Characterizations of Scalar Curvature in Two-Dimensional Finsler Spaces with Cubic Metrics

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Abstract

The (h)-scalar curvature is a fundamental invariant in Finsler geometry which is used to analyze the geometric structure of Finsler spaces. In this paper, we derive an explicit expression for the (h)-scalar curvature R of a two-dimensional Finsler space equipped with a cubic metric $L^3 = a_{ijk}(x)y^iy^jy^k$. To investigate its variation, we consider specific forms of the cubic metric and compute the (h)-scalar curvature R for all possible non-vanishing components in two-dimensional space. The dependency of R on both positional coordinates x and directional variables y is further examined through Python-based graphical visualizations, revealing distinct geometric patterns for various forms of cubic metric. The analysis demonstrates that scalar curvature in Finsler geometry is inherently direction-dependent and that singular behaviors emerge along specific directions determined by the structure of the cubic metric.

Keywords: Cubic metric, scalar curvature R, Berwlad connection, Berwald frame.

Mathematics Subject Classification: 53B40 , 53C60

1 Introduction

The concept of the m-th root Finsler metric was introduced by the renowned mathematician Shimada [1] as a natural extension of Riemannian metrics, and it is defined as follows:

$$L\left(x,y\right)=\left[a_{i_1\ldots i_m}\left(x\right)y^{i_1}\ldots y^{i_m}\right]^{\frac{1}{m}},$$

where $a_{i_1\ldots i_m}(x)$ denotes the components of a symmetric $(0,m)$ -tensor field depending only on the position coordinates x and y^i represents the directional components the metric. For $m=2$, this metric reduces to the classical Riemannian metric, while $m=3$ and $m=4$, it yields the cubic and quartic Finsler metrics, respectively. This metric plays a significant role in the study of Finsler spaces due to the nonvanishing nature of its Cartan torsion tensor. Furthermore, various authors [2–12] have investigated the geometric properties of cubic, quartic and m-th root metrics, identifying conditions under which Finsler spaces with these metrics become Berwald or Landsberg spaces, and analyzing their projective and metrical properties, scalar curvature and constant curvature. These studies highlight the significant contributions of these metrics to the geometric structure of Finsler spaces. Additionally, such metrics find applications in diverse fields, including biology (ecological metrics), physics, space-time theory, gravitation, general relativity, and seismic ray theory [13–15].

The (h)-scalar curvature is a fundamental invariant in Finsler geometry, used to investigate the geometric structure of Finsler spaces associated with the Berwald connection. Several authors [16–23] have studied Finsler spaces in the context of the Berwald connection, defining related concepts such as the Berwald scalar curvature, constant curvature, isotropic Berwald curvature and constant flag curvature. To highlight its significance and examine its variation in the two-dimensional case, Berwald?s theory of two-dimensional Finsler spaces [24] can be applied, as the scalar curvature in such spaces provides a useful tool for classifying geometry and analyzing the divergence of geodesics. This theory is based on an intrinsic orthonormal frame consisting of the normalized supporting element l^i, and the normalized Cartan torsion vector $\frac{C_i}{C}=m_i$, where $C_i=C_{ijk}g^{ij}$ and g^{ij} is the inverse of the fundamental metric tensor. Importantly, m_i is orthogonal to contravariant component l^i of the normalized supporting element with respect to the Finsler metric. In a two-dimensional Finsler space, the $v(h)$ torsion tensor R^i_{jk} can be expressed in terms of the Berwald frame (l,m) as

$$R^i_{jk}=LRm^i(l_jm_k-l_km_j), \tag{1}$$

where R denotes the h-scalar curvature, $m^i=g^{ij}m_j$ and the indices i,j range over 1 and 2.

The aim of the present paper is to derive an explicit expression for the (h)-scalar curvature R of a two-dimensional Finsler space equipped with a cubic metric $L^3=a_{ijk}(x)y^iy^jy^k$. The main results are presented in theorem 1, with special cases discussed in Theorems 2 - 4 and 6 - 8. To analyze the variation of R we consider specific forms of the cubic metric and compute the (h)-scalar curvature for all non-vanishing components in two dimensions. The dependence of R on both positional coordinates x and directional variables y is further explored through Python-based graphical visualizations, highlighting distinct geometric patterns for various possible forms of cubic metrics.

2 Basic Tensors of Cubic Finsler Space

The cubic metric is a special case of the m-th root Finsler metric with $m = 3$, characterized by a non-vanishing Cartan torsion. This makes it particularly significant in the study of Berwald and Landsberg spaces. The metric is defined by

$$L^3 = a_{ijk}(x)y^i y^j y^k \tag{2}$$

where $a_{ijk}(x)$ are the components of a symmetric $(0, 3)$ -tensor field depending only on the positional coordinates x. The basic tensors of a Finsler space equipped with a cubic metric [4, 24] are given by

$$\left.\begin{aligned}
&\text{a) } l_i = a_i, \\
&\text{b) } h_{ij} = 2(a_{ij} - a_i a_j), \\
&\text{c) } g_{ij} = 2a_{ij} - a_i a_j, \\
&\text{d) } LC_{ijk} = (a_{ijk} - a_{jk}a_i - a_{ki}a_j - a_{ij}a_k + 2a_i a_j a_k).
\end{aligned}\right\} \tag{3}$$

where l_i, h_{ij}, g_{ij}, C_{ijk} denote the normalized supporting element, the angular metric tensor, the fundamental tensor, and the Cartan torsion tensor, respectively, with respect to the Cartan connection of the Finsler space endowed with a cubic metric. Further the functions a_i and a_{ij} which depend on both position and direction, are given by $a_{ij}(x, y) = \frac{a_{ijk}(x)y^k}{L}$ and $a_i(x, y) = \frac{a_{ijk}(x)y^j y^k}{L^2}$ respectively.

The tensor $a_{ij}(x, y)$ referred to as the basic tensor, plays a key role in determining whether the metric L is regular. The metric is regarded as regular whenever the determinant of this tensor is nonzero. Throughout our study of cubic root metrics, we shall assume that this regularity condition is satisfied.

Let $a^{ij}(x, y)$ denote the reciprocal tensor of $a_{ij}(x, y)$. Then the reciprocal of $g_{ij}(x, y)$ denoted by $g^{ij}(x, y)$ [24], is given by

$$2g^{ij} = a^{ij} + a^i a^j \quad \text{and} \quad l^i = a^i,$$

where

$$a_i a^i = 1, \quad a^i = a^{ij} a_j, \quad l^i = y^i / L.$$

3 The Berwald connection coefficients in a two-dimensional Finsler space

In Finsler geometry, a connection provides a systematic framework for describing the variation of vectors along curves on a manifold. The Berwald connection is a distinguished Finsler connection characterized by minimal torsion, with only the $v(h)$-torsion R^i_{jk} remaining non-zero, which facilitates the analysis of geodesics, curvature and parallel transport. It plays a central role in special Finsler spaces, such as Berwald spaces, where the connection coefficients depend solely on position, and Landsberg spaces, where the horizontal covariant derivative of the Cartan torsion vanishes. By preserving the essential geometric structure while reducing torsion, the Berwald connection provides a streamlined framework for studying the curvature and intrinsic properties of Finsler spaces. It is obtained from Rund?s connection through the P^1 -process, which eliminates the torsion tensor P^i_{jk} resulting in its torsion-minimized form. For an n-dimensional Finsler space F^n the Berwald connection is expressed as $B\Gamma = (G^i_{jk}, G^i_j, 0)$ with components $G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i$ and $G^i_j = \dot{\partial}_j G^i$, where, $\dot{\partial}_j$ denotes differentiation with respect to the directional coordinates y^j. In this context, the only non-vanishing torsion tensor is the $v(h)$-torsion tensor R^i_{jk} is defined by

$$R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k, \tag{4}$$

where $\delta_k = \partial_k - G^r_k \dot{\partial}_r$.

The geometry of a two-dimensional Finsler space can be conveniently described using the Berwald frame (l, m) where the vectors l and m are mutually orthogonal. From equation (1), the only non-zero components of the R^i_{jk} are $R^1_{12} = -R^1_{21}$ and $R^2_{12} = -R^2_{21}$. In the present study, we examine both non-vanishing components of R^i_{jk} in a two-dimensional Finsler space.

In a two-dimensional Finsler space, the cubic metric (2) can be expressed as

$$L^3 = c_0 \dot{x}^3 + 3c_1 \dot{x}^2 \dot{y} + 3c_2 \dot{x} \dot{y}^2 + c_3 \dot{y}^3. \tag{5}$$

where $(c_0, c_1, c_2, c_3) = (a_{111}, a_{112}, a_{122}, a_{222})$ are functions of the positional coordinates $(x^1, x^2) = (x, y)$ and $(y^1, y^2) = (\dot{x}, \dot{y})$ denote the directional components in the two-dimensional Finsler space.

The third-order Christoffel symbols for a cubic Finsler space [24] are defined by

$$\{ijk, h\} = \frac{1}{4}(\partial_i a_{jkh} + \partial_j a_{ikh} + \partial_k a_{ijh} - \partial_h a_{ijk}). \tag{6}$$

In the two-dimensional case, the indices i, j, k, h range over 1 and 2.Let us define $\frac{\partial c_a}{\partial x} = c_{a1}$, $\frac{\partial c_a}{\partial y} = c_{a2}$, where $a = 0, 1, 2, 3$. Using equation (6), the third-order Christoffel symbols for the two-dimensional case can be computed which are given below:

$$\{111, 1\} = \frac{1}{4}(2\partial_1 a_{111}) = \frac{1}{2}(\frac{\partial c_0}{\partial x}) = \frac{c_{01}}{2}.$$

Similarly,

$$\{111,2\} = \frac{3c_{11}-c_{02}}{4}, \qquad \{112,1\} = \frac{c_{02}+c_{11}}{4}, \qquad \{112,2\} = \frac{c_{21}}{2},$$
$$\{122,1\} = \frac{c_{12}}{2}, \qquad \{122,2\} = \frac{c_{22}+c_{31}}{4}, \qquad \{222,1\} = \frac{3c_{22}-c_{31}}{4},$$
$$\{222,2\} = \frac{c_{32}}{2}.$$

For a cubic metric to be regular, the determinant of the basic tensor $a_{ij}(x,y)$ must be nonzero. Using $a_{ij}(x,y) = \frac{a_{ijk}(x)y^k}{L}$ the determinant of $a_{ij}(x,y)$ is expressed as

$$|a_{ij}| = \frac{A\dot{x}^2 + B\dot{x}\dot{y} + C\dot{y}^2}{L^2}$$

where $\quad A = c_0c_2 - c_1^2, \quad B = c_0c_3 - c_1c_2, \quad C = c_1c_3 - c_2^2.$

The above equation can be rewritten as

$$H = A\dot{x}^2 + B\dot{x}\dot{y} + C\dot{y}^2, \tag{7}$$

where $H = L^2\,|a_{ij}|$.

