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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 144–145 (2023) УДК 510.67
https://doi.org/10.33048/semi.2023.20.??? MSC 03C64, 03C40, 03C45

ALGEBRAS OF BINARY FORMULAS FOR CIRCULARLY ORDERED STRUCTURES: PIECEWISE MONOTONIC CASE

B.SH. KULPESHOV

Communicated by P.P. PETROV

Abstract: This article concerns the notion of weak circular minimality being a variant of o-minimality for circularly ordered structures. Algebras of binary isolating formulas are studied for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a piecewise monotonic-to-left function to the definable completion of a structure. On the basis of the study, we present a description of these algebras. It is shown that for this case there exist only non-commutative algebras. A strict s-deterministicity of such algebras for some natural number s is also established.

Keywords: algebra of binary formulas, weak circular minimality, $\aleph_0$-categorical theory, circularly ordered structure, convexity rank.

1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of an one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating*

KULPESHOV B.SH., ALGEBRAS OF BINARY FORMULAS FOR CIRCULARLY ORDERED STRUCTURES: PIECEWISE MONOTONIC CASE.

The work is supported by Science Committee of Ministry of Science and Higher Education of the Republic of Kazakhstan (grant BR31714735).

Received January, 1, 2023, Published December, 31, 2023.

formula is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [22, 23]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [1], [6]–[15].

Let L be a countable first-order language. Throughout we consider L-structures and assume that L contains a ternary relational symbol K, interpreted as a circular order in these structures (unless otherwise stated).

The *circular order* is described by a ternary relation K satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1. [3] *(i) If $\langle M, \leq \rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule*

$$K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$$

then K is a circular order relation on M.

(ii) If $\langle N, K \rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered.

Example 1. [4, 5] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L \rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi, \ (a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise. Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was studied initially in [16]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. The study of weakly circularly minimal structures was continued in the papers [2], [16]–[18].

Let M be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G-invariant equivalence relation on M. The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M.

Notation 1. (1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \ldots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \ldots, u_n \rangle$ having the length 3 (in ascending order) satisfy K; similar notations are used for K_0.

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further we need the notion of the definable completion of a circularly ordered structure, introduced in [16]. Its linear analog was introduced in [21]. A *cut* $C(x)$ in a circularly ordered structure M is maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\overline{M}$ consists of endpoints of definable subsets of the structure M).

Notation 2. [16] Let $F(x, y)$ be an L-formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^\ell(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \to \neg F(t_1, y)) \wedge$$
$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \to F(t_2, y))].$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x, y) \to F^l(y) \wedge \forall z (K(y, z, x) \to F(z, y))].$$

If $F_1(x, y), F_2(x, y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M, a) \subset F_2(M, a)$. If M is 1-transitive and this holds for some a, it holds for all a. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x, y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M,a)$ for arbitrary $a \in M$. In general, $F(M,a)$ has no the right endpoint in M. For example, if $dcl(\{a\}) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x,y)$ and any $a \in M$ the formula $F(M,a)$ has no the right endpoint in M. We write $f(y) :=$ rend $F(M,y)$, assuming that $f(y)$ is the right endpoint of the set $F(M,y)$ that lies in general in the definable completion $\overline{M}$ of M. Then f is a function mapping M in $\overline{M}$.

Notation 3. Let $E(x,y)$ be an $\emptyset$–definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (non-obligatory in M). Then

$$E^*(x,y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge \forall t (K(y_1,t,y_2) \to E(t,x)) \wedge K_0(y_1,y,y_2)].$$

Let M, N be circularly ordered structures. The 2-*reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$-definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic to* N if the 2-reduct of M is isomorphic to the 2-reduct of N.

Let f be a unary function from M to $\overline{M}$. We say that f is *monotonic-to-right (left) on* M if it preserves (reverses) the relation K_0, i.e. for any $a,b,c \in M$ such that $K_0(a,b,c)$, we have $K_0(f(a),f(b),f(c))$ $(K_0(f(c),f(b),f(a)))$.

We also say that f is *piecewise monotonic-to-right (left) on* M if there exists an $\emptyset$-definable non-trivial equivalence relation $E(x,y)$ partitioning M into finitely many infinite convex classes so that f is monotonic-to-right on each E-class and f is not monotonic-to-left (right) on M/E, where by M/E we denote the set of representatives of E-classes in M.

