A magneto-electro-elastic-visco-plastic problem with a subdifferential contact condition

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Abstract

The aim of this paper is to present a new result in the study of a quasistatic frictional contact problem between a magneto-electro-elastic-visco-plastic body and a foundation. The contact is modelled with the Signorini unilateral contact condition and a friction law assumed to be slip-dependent and expressed in terms of a subdifferential of a nonconvex potential function. We derive a variational formulation for the model, which couples a variational-hemivariational inequality with a variational equation. Then, we use arguments involving pseudomonotone operators and the Banach fixed point theorem to prove its unique solvability.

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1 Introduction

Elastic electromagnetic effects result from the coupling between elastic, piezoelectric, and piezomagnetic fields. This interaction leads to the simultaneous

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induction of both ferromagnetic moment and electric polarization by the application of mechanical stress. Conversely, the presence of an electric or magnetic field induces mechanical stress. Furthermore, a coupling occurs between magnetic and electric subsystems. This allows for the control of electric polarization by a magnetic field (direct effect) and the manipulation of ferromagnetic moment by an electric field (converse effect).

Recently developed composite materials, which combine piezoelectric and piezomagnetic effects, have been shown to achieve very strong magnetoelectric effects even at room temperature. This makes them promising tools for various technological applications. For example, in medicine, it is essential to develop a method that is as minimally invasive as possible to locally stimulate individual neurons. Furthermore, the redistribution of biomolecules in damaged cells requires relatively high efficiency and sufficient spatial and temporal resolution. Although their applications in biology and medicine are still in their early stages, the multifunctional properties of electromagnetic materials enable the design of new electronic devices for various sensing, switching and memory applications (see [4, 14, 22]).

Works relating to piezoelectric and piezoelectromagnetic materials are available, for example, in references [1, 3, 11, 12, 13, 15, 16, 18, 19, 23, 25]. In [1, 15], the uniqueness and reciprocity theorems for thermo-electro-magneto-elastic problems were discussed. In [25], the asymptotic homogenization method was applied in the study of a set of thermo-electro-magneto-elastic problems with rapidly oscillating coefficients and comparisons with other previously published results were presented. In [12], the quasistatic frictional contact problem with normal compliance for electro-elastic-viscoplastic materials with slip-dependent friction and damage was studied. A bilateral contact problem with nonlocal friction law, adhesion and damage was investigated for electroelastic viscoplastic materials in [13]. There, the unique solvability of a weak solution with a regularity result was proved, and the study of the dependence of the solution on the data was performed. The quasistatic frictionless contact problem for electro-elastic–visco-plastic materials with subdifferential boundary conditions was considered in [18]. In this last reference, within the framework of hemivariational inequalities, a weak formulation of the corresponding problem was derived and a solvability result of the solution was established.

The theory of hemivariational inequalities relies on the properties of Clarke's subdifferential of locally Lipschitz functions, which can be nonconvex (see [5, 6, 7, 17, 20]). Variational-hemivariational inequalities constitute a special class of inequalities that combine both convex and nonconvex functions (see [2, 9, 10, 26]). In [2], a class of elliptic variational-hemivariational inequalities was introduced in the study of a family of contact problems for ideally locked elastic materials, and corresponding numerical results were reported. The latest techniques for numerical analysis of variational-hemivariational inequalities in contact mechanics can be found in [9, 10]. In [26], existence, uniqueness, and convergence results for various classes of variational-hemivariational inequalities were studied, and applications of these inequalities in the study of concrete mathematical models were presented.

The novelty of this paper lies in the magneto-