Multi-agent Logics with Frozen States, Admissibility via Projectivity

Vladimir V. Rybakov
Institute of Mathematics and Computer Science,
Siberian Federal University, Krasnoyarsk;
and Institute of Informatics Systems Ac. Sci., Novosibirsk
Vladimir_Rybakov@mail.ru

Subject Classification: 510.64,510.65,510.66

Abstract. [1]

In this paper we study multi-agent temporal logics whose models allow frozen states if information nodes (information states). Models for these logics are similar to standard ones for multi-modal temporal logics, but the current time clusters may posses states with broken connections of information channels. Such look at models (as we feel) stays close to real world situation with networks of agents' information. In order to solve open problems of decidability in such logics and recognizing admissible rules we use some elements of unification and projective formulas techniques and find positive solution of these problems.

Key words: *temporal logics, multi-agent logics, information, knowledge, deciding algorithms*

1 Introduction

Multi-agent logics and their combination with modal logics made active research area in AI and social sciences. E.g. the book [1] suggests approach used multi-dimensional Kripke models $M = \langle W, R_1, \ldots, R_n, V \rangle$, where W models some amount of states and R_j represents agents' accessibility relations between states. Then the modal-like formula $K_i A$ is interpreted as – the agent i knows the information ((or the knowledge) described by formula A. That approach may be extended by usage instead the formula $K_i A$ a more informative ones, which means that the agent is simply informed (a formula $A_i(x)$ illustrates that *the agent i knows (may know) information x.* That ideas were developed in Rybakov [2], in particular, there we find algorithm for decidability such logics,

[1]This work was supported by the Russian Science Foundation and Krasnoyarsk Regional Fund of Science (Project No 25-21-20011, https://rscf.ru/en/project/25-21-20011/).

several applications been suggested. To nowadays many works in application non-standard logic to AI and computation logical information appeared in press (cf. for instance [3, 4, 5, 6, 7, 8]).

In this our paper we develop some new logical techniques in order to solve open problems for the case of multi-agent logics whose models allow frozen states if information nodes (information states). The models look as standard multi-modal temporal logics, but the current time clusters may posses states with broken connection of information channels. All accessibility relations of agents' - relations R_j – in each time cluster $C(i)$ defined separately and independently. This approach corresponds close to real world situation with networks of agents' information. In order to solve open problems of decidability in such logics and recognizing admissible rules we use some elements of unification and projective formulas techniques. That not only allows to essentially short proofs but also helps to solve affirmatively the problems.

2 Short recall on projective formulas

We start by short recall known technique which we will use longer. Basic knowledge in non-classical logic is assumed. We need to report basics of unification. For a set of propositional letters P a substitution is a mapping ε of P into set of all formulas For; any such substitution may be extended to set of all formulas by $\varepsilon(\varphi(x_1, \ldots, x_n)) := \varphi(\varepsilon(x_1), \ldots, \varepsilon(x_n))$. A formula is said to be unifiable in a logic L if there is a substitution ε (which is said to be an unifier for φ) such that $\varepsilon(\varphi) \in L$.

Definition 2.1. *A unifier ε for a formula φ in a logic L is said to be more general then a unifier ε_1 if there is a substitution δ such that for any letter x, $[\varepsilon_1(x) \equiv \delta(\varepsilon(x))] \in L$.*

In usual, if a logic L is decidable, to verify unifiability of a formula is an easy task (in theoretical means, since complexity of computation may be very high) : actually it is sufficient to verify only substitution of letters from the set $\{\bot, \top\}$ instead propositional letters - variables. At the same time, the question of finding all unifiers is not simple.

Definition 2.2. *A set of unifiers CU for a formula φ in a logic L is complete set of unifiers if the following holds. For any unifier σ of φ in L, there is a unifier σ_1 from CU, such that σ_1 is more general then σ.*

For modal logics, for example extending modal logic $S4$, the definition of projective formulas is as follows:

Definition 2.3. *A formula φ is projective in a logic L if the following holds. There exists a substitution σ (which is called projective substitution for formula φ) such that σ is a unifier for φ in L and $\Box\varphi \to [x_i \equiv \sigma(x_i)] \in L$ for any propositional letter x_i from φ.*

Lemma 2.4. *If a substitution σ_p is projective for φ in L, then $\{\sigma_p\}$ is a complete set of unifiers for φ (that is σ_p is more general unifier).*

Proof. Assume that σ is some unifier for φ in L. Since σ_p is projective for φ in L, we get $\Box\varphi \rightarrow [x_i \equiv \sigma_p(x_i)] \in L$ for any letter x_i from φ. Applying σ to formulas above we get $\sigma(\Box\varphi) \rightarrow [\sigma(x_i) \equiv \sigma(\sigma_p(x_i))] \in L$, that is $\sigma(x_i) \equiv \sigma(\sigma_p(x_i)) \in L$. Q.E.D.