The components G^i of the Berwald connection [24] can be expressed in terms of the third-order Christoffel symbols as

$$a_{ir}G^r = \tfrac{1}{3L}\{jkh, i\}y^jy^ky^h$$

In a two-dimensional Finsler space, the Berwald connection has two components, G^1 and G^2. Using the relation for $a_{ir}G^r$ with indices r, i, j, k, h ranging over 1 and 2, and substituting the value of third-order Christoffel symbols for a cubic metric, we obtain

$$a_{11}G^1 + a_{12}G^2 = \frac{1}{6L}\{c_{01}\dot{x}^3 + \frac{3}{2}(c_{11}+c_{02})\dot{x}^2\dot{y} + 3c_{12}\dot{x}\dot{y}^2 + \frac{1}{2}(3c_{22}-c_{31})\dot{y}^3\}. \tag{8}$$

$$a_{21}G^1 + a_{22}G^2 = \frac{1}{6L}\{\frac{1}{2}(3c_{11}-c_{02})\dot{x}^3 + 3c_{21}\dot{x}^2\dot{y} + \frac{3}{2}(c_{22}+c_{31})\dot{x}\dot{y}^2 + c_{32}\dot{y}^3\}. \tag{9}$$

These equations provide the explicit linear system for G^1 and G^2 in terms of the metric coefficients and directional components. Solving the above system of equations, we obtain

$$(3H)(2G^1) = a\dot{x}^4 + b\dot{x}^3\dot{y} + c\dot{x}^2\dot{y}^2 + d\dot{x}\dot{y}^3 + e\dot{y}^4, \tag{10}$$

where

$$a = \frac{1}{2}c_1(c_{02}-3c_{11}) + c_2c_{01}, \quad b = 2c_2c_{02} + c_3c_{01} - 3c_1c_{21},$$
$$c = 3\left\{c_2(c_{12}-c_{21}) + \frac{1}{2}c_3(c_{02}+c_{11}) - \frac{1}{2}c_1(c_{22}+c_{31})\right\},$$

$$d = 3c_3c_{12} - c_1c_{32} - 2c_2c_{31}, \quad e = \frac{1}{2}c_3(3c_{22} - c_{31}) - c_2c_{32},$$

and

$$(3H)(2G^2) = f\dot{x}^4 + g\dot{x}^3\dot{y} + h\dot{x}^2\dot{y}^2 + k\dot{x}\dot{y}^3 + l\dot{y}^4, \tag{11}$$

where

$$f = \frac{1}{2}c_0(3c_{11} - c_{02}) - c_1c_{01}, \quad g = 3c_0c_{21} - 2c_1c_{02} - c_2c_{01},$$
$$h = 3\left\{\frac{1}{2}c_0(c_{22} + c_{31}) + c_1(c_{21} - c_{21}) - \frac{1}{2}c_2(c_{02} + c_{11})\right\},$$
$$k = c_0c_{32} + 2c_1c_{31} - 3c_2c_{12}, \quad l = c_1c_{32} + \frac{1}{2}c_2(c_{31} - 3c_{22}).$$

4 Scalar Curvature of Two-Dimensional Cubic Finsler Space

Since there are only two non-vanishing components of the $v(h)$-torsion tensor, namely R^1_{12} and R^2_{12}, we can express them in terms of the Berwald connection using equation (4) which are given below:

$$R^1_{12} = \frac{\partial G^1_1}{\partial x^2} - G^r_2\frac{\partial G^1_1}{\partial y^r} - \frac{\partial G^1_2}{\partial x^1} + G^r_1\frac{\partial G^1_2}{\partial y^r}.$$

$$R^2_{12} = \frac{\partial G^2_1}{\partial x^2} - G^r_2\frac{\partial G^2_1}{\partial y^r} - \frac{\partial G^2_2}{\partial x^1} + G^r_1\frac{\partial G^2_2}{\partial y^r}.$$

where $G^1_1 = \frac{\partial G^1}{\partial \dot{x}}, G^1_2 = \frac{\partial G^1}{\partial \dot{y}}, G^2_1 = \frac{\partial G^2}{\partial \dot{x}}$ and $G^2_2 = \frac{\partial G^2}{\partial \dot{y}}$. Therefore,

$$R^1_{12} = \frac{\partial G^1_1}{\partial y} - (G^1_2\frac{\partial G^1_1}{\partial \dot{x}} + G^2_2\frac{\partial G^1_1}{\partial \dot{y}}) - \frac{\partial G^1_2}{\partial x} + (G^1_1\frac{\partial G^1_2}{\partial \dot{x}} + G^2_1\frac{\partial G^1_2}{\partial \dot{y}}). \tag{12}$$

$$R^2_{12} = \frac{\partial G^2_1}{\partial y} - (G^1_2\frac{\partial G^2_1}{\partial \dot{x}} + G^2_2\frac{\partial G^2_1}{\partial \dot{y}}) - \frac{\partial G^2_2}{\partial x} + (G^1_1\frac{\partial G^2_2}{\partial \dot{x}} + G^2_1\frac{\partial G^2_2}{\partial \dot{y}}). \tag{13}$$

From (1), we have

$$\left.\begin{aligned} R^1_{12} = LRm^1(l_1m_2 - l_2m_1) \\ R^2_{12} = LRm^2(l_1m_2 - l_2m_1) \end{aligned}\right\} \tag{14}$$

Initially, we consider equation (14) for the cubic metric (5). The components a_i and a_{ij} defined by $a_{ij}\,(x, y) = \frac{a_{ijk}(x)y^k}{L}$ and $a_i\,(x, y) = \frac{a_{ijk}(x)y^jy^k}{L^2}$ can be directly written for the cubic metric (5) as

$$La_{11} = c_0\dot{x} + c_1\dot{y}, \quad La_{12} = c_1\dot{x} + c_2\dot{y}, \quad La_{22} = c_2\dot{x} + c_3\dot{y},$$
$$L^2a_1 = c_0\dot{x}^2 + 2c_1\dot{x}\dot{y} + c_2\dot{y}^2, \quad L^2a_2 = c_1\dot{x}^2 + 2c_2\dot{x}\dot{y} + c_3\dot{y}^2.$$

With the help of the above equations, the expression (3(c)) may be expressed as

$$\left.\begin{aligned} L^4 g_{11} &= c_0^2\dot{x}^4 + 4c_0c_1\dot{x}^3\dot{y} + 2(2c_0c_2 + c_1^2)\dot{x}^2\dot{y}^2 + 2(c_0c_3 + c_1c_2)\dot{x}\dot{y}^3 \\ &\quad + (2c_1c_3 - c_2^2)\dot{y}^4 \\ L^4 g_{12} &= c_0c_1\dot{x}^4 + 4c_1^2\dot{x}^3\dot{y} + (7c_1c_2 - c_0c_3)\dot{x}^2\dot{y}^2 + 4c_2^2\dot{x}\dot{y}^3 \\ &\quad + c_2c_3\dot{y}^4 = L^4 g_{21} \\ L^4 g_{22} &= (2c_0c_2 - c_1^2)\dot{x}^4 + 2(c_1c_2 + c_0c_3)\dot{x}^3\dot{y} + 2(c_2^2 + 2c_1c_3)\dot{x}^2\dot{y}^2 \\ &\quad + 4c_2c_3\dot{x}\dot{y}^3 + c_3^2\dot{y}^4 \end{aligned}\right\} \tag{15}$$

Also, since $l_i = g_{ij}l^j$, we have

$$\left.\begin{aligned} Ll_1 &= g_{11}\dot{x} + g_{12}\dot{y} \\ Ll_2 &= g_{21}\dot{x} + g_{22}\dot{y} \end{aligned}\right\} \tag{16}$$

Then, using equation (15), equation (16) can be expressed in the form:

$$\left.\begin{aligned} L^5 l_1 &= c_0^2\dot{x}^5 + 5c_0c_1\dot{x}^4\dot{y} + (4c_0c_2 + 6c_1^2)\dot{x}^3\dot{y}^2 + (c_0c_3 + 9c_1c_2)\dot{x}^2\dot{y}^3 \\ &\quad + (2c_1c_3 + 3c_2^2)\dot{x}\dot{y}^4 + c_2c_3\dot{y}^5 \\ L^5 l_2 &= c_0c_1\dot{x}^5 + (2c_0c_2 + 3c_1^2)\dot{x}^4\dot{y} + (c_0c_3 + 9c_1c_2)\dot{x}^3\dot{y}^2 \\ &\quad + (6c_2^2 + 4c_1c_3)\dot{x}^2\dot{y}^3 + 5c_2c_3\dot{x}\dot{y}^4 + c_3^2\dot{y}^5 \end{aligned}\right\} \tag{17}$$

