4, \delta_4)$ of 0. Let P_2 denote the Taylor polynomial of order mq of g_2 at 0. Then

$$g_2(t) - P_2(t) = o(t^{mq}) \quad \text{as } t \rightarrow 0^+. \quad (3)$$

For $t > 0$ small enough, we obtain

$$\begin{aligned} |f(t) - P_2 \circ P^{-1}(t)| &= |g_2 \circ g_1^{-1}(t) - P_2 \circ P^{-1}(t)| \\ &\leq |g_2 \circ g_1^{-1}(t) - g_2 \circ P^{-1}(t)| + |g_2 \circ P^{-1}(t) - P_2 \circ P^{-1}(t)| \\ &\leq L |g_1^{-1}(t) - P^{-1}(t)| + |(g_2 - P_2) \circ P^{-1}(t)|. \end{aligned}$$

Using (2) and (3) together with (1), we conclude that

$$f(t) - P_2 \circ P^{-1}(t) = o(t^m) \quad \text{as } t \rightarrow 0^+.$$

Set $\varphi := P_2 \circ P^{-1}$, which is clearly a Nash function.

Corollary 2.4 Let M be one of the o-minimal structures of [18]; then H^ω is dense in H_M .

We do not know whether $H^\omega \subset H_M$ is dense for other o-minimal structures without analytic cell decomposition. It would be interesting to know whether the Pfaffian closure of an o-minimal expansion preserves the density of H^ω in H_M .

3 Approximation

We prove that the statements (a) and (b) of Theorem 1.1 are equivalent. The implication (b) \Rightarrow (a) is evident.

First we introduce a subfield of H^ω , and demonstrate its density in H_M under certain conditions.

Definition 3.1 By A_M we denote the set of germs at $+\infty$ of definable analytic functions from $(0, \infty)$ to \mathbb{R} .

Proposition 3.2 Let H^ω be a dense subset of H_M . Then $A_M \subseteq H_M$ is dense.

Proof It is sufficient to prove the density of A_M in H . Let $g, h : (0, \infty) \rightarrow \mathbb{R}$ be definable functions. As $H^\omega \subseteq H$ is dense, there is an $\varepsilon > 0$ and a definable analytic mapping $f : (0, \varepsilon) \rightarrow \mathbb{R}$ such that

$$g(t) < f(t) < h(t) \quad \text{for small } t > 0.$$

We shall find a definable analytic function $\psi : (0, \infty) \rightarrow (0, \varepsilon)$ such that

$$g(t) < f \circ \psi(t) < h(t), \quad t \in (0, \varepsilon).$$

It suffices to construct a definable analytic mapping $\varphi : (0, \infty) \rightarrow (0, \varepsilon)$ which is an arbitrarily close approximation of the identity function near 0. Let $\varphi_1 : (0, \delta) \rightarrow (0, \infty)$ be a definable analytic function with $\lim_{t \rightarrow 0^+} \varphi_1(t) = 0$. By choosing δ small enough, we may assume that φ_1 is strictly increasing. Let $\varphi_2 : (0, \infty) \rightarrow (0, \infty)$ be defined by

$$\varphi_2(t) = \frac{t\varphi_1(\delta t)}{1+t}.$$

Then $0 < \varphi_2(t) < \varphi_1(t)$ for small t , and φ_2 is a strictly increasing definable analytic function whose range is $(0, \infty)$. Define

$$\psi : (0, \infty) \rightarrow (0, \infty), \quad \psi(t) = t\varphi_2^{-1}(\varphi_2(1) + \varphi_2(t)).$$

Then, by Taylor's theorem,

$$\begin{aligned} t - \psi(t) &= t - t\varphi_2^{-1}(\varphi_2(1) + \varphi_2(t)) \\ &= -t \cdot \varphi_2^{-1'}(r) \varphi_2(t) + o(t\varphi_2(t)) \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

for some r between 1 and $1+t$. As φ_1 bounds φ_2 near 0, and as φ_1 may be chosen arbitrarily close to 0, the proposition is proved.

We obtain the following corollaries.

Corollary 3.3 Let $H^\omega \subseteq H_M$ be dense. Then, for every continuous function

$f : (0, \varepsilon] \rightarrow (0, \infty)$ there is a strictly increasing definable analytic function $g : (0, \infty) \rightarrow (0, \infty)$ such that

$$0 < g(t) < f(t), \quad t \in (0, \varepsilon).$$

Proposition 3.2 provides us with a large class of o-minimal structures, for which $A_M \subseteq H_M$ is dense.