It is easy to see that, if for a decidable modal logic L, we can verify if a formula φ is projective and can compute its projective unifier, then **admissibility problem in logic L is decidable**. Indeed, recall *the definition.* An inference rule $\varphi_1, \dots \varphi_n/\psi$ is admissible in L iff for any unifier ϵ for $\varphi_1 \wedge \dots \wedge \varphi_n$ this ϵ is also a unifier for ψ.

To show decidability of the admissibility problem then, admit that an inference rule $\varphi_1, \dots \varphi_n/\psi$ is admissible in L, Admit that there is a unifier for $\varphi_1, \dots \varphi_n$. Then by our assumption above we may compute its projective unifier and it is enough just to verify if this projective unifier (which is then most general unifier) is a unifier for ψ, and the problem is solved. This fruitful idea was found by Silvio Ghilardi [9,10]. In this our paper we will apply elements of this technique below.

3 Decidability by Projectivity

All definitions an facts recalled above are true (and applicable) to our logic introduced below. As we noted above, applications of projective technique was found by Silvio Ghilardi [9,10] and later on it was applied to the linear modal logics in works of Wojciech Dzik and Piotr Wojtylak for modal logics extending S.4.3 cf. [11]. Recently we used similar technique for multi-modal logics studying transfer information and its' reliability (cf. [12]). We start now by definition of semantics for our logic - Kripke-like frames of our logic.

Definition 3.1. *Frames $F := \langle \bigcup_{i \in N} C(i), \leq, Prev, R_j, j \in J \rangle$ are structures where $C(i), i \in N$ are infinite sequence of not-intersecting sets $C(i)$ of states, which may be thought as clusters of time.*

The definitions of operations used in the one above are explained below. Here, each R_j is a binary reflexive and transitive accessibility relation on each $C(i)$, – the agents' accessibility relations (if aR_jb there is $C(i)$ where $a, b \in C(i)$). The binary relation $\leq$ is the linear-like order of time: For all $a, b \in \bigcup_{i \in N} C(i)$, $a \leq b$ iff $a \in C(i)$ for some $C(i)$ and $b \in C(i_1)$, where $i \leq i_1$. $Prev$ is the binary relation introduced to distinguish the previous time cluster, its definition is as follows: $\forall a, b \in\in \bigcup_{i \in N} C(i)$, $aPrevb \Leftrightarrow b \in C(i+1), a \in C(i)$ or $a = b$ (the latter one is very important to our research technique would work).

A model M for our logic to be obtained from a frame F by choice a valuation V a set of propositional letters in states of the frame. A valuation V of a set of propositional letters (variables) P in F is a mapping V of P in the set of all subsets of $\bigcup_{i \in N} C(i)$, so $\forall p \in P, V(p) \subseteq \bigcup_{i \in N} C(i)$. We write for $a \in F$ that $a \Vdash_V p$ iff $a \in V(P)$, and we say then that p is true at a w.r.t. V.

The language of our temporal multi-agent logic consists of propositional letters, usual Boolean logical operations, temporal-modal operations $\Box$ and $\Diamond$, and agents' operations $\Box_i, \Diamond_j, j \in J$ and the previous-time unary operation $\Diamond_{Prev}$. Rules for constructions of formulas by these operations are standard.

For any Kripke model M, the truth values can be extended from propositions letters P to arbitrary formulas constructed from these propositions as follows:

$\forall p \in Prop\ (M,a) \Vdash_V p \Leftrightarrow a \in \bigcup_{i\in N} C(i) \wedge\ a \in V(p)$;

$(M,a) \Vdash_V (\varphi \wedge \psi) \Leftrightarrow (M,a) \Vdash_V \varphi\ \wedge\ (M,a) \Vdash_V \psi$;

$(M,a) \Vdash_V (\varphi \vee \psi) \Leftrightarrow (M,a) \Vdash_V \varphi\ \vee\ (M,a) \Vdash_V \psi$;

$(M,a) \Vdash_V \neg\varphi \Leftrightarrow not[(M,a) \Vdash_V \varphi]$;

$(M,a) \Vdash_V \Diamond\varphi \Leftrightarrow \exists b[(a \leq b)\&(M,b) \Vdash_V \varphi]$;