The components m^i and m_i $(i = 1, 2)$ the Berwald frame are related to l^i and l_i through the following relations [24]:

$$\left.\begin{aligned} (m^1, m^2) &= \frac{1}{\sqrt{g}}(-l_2, l_1) \\ (m_1, m_2) &= \sqrt{g}(-l^2, l^1) \end{aligned}\right\} \tag{18}$$

where $g = |g_{ij}|$.
Using equation (18), equation (14) reduces to:

$$R^1_{12} = LRm^1(l_1m_2 - l_2m_1) = -R_1 l_2(l_1\dot{x} + l_2\dot{y}) \tag{19}$$

$$R^2_{12} = LRm^2(l_1m_2 - l_2m_1) = R_2 l_1(l_1\dot{x} + l_2\dot{y}) \tag{20}$$

where R_1 and R_2 denote the (h)-scalar curvature corresponding to the non-vanishing components R^1_{12} and R^2_{12}, respectively, and l_1 and l_2 are given by equation (17). Now, using equation (7), we obtain:

$$\left.\begin{aligned}
\dot{H}_1 &= \frac{\partial H}{\partial \dot{x}} = 2A\dot{x} + B\dot{y}, \qquad \dot{H}_2 = \frac{\partial H}{\partial \dot{y}} = B\dot{x} + 2C\dot{y},\\
\dot{H}_{11} &= \frac{\partial^2 H}{\partial \dot{x}^2} = 2A, \qquad \dot{H}_{22} = \frac{\partial^2 H}{\partial \dot{y}^2} = 2C,\\
\dot{H}_{12} &= \frac{\partial^2 H}{\partial \dot{x}\partial \dot{y}} = B = \dot{H}_{21},\\
H_1 &= \frac{\partial H}{\partial x} = A_1\dot{x}^2 + B_1\dot{x}\dot{y} + C_1\dot{y}^2,\\
H_2 &= \frac{\partial H}{\partial y} = A_2\dot{x}^2 + B_2\dot{x}\dot{y} + C_2\dot{y}^2,\\
H_{12} &= \frac{\partial \dot{H}_1}{\partial y} = 2A_2\dot{x} + B_2\dot{y}, \qquad H_{21} = \frac{\partial \dot{H}_2}{\partial x} = B_1\dot{x} + 2C_1\dot{y},
\end{aligned}\right\} \tag{21}$$

where

$$A_1 = \frac{\partial A}{\partial x}, \quad B_1 = \frac{\partial B}{\partial x}, \quad C_1 = \frac{\partial C}{\partial x}, \quad A_2 = \frac{\partial A}{\partial y}, \quad B_2 = \frac{\partial B}{\partial y}, \quad \text{and} \quad C_2 = \frac{\partial C}{\partial y}.$$

Since $G^i_j = \dot{\partial}_j G^i$ and $G^i_{jk} = \dot{\partial}_j\dot{\partial}_k G^i$, therefore, from equation (10), we have

$$\left.\begin{aligned}
6HG^1_1 &= 4a\dot{x}^3 + 3b\dot{x}^2\dot{y} + 2c\dot{x}\dot{y}^2 + d\dot{y}^3 - 6\dot{H}_1G^1\\
6HG^1_{11} &= 12a\dot{x}^2 + 6b\dot{x}\dot{y} + 2c\dot{y}^2 - 6\dot{H}_{11}G^1 - 12\dot{H}_1G^1_1\\
6HG^1_{12} &= 3b\dot{x}^2 + 4c\dot{x}\dot{y} + 3d\dot{y}^2 - 6\dot{H}_{12}G^1 - 6\dot{H}_1G^1_2 - 6\dot{H}_2G^1_2\\
6HG^1_2 &= b\dot{x}^3 + 2c\dot{x}^2\dot{y} + 3d\dot{x}\dot{y}^2 + 4e\dot{y}^3 - 6\dot{H}_2G^1\\
6HG^1_{21} &= 3b\dot{x}^2 + 4c\dot{x}\dot{y} + 3d\dot{y}^2 - 6\dot{H}_{21}G^1 - 6\dot{H}_2G^1_1 - 6\dot{H}_1G^1_2\\
6HG^1_{22} &= 2c\dot{x}^2 + 6d\dot{x}\dot{y} + 12e\dot{y}^2 - 6\dot{H}_{22}G^1 - 12\dot{H}_2G^1_2
\end{aligned}\right\} \tag{22}$$

Also, from equation (11), we have

$$\left.\begin{aligned}
6HG^2_1 &= 4f\dot{x}^3 + 3g\dot{x}^2\dot{y} + 2h\dot{x}\dot{y}^2 + k\dot{y}^3 - 6\dot{H}_1G^2\\
6HG^2_{11} &= 12f\dot{x}^2 + 6g\dot{x}\dot{y} + 2h\dot{y}^2 - 6\dot{H}_11G^2 - 12\dot{H}_1G^2_1\\
6H^2_{12} &= 3g\dot{x}^2 + 4h\dot{x}\dot{y} + 3k\dot{y}^2 - 6\dot{H}_2G^2_1 - 6\dot{H}_{12}G^2 - 6\dot{H}_1G^2_2\\
6HG^2_2 &= g\dot{x}^3 + 2h\dot{x}^2\dot{y} + 3k\dot{x}\dot{y}^2 + 4l\dot{y}^3 - 6\dot{H}_2G^2\\
6HG^2_{22} &= 2h\dot{x}^2 + 6k\dot{x}\dot{y} + 12l\dot{y}^2 - 12\dot{H}_2G^2_2 - 6\dot{H}_{22}G^2\\
6HG^2_{21} &= 3g\dot{x}^2 + 4h\dot{x}\dot{y} + 3k\dot{y}^2 - 6\dot{H}_2G^2_1 - 6\dot{H}_1G^2_2 - 6\dot{H}_{21}G^2
\end{aligned}\right\} \tag{23}$$

Differentiating partially G^1_1, G^1_2 and G^2_1, G^2_2 from (22) and (23) w.r.t. $x, y, \dot{x}, \dot{y}$, we get

$$\left.\begin{aligned}
6H\frac{\partial G^1_1}{\partial y} &= 4a_2\dot{x}^3 + 3b_2\dot{x}^2\dot{y} + 2c_2\dot{x}\dot{y}^2 + d_2\dot{y}^3 - 6H_2G^1_1 - 6\dot{H}_1D_2 \\
&\quad - 6H_{12}G^1 \\
6H\frac{\partial G^1_2}{\partial x} &= b_1\dot{x}^3 + 2c_1\dot{x}^2\dot{y} + 3d_1\dot{x}\dot{y}^2 + 4e_1\dot{y}^3 - 6H_1G^1_2 - 6\dot{H}_2D_1 \\
&\quad - 6H_{21}G^1 \\
6H\frac{\partial G^1_1}{\partial \dot{x}} &= 12a\dot{x}^2 + 6b\dot{x}\dot{y} + 2c\dot{y}^2 - 12\dot{H}_1G^1_1 - 6\dot{H}_{11}G^1 \\
6H\frac{\partial G^1_1}{\partial \dot{y}} &= 3b\dot{x}^2 + 4c\dot{x}\dot{y} + 3d\dot{y}^2 - 6\dot{H}_2G^1_1 - 6\dot{H}_1G^1_2 - 6\dot{H}_{12}G^1 \\
6H\frac{\partial G^1_2}{\partial \dot{x}} &= 3b\dot{x}^2 + 4c\dot{x}\dot{y} + 3d\dot{y}^2 - 6\dot{H}_1G^1_2 - 6\dot{H}_2G^1_1 - 6\dot{H}_{21}G^1 \\
6H\frac{\partial G^1_2}{\partial \dot{y}} &= 2c\dot{x}^2 + 6d\dot{x}\dot{y} + 12e\dot{y}^2 - 12\dot{H}_2G^1_2 - 6\dot{H}_{22}G^1
\end{aligned}\right\} \tag{24}$$

where

$$\begin{aligned}
&b_1 = \partial_1 b, \quad c_1 = \partial_1 c, \quad d_1 = \partial_1 d, \quad e_1 = \partial_1 e, \quad D_1 = \partial_1 G^1, \\
&a_2 = \partial_2 a, \quad b_2 = \partial_2 b, \quad c_2 = \partial_2 c, \quad d_2 = \partial_2 d, \quad D_2 = \partial_2 G^1.
\end{aligned}$$

and

$$\left.\begin{aligned}
6H\frac{\partial G^2_1}{\partial y} &= 4f_2\dot{x}^3 + 3g_2\dot{x}^2\dot{y} + 2h_2\dot{x}\dot{y}^2 + k_2\dot{y}^3 - 6H_2G^2_1 - 6\dot{H}_1E_2 \\
&\quad - 6H_{12}G^2 \\
6H\frac{\partial G^2_2}{\partial x} &= g_1\dot{x}^3 + 2h_1\dot{x}^2\dot{y} + 3k_1\dot{x}\dot{y}^2 + 4l_1\dot{y}^3 - 6H_1G^2_2 - 6\dot{H}_2E_1 \\
&\quad - 6H_{21}G^2 \\
6H\frac{\partial G^2_1}{\partial \dot{x}} &= 12f\dot{x}^2 + 6g\dot{x}\dot{y} + 2h\dot{y}^2 - 12\dot{H}_1G^2_1 - 6\dot{H}_{11}G^2 \\
6H\frac{\partial G^2_1}{\partial \dot{y}} &= 3g\dot{x}^2 + 4h\dot{x}\dot{y} + 3k\dot{y}^2 - 6\dot{H}_2G^2_1 - 6\dot{H}_1G^2_2 - 6\dot{H}_{12}G^2 \\
6H\frac{\partial G^2_2}{\partial \dot{x}} &= 3g\dot{x}^2 + 4h\dot{x}\dot{y} + 3k\dot{y}^2 - 6\dot{H}_1G^2_2 - 6\dot{H}_2G^1_2 - 6\dot{H}_{21}G^2 \\
6H\frac{\partial G^2_2}{\partial \dot{y}} &= 2h\dot{x}^2 + 6k\dot{x}\dot{y} + 12l\dot{y}^2 - 12\dot{H}_2G^2_2 - 6\dot{H}_{22}G^2
\end{aligned}\right\} \tag{25}$$

where

$$\begin{aligned}
&g_1 = \partial_1 g, \quad h_1 = \partial_1 h, \quad k_1 = \partial_1 k, \quad l_1 = \partial_1 l, \quad E_1 = \partial_1 G^2, \\
&f_2 = \partial_2 f, \quad g_2 = \partial_2 g, \quad h_2 = \partial_2 h, \quad k_2 = \partial_2 k, \quad E_2 = \partial_2 G^2.
\end{aligned}$$