Corollary 3.4 Let M admit analytic cell decomposition. Then A_M is a dense subset of H_M .

3.1 Partition of unity

We construct a definable analytic partition of unity on \mathbb{R} . Here, we generally assume that H^ω is dense in H_M .

Lemma 3.5 Let $0 < a < b < 1$, and $\Delta > 0$. Let $\varphi, \varepsilon : (0, 1) \rightarrow (0, \infty)$ be definable continuous functions. Then there is a definable analytic function $\psi : (0, 1) \rightarrow (0, \infty)$ such that

- (a) $\psi(t) > 1$ for $t \in (a, b)$,
- (b) $\psi(t)\varphi(t) < \varepsilon(t)$ for $t \in (0, a - \Delta/2) \cup (b + \Delta/2, 1)$.

Proof Consider the continuous semilinear map $s : (0, 1) \rightarrow (0, \infty)$ defined by

$$s(t) = \frac{1}{\min\{|t - a|, |t - b|\}}.$$

By the Łojasiewicz inequality, there is a definable continuous strictly increasing bijective function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ with $\rho(0) = 0$ such that for all $t \in (0, 1)$,

$$\rho(s(t)) \geq 1 \quad \text{for } t \in (a, b).$$

By Corollary 3, we may assume that ρ is analytic on $(0, \infty)$. Let $K := \rho^{-1}(1)$. We define a further continuous semilinear function $S : (0, 1) \rightarrow (0, \infty)$ as follows:

$$S(t) = \begin{cases} s(t), & t \in (0, a - \frac{\Delta}{2}) \cup (b + \frac{\Delta}{2}, 1), \\ K, & t \in (a, b). \end{cases}$$

Then S satisfies

$$\rho \circ S(t) < \rho \circ s(t), \quad t \notin (a - \frac{\Delta}{2}, b + \frac{\Delta}{2}),$$

and $\rho \circ S(t) > 1$ for $t \in (a, b)$. We apply Shiota's Approximation Theorem, cf. [19, 20], to S and $\tilde{\varepsilon} : (0, 1) \rightarrow (0, \infty)$, which is defined by

$$\tilde{\varepsilon}(t) = \frac{\varepsilon(t)}{\varphi(t) + 1},$$

and obtain a Nash function $g : (0, 1) \rightarrow (0, \infty)$ such that

$$|g(t) - S(t)| < \tilde{\varepsilon}(t), \quad t \in (0, 1).$$

Hence,

$$\rho \circ g(t) < \rho \circ s(t) \quad \text{outside } (a - \frac{\Delta}{2}, b + \frac{\Delta}{2}),$$

and

$$\rho \circ g(t) > 1 \quad \text{for } t \in (a, b).$$

Set $\psi := \rho \circ g$.

The topological closure of a set U is denoted by $\text{cl}(U)$, and its boundary by ∂U . The previous lemma implies the following kind of definable analytic partition of unity.

Corollary 3.6 Let $U_1, \dots, U_n \subseteq \mathbb{R}$ be definable open sets, and let $U = \bigcup_{i=1}^n U_i$. Let $\varphi_i : U \rightarrow (0, \infty)$, $i = 1, \dots, n$, and $\varepsilon : U \rightarrow (0, \infty)$ be definable continuous functions. Then there are definable analytic functions $\phi_i : U \rightarrow (0, 1)$, $i = 1, \dots, n$, such that

- (a) $\sum_{i=1}^n \phi_i(t) = 1$, for all $t \in U$,
- (b) $\phi_i(t)\varphi_i(t) < \varepsilon(t)$, for all $t \in U \setminus U_i$.

Proof Let $V_1, \dots, V_n \subset U$ be definable open sets covering U , such that $\text{cl}(V_i) \subset U_i$ for $i = 1, \dots, n$. Choose for every $i = 1, \dots, n$ a definable analytic function $\psi_i : U \rightarrow (0, \infty)$ with

- (a) $\psi_i(t) > 1$ for $t \in V_i$,
- (b) $\psi_i(t)\varphi_i(t) < \varepsilon(t)$ for $t \in U \setminus U_i$.