$(M,a) \Vdash_V \Box\varphi \Leftrightarrow \forall b[(a \leq b)\Rightarrow(M,b) \Vdash_V \varphi]$;

$(M,a) \Vdash_V \Diamond_j\varphi \Leftrightarrow \exists b[(aR_jb)\&(M,b) \Vdash_V \varphi]$;

$(M,a) \Vdash_V \Box_j\varphi \Leftrightarrow \forall b[(aR_jb)\Rightarrow(M,b) \Vdash_V \varphi]$;

$(M,a) \Vdash_V \Diamond_{Prev}\varphi \Leftrightarrow \exists b[(bPreva)\&(M,b) \Vdash_V \varphi]$;

For a Kripke model $M := \langle F, V\rangle$ and a formula φ with letters from the domain of V, φ is true in M (denotation – $M \Vdash \varphi$) if, for any state b from F, the formula φ is true at b (that is: $(M,b) \Vdash_V \varphi$). For a frame $F := \langle W, R\rangle$, we say that a formula φ is true at F, (and we will write $F \Vdash \varphi$) if φ is true at any model based at F.

Let $P_n := \{p_1, \ldots, p_n\}$ be a fixed set of propositional letters. Let Pos_n be a non-empty set of some subsets Y_n of P_n. Let $\psi(Y_n) := [\bigwedge_{p_j \in Y_n} p_j \wedge \bigwedge_{p_j \notin Y_n} \neg p_j]$. We now impose some additional requirement on the structure of all models M of our logic in order to express all possible situations for agents accessibility relations.

Definition 3.2. *We say that a model M has frozen states w.r.t. P_n if for any $C(i)$ there is some unique set $P_n[C(i)]$ of some sets Pos_n each of which consists of letters from P_n with following properties. For any $Y_n \in Pos_n$ there are some states $a_{Y_n} \in C(i)$ such that $(a_{Y_n} \Vdash_V [\psi(Y_n) \wedge \bigwedge_j \Box_j \psi(Y_m)])$. Besides, for all subsets Y_m of $P_n := \{p_1, \ldots, p_n\}$ not occurring in the set $P_n[C(i)]$, $\forall b \in C(i)(b \nVdash_V [\psi(Y_m) \wedge \bigwedge_j \Box_j \psi(Y_m)])$.*

So, states a_{Y_n} have frozen agents' accessibility relations R_J, so to say broken information channels. All accessibility relations of agents' - relations R_j – in each time cluster $C(i)$ defined separately and independently. This approach corresponds close to real world situation with networks of agents' information.

Our logic L is *the set of all formulas $\varphi(p_1, \ldots, p_n)$ which are true at any state of any model M with frozen sets w.r.t. P_n and any given valuation.*

Recall that a formula $\varphi(p_1, \dots, p_n)$ is said to be unifiable in a logic L if there is a substitution $\varepsilon(x_i) := \beta_i$ arbitrary formulas β_i instead of propositional letters x_i such that $\varphi(\beta_1, \dots, \beta_n) \in L$.

Possible case for choice of our models for the logic L is the model M_0 when for all i $C(i) = \{a_{0,i}\}$ and when the valuation is $V_0(p_k) = \{a_{0,i} \mid 0, i\}$ for all p_k. That is for all $a_{0,i}$, $a_{0,i} \Vdash_{V_0} p_k$ for all k.

Lemma 3.3. *There is an algorithm, which for any formula $\varphi(p_1, \dots, p_n)$, constructed out of letters $p_1, \dots, p_n$, verifies if this formula is unifiable in L and if yes constructs a substitution for letters $p_1, \dots, p_n$ which replaces p_j on y_j which is a unifier for $\varphi(p_1, \dots p_n)$ in L and any y_j is either $\top$ or $\bot$. We will denote in the sequel $\epsilon(p_j) := y_j$.*

Proof. If a formula $\varphi(p_1, \dots, p_n)$ is unifiable in L then there is a substitution $p_j \Rightarrow \psi_j$, such that the formula $\varphi(\psi_1, \dots, \psi_n)$ is true by any valuation V at any state of any model for L with some frozen sets w.r.t all propositional letters from $\varphi(\psi_1, \dots, \psi_n)$. In particular this holds at the model M_o w.r.t. $Pos_n = \{P_n\}$ and P_n to be the set of all propositional letters from $\varphi(\psi_1, \dots, \psi_n)$, $M_o \Vdash \varphi(\psi_1, \dots, \psi_n)$.