Example 2. [17] Let $M := \langle M, K, E^2, f^1 \rangle$ be a circularly ordered structure, where M is a disjoint union of $\mathbb{Q}_1$, $\mathbb{Q}_2$, …, $\mathbb{Q}_6$, where $\mathbb{Q}_i$ is a copy of the ordering of rational numbers $\mathbb{Q}$. The symbol E interprets an equivalence relation on M as follows: $E(a,b)$ iff there is $1 \leq i \leq 6$ with $a,b \in \mathbb{Q}_i$. The symbol f interprets a function on M as follows: $f(\mathbb{Q}_i) = \mathbb{Q}_{i+3}$ for each $1 \leq i \leq 3$, $f(\mathbb{Q}_j) = \mathbb{Q}_{j-3}$ for each $4 \leq j \leq 6$, and $f(q) = -q$ for all $q \in \mathbb{Q}$.

It can be proved that M is an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, f is a bijection on M so that $f^2(a) = a$ for all $a \in M$, f is monotonic-to-left on each E-class and f is monotonic-to-right on M/E, i.e. f is piecewise monotonic-to-left on M.

The following definition can be used in a circular ordered structure as well.

Definition 1. [19], [20] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$. *The rank of convexity of the set* A $(RC(A))$ is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if A is finite and non-empty.

3) $RC(A) \geq 1$ if A is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures M of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) := \text{rend } R(M, y)$ is piecewise monotonic-to-left on M:

Theorem 1. [18] *Let M be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1, $dcl(\{a\}) = \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula $R(x, y)$ such that $r(y) := \text{rend } R(M, y)$ is piecewise monotonic-to-left on M. Then M is isomorphic up to binarity to*

$$M'_{s,m,k} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

where M is a circularly ordered structure, M is densely ordered, $s \geq 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation partitioning every E_{i+1}-class into infinitely many infinite convex E_i-subclasses without endpoints so that the induced order on E_i-subclasses is dense without endpoints; $R(M, a)$ has no right endpoint in M and $r^k(a) = a$ for all $a \in M$ and some $k \geq 2$, where $r^k(y) := r(r^{k-1}(y))$; for every $1 \leq i \leq s + 1$ and any $a \in M$

$$M'_{s,m,k} \models \neg E_i^*(a, r(a)) \land \forall y(E_i(y, a) \to \exists u[E_i^*(u, r(a)) \land E_i^*(u, r(y))]),$$

$m \geq 4$, k is even and k divides m; r is monotonic-to-left on every E_{s+1}-class and r is monotonic-to-right on M/E_{s+1}.

In [12] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In

[13]–[14] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [15] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure. Here we describe algebras of binary isolating formulas for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure having a piecewise monotonic-to-left function to the definable completion of a structure for arbitrary s and $m = k = 4$.

2 Results

Let M be an 1-transitive structure. We denote every binary isolating formula acting in M by a label $u \in \rho_M$, where ρ_M denotes the set of all labels for the algebra $\mathcal{P}_M$ of binary isolating formulas of the structure M.

Definition 2. [23] The algebra $\mathcal{P}_M$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_M$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_M$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_M$. We also say that an m-deterministic algebra $\mathcal{P}_M$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic.

Example 3. Consider the structure $M'_{1,4,4} := \langle M, K^3, E_1^2, E_2^2, R^2 \rangle$ from Theorem 1 with the condition that the function $r(y) := \text{rend } R(M, y)$ is piecewise monotonic-to-left on M.

We assert that $Th(M'_{1,4,4})$ has seventeen binary isolating formulas:

$$\theta_0(x,y) := x = y,$$
$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$
$$\theta_2(x,y) := K_0(x,y,r(x)) \wedge E_2(x,y) \wedge \neg E_1(x,y),$$
$$\theta_3(x,y) := K_0(x,y,r(x)) \wedge E_2^*(y,r(x)) \wedge \neg E_1^*(y,r(x)),$$
$$\theta_4(x,y) := K_0(x,y,r(x)) \wedge E_1^*(y,r(x)),$$
$$\theta_5(x,y) := K_0(r(x),y,r^2(x)) \wedge E_1^*(y,r(x)),$$
$$\theta_6(x,y) := K_0(r(x),y,r^2(x)) \wedge E_2^*(y,r(x)) \wedge \neg E_1^*(y,r(x)),$$
$$\theta_7(x,y) := K_0(r(x),y,r^2(x)) \wedge E_2^*(y,r^2(x)) \wedge \neg E_1^*(y,r^2(x)),$$
$$\theta_8(x,y) := K_0(r(x),y,r^2(x)) \wedge E_1^*(y,r^2(x)),$$
$$\theta_9(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E_1^*(y,r^2(x)),$$
$$\theta_{10}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E_2^*(y,r^2(x)) \wedge \neg E_1^*(y,r^2(x)),$$
$$\theta_{11}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E_2^*(y,r^3(x)) \wedge \neg E_1^*(y,r^3(x)),$$
$$\theta_{12}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E_1^*(y,r^3(x)),$$
$$\theta_{13}(x,y) := K_0(r^3(x),y,x) \wedge E_1^*(y,r^3(x)),$$