Note that $\psi_1(t) + \dots + \psi_n(t) > 1$ for all $t \in U$. Therefore, the functions

$$\phi_i := \frac{\psi_i}{\sum_{j=1}^n \psi_j}, \quad i = 1, \dots, n,$$

satisfy the desired properties.

Next we prove that (a) \Rightarrow (b) of Theorem 1.1.

Proof of (a) \Rightarrow (b) of Theorem 1.1. By applying the function $t \mapsto \frac{t}{1+t^2}$ to U we may assume that U is bounded. Moreover, we may assume that U is connected, that is, $U = (a, b)$. By Proposition 3, there exist a $\delta > 0$ and definable analytic functions $p_1 : U \rightarrow \mathbb{R}$ and $p_3 : U \rightarrow \mathbb{R}$ such that

$$|f(t) - p_1(t)| < \frac{1}{6}\varepsilon(t) \quad \text{for } t \in (a, a + \delta),$$

$$|f(t) - p_3(t)| < \frac{1}{6}\varepsilon(t) \quad \text{for } t \in (b - \delta, b).$$

By the Weierstrass Approximation Theorem, cf. [?, page 33, Thm. 1.6.2], applied to f restricted to the compact interval $[a + \delta/2, b - \delta/2]$, there is a polynomial p_2 such that

$$|f(t) - p_2(t)| < \frac{1}{6}\varepsilon(t) \quad \text{for } t \in [a + \delta/2, b - \delta/2].$$

Define $U_1 := (a, a + \delta)$, $U_2 := (a + \delta/2, b - \delta/2)$ and $U_3 := (b - \delta, b)$. Select by Corollary ?? some definable analytic functions $\psi_i : U \rightarrow (0, 1]$ such that

$$\psi_1(t) + \psi_2(t) + \psi_3(t) = 1, \quad t \in U,$$

and

$$|\psi_i(t)p_i(t)| < \frac{1}{6}\varepsilon(t), \quad t \in U \setminus U_i.$$

Then

$$g := \psi_1 p_1 + \psi_2 p_2 + \psi_3 p_3$$

is definable, analytic, and satisfies

$$|g(t) - f(t)| < \varepsilon(t), \quad t \in U.$$

4 Separation of sets

4.1 Separation for special cases

In general, we cannot approximate unary definable C^1 functions by Nash functions, as Nash functions can be bounded by rational functions. However, if a function is additionally Lipschitz continuous, we can Nash approximate

with sufficiently good quality.

Lemma 4.1 Let $f : (a, b) \rightarrow \mathbb{R}$ be a definable Lipschitz continuous C^1 function. Then, for every continuous semilinear function $\varepsilon : (a, b) \rightarrow (0, \infty)$ there is a Lipschitz continuous Nash function $g : (a, b) \rightarrow \mathbb{R}$ such that

$$f(t) - \varepsilon(t) < g(t) < f(t), \quad t \in (a, b). \quad (4)$$

Proof: We may assume that

$$\varepsilon(t) < \min\{t - a, b - t\}. \quad (5)$$

Since f' is a bounded function, both

$$\lim_{t \rightarrow a^+} f'(t) \quad \text{and} \quad \lim_{t \rightarrow b^-} f'(t)$$

exist in \mathbb{R} , and we denote them by c and d , respectively. Select a $\Delta > 0$ so small that for some $\delta > 0$,

$$f(t) - \varepsilon(t) < f(a) + (c - \Delta)(t - a) < f(t) - \frac{1}{6}\varepsilon(t), \quad t \in (a, a + \delta),$$

and

$$f(t) - \varepsilon(t) < f(b) + (d - \Delta)(t - b) < f(t) - \frac{1}{6}\varepsilon(t), \quad t \in (b - \delta, b).$$

We apply the Weierstrass Approximation Theorem to $f - \frac{\varepsilon}{3}$ restricted to $[a + \delta/2, b - \delta/2]$, and obtain a polynomial p such that

$$|f(t) - \frac{1}{3}\varepsilon(t) - p(t)| < \frac{1}{6}\varepsilon(t), \quad t \in [a + \delta/2, b - \delta/2].$$