Then the formula $\varphi(\epsilon(p_1), \dots, \epsilon(p_n))$ is true in the one element model w.r.t. any valuation V_1 when w.r.t. V_1 truth value of any letter p_j is $\epsilon(p_j)$ where $\epsilon(p_j) = \top$ or $\epsilon(p_j) = \bot$ w.r.t. how ψ_j was true in M_o at any state.

For some substitutions $\epsilon_1(p_j) \in \{\top, \bot\}$, if the formula $\varphi(\epsilon_1(p_1), \dots, \epsilon_1(p_n))$ is true in the one element model it is easy to calculate. If we found a such substitution, then the formula $\varphi(\epsilon_1(p_1), \dots, \epsilon_1(p_n)$ is true in any model with frozen sets, so φ is unifiable. Q.E.D.

Theorem 3.4. *Any unifiable formula φ is projective in L. There is an algorithm verifying unifiability in L and constructing a projective unifier for φ if φ is unifiable (so, then the admissibility of inference rules in L is decidable).*

Proof. Given a formula $\varphi(p_1, \dots, p_n)$ constructed out of propositional letters p_i. By **Lemma 3.3.** we verify if this formula is unifiable in L and if yes construct a shown in this Lemma unifier $\epsilon(p_j) := y_j$. We now will try to prove that this formula is projective.

Let $Var(\Box\varphi)$ be the set $\{p_1, \dots, p_n\}$ of all propositional letters from $\Box\varphi$ (so in other notation, $Var(\Box\varphi) = P_n$. To fix notation, let now SV be the set of all subsets of the set $Var(\Box\varphi)$.

Let for any $SV_p \subseteq SV$, ASV_p be a fixed set from SV_p. Else let for $p_k \in Var(\Box\varphi)$, $tv(SV_p, p_k) = \top$ if $p_k \in ASV_p$ and $tv(SV_p, p_k) = \bot$ otherwise.

We introduce now our substitution. The structure of this substitution (as well as understanding of its value and actions) is crucial for the rest of the proof below; the expressed by this substitution properties are important via joining all of them together and cannot be commented separately. For any letter p_i from P_n, let

$$\sigma(p_i) := [(\Box\varphi) \wedge p_i] \vee [\neg\Diamond\Box\varphi \wedge \epsilon(p_i)] \vee [\neg\Box\varphi \wedge \Diamond\Box\varphi \wedge$$

$$(\bigvee_{SV_P \subseteq SV} [\bigwedge_{A \in SV_P} ([\bigwedge_{Y_n \in A} [\Diamond[\Box\varphi \wedge \Diamond_{Prev}\neg\Box\varphi \wedge \psi(Y_n)] \wedge$$

$$\Box[\Box\varphi \wedge \Diamond_{Prev}\neg\Box\varphi \wedge [\psi(Y_n) \rightarrow \bigwedge_{j \in J} \Box_j \psi(Y_n)]]\big) \wedge tv(SV_p, p_i)\big) \wedge$$

$$\bigwedge_{Y_r \subseteq P_n, Y_r \not\in SV_p} \neg\Diamond[\Box\varphi \wedge \Diamond_{Prev}\neg\Box\varphi \wedge \psi(Y_r) \wedge \bigwedge_j \Box_j \psi(Y_r)]]\big),$$

Now we will prove that this substitution is a projective unifier for φ in logic L. Choose any model $M = \langle F, V \rangle$ of our logic with a frame F and given valuation of the set of all letters P_n from φ, which is frozen w.r.t. P_n. Take any state $a \in F$.

Consider first the case when $(F, a) \Vdash_V \neg\Diamond\Box\varphi$. By our definition of the unifier $\sigma(p_i)$ and Lemma 3.3. we have $\sigma(p_i)$ is equivalent to $\epsilon(p_i)$, where $\epsilon(p_i) = y_i$, and then σ it is a unifier for formula φ in our logic. So,

$$\forall b \in F, \ (F, b) \Vdash_V \sigma(\varphi)).$$

Let now $(F, a) \Vdash_V \Box\varphi$. Then the substitution σ does not change truth values of letters p_i in a and all b, where $a \leq b$, comparing with their original truth values w.r.t. V. Therefore

$$(F, a) \Vdash_V p_i \Leftrightarrow (F, a) \Vdash_V \sigma(p_i), \ \forall b \geq a \ (F, b) \Vdash_V p_i \Leftrightarrow (F, b) \Vdash_V \sigma(p_i).$$

This implies

$$(F, a) \Vdash_V \sigma(\Box\varphi).$$

It remains only to consider the case when

$$(F, a) \Vdash_V \neg\Box\varphi \wedge \Diamond\Box\varphi.$$

Then let we look close to our model. The model M is based at F is a model with frozen states w.r.t. P_n, then by definition of frozen, for any $C(i)$ there is a unique set $P_n[C(i)]$ of some sets Pos_n of letters from $Var(\Box\varphi)$ with all required properties.