$$\theta_{14}(x,y) := K_0(r^3(x),y,x) \wedge E_2^*(y,r^3(x)) \wedge \neg E_1^*(y,r^3(x)),$$
$$\theta_{15}(x,y) := K_0(r^3(x),y,x) \wedge E_2(x,y) \wedge \neg E_1(x,y),$$
$$\theta_{16}(x,y) := K_0(r^3(x),y,x) \wedge E_1(x,y)$$

and the following holds for any $a \in M$:

$$K_0(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\ldots,\theta_{14}(a,M),\theta_{15}(a,M),\theta_{16}(a,M)).$$

It can be proved that the algebra $\mathfrak{P}_{M'_{1,4,4}}$ is non-commutative and strictly 5-deterministic.

Theorem 2. *The algebra* $\mathfrak{P}_{M'_{s,4,4}}$ *of binary isolating formulas with piecewise monotonic-to-left function* r *has* $8s+9$ *labels, is non-commutative and strictly* $(2s+3)$*-deterministic for every* $s \geq 1$.

Proof. We assert that the algebra $\mathfrak{P}_{M'_{s,4,4}}$ has $8s+9$ binary isolating formulas:

$$\theta_0(x,y) := x = y,$$
$$\theta_1(x,y) := K_0(x,y,r(x)) \wedge E_1(x,y),$$
$$\theta_{l_1}(x,y) := K_0(x,y,r(x)) \wedge E_{l_1}(x,y) \wedge \neg E_{l_1-1}(x,y), \text{ where } 2 \leq l_1 \leq s+1,$$
$$\theta_{l_2}(x,y) := K_0(x,y,r(x)) \wedge E^*_{2s+3-l_2}(y,r(x)) \wedge \neg E^*_{2s+2-l_2}(y,r(x)),$$
$$\text{where } s+2 \leq l_2 \leq 2s+1,$$
$$\theta_{2s+2}(x,y) := K_0(x,y,r(x)) \wedge E_1^*(y,r(x)),$$
$$\theta_{2s+3}(x,y) := K_0(r(x),y,r^2(x)) \wedge E_1^*(y,r(x)),$$
$$\theta_{l_3}(x,y) := K_0(r(x),y,r^2(x)) \wedge E^*_{l_3-(2s+2)}(y,r(x)) \wedge \neg E^*_{l_3-(2s+3)}(y,r(x)),$$
$$\text{where } 2s+4 \leq l_3 \leq 3s+3,$$
$$\theta_{l_4}(x,y) := K_0(r(x),y,r^2(x)) \wedge E^*_{4s+5-l_4}(y,r^2(x)) \wedge \neg E^*_{4s+4-l_4}(y,r^2(x)),$$
$$\text{where } 3s+4 \leq l_2 \leq 4s+3,$$
$$\theta_{4s+4}(x,y) := K_0(r(x),y,r^2(x)) \wedge E_1^*(y,r^2(x)),$$
$$\theta_{4s+5}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E_1^*(y,r^2(x)),$$
$$\theta_{l_5}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E^*_{l_5-(4s+4)}(y,r^2(x)) \wedge \neg E^*_{l_5-(4s+5)}(y,r^2(x)),$$
$$\text{where } 4s+6 \leq l_5 \leq 5s+5,$$
$$\theta_{l_6}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E^*_{6s+7-l_6}(y,r^3(x)) \wedge \neg E^*_{6s+6-l_6}(y,r^3(x)),$$
$$\text{where } 5s+6 \leq l_6 \leq 6s+5,$$
$$\theta_{6s+6}(x,y) := K_0(r^2(x),y,r^3(x)) \wedge E_1^*(y,r^3(x)),$$
$$\theta_{6s+7}(x,y) := K_0(r^3(x),y,x) \wedge E_1^*(y,r^3(x)),$$
$$\theta_{l_7}(x,y) := K_0(r^3(x),y,x) \wedge E^*_{l_7-(6s+6)}(y,r^3(x)) \wedge \neg E^*_{l_7-(6s+7)}(y,r^3(x)),$$
$$\text{where } 6s+8 \leq l_7 \leq 7s+7,$$
$$\theta_{l_8}(x,y) := K_0(r^3(x),y,x) \wedge E_{8s+9-l_8}(y,x) \wedge \neg E_{8s+8-l_8}(y,x),$$
$$\text{where } 7s+8 \leq l_8 \leq 8s+7,$$
$$\theta_{8s+8}(x,y) := K_0(r^3(x),y,x) \wedge E_1(x,y).$$

Thus, we have $2+2s+2+2s+2+2s+2+2s+1 = 8s+9$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a,M), \theta_1(a,M), \theta_2(a,M), \ldots, \theta_{8s+6}(a,M), \theta_{8s+7}(a,M), \theta_{8s+8}(a,M)).$$

Prove now that the algebra $\mathfrak{P}_{M'_{s,4,4}}$ is non-commutative and strictly $(2s+3)$-deterministic for every $s \geq 1$.