Choose a semialgebraic continuous partition of unity $\varphi_1, \varphi_2, \varphi_3 : (a, b) \rightarrow [0, \infty)$ subordinate to the sets

$$(a, a + \delta), \quad [a + \delta/2, b - \delta/2], \quad (b - \delta, b).$$

Define

$$h(t) := \varphi_1(t)(f(a) + (c - \Delta)(t - a)) + \varphi_2(t)p(t) + \varphi_3(t)(f(b) + (d - \Delta)(t - b)).$$

The function h is continuous, semialgebraic, and satisfies

$$f(t) - \frac{2}{3}\varepsilon(t) < h(t) < f(t) - \frac{1}{6}\varepsilon(t), \quad t \in (a, b).$$

By Shiota's Approximation Theorem, there is a Nash function $g : (a, b) \rightarrow \mathbb{R}$ such that

$$|g(t) - h(t)| < \frac{1}{6}\varepsilon(t), \quad t \in (a, b),$$

and such that g satisfies inequality (4). Note that by definability, the function ε is C^1 smooth near a and b . By l'Hospital's rule,

$$\lim_{t \rightarrow a^+} \frac{g(t) - f(a)}{t - a}, \quad \lim_{t \rightarrow b^-} \frac{g(t) - f(b)}{t - b}$$

exist and are finite. Therefore, g' extends continuously to $[a, b]$. Thus g' is bounded as $[a, b]$ is compact, so the function g is Lipschitz continuous.

Note that a regular open set U is a set that equals the interior of its closure.

Definition 4.2 A bounded definable regular open set $U \subseteq \mathbb{R}^2$, whose boundary is a C^1 submanifold, is called *nice*.

Next we prove definable analytic separation of sets for special situations.

Lemma 4.3 Let U be a nice set. Let $A, B \subseteq U$ be definable disjoint sets, which are closed in U , such that for every point $\xi \in \text{cl}(A) \cap \text{cl}(B)$, and every definable path $\psi : (0, 1) \rightarrow A$ with $\lim_{t \rightarrow 0} \psi(t) = \xi$,

$$\text{dist}(\psi(t), B) \geq |\psi(t) - \xi| \tag{6}$$

for t small enough. Then there is a definable analytic mapping $\varphi : U \rightarrow \mathbb{R}$ that separates A and B .

Proof:

We shall prove that there is an open semialgebraic neighbourhood V of U , and disjoint semialgebraic neighbourhoods $W(A)$ and $W(B)$ of A and B , respectively, which are closed in V . Then, by Mostowski's Theorem, cf. [1, page 49, Thm. 2.7.7]1, there is a Nash function $\varphi : V \rightarrow \mathbb{R}$ that separates $W(A)$ and $W(B)$, and therefore $\varphi|_U$ separates A and B .

By Lipschitz stratification, cf. [8, Theorem 1.4], every definable set in \mathbb{R}^2 partitions into finitely many Lipschitz cells, which are, after a possible rotation, sets of the form

- (a) a singleton,

- (b) $\{a\} \times I$ where $I \subseteq \mathbb{R}$ is an open interval,
- (c) $(h)I := \{(t, h(t)) : t \in I\}$, where $I \subseteq \mathbb{R}$ is an open interval and $h : I \rightarrow \mathbb{R}$ is a definable Lipschitz continuous C^1 function,
- (d) $(f, g)I := \{(t, y) : t \in I, f(t) < y < g(t)\}$, where $I \subseteq \mathbb{R}$ is an open interval and $f, g : I \rightarrow \mathbb{R}$ are definable Lipschitz continuous C^1 functions, such that $f(t) < g(t)$ for all $t \in I$.

Moreover, we may assume that for any cell C , either $C \cap X = C$ or $C \cap X = \emptyset$ whenever X is one of the sets $U, \partial U, A, \partial A, B, \partial B$. We consider a cell contained in A . In the cases (a) and (b), nothing has to be done. In the case (c), by inequality (6), there is a positive semilinear function $\Delta : I \rightarrow \mathbb{R}$ which bounds

$$t \mapsto \frac{1}{\text{dist}(h(t), B)} \quad (7)$$

from below. We may assume that $\Delta(t) \rightarrow 0$ as $t \rightarrow a$ or $t \rightarrow b$. Hence, by Lemma 4.1, there are two semialgebraic functions $h^-, h^+ : I \rightarrow \mathbb{R}$, such that

$$h(t) - \Delta(t) < h^-(t) < h(t) < h^+(t) < h(t) + \Delta(t).$$

Therefore,

$$\{(t, h(t)) : t \in I\} \subset (h^-, h^+)I.$$

Similarly, we find open semialgebraic neighbourhoods of cells of the form (d). For B we do the same. So we can cover A and B by open semialgebraic neighbourhoods A_r and B_r , such that $\text{cl}(A_r) \cap \text{cl}(B_r)$ is a finite set contained in ∂U . Then there are also cells in ∂U that do not bound A or B . Construct similar semialgebraic neighbourhoods for them.