Using equations (7), (21)?(24), and (25), equations (12) and (13) reduce to:

$$R^1_{12} = \frac{1}{36H^2}\Big\{\mathcal{A}_1\dot{x}^5 + \mathcal{B}_1\dot{x}^4\dot{y} + \mathcal{C}_1\dot{x}^3\dot{y}^2 + \mathcal{D}_1\dot{x}^2\dot{y}^3 + \mathcal{E}_1\dot{x}\dot{y}^4 + \mathcal{F}_1\dot{y}^5 - \mathcal{G}_1\dot{x}^3 - \mathcal{H}_1\dot{x}^2\dot{y} - \mathcal{I}_1\dot{x}\dot{y}^2 - \mathcal{J}_1\dot{y}^3 - \mathcal{K}_1\dot{x}^2 - \mathcal{L}_1\dot{x}\dot{y} - \mathcal{M}_1\dot{y}^2 + \mathcal{N}_1 + \mathcal{O}_1 - \mathcal{P}_1 - \mathcal{Q}_1\Big\}, \tag{26}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= 24a_2A - 6b_1A + 8cf - 3bg,\\
\mathcal{B}_1 &= 24a_2B - 6b_1B + 18b_2A - 12c_1A + 3b^2 - 8ac + 2gc + 24df - 6bh,\\
\mathcal{C}_1 &= 24a_2C - 6b_1C + 18b_2B - 12c_1B + 12c_2A - 18d_1A + 4bc - 24ad\\
&\quad - 4ch + 15dg + 48ef - 9bk,\\
\mathcal{D}_1 &= 18b_2C - 12c_1C + 12c_2B - 18d_1B + 6d_2A - 24e_1A - 6bd + 4c^2\\
&\quad - 10ck + 6dh + 36eg - 48ae - 12bl,\\
\mathcal{E}_1 &= 12c_2C - 18d_1C + 6d_2B - 24e_1B + 4cd - 3dk + 24eh - 24be - 16cl,
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_1 &= 6d_2C - 24e_1C + 3d + 12ek - 8ce - 12dl,\\
\mathcal{G}_1 &= 24a(\dot{H}_{21}G^1 + \dot{H}_2G^1_1 + \dot{H}_1G^1_2) + 24f(\dot{H}_{22}G^1 + 2\dot{H}_2G^1_2)\\
&\quad - 6b(\dot{H}_{11}G^1 + 2\dot{H}_1G^1_1) - 6g(\dot{H}_{12}G^1 + \dot{H}_1G^1_2 + \dot{H}_2G^1_1),\\
\mathcal{H}_1 &= 18b(\dot{H}_{21}G^1 + \dot{H}_2G^1_1 + \dot{H}_1G^1_2) + 18g(\dot{H}_{22}G^1 + 2\dot{H}_2G^1_2)\\
&\quad - 12c(\dot{H}_{11}G^1 + 2\dot{H}_1G^1_1) - 12h(\dot{H}_{12}G^1 + \dot{H}_1G^1_2 + \dot{H}_2G^1_1),\\
\mathcal{I}_1 &= 12c(\dot{H}_{21}G^1 + \dot{H}_2G^1_1 + \dot{H}_1G^1_2) + 12h(\dot{H}_{22}G^1 + 2\dot{H}_2G^1_2)\\
&\quad - 18d(\dot{H}_{11}G^1 + 2\dot{H}_1G^1_1) - 18k(\dot{H}_{12}G^1 + \dot{H}_1G^1_2 + \dot{H}_2G^1_1),\\
\mathcal{J}_1 &= 6d(\dot{H}_{21}G^1 + \dot{H}_2G^1_1 + \dot{H}_1G^1_2) + 6k(\dot{H}_{22}G^1 + 2\dot{H}_2G^1_2)\\
&\quad - 24e(\dot{H}_{11}G^1 + 2\dot{H}_1G^1_1) - 24l(\dot{H}_{12}G^1 + \dot{H}_1G^1_2 + \dot{H}_2G^1_1),\\
\mathcal{K}_1 &= 18b\dot{H}_1G^1 + 12c\dot{H}_1G^2 - 72a\dot{H}_2G^1 - 18b\dot{H}_2G^2\\
&\quad + 36A(H_{12}G^1 - H_{21}G^1 + \dot{H}_1D_2 + H_2G^1_1 - \dot{H}_2D_1 - H_1G^1_2),\\
\mathcal{L}_1 &= 24c\dot{H}_1G^1 + 36d\dot{H}_1G^2 - 36b\dot{H}_2G^1 - 24c\dot{H}_2G^2\\
&\quad + 36B(H_{12}G^1 - H_{21}G^1 + \dot{H}_1D_2 + H_2G^1_1 - \dot{H}_2D_1 - H_1G^1_2),\\
\mathcal{M}_1 &= 18d\dot{H}_1G^1 + 72e\dot{H}_1G^2 - 12c\dot{H}_2G^1 - 18d\dot{H}_2G^2\\
&\quad + 36C(H_{12}G^1 - H_{21}G^1 + \dot{H}_1D_2 + H_2G^1_1 - \dot{H}_2D_1 - H_1G^1_2),\\
\mathcal{N}_1 &= 36\dot{H}_1G^1(\dot{H}_{21}G^1 + \dot{H}_1G^1_2),\\
\mathcal{O}_1 &= 36\dot{H}_1G^2(\dot{H}_{22}G^1 + \dot{H}_2G^1_2),\\
\mathcal{P}_1 &= 36\dot{H}_2G^1(\dot{H}_{11}G^1 + \dot{H}_1G^1_1),
\end{aligned}$$

$$\mathcal{Q}_1 = 36\dot{H}_2 G^2(\dot{H}_{21}G^1 + \dot{H}_2 G^1_1).$$

and

$$R^2_{12} = \frac{1}{36H^2}\Big\{\mathcal{A}_2\dot{x}^5 + \mathcal{B}_2\dot{x}^4\dot{y} + \mathcal{C}_2\dot{x}^3\dot{y}^2 + \mathcal{D}_2\dot{x}^2\dot{y}^3 + \mathcal{E}_2\dot{x}\dot{y}^4 + \mathcal{F}_2\dot{y}^5 - \mathcal{G}_2\dot{x}^3 - \mathcal{H}_2\dot{x}^2\dot{y} - \mathcal{I}_2\dot{x}\dot{y}^2 - \mathcal{J}_2\dot{y}^3 - \mathcal{K}_2\dot{x}^2 - \mathcal{L}_2\dot{x}\dot{y} - \mathcal{M}_2\dot{y}^2 + \mathcal{N}_2 + \mathcal{O}_2 - \mathcal{P}_2 - \mathcal{Q}_2\Big\}, \tag{27}$$

where

$$\begin{aligned}
\mathcal{A}_2 &= 24f_2A - 6g_1A - 12bf - 3g^2 + 12ga + 8fh,\\
\mathcal{B}_2 &= 24f_2B - 6g_1B + 18g_2A - 12h_1 + 3bg - 24cf - 4gh + 16ah + 24fk,\\
\mathcal{C}_2 &= 24f_2C - 6g_1C + 18g_2B - 12h_1B + 12h_2A - 18k_1A + 10bh - 12cg\\
&\quad + 36df + 6gk - 4h^2 + 12ak + 6gh + 48fl,\\
\mathcal{D}_2 &= 18g_2C - 12h_1C + 12h_2B - 18k_1B + 6k_2A - 24l_1A + 4ch - 15dg\\
&\quad - 48ef - 4kh + 24gl + 9bk,\\
\mathcal{E}_2 &= 12h_2C - 18k_1C + 6k_2B - 24l_1B - 2dh - 24eg + 8hl - 3k^2 + 6ck,\\
\mathcal{F}_2 &= 6k_2C - 24l_1C - 8eh + 3dk,\\
\mathcal{G}_2 &= 24a(\dot{H}_1G^2_2 + \dot{H}_{12}G^2 + \dot{H}_2G^2_1) + 24f(2\dot{H}_2G^2_2 + \dot{H}_{22}G^2)\\
&\quad - 6b(\dot{H}_{11}G^2 + 2\dot{H}_1G^2_1) - 6g(\dot{H}_{21}G^2 + \dot{H}_1G^2_2 + \dot{H}_2G^2_1),\\
\mathcal{H}_2 &= 18b(\dot{H}_1G^2_2 + \dot{H}_{12}G^2 + \dot{H}_2G^2_1) + 18g(2\dot{H}_2G^2_2 + \dot{H}_{22}G^2)\\
&\quad - 12c(\dot{H}_{11}G^2 + 2\dot{H}_1G^2_1) - 12h(\dot{H}_{21}G^2 + \dot{H}_1G^2_2 + \dot{H}_2G^2_1),\\
\mathcal{I}_2 &= 12c(\dot{H}_1G^2_2 + \dot{H}_{12}G^2 + \dot{H}_2G^2_1) + 12h(2\dot{H}_2G^2_2 + \dot{H}_{22}G^2)\\
&\quad - 18d(\dot{H}_{11}G^2 + 2\dot{H}_1G^2_1) - 18k(\dot{H}_{21}G^2 + \dot{H}_1G^2_2 + \dot{H}_2G^2_1),\\
\mathcal{J}_2 &= 6d(\dot{H}_1G^2_2 + \dot{H}_{12}G^2 + \dot{H}_2G^2_1) + 6k(2\dot{H}_2G^2_2 + \dot{H}_{22}G^2)\\
&\quad - 24e(\dot{H}_{11}G^2 + 2\dot{H}_1G^2_1) - 24l(\dot{H}_{21}G^2 + \dot{H}_1G^2_2 + \dot{H}_2G^2_1),\\
\mathcal{K}_2 &= 36A(H_2G^2_1 + H_{12}G^2 + \dot{H}_1E_2 - H_1G^2_2 - H_{21}G^2 - \dot{H}_2E_1) - 72f\dot{H}_2G^1\\
&\quad - 18g\dot{H}_2G^2 + 18g\dot{H}_1G^1 + 12h\dot{H}_1G^2,\\
\mathcal{L}_2 &= 36B(H_2G^2_1 + H_{12}G^2 + \dot{H}_1E_2 - H_1G^2_2 - H_{21}G^2 - \dot{H}_2E_1) - 36g\dot{H}_2G^1\\
&\quad - 24h\dot{H}_2G^2 + 24h\dot{H}_1G^1 + 36k\dot{H}_1G^2,\\
\mathcal{M}_2 &= 36C(H_2G^2_1 + H_{12}G^2 + \dot{H}_1E_2 - H_1G^2_2 - H_{21}G^2 - \dot{H}_2E_1) - 12h\dot{H}_2G^1\\
&\quad - 18k\dot{H}_2G^2 + 18k\dot{H}_1G^1 + 72l\dot{H}_1G^2,\\
\mathcal{N}_2 &= 36\dot{H}_1G^1(\dot{H}_{21}G^2 + \dot{H}_1G^2_2),\\
\mathcal{O}_2 &= 36\dot{H}_1G^2(\dot{H}_{22}G^2 + \dot{H}_2G^2_2),\\
\mathcal{P}_2 &= 36\dot{H}_2G^1(\dot{H}_{11}G^2 + \dot{H}_1G^2_1),
\end{aligned}$$