Firstly, obviously that $0 \cdot l = l \cdot 0 = \{l\}$ for any $0 \leq l \leq 8s+8$.

Suppose further that $l_1 > 0$ and $l_2 > 0$. Consider the following formula:

$$\exists t[\theta_{l_1}(x,t) \wedge \theta_{l_2}(t,y)].$$

Case 1: $l_1 = 1$.

We have: $K_0(x,t,r(x))$ and $E_1(x,t)$. Let also $l_2 = 1$, i.e. $K_0(t,y,r(t))$ and $E_1(t,y)$, whence we obtain: $K_0(x,y,r(x)) \wedge E_1(x,y)$, i.e. $l_1 \cdot l_2 = \{l_2\}$.

Let now $2 \leq l_2 \leq s+1$, i.e. $K_0(t,y,r(t)) \wedge E_{l_2}(t,y) \wedge \neg E_{l_2-1}(t,y)$. We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $s+2 \leq l_2 \leq 2s+1$, i.e. $K_0(t,y,r(t)) \wedge E^*_{2s+3-l_2}(y,r(t)) \wedge \neg E^*_{2s+2-l_2}(y,r(t))$. We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 2s+2$, i.e. $K_0(t,y,r(t)) \wedge E^*_1(y,r(t))$. We have: $l_1 \cdot l_2 = \{2s+2\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{2s+2, 2s+3\}$.

Thus, we conclude that $\mathfrak{P}_{M'_{s,4,4}}$ is not commutative for every $s \geq 1$.

Let now $l_2 = 2s+3$, i.e. $K_0(r(t),y,r^2(t)) \wedge E^*_1(y,r(t))$. We have: $l_1 \cdot l_2 = \{2s+2, 2s+3\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{2s+3\}$.

Let now $2s+4 \leq l_2 \leq 3s+3$, i.e.

$$K_0(r(t),y,r^2(t)) \wedge E^*_{l_2-(2s+2)}(y,r(t)) \wedge \neg E^*_{l_2-(2s+3)}(y,r(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $3s+4 \leq l_2 \leq 4s+3$, i.e.

$$K_0(r(t),y,r^2(t)) \wedge E^*_{4s+5-l_2}(y,r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y,r^2(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t),y,r^2(t)) \wedge E^*_1(y,r^2(t))$. We have: $l_1 \cdot l_2 = \{4s+4, 4s+5\}$. Consider $l_2 \cdot l_1$. We also have that $l_2 \cdot l_1 = \{4s+4, 4s+5\}$.

Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t),y,r^3(t)) \wedge E^*_1(y,r^2(t))$. We have: $l_1 \cdot l_2 = \{4s+5\}$. Consider $l_2 \cdot l_1$. We also have that $l_2 \cdot l_1 = \{4s+5\}$.

Let now $4s+6 \leq l_2 \leq 5s+5$, i.e.

$$K_0(r^2(t),y,r^3(t)) \wedge E^*_{l_2-(4s+4)}(y,r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y,r^2(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t),y,r^3(t)) \wedge E^*_{6s+7-l_2}(y,r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y,r^3(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t),y,r^3(t)) \wedge E^*_1(y,r^3(t))$. We have: $l_1 \cdot l_2 = \{6s+6\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{6s+6, 6s+7\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. We have: $l_1 \cdot l_2 = \{6s+6, 6s+7\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{6s+7\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t)$. We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(t, y)$. We have: $l_1 \cdot l_2 = \{0, 1, 8s+8\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{0, 1, 8s+8\}$.

Case 2: $2 \leq l_1 \leq s+1$.

We have: $K_0(x, t, r(x)) \wedge E_{l_1}(x, t) \wedge \neg E_{l_1-1}(x, t)$.

Let also $2 \leq l_2 \leq s+1$, i.e. $K_0(t, y, r(t)) \wedge E_{l_2}(t, y) \wedge \neg E_{l_2-1}(t, y)$. Then we have the following: if $l_1 \geq l_2$ then $l_1 \cdot l_2 = \{l_1\}$; if $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2\}$.

Let now $s+2 \leq l_2 \leq 2s+1$, i.e.

$$K_0(t, y, r(t)) \wedge E^*_{2s+3-l_2}(y, r(t)) \wedge \neg E_{2s+2-l_2}(y, r(t)).$$

We have the following: if $l_1 < 2s+3-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 2s+3-l_2$ then $l_1 \cdot l_2 = \{l_2\}$ and $l_2 \cdot l_1 = \{l_2, \ldots, 4s+5-l_2\}$.