Let $C(i)$ be the lowest cluster where the formula $\Box\varphi$ is true w.r.t. the original valuation V. Then for $SV_p = P_n[C(i)]$ the disjunct member of the definition of $\sigma(p_i)$ above is true at all states b situated strictly below $C(i)$. And for SV_p distinct from $P_n[C(i)]$ this is not the case for any such b.

Then the truth value of all $\sigma(p_i)$ at all such b is exactly the same as the truth value of p_i w.r.t. the original frame valuation V in a state a_{Y_k} from $C(i)$ where $(a_{Y_k} \Vdash_V \psi(Y_k))$ where $Y_k \in ASV_p$ (and, recall, $ASV_p \in SV_p = P_n[C(i)]$.

Lemma 3.5. *Truth value of any formula $\sigma(\alpha)$, where α is a formula constructed out of letters $p_1, \ldots, p_n$, by Boolean operations and operations $\Diamond_j$, at the states b situated strictly below $C(i)$ w.r.t. V is the same as the truth of this formula at a_{Y_k} w.r.t. V.*

Proof. Here it is sufficient to use standard induction by the length of formula, for Boolean logical operation it is evident. For modal operations it is again trivial because we consider in this lemma truth of formulas strictly below $C(i)$.

Lemma 3.6. *Truth value of any formula $\sigma(\alpha)$, where α is a formula constructed out of letters $p_1, \ldots, p_n$, by Boolean operations and operations $\Diamond_j$, $\Diamond$ and $\Diamond_{Prev}$ at the states b situated strictly below $C(i)$ w.r.t. V is the same as truth value of $\sigma(\alpha)$ at a_{Y_k} w.r.t. V.*

Proof. Again, as in the lemma above we use standard induction by the length of formula, and for more used modal operation it is again directly follows from the case that by condition of this current lemma states b situated strictly below $C(i)$ w.r.t. V.

So, σ is a unifier for φ and it is evident that σ is projective for φ (that is $\Box\varphi \to [x_i \equiv \sigma(x_i)] \in L$). Theorem is proved.

References

[1] *Ronald Fagin, Joseph Y. Halpern, Yoram Moses and Moshe Vardi* (Reasoning About Knowledge, MIT, 1995).

[2] *V. V. Rybakov. Refined common knowledge logics or logics of common information.- Archive for mathematical Logic, 42 (2), 2003, 179 – 200.*

[3] *S. Artemov* Explicit Generic-Common Knowledge, Lect. Notes in CS, LFCS 2013: Logical Foundations of Computer Science 2013, pp 1628.

[4] *S. Artemov* Justification awareness. – Journal of Logic and Computation 30 (8), 2020, 1431 – 1446.

[5] *V Rybakov Temporal Multi-Agents Logic, Knowledge, Uncertainty, and Plausibility.* – Agents and Multi-Agent Systems: Technologies and Applications, LNCS, 2021, 2005 - 2014.

[6] Baader F., Kusters R. *Unification in a description logic with transitive closure of roles.* In: Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2001. Vol. 2250. LNCS, Springer (2001), 217 – 232.

[7] Rybakov V.V., *Multiagent temporal logics with multivaluations*, Siberian. Math. J., 59, no 4, (2018), 710 - 720.

[8] Rybakov V. V. *Projective formulas and unification in linear temporal logic $LTLU$*, Logic Journal of the IGPL, Oxford Press. – 22 (4), (2014), 665 – 672.

[9] Ghilardi S. *Unification Through Projectivity,* – J. of Logic and Computation, Oxford Press, 7(6) (1997), 733 – 752.

[10] Ghilardi S. *Unification, finite duality and projectivity in varieties of Heyting algebras,* – Annals of Pure and Applied Logic, 127(1-3), (2004), 99 – 115.

[11] Dzik W., Wojtylak P. *Projective unification in modal logic.* – Logic Journal of IGPL, – 20, (1), (2012), 121 – 153.

[12] Rybakov V. V., *Multi-agent logics with interaction, unifiability and projectivity*, – Siberian Electronic Mathematical Reports, – 21, (2), (2024), 1370-1384.