$$\mathcal{Q}_2 = 36\dot{H}_2 G^2(\dot{H}_{21} G^2 + \dot{H}_2 G_1^2).$$

Based on equation (19) and (26), we have

$$R_1 = -\frac{X}{36H^2 l_2(l_1\dot{x} + l_2\dot{y})}, \tag{28}$$

where

$$\begin{aligned} X = \Big\{ &\mathcal{A}_1\dot{x}^5 + \mathcal{B}_1\dot{x}^4\dot{y} + \mathcal{C}_1\dot{x}^3\dot{y}^2 + \mathcal{D}_1\dot{x}^2\dot{y}^3 + \mathcal{E}_1\dot{x}\dot{y}^4 + \mathcal{F}_1\dot{y}^5 - \mathcal{G}_1\dot{x}^3 \\ &- \mathcal{H}_1\dot{x}^2\dot{y} - \mathcal{I}_1\dot{x}\dot{y}^2 - \mathcal{J}_1\dot{y}^3 - \mathcal{K}_1\dot{x}^2 - \mathcal{L}_1\dot{x}\dot{y} - \mathcal{M}_1\dot{y}^2 + \mathcal{N}_1 \\ &+ \mathcal{O}_1 - \mathcal{P}_1 - \mathcal{Q}_1 \Big\}, \end{aligned}$$

and based on equation (20) and (27), we have

$$R_2 = \frac{Y}{36H^2 l_1(l_1\dot{x} + l_2\dot{y})}, \tag{29}$$

where

$$\begin{aligned} Y = \Big\{ &\mathcal{A}_2\dot{x}^5 + \mathcal{B}_2\dot{x}^4\dot{y} + \mathcal{C}_2\dot{x}^3\dot{y}^2 + \mathcal{D}_2\dot{x}^2\dot{y}^3 + \mathcal{E}_2\dot{x}\dot{y}^4 + \mathcal{F}_2\dot{y}^5 - \mathcal{G}_2\dot{x}^3 \\ &- \mathcal{H}_2\dot{x}^2\dot{y} - \mathcal{I}_2\dot{x}\dot{y}^2 - \mathcal{J}_2\dot{y}^3 - \mathcal{K}_2\dot{x}^2 - \mathcal{L}_2\dot{x}\dot{y} - \mathcal{M}_2\dot{y}^2 + \mathcal{N}_2 \\ &+ \mathcal{O}_2 - \mathcal{P}_2 - \mathcal{Q}_2 \Big\}. \end{aligned}$$

Theorem 1 *The (h)-scalar curvature of a two-dimensional Finsler space with a cubic metric, associated with the non-vanishing components R^1_{12} and R^2_{12} of the $v(h)$-torsion tensor is expressed by equations (28) and (29).*

5 Variation of (h)-scalar curvature for special cubic metric in two-dimensional Finsler space

The expressions for the (h)-scalar curvatures associated with the non-vanishing components R^1_{12} and R^2_{12}, given in equations (28) and (29), are highly complex, making their direct analysis analytically difficult. As a result, conducting a general study of the geometric properties of two-dimensional Finsler spaces equipped with this cubic metric becomes challenging. To enable a more manageable investigation of curvature behavior, this section focuses on special forms of the cubic Finsler metric.

In particular, by imposing the symmetry conditions $c_0 = c_3$ and $c_1 = c_2$, the cubic metric given in equation (5) reduces to the symmetric form

$$L^3 = c_0\left(\dot{x}^3 + \dot{y}^3\right) + 3c_1\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right). \tag{30}$$

which provides a more tractable framework for analyzing the corresponding (h)-scalar curvatures. To simplify the analysis further, we consider two special forms of the cubic metric given in equation (30). In the first case, we assume $c_1 = 0$, and in the second case, we impose $c_0 = 0$. These reductions yield two distinct types of cubic metrics, each offering a more manageable structure for studying the associated curvature properties.

5.1 Special cubic Finsler metric for the case $c_1 = 0$)

Let $c_1 = 0$. In this case, the cubic metric given in equation (30) reduces to $L^3 = c_0\left(\dot{x}^3 + \dot{y}^3\right)$, and the corresponding function becomes $H = c_0^2\dot{x}\dot{y}$. Substituting these expressions into equations (28) and (29), we get:

$$R_1 = \frac{L}{48c_0^3\dot{y}^2}\Bigg\{(2c_{01}c_{02} - 4c_0c_{012})\dot{x} + c_{02}^2\frac{\dot{x}^3}{\dot{y}^2} + (13c_{02}^2 - 12c_0c_{022})\dot{y} + (13c_{01}^2 - 12c_0c_{011})\frac{\dot{y}^2}{\dot{x}} + (2c_{01}c_{02} - 4c_0c_{012})\frac{\dot{y}^3}{\dot{x}^2} + c_{01}^2\frac{\dot{y}^5}{\dot{x}^4}\Bigg\} \tag{31}$$

and

$$R_2 = \frac{L}{48c_0^3\dot{x}^2}\Bigg\{\frac{-3}{2}c_{01}c_{02}\frac{\dot{x}^3}{\dot{y}^2} + (13c_{02}^2 - 9c_0c_{022})\frac{\dot{x}^2}{\dot{y}} + (13c_{01}^2 - 9c_0c_{011})\dot{x} + (-18c_{01}c_{02} - 4c_0c_{012})\dot{y} + c_{01}^2\frac{\dot{y}^3}{\dot{x}^2} + c_{02}^2\frac{\dot{x}^5}{\dot{y}^4}\Bigg\}, \tag{32}$$

where $\quad c_{ab1} = \frac{\partial c_{ab}}{\partial x}, \quad c_{ab2} = \frac{\partial c_{ab}}{\partial y}, \quad a = 0,1,2,3 \quad$ and $\quad b = 1,2.$

The indicatrix of a curve is a geometric device that captures how a given property varies with direction at a point. For the metric $L^3 = c_0\left(\dot{x}^3 + \dot{y}^3\right)$, the coefficient c_0 depends on the positional coordinates and, in the simplest situations, may vary with x, y or be a constant. Accordingly, the indicatrix is examined for these three forms of c_0 in the orthonormal coordinate system $(\dot{\bar{x}}, \dot{\bar{y}})$, obtained by rotating $(\dot{x}, \dot{y})$ by $-45°$. In all three cases of c_0, the resulting indicatrix curve is symmetric about the $\dot{\bar{y}}$-axis and approaches the $\dot{\bar{x}}$-axis asymptotically as a point moves along the curve from $A \to C \to D$, as shown in the first image of Figure 1. The second image of Figure 1 describes how the magnitude of the indicatrix varies at a fixed point. This variation depends solely on the parameter $t = \frac{\dot{y}^3}{\dot{x}^3}$. The magnitude achieves its maximum value of 1 when $t = 0$. For $\dot{x} > \dot{y}$ $(t < 1)$, $|I|(t)$ is relatively large and increases monotonically toward 1. When $\dot{x} < \dot{y}$ $(t > 1)$, $|I|(t)$ decreases rapidly and approaches zero asymptotically. This behavior is consistent across all three choices of c_0.

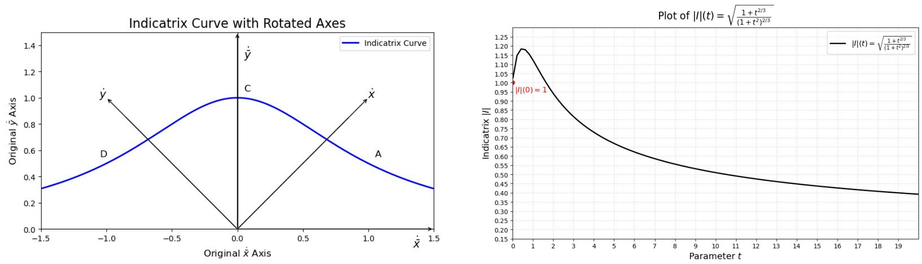

Fig. 1 Indicatrix curve with a rotated axis and its magnitude variation as a function of t for the metric $L^3 = c_0\left(\dot{x}^3 + \dot{y}^3\right)$.