If $l_1 > 2s+3-l_2$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $l_2 = 2s+2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. Then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$. Let now $l_2 = 2s+3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Also, then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $2s+4 \leq l_2 \leq 3s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{l_2-(2s+2)}(y, r(t)) \wedge \neg E^*_{l_2-(2s+3)}(y, r(t)).$$

We have the following: if $l_1 < l_2-(2s+2)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = l_2-(2s+2)$ then $l_1 \cdot l_2 = \{2s+3-l_1, \ldots, l_2\}$ and $l_2 \cdot l_1 = \{l_2\}$.

If $l_1 > l_2-(2s+2)$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $3s+4 \leq l_2 \leq 4s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{4s+5-l_2}(y, r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y, r^2(t)).$$

We have the following: if $l_1 < 4s+5-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 4s+5-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2, \ldots, 8s+9-l_2\}$.

If $l_1 > 4s+5-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$. Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Also, then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $4s+6 \leq l_2 \leq 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: if $l_1 \leq l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 > l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: if $l_1 < 6s+7-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_2\}$ and $l_2 \cdot l_1 = \{l_2, \ldots, 6s+6+l_1\}$.

If $l_1 > 6s+7-l_2$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then also $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $l_1 < l_2-(6s+6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = l_2-(6s+6)$ then $l_1 \cdot l_2 = \{6s+7-l_1, \ldots, l_2\}$ and $l_2 \cdot l_1 = \{l_2\}$.

If $l_1 > l_2-(6s+6)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 < 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$. If $l_1 = 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2, \ldots, 8s+8, 0, 1, \ldots, l_1\}$. If $l_1 > 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$. Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(t, y)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 3. $s+2 \leq l_1 \leq 2s+1$.

We have: $K_0(x, t, r(x)) \wedge E^*_{2s+3-l_1}(t, r(x)) \wedge \neg E^*_{2s+2-l_1}(t, r(x))$.

Let also $s+2 \leq l_2 \leq 2s+1$, i.e.

$$K_0(t, y, r(t)) \wedge E^*_{2s+3-l_2}(y, r(t)) \wedge \neg E^*_{2s+2-l_2}(y, r(t)).$$

Then we have the following: if $l_1 < l_2$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$. If $l_1 = l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{2s+2+l_1, \ldots, 6s+7-l_1\}$. If $l_1 > l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $l_2 = 2s+2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $l_2 = 2s+3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Then we also have $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $2s+4 \leq l_2 \leq 3s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{l_2-(2s+2)}(y, r(t)) \wedge \neg E^*_{l_2-(2s+3)}(y, r(t)).$$

We have the following: if $2s+3-l_1 = l_2-(2s+2)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1, \ldots, 6s+7-l_1\}$. If $2s+3-l_1 < l_2-(2s+2)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$. If $2s+3-l_1 > l_2-(2s+2)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $3s+4 \leq l_2 \leq 4s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{4s+5-l_2}(y, r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y, r^2(t)).$$

We have the following: if $2s+3-l_1 = 4s+5-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{2s+2+l_1, \ldots, 10s+11-l_2\}$.

If $2s+3-l_1 < 4s+5-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

If $2s+3-l_1 > 4s+5-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we also have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $4s+6 \leq l_2 \leq 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: if $2s+3-l_1 > l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$. If $2s+3-l_1 = l_2-(4s+4)$ then $l_1 \cdot l_2 = \{4s+4+l_1, \ldots, 2s+2+l_2\}$ and $l_2 \cdot l_1 = \{4s+4+l_1\}$. If $2s+3-l_1 < l_2-(4s+4)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: if $2s+3-l_1 > 6s+7-l_2$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$. If $2s+3-l_1 = 6s+7-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{2s+2+l_2, \ldots, 8s+8, 0, 1, \ldots, 6s+7-l_2\}$. If $2s+3-l_1 < 6s+7-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we also have $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $2s+3-l_1 \geq l_2-(6s+6)$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$. If $2s+3-l_1 < l_2-(6s+6)$ then $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{8s+9-l_2}(y, t) \wedge \neg E^*_{8s+8-l_2}(y, t).$$

We have the following: if $2s+3-l_1 > 8s+9-l_2$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$. If $2s+3-l_1 = 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_1\}$ and $l_2 \cdot l_1 = \{l_1, \ldots, 4s+5-l_1\}$. If $2s+3-l_1 < 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 4. $l_1 = 2s+2$.

We have: $K_0(x, t, r(x)) \wedge E_1^*(t, r(x))$.