Define V as the union of U and all the constructed neighbourhoods, and set

$$W(A) := \text{cl}(A_r) \cap V, \quad W(B) := \text{cl}(B_r) \cap V.$$

The sets $W(A)$ and $W(B)$ do not intersect, because they are, by equation (7), sufficiently small neighbourhoods of A and B .

Proof of (a) \Rightarrow (c) of Theorem 1.1. Let $U \subset \mathbb{R}^2$ be a definable open set, and let $A_0, B_0 \subset U$ be definable disjoint sets closed in U . Then there exist a nice set U and a definable C^ω diffeomorphism $\varphi : U \rightarrow U$. We have to show that we can choose φ so that $U, \Delta_\varphi := \varphi(A_0)$ and $B := \varphi(B_0)$ satisfy

the conditions of Lemma 4.3. Let $\xi \in \text{cl}(A) \cap \text{cl}(B)$. Then ξ is contained in the boundary of U . By applying the procedure of step 2 of the proof of Proposition 4.4, the point ξ satisfies the condition of Lemma 4.3. The set U is nice. Hence, the set $\text{cl}(A) \cap \text{cl}(B)$ is finite, and this procedure ends after finitely many steps

4.3 Proof of (c) \Rightarrow (a) of *Theorem 1.1*

Proof of (c) \Rightarrow (a) of *Theorem 1.1*. Suppose that definable analytic separation holds. Let $f : (r, \infty) \rightarrow \mathbb{R}$ and $\varepsilon : (r, \infty) \rightarrow (0, \infty)$ be continuous definable functions. Define $U = (r, \infty) \times \mathbb{R}$ and

$$A := \{(x, y) : x > r, y \geq f(x) + \varepsilon(x)/2\}$$

$$B := \{(x, y) : x > r, y \leq f(x) - \varepsilon(x)/2\}.$$

Let $\phi : U \rightarrow \mathbb{R}$ be a definable analytic mapping with $A \subset \{\phi > 0\}$ and $B \subset \{\phi < 0\}$. Then

$$\dim(\{\phi = 0\}) = 1, \tag{8}$$

as ϕ is analytic. Select a C^1 cell stratification, cf. [2, page 68] in connection with [2, page 115], partitioning $\{\phi = 0\}$. Let C be any cell contained in $\{\phi = 0\}$ of the form $C = (h)_{(s, \infty)}$, which exists, since ϕ separates A and B . By the axioms of stratification, there is no branching point of $\{\phi = 0\}$ contained in C , so C is an analytic manifold. Hence, h is a C^1 function, whose graph is an analytic manifold. Therefore, h is analytic. Moreover,

$$|h(t) - f(t)| < \varepsilon(t), \quad t > s.$$

5. Proof of *Theorem 1.2*

For the proof of *Theorem 1.2* we only have to pattern the proofs we performed for *Theorem 1.1*.

To be more precise: We let $A_M^\infty \subseteq H_M$ denote the Hardy subfield of H_M which consists of all definable C^∞ functions $f_* : (0, \infty) \rightarrow \mathbb{R}$. In Proposition

3.2, Corollary 3.3, Lemma 3.5, Corollary 3.6 and in the proof of Theorem 1.1 (a) \Rightarrow (b) we use A_M^∞ , H_M^∞ and C^∞ in place of A_M , H_M and analytic, respectively. The same applies to Lemma 4.3, Proposition 4.4 and Proof of Theorem 1.1 (a) \Rightarrow (b)

There is one exception: we do not know whether the implication (c) \Rightarrow (b) of Theorem 1.2 holds. This is due to equation (8) which does not hold true for definable C^∞ functions if M is not polynomially bounded. In polynomially bounded o-minimal structures, definable C^∞ functions are quasianalytic, cf. [16], and a corresponding equation (8) holds for definable C^∞ functions.

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