The expressions in (31) and (32) are still rather complicated. We now examine how the (h)-scalar curvature varies for the three possible forms of c_0, as described below:

5.1.1 When $c_0 = x$

When $c_0 = x$, then the metric L is given by

$$L^3 = x\left(\dot{x}^3 + \dot{y}^3\right).$$

Using equations (31) and (32), both (h)-scalar curvature R_1 and R_2 become equivalent, and are expressed as

$$R_x(t) = R_1^3 = R_2^3 = \frac{(t+1)(t+13)^3}{(48)^3 x^8}, \tag{33}$$

where $t = \frac{(\dot{y})^3}{(\dot{x})^3}$ and $R_x(t)$ represents the (h)-scalar curvature when $c_0 = x$. Thus we have

Theorem 2 *The (h)-scalar curvature of a two-dimensional Finsler space with the cubic metric $L^3 = x\left(\dot{x}^3 + \dot{y}^3\right)$ is given by equation (33).*

The variation of the (h)-scalar curvature given by equation (33) for different values of x such as $x = 1, 1.1, 1.2$, is shown in the Figure 2 and its analytical analysis is given below:

$$t = -1 \Rightarrow \left(\frac{\dot{y}^3}{\dot{x}^3}\right) = -1 \Rightarrow \dot{y} = -\dot{x} \Rightarrow \dot{x} + \dot{y} = 0 \Rightarrow \dot{\bar{y}} = 0, \text{ which corresponds to } A.$$

$$t = 0 \Rightarrow \left(\frac{\dot{y}^3}{\dot{x}^3}\right) = 0 \Rightarrow \dot{\bar{x}} = \dot{\bar{y}}, \text{ which corresponds to } B.$$

$$t = 1 \Rightarrow \left(\frac{\dot{y}^3}{\dot{x}^3}\right) = 1 \Rightarrow \dot{y} = \dot{x} \Rightarrow \dot{x} - \dot{y} = 0 \Rightarrow \dot{\bar{x}} = 0, \text{ which corresponds to } C.$$

$$t \to \infty \Rightarrow \dot{x} = 0 \Rightarrow \dot{\bar{y}} = -\dot{\bar{x}}, \text{ which corresponds to } D.$$

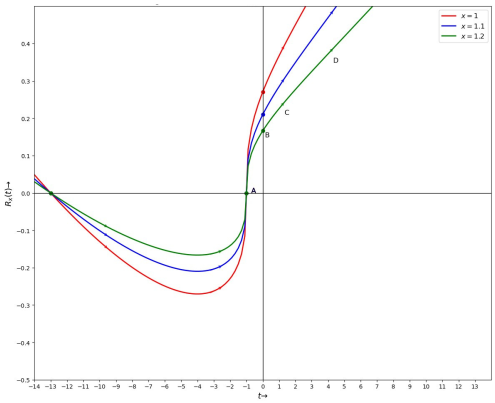

Fig. 2 Scalar Curvature $R_x(t)$ with various value of x

Thus, from Figure 2, we observe that the scalar curvature $R_x(t)$ exhibits smooth behavior for different values of x. The curvature crosses zero at $t = 0$ (point A) for all x. For negative t, $R_x(t)$ decreases to a minimum before rising toward zero, with the minimum becoming less pronounced as x increases. For positive t, the curvature increases steadily, with larger x values showing a slightly slower rise. Overall, increasing x results in a smoother curvature profile with reduced negative dips and a more gradual increase for positive t. Furthermore, as a point moves along with the indicatrix from $A \to C \to D$ in the first image of Figure 1, the corresponding variation in scalar curvature is depicted in Figure 2, where the point progresses from $A \to B \to C \to D$.

5.1.2 When $c_0 = y$

Let $c_0 = y$, then the metric L is given by

$$L^3 = y\left(\dot{x}^3 + \dot{y}^3\right).$$

Using equations (31) and (32), again the both (h)-scalar curvature R_1 and R_2 become equivalent, and are expressed as

$$R_y(t) = R_1^3 = R_2^3 = \frac{\left(\frac{1}{t}+1\right)\left(\frac{1}{t}+13\right)^3}{(48)^3 y^8} \tag{34}$$

where $t = \frac{(\dot{y})^3}{(\dot{x})^3}$ and $R_y(t)$ represents the (h)-scalar curvature when $c_0 = y$. Thus we have

Theorem 3 *The (h)-scalar curvature of a two-dimensional Finsler space with the cubic metric $L^3 = y\left(\dot{x}^3 + \dot{y}^3\right)$ is given by equation (34).*

The variation of the (h)-scalar curvature given by equation (34) for different values of Specifically, at $y = 0.5$, 1 and 1.5 is shown in the Figure 3 and its analytical analysis is given below:

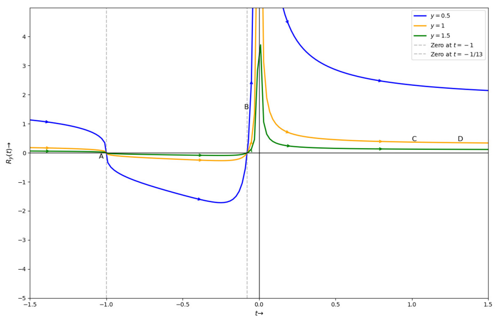

Fig. 3 Scalar Curvature $R_y(t)$ for various value of y

$$t = -1 \Rightarrow \left(\frac{\dot{y}^3}{\dot{x}^3}\right) = -1 \Rightarrow \dot{y} = -\dot{x} \Rightarrow \dot{x} + \dot{y} = 0 \Rightarrow \dot{\bar{y}} = 0, \text{ which corresponds to } A.$$

$$t = 0 \Rightarrow \left(\frac{\dot{y}^3}{\dot{x}^3}\right) = 0 \Rightarrow \dot{\bar{x}} = \dot{\bar{y}} \Rightarrow R_y(t) \to \infty, \text{ which corresponds to } B.$$

$$t = 1 \Rightarrow \left(\frac{\dot{y}^3}{\dot{x}^3}\right) = 1 \Rightarrow \dot{y} = \dot{x} \Rightarrow \dot{x} - \dot{y} = 0 \Rightarrow \dot{\bar{x}} = 0, \text{ which corresponds to } C.$$

$$t \to \infty \Rightarrow \dot{x} = 0 \Rightarrow \dot{\bar{y}} = -\dot{\bar{x}}, \text{ which corresponds to } D.$$

Figure 3 concludes that the variation of scalar curvature $R_y(t)$ for different values of y. A singular peak occurs at $t = 0$, with smaller y showing a stronger singularity. For $t < 0$, curvature decreases toward the singularity, exhibiting larger variation for smaller y, while larger y remain near zero. For $t > 0$, all curves drop sharply and flatten, with larger y converging faster to stable, small values. This corresponds to a point moving along the indicatrix in Figure 1 from $A \to C \to D$, with the scalar curvature varying from $A \to B \to C \to D$.

5.1.3 When $c_0 = constant$

If c_0 = constant, then the metric L is given by L^3 = constant $\left(\dot{x}^3 + \dot{y}^3\right)$. Using equations (31) and (32), both (h)-scalar curvature R_1 and R_2 vanishes identically. Thus we have

Theorem 4 *The (h)-scalar curvature of a two-dimensional Finsler space with the cubic metric L^3 = constant $\left(\dot{x}^3 + \dot{y}^3\right)$ has vanished identically.*

Thus, from above theorem we have

Corollary 5 *A two-dimensional Finsler space with a cubic metric of the form* $L^3 = constant \left(\dot{x}^3 + \dot{y}^3\right)$ *is flat.*

5.2 Special cubic Finsler metric (when $c_0 = 0$)

Let $c_0 = 0$, so the metric above reduces to $L^3 = 3c_1\left((\dot{x})^2\dot{y} + \dot{x}(\dot{y})^2\right)$, and the function $H = -c_1^2\left((\dot{x})^2 + \dot{x}\dot{y} + (\dot{y})^2\right)$. Putting these values in (28) and (29), we get

$$R_1 = \frac{-L}{36c_1^3\alpha_1}\Big\{\mathcal{S}_1\dot{x}^4 + \mathcal{T}_1\dot{x}^3\dot{y} + \mathcal{U}_1\dot{x}^2\dot{y}^2 + \mathcal{V}_1\dot{x}\dot{y}^3 + \mathcal{W}_1\dot{y}^4 + \frac{1}{(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2)}\mathcal{X}_1 + \frac{1}{(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2)^2}\mathcal{Y}_1 + \frac{1}{(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2)^3}\mathcal{Z}_1\Big\}, \tag{35}$$

where

$$\begin{aligned}
\alpha_1 &= \dot{x}^5 + 4\dot{x}^4\dot{y} + 7\dot{x}^3\dot{y}^2 + 8\dot{x}^2\dot{y}^3 + 5\dot{x}\dot{y}^4 + 2\dot{y}^5,\\
\mathcal{S}_1 &= 36\left(c_1c_{112} + c_{11}c_{12}\right) - 18\left(c_1c_{111} + c_{11}^2\right),\\
\mathcal{T}_1 &= 108c_1c_{112} + 72c_{11}c_{12} - 54c_1c_{111} - 36c_{11}^2,\\
\mathcal{U}_1 &= 144c_1c_{112} - 54c_1c_{111} - 18c_1c_{122},\ \mathcal{V}_1 = 108c_1c_{112} - 36c_1c_{111} - 18c_1c_{122} + 36c_{12}^2,\\
\mathcal{W}_1 &= -18c_1c_{122} + 36c_1c_{112} - 36c_{11}c_{12} + 18c_{12}^2,\\
\mathcal{X}_1 &= \left(-18c_{11}c_{12} + 27c_{11}^2 + 9c_1c_{111} + 81c_1c_{112}\right)\dot{x}^6 + \left(-46c_{11}c_{12} + 108c_{11}^2\right.\\
&\quad \left.+45c_1c_{111} + 63c_1c_{112}\right)\dot{x}^5\dot{y} + \left(\frac{-281}{2}c_{11}c_{12} + \frac{531}{2}c_{11}^2 + 72c_{12}^2 + 99c_1c_{111}\right.\\
&\quad \left.+108c_1c_{112} - 18c_1c_{122}\right)\dot{x}^4\dot{y}^2 + \left(53c_{11}c_{12} + 351c_{11}^2 + 54c_{12}^2 + 126c_1c_{111}\right.\\
&\quad \left.+90c_1c_{112} - 27c_1c_{122}\right)\dot{x}^3\dot{y}^3 + \left(\frac{405}{2}c_{11}c_{12} + 180c_{11}^2 - \frac{27}{2}c_{12}^2 + 90c_1c_{111}\right.\\
&\quad \left.+45c_1c_{112} - 27c_1c_{122}\right)\dot{x}^2\dot{y}^4 + \left(216c_{11}c_{12} + 72c_{11}^2 - \frac{117}{2}c_{12}^2 + 36c_1c_{111}\right.\\
&\quad \left.-9c_1c_{122}\right)\dot{x}\dot{y}^5 + \left(180c_{11}c_{12} - 54c_{12}^2\right)\dot{y}^6,\\
\mathcal{Y}_1 &= \left(\frac{-99}{4}c_{11}^2 + 36c_{11}c_{12}\right)\dot{x}^7\dot{y} + \left(126c_{11}c_{12} - \frac{81}{4}c_{11}^2 - 36c_{12}^2\right)\dot{x}^6\dot{y}^2\\
&\quad + \left(\frac{873}{2}c_{11}c_{12} + 358c_{11}^2 - 18c_{12}^2\right)\dot{x}^5\dot{y}^3 + \left(\frac{2079}{2}c_{11}c_{12} + 81c_{11}^2 + 54c_{12}^2\right)\dot{x}^4\dot{y}^4\\
&\quad + \left(261c_{11}c_{12} + 27c_{11}^2 + \frac{153}{4}c_{12}^2\right)\dot{x}^3\dot{y}^5 + \left(333c_{11}c_{12} + 45c_{11}^2 - \frac{117}{4}c_{12}^2\right)\dot{x}^2\dot{y}^6\\
&\quad + \left(198c_{11}c_{12} + 36c_{11}^2 - 54c_{12}^2\right)\dot{x}\dot{y}^7 + \left(72c_{11}c_{12} - 36c_{12}^2\right)\dot{y}^8.
\end{aligned}$$