Let also $l_2 = 2s+2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{4s+4, 4s+5\}$. Let now $l_2 = 2s+3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{4s+5\}$ and $l_2 \cdot l_1 = \{4s+4\}$.

Let now $2s+4 \leq l_2 \leq 3s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{l_2-(2s+2)}(y, r(t)) \wedge \neg E^*_{l_2-(2s+3)}(y, r(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $3s+4 \le l_2 \le 4s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{4s+5-l_2}(y, r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s+6\}$ and $l_2 \cdot l_1 = \{6s+6, 6s+7\}$.

Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s+6, 6s+7\}$ and $l_2 \cdot l_1 = \{6s+6\}$.

Let now $4s+6 \le l_2 \le 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{8s+8, 0, 1\}$. Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{1\}$ and $l_2 \cdot l_1 = \{8s+8\}$.

Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{2s+2\}$ and $l_2 \cdot l_1 = \{2s+2, 2s+3\}$.

Case 5. $l_1 = 2s+3$.

We have: $K_0(r(x), t, r^2(x)) \wedge E^*_1(t, r(x))$.

Let also $l_2 = 2s+3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E^*_1(y, r(t))$. Then we have $l_1 \cdot l_2 = \{4s+4, 4s+5\}$. Let now $2s+4 \le l_2 \le 3s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{l_2-(2s+2)}(y, r(t)) \wedge \neg E^*_{l_2-(2s+3)}(y, r(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $3s+4 \le l_2 \le 4s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{4s+5-l_2}(y, r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s+6, 6s+7\}$ and $l_2 \cdot l_1 = \{6s+7\}$.

Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s+7\}$ and $l_2 \cdot l_1 = \{6s+6, 6s+7\}$.

Let now $4s+6 \le l_2 \le 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{8s+8\}$ and $l_2 \cdot l_1 = \{1\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{8s+8, 0, 1\}$. Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{2s+2, 2s+3\}$ and $l_2 \cdot l_1 = \{2s+3\}$.

Case 6. $2s+4 \le l_1 \le 3s+3$.

We have: $K_0(r(x), t, r^2(x)) \wedge E^*_{l_1-(2s+2)}(t, r(x)) \wedge \neg E^*_{l_1-(2s+3)}(t, r(x))$.

Let also $2s+4 \le l_2 \le 3s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{l_2-(2s+2)}(y, r(t)) \wedge \neg E^*_{l_2-(2s+3)}(y, r(t)).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{6s+7-l_2, \ldots, 2s+2+l_2\}$.

If $l_1 < l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{2s+2+l_1\}$.

Let now $3s+4 \le l_2 \le 4s+3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{4s+5-l_2}(y, r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y, r^2(t)).$$

We have the following: if $l_1-(2s+2) < 4s+5-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$. If $l_1-(2s+2) = 4s+5-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2, \ldots, 10s+11-l_2\}$ and $l_2 \cdot l_1 = \{4s+4+l_2\}$. If $l_1-(2s+2) > 4s+5-l_2$ then $l_1 \cdot l_2 = \{4s+4+l_1\}$ and $l_2 \cdot l_1 = \{8s+9-l_1\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^2(t))$. Then we also have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $4s+6 \le l_2 \le 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: if $l_1-(2s+2) < l_2-(4s+4)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$. If $l_1-(2s+2) = l_2-(4s+4)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$

and $l_2 \cdot l_1 = \{10s + 11 - l_2, \ldots, 2s + 2 + l_2\}$. If $l_1 - (2s + 2) > l_2 - (4s + 4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: if $l_1-(2s+2) \leq 6s+7-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$. If $l_1-(2s+2) > 6s+7-l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s + 2)\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{10s + 11 - l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s + 2)\}$. Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E^*_1(y, r^3(t))$. Then we also have $l_1 \cdot l_2 = \{10s + 11 - l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s + 2)\}$. Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $l_1 - (2s + 2) < l_2 - (6s + 6)$ then $l_1 \cdot l_2 = \{l_2 - (6s + 6)\}$ and $l_2 \cdot l_1 = \{14s + 15 - l_2\}$. If $l_1 - (2s + 2) = l_2 - (6s + 6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{10s+11-l_1, \ldots, 0, \ldots, l_1-(2s+2)\}$. If $l_1-(2s+2) > l_2 - (6s + 6)$ then $l_1 \cdot l_2 = \{10s + 11 - l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s + 2)\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1-(2s+2) < 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$. If $l_1 - (2s + 2) = 8s + 9 - l_2$ then $l_1 \cdot l_2 = \{4s + 5 - l_1, \ldots, l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$. If $l_1 - (2s + 2) > 8s + 9 - l_2$ then $l_1 \cdot l_2 = \{10s + 11 - l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 7. $3s + 4 \leq l_1 \leq 4s + 3$.

We have: $K_0(r(x), t, r^2(x)) \wedge E^*_{4s+5-l_1}(t, r^2(x)) \wedge \neg E^*_{4s+4-l_1}(t, r^2(x))$.

Let also $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E^*_{4s+5-l_2}(y, r^2(t)) \wedge \neg E^*_{4s+4-l_2}(y, r^2(t)).$$

We have the following: if $l_1 \leq l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_2\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^2(t))$. Then we also have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$. Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: if $4s+5-l_1 < l_2-(4s+4)$ then $l_1 \cdot l_2 = \{14s+5-l_2\}$ and $l_2 \cdot l_1 = \{l_2 - (4s+4)\}$. If $4s+5-l_1 = l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1, \ldots, 40, 0, \ldots, 4s + 5 - l_1\}$. If $4s + 5 - l_1 > l_2 - (4s + 4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: if $4s+5-l_1 < 6s+7-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 = 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_2-(4s+4), \dots, 6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 > 6s+7-l_2$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E^*_1(y, r^3(t))$. Then we also have $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $4s+5-l_1 < l_2-(6s+6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 = l_2-(6s+6)$ then $l_1 \cdot l_2 = \{l_2-(4s+4)\}$ and $l_2 \cdot l_1 = \{8s+9-l_2, \dots, l_2-(4s+4)\}$. If $4s+5-l_1 > l_2-(6s+6)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $4s+5-l_1 < 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 \ge 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 8. $l_1 = 4s+4$.

We have: $K_0(r(x), t, r^2(x)) \wedge E^*_1(t, r^2(x))$.

Let also $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{8s+8\}$. Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{0, 1, 8s+8\}$ and $l_2 \cdot l_1 = \{2s+2\}$.

Let now $4s+6 \le l_2 \le 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{6s+7-l_2\}$ and $l_2 \cdot l_1 = \{l_2-(4s+4)\}$.

Let now $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2-(4s+4)\}$ and $l_2 \cdot l_1 = \{l_2-(3s+2)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+2, 2s+3\}$ and $l_2 \cdot l_1 = \{2s+2\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E^*_1(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+3\}$ and $l_2 \cdot l_1 = \{2s+2, 2s+3\}$.

Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y,t) \wedge \neg E_{8s+8-l_2}(y,t).$$

We also have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y,t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4\}$.

Case 9. $l_1 = 4s+5$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E_1^*(t, r^2(x))$.

Let also $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{1\}$. Let now $4s+6 \le l_2 \le 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

Let now $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+2\}$ and $l_2 \cdot l_1 = \{2s+2, 2s+3\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+2, 2s+3\}$ and $l_2 \cdot l_1 = \{2s+3\}$.

Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y,t) \wedge \neg E^*_{8s+8-l_2}(y,t).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y,t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4, 4s+5\}$.

Case 10. $4s+6 \le l_1 \le 5s+5$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E^*_{l_1-(4s+4)}(t, r^2(x)) \wedge \neg E^*_{l_1-(4s+5)}(t, r^2(x))$.

Let also $4s+6 \le l_2 \le 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{l_2-(4s+4)}(y, r^2(t)) \wedge \neg E^*_{l_2-(4s+5)}(y, r^2(t)).$$

We have the following: if $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2 - (4s+4)\}$, if $l_1 \ge l_2$ then $l_1 \cdot l_2 = \{l_1 - (4s+4)\}$. Let now $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^3(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^3(t)).$$

We have the following: if $l_1 - (4s+4) < 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_2 - (4s+4)\}$, if $l_1 - (4s+4) \ge 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_1 - (3s+4)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{l_1 - (3s+4)\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{l_1 - (3s+4)\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $l_1 - (4s+4) = l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{l_1 - (4s+3), l_1 - (4s+2), \ldots, l_1 - (2s+2)\}$ and $l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

If $l_1 - (4s+4) < l_2 - (6s+6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

If $l_1 - (4s+4) > l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_2 - (2s+2)\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{8s+9-l_2}(y, t) \wedge \neg E^*_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 - (4s+4) = 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{8s+9-l_1, 8s+8-l_1, \ldots, l_1\}$.

If $l_1 - (4s+4) < 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

If $l_1 - (4s+4) > 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 11. $5s+6 \le l_1 \le 6s+5$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E^*_{6s+7-l_1}(t, r^2(x)) \wedge \neg E^*_{6s+6-l_1}(t, r^2(x))$.

Let also $5s+6 \le l_2 \le 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E^*_{6s+7-l_2}(y, r^2(t)) \wedge \neg E^*_{6s+6-l_2}(y, r^2(t)).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$. If $l_1 < l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$. If $l_1 > l_2$ then $l_1 \cdot l_2 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $6s+8 \le l_2 \le 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $6s+7-l_1 = l_2-(6s+6)$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$. If $6s+7-l_1 < l_2-(6s+6)$ then $l_1 \cdot l_2 = \{l_2-(2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$. If $6s+7-l_1 > l_2-(6s+6)$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{8s+9-l_2}(y, t) \wedge \neg E^*_{8s+8-l_2}(y, t).$$

We have the following: if $6s+7-l_1 = 8s+9-l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1, l_1+1, \ldots, 14s+15-l_2(12s+13-l_1)\}$. If $6s+7-l_1 < 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_2-(2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$. If $6s+7-l_1 > 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 12. $l_1 = 6s + 6$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E_1^*(t, r^3(x))$. Let also $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{4s + 4, 4s + 5\}$.

Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$.

We have the following: $l_1 \cdot l_2 = \{4s + 5\}$ and $l_2 \cdot l_1 = \{4s + 4\}$.

Let now $6s + 8 \le l_2 \le 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $7s + 8 \le l_2 \le 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2)}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{6s + 6\}$ and $l_2 \cdot l_1 = \{6s + 6, 6s + 7\}$.

Case 13. $l_1 = 6s + 7$.

We have: $K_0(r^3(x), t, x) \wedge E_1^*(t, r^3(x))$. Let also $l_2 = 6s + 7$,
i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{4s + 4, 4s + 5\}$.

Let now $6s + 8 \le l_2 \le 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $7s + 8 \le l_2 \le 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2)}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{6s + 6, 6s + 7\}$ and $l_2 \cdot l_1 = \{6s + 7\}$.

Case 14. $6s + 8 \le l_1 \le 7s + 7$.

We have: $K_0(r^3(x), t, x) \wedge E^*_{l_1-(6s+6)}(t, r^3(x)) \wedge \neg E^*_{l_1-(6s+7)}(t, r^3(x))$.

Let also $6s + 8 \le l_2 \le 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E^*_{l_2-(6s+6)}(y, r^3(t)) \wedge \neg E^*_{l_2-(6s+7)}(y, r^3(t)).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1, \dots, l_1-(2s+2)\}$.

If $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2 - (2s + 2)\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = \{10s + 11 - l_2\}$ and $l_2 \cdot l_1 = \{l_2 - (2s + 2)\}$.

Let now $7s + 8 \le l_2 \le 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2)}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 - (6s + 6) = 8s + 9 - l_2$ then $l_1 \cdot l_2 = \{12s + 13 - l_1, \dots, l_1\}$ and $l_2 \cdot l_1 = \{14s + 15 - l_2\}$. If $l_1 - (6s + 6) < 8s + 9 - l_2$ then $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$. If $l_1 - (6s+6) > 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{10s + 11 - l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$.

Case 15. $7s+8 \le l_1 \le 8s+7$.
We have: $K_0(r^3(x),t,x) \wedge E_{8s+9-l_1)}(t,x) \wedge \neg E_{8s+8-l_1}(t,x)$.
Let also $7s+8 \le l_2 \le 8s+7$, i.e.

$$K_0(r^3(t),y,t) \wedge E_{8s+9-l_2)}(y,t) \wedge \neg E_{8s+8-l_2}(y,t).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{l_1\}$. If $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2\}$. If $l_1 > l_2$ then $l_1 \cdot l_2 = \{l_1\}$.
Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t),y,t) \wedge E_1(y,t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 16. $l_1 = 8s+8$.
We have: $K_0(r^3(x),t,x) \wedge E_1(t,x)$. Let also $l_2 = 8s+8$, i.e. $K_0(r^3(t),y,t) \wedge E_1(y,t)$. Then we have $l_1 \cdot l_2 = \{l_1\}$.
Thus, we established that the algebra $\mathfrak{P}_{M'_{s,m,k}}$ is commutative and strictly $(2s+3)$-deterministic for all valid values s, m and k. $\square$

3 Conclusion

We investigated algebras of binary isolating formulas for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a non-trivial piecewise monotonic-to-left function acting on the universe of a structure with an arbitrary s and $m = k = 4$. We also proved their non-commutativity and established their strict l-deterministicity for some natural l. It would now be interesting to describe the corresponding algebras for such theories having arbitrary s, m and k.

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Beibut Shaiykovich Kulpeshov
Institute of Mathematics and Mathematical Modeling,
Shevchenko str., 28,
050010, Almaty, Kazakhstan
Kazakh-British Technical University,
Tole bi str., 59,
050000, Almaty, Kazakhstan
E-mail address: kulpesh@mail.ru, b.kulpeshov@kbtu.kz