$$\mathcal{Z}_1 = \left(-18c_{11}c_{12} - 18c_{11}^2\right)\dot{x}^9\dot{y} + \left(-63c_{11}c_{12} - 18c_{11}^2\right)\dot{x}^8\dot{y}^2 + \left(-90c_{11}c_{12} + \frac{381}{2}c_{11}^2\right)\dot{x}^7\dot{y}^3$$
$$+ \left(-81c_{11}c_{12} - \frac{873}{2}c_{11}^2 - 45c_{12}^2\right)\dot{x}^6\dot{y}^4 + \left(-198c_{11}c_{12} - 522c_{11}^2 - 81c_{12}^2\right)\dot{x}^5\dot{y}^5$$
$$+ \left(-405c_{11}c_{12} + 171c_{11}^2 - \frac{117}{2}c_{12}^2\right)\dot{x}^4\dot{y}^6 + \left(-531c_{11}c_{12} - 90c_{11}^2 + 36c_{12}^2\right)\dot{x}^3\dot{y}^7$$
$$+ \left(-423c_{11}c_{12} - \frac{189}{2}c_{12}^2\right)\dot{x}^2\dot{y}^8 + \left(-180c_{11}c_{12} + 63c_{12}^2\right)\dot{x}\dot{y}^9$$
$$+ \left(-36c_{11}c_{12} + 18c_{12}^2\right)\dot{y}^{10}$$

and

$$R_2 = \frac{L}{36c_1^3\alpha_2}\Big\{\mathcal{S}_2\dot{x}^4 + \mathcal{T}_2\dot{x}^3\dot{y} + \mathcal{U}_2\dot{x}^2\dot{y}^2 + \mathcal{V}_2\dot{x}\dot{y}^3 + \mathcal{W}_2\dot{y}^4 + \frac{1}{\left(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2\right)^2}\mathcal{X}_2 + \frac{1}{\left(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2\right)}\mathcal{Y}_2 + \frac{1}{\left(\dot{x}^2+\dot{x}\dot{y}+\dot{y}^2\right)^3}\mathcal{Z}_2\Big\}, \tag{36}$$

where

$$\alpha_2 = 2\dot{x}^5 + 5\dot{x}^4\dot{y} + 8\dot{x}^3\dot{y}^2 + 7\dot{x}^2\dot{y}^3 + 4\dot{x}\dot{y}4 + \dot{y}^5,$$
$$\mathcal{S}_2 = 18c_1c_{111} - 36c_1c_{112} - 18c_{11}^2 + 36c_{11}12,$$
$$\mathcal{T}_2 = -36c_{11}^2 + 18c_1c_{111} + 72c_{11}c_{12} - 72c_1c_{112},$$
$$\mathcal{U}_2 = 18c_1c_{111} - 108c_1c_{112} + 18c_1c_{122},$$
$$\mathcal{V}_2 = -72c_{11}c_{12} - 108c_1c_{112} + 36c_{12}^2 + 18c_1C_{122},$$
$$\mathcal{W}_2 = 18c_{12}^2 + 18c_1c_{122} - 36c_{11}c_{12} - 36c_1c_{112},$$
$$\mathcal{X}_2 = c_{11}c_{12}(36\dot{x}^6 + 180\dot{x}^5\dot{y} + \frac{675}{2}\dot{x}^4\dot{y}^2 + 495\dot{x}^3\dot{y}^3 + \frac{891}{2}\dot{x}^2\dot{y}^4 + 207\dot{x}\dot{y}^5 + 54\dot{y}^6) + c_{11}^2(-18\dot{x}^6 - \frac{99}{2}\dot{x}^5\dot{y} - \frac{117}{2}\dot{x}^4\dot{y}^2 - 54\dot{x}^3\dot{y}^3 - 36\dot{x}^2\dot{y}^4) + c_{11}c_{12}(18\dot{a}^5\dot{y} + 45\dot{x}^4\dot{y}^2 + 99\dot{x}^3\dot{y}^3 + 108\dot{x}^2\dot{y}^4 + 63\dot{x}\dot{y}^5 18\dot{y}^6),$$
$$\mathcal{Y}_2 = c_{11}c_{12}(-72\dot{x}^8 - \frac{729}{2}\dot{x}^7\dot{y} - \frac{1431}{2}\dot{x}^6\dot{y}^2 - 112\dot{x}^5\dot{y}^3\frac{1071}{2}\dot{x}^4\dot{y}^4 - 263\dot{x}^3\dot{y}^5 - \frac{619}{4}\dot{x}^2\dot{y}^6 - 27\dot{x}\dot{y}^7) + c_{11}^2(36\dot{x}^8 + \frac{369}{2}\dot{x}^7\dot{y} + \frac{413}{4}\dot{x}^6\dot{y}^2 + \frac{9}{4}\dot{x}^5\dot{y}^3 + \frac{63}{4}\dot{x}^4\dot{y}^4 + \frac{153}{2}\dot{x}^3\dot{y}^5 + 36\dot{x}^2\dot{y}^6) + c_{12}^2(72\dot{x}^7\dot{y} + 261\dot{x}^6\dot{y}^2 + \frac{911}{2}\dot{x}^5\dot{y}^3 + \frac{369}{2}\dot{x}^4\dot{y}^4 - \frac{299}{4}\dot{x}^3\dot{y}^5 - \frac{265}{4}\dot{x}^2\dot{y}^6 + \frac{279}{4}\dot{x}\dot{y}^7 + \frac{45}{4}\dot{y}^8),$$

$$\mathcal{Z}_2 = c_{11}c_{12}(36\dot{x}^{10} + 180\dot{x}^9\dot{y} + 459\dot{x}^8\dot{y}^2 + 576\dot{x}^7\dot{y}^3 + 393\dot{x}^6\dot{y}^4 + 126\dot{x}^5\dot{y}^5 + 72\dot{x}^3\dot{y}^7 + 63\dot{x}^2\dot{y}^8 + 18\dot{x}\dot{y}^9) + c_{11}^2(-18\dot{x}^10 - 63\dot{x}^9\dot{y} - 90\dot{x}^8\dot{y}^2$$

$$- 27\dot{x}^7\dot{y}^3 + 63\dot{x}^6\dot{y}^4 + 90\dot{x}^5\dot{y}^5 + 45\dot{x}^4\dot{y}^6) + c_{12}^2(-72\dot{x}^8\dot{y}^2 - 126\dot{x}^7\dot{y}^3 + 9\dot{x}^6\dot{y}^4 + 279\dot{x}^5\dot{y}^5 + 432\dot{x}^4\dot{y}^6 + 315\dot{x}^3\dot{y}^7 + 117\dot{x}^2\dot{y}^8 + 18\dot{x}\dot{y}^9)$$

and $\quad c_{ab1} = \frac{\partial c_{ab}}{\partial x}, \quad c_{ab2} = \frac{\partial c_{ab}}{\partial y}, \quad a = 0, 1, 2, 3 \quad$ and $\quad b = 1, 2.$

Figure 4 presents the indicatrix curve corresponding to the metric $L^3 = 3c_1\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$ for all three cases of c_1- when it depends on x, on y, or is constant. The first image shows that these cases produce similar indicatrix curves, while the second image illustrates that the magnitude of the indicatrix as a function of r is essentially the same for all three choices of c_1. The second image of Figure 4 describes the variation of the magnitude of the indicatrix at a fixed point which depends solely on the parameter $r = \frac{\dot{y}}{\dot{x}}$. The indicatrix curve plunges steeply from infinity at $r \to 0$ to a minimum value when $r = 1$, then again rises slowly towards infinity as $r \to \infty$.

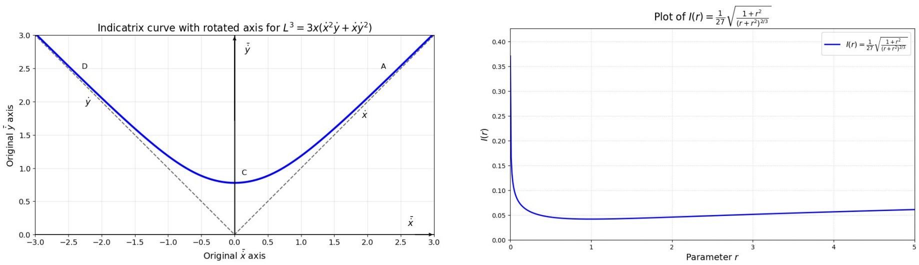

Fig. 4 Indicatrix curve with a rotated axis and its magnitude variation as a function of r for the metric $L^3 = 3c_1\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$.

Now, the above expression is again quite complex. To describe the variation of the (h)-scalar curvature, we can substitute different possible forms of c_1. Since c_1 is also a function of the position coordinates, in the simplest cases it may depend on x, y or be a constant. We will now analyze the variation of the (h)-scalar curvature for these three cases of c_1, as given below:

5.2.1 When $c_1 = x$

Let $c_1 = x$, so the metric L is given by

$$L^3 = 3x\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right).$$

Using equations (35) and (36), both (h)-scalar curvature R_1 and R_2 become equivalent, and are expressed as

$$R_x(r) = R_1^3 = R_2^3 = \frac{-r\,(1+r)}{12 \times 36^2 x^8 \beta^3} \Big\{ -18 - 36r + \frac{r}{(1+r+r^2)} I + \frac{1}{(1+r+r^2)^2} J + \frac{r}{(1+r+r^2)^3} K \Big\}^3, \tag{37}$$

where

$$\begin{aligned}
r &= \frac{\dot{y}}{\dot{x}}, \\
\beta &= 1 + 4r + 7r^2 + 8r^3 + 5r^4 + 2r^5, \\
I &= 27 + \frac{261}{2} r + \frac{531}{2} r^2 + 351 r^3 + 180 r^4 + 72 r^5, \\
J &= \frac{-99}{4} - \frac{81}{4} r + 358 r^2 + 81 r^3 + 27 r^4 + 45 r^5 + 36 r^6, \\
K &= -18 - 18r + \frac{381}{2} r^2 - \frac{873}{2} r^3 - 522 r^4 + 171 r^5 - 90 r^6, \\
R_x(r) &= \text{(h)-scalar curvature when } c_1 = x.
\end{aligned}$$

Theorem 6 *The (h)-scalar curvature of a two-dimensional Finsler space with the cubic metric $L^3 = 3x\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$ is given by equation (37).*

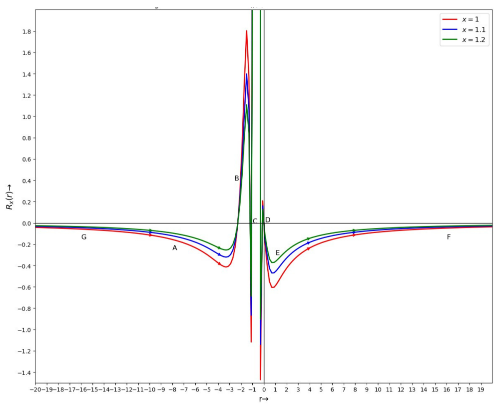

Fig. 5 Scalar Curvature $R_x(r)$ with various value of x

The variation of the (h)-scalar curvature given by equation (37) for different values of Specifically, at $x = 1$, 1.1 and 1.2 is shown in the Figure 5 and its analytical analysis is given below:

$$r = -1 \Rightarrow \frac{\dot{y}}{\dot{x}} = -1 \Rightarrow \dot{y} = -\dot{x} \Rightarrow \dot{x} + \dot{y} = 0 \Rightarrow \dot{\bar{y}} = 0, \text{ which corresponds to } C.$$

$$r = 0 \Rightarrow \frac{\dot{y}}{\dot{x}} = 0 \Rightarrow \dot{\bar{x}} = \dot{\bar{y}}, \text{ which corresponds to } D.$$
$$r = 1 \Rightarrow \frac{\dot{y}}{\dot{x}} = 1 \Rightarrow \dot{y} = \dot{x} \Rightarrow \dot{x} - \dot{y} = 0 \Rightarrow \dot{\bar{x}} = 0, \text{ which corresponds to } E.$$
$$r \to \infty \Rightarrow \dot{x} = 0 \Rightarrow \dot{\bar{y}} = -\dot{\bar{x}}, \text{ which corresponds to } F.$$

Figure 5 shows that the scalar curvature $R_x(r)$ exhibits a central singularity at $r = 0$, which becomes sharper and stronger as x increases. The curvature displays symmetric dips around the singularity, transitioning from negative to positive and back. For $r < 0$, it decreases to a negative minimum before rising toward the singularity, with smaller x producing broader dips and larger x sharper changes. For $r > 0$, a secondary negative dip forms and then gradually rises, with larger x leading to faster stabilization. The variation in scalar curvature is depicted in Figure 5, where the point progresses from $C \to D \to E \to F$

5.2.2 When $c_1 = y$

Let $c_1 = y$, so the metric L is given by

$$L^3 = 3y\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$$

Using equations (35) and (36), both (h)-scalar curvature R_1 and R_2 become equivalent, and are expressed as

$$\begin{aligned} R_y(r) = R_1^3 = R_2^3 = \frac{-\left(\frac{1}{r}\right)\left(1+\frac{1}{r}\right)}{12 \times 36^2 y^8 \gamma^3} \Bigg\{ 18 + 36\left(\frac{1}{r}\right) + \frac{1}{\left[1+\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2\right]} L \\ + \frac{1}{\left[1+\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2\right]^2} M + \frac{1}{\left[1+\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2\right]^3} N \Bigg\}^3, \end{aligned} \tag{38}$$

where

$$r = \frac{\dot{y}}{\dot{x}},$$
$$\gamma = 2 + 5\left(\frac{1}{r}\right) + 8\left(\frac{1}{r}\right)^2 + 7\left(\frac{1}{r}\right)^3 + 4\left(\frac{1}{r}\right)^4 + \left(\frac{1}{r}\right)^5,$$
$$L = -54 - \frac{117}{2}\left(\frac{1}{r}\right) - \frac{27}{2}\left(\frac{1}{r}\right)^2 + 54\left(\frac{1}{r}\right)^3 + 72\left(\frac{1}{r}\right)^4,$$
$$M = -36 - 54\left(\frac{1}{r}\right) - \frac{117}{4}\left(\frac{1}{r}\right)^2 + \frac{153}{4}\left(\frac{1}{r}\right)^3 + 54\left(\frac{1}{r}\right)^4 - 18\left(\frac{1}{r}\right)^5 - 36\left(\frac{1}{r}\right)^6,$$
$$N = 18 + 63\left(\frac{1}{r}\right) - \frac{189}{2}\left(\frac{1}{r}\right)^2 + 36\left(\frac{1}{r}\right)^3 - \frac{117}{2}\left(\frac{1}{r}\right)^4 - 81\left(\frac{1}{r}\right)^5 - 45\left(\frac{1}{r}\right)^6$$

$R_y(r) =$ (h)-scalar curvature when $c_1 = y$.

Theorem 7 *The (h)-scalar curvature of a two-dimensional Finsler space with the cubic metric $L^3 = 3y\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$ is given by equation (38).*

Figure 6 presents the behavior of the (h)-scalar curvature as determined by equation (38) for the selected values $y = 1$, 1.5 and 2. The analytical examination of these variations is presented below:

$$r = -1 \Rightarrow \frac{\dot{y}}{\dot{x}} = -1 \Rightarrow \dot{y} = -\dot{x} \Rightarrow \dot{x} + \dot{y} = 0 \Rightarrow \dot{\bar{y}} = 0, \text{ which corresponds to } B.$$
$$r = 0 \Rightarrow \frac{\dot{y}}{\dot{x}} = 0 \Rightarrow \dot{\bar{x}} = \dot{\bar{y}}, \text{ which corresponds to } D.$$
$$r = 1 \Rightarrow \frac{\dot{y}}{\dot{x}} = 1 \Rightarrow \dot{y} = \dot{x} \Rightarrow \dot{x} - \dot{y} = 0 \Rightarrow \dot{\bar{x}} = 0, \text{ which corresponds to } E.$$
$$r \to \infty \Rightarrow \dot{x} = 0 \Rightarrow \dot{\bar{y}} = -\dot{\bar{x}}, \text{ which corresponds to } F.$$

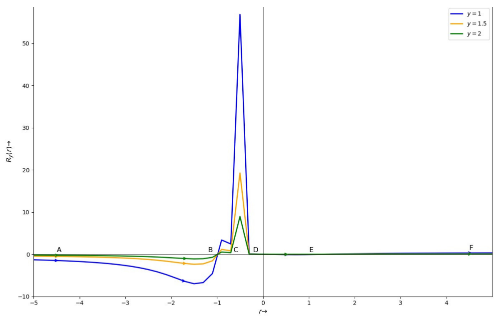

Fig. 6 Scalar Curvature $R_y(r)$ with various value of y

Figure 6 shows that the scalar curvature $R_y(r)$ exhibits a sharp singular peak at $r \approx 0.5$ for all values of y, with the peak intensity decreasing as y increases. For $r < -0.5$, the curvature is slightly negative, more pronounced for smaller y, while for $r > -0.5$, it rapidly stabilizes near zero. Larger y values correspond to weaker singularities and faster stabilization. The variation in scalar curvature is depicted in Figure 6, where the point progresses from $B \to D \to E \to F$.

5.2.3 When $c_1 = constant$

If c_1 = constant, then the metric L is given by L^3 = constant $\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$. Using equations (35) and (36), both (h)-scalar curvature R_1 and R_2 vanishes identically. Thus, we have

Theorem 8 *The (h)-scalar curvature of a two-dimensional Finsler space with the cubic metric L^3 = constant $\left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$ has vanished identically.*

Thus, form above theorem we have

Corollary 9 *A two-dimensional Finsler space with a cubic metric of the form* $L^3 = constant \left(\dot{x}^2\dot{y} + \dot{x}\dot{y}^2\right)$ *is flat.*

6 Conclusion

In this paper, we derived the (h)-scalar curvature R with respect to the Berwald frame for a two-dimensional Finsler space with a cubic metric and found it to be highly complex, making a direct geometric analysis difficult. To obtain a tractable approach, we focused on a special class of cubic metrics discussed in Section 5, considering two distinct cases and deriving explicit, simplified formulas for the scalar curvature. The key results are presented in Theorem 1, with special cases detailed in Theorems 2-4 and 6-8. A comprehensive analysis of the scalar curvatures variation with respect to the coordinate variables was conducted, and its behavior was illustrated graphically using Python. These explicit results provide a necessary foundation for the geometric classification of cubic Finsler spaces and open new avenues for research, particularly regarding the derivation of the fundamental (h)-scalar curvature with respect to the Berwald connection.

Declaration

On behalf of all authors, the corresponding author states that there is no conflict of interest and it meets all the ethical standard.

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Conflict of interest

The authors declare that there are no conflict of interest related to this research.

Author Contribution

All authors equally contributed to write and review the manuscript.

Data availability

Not Applicable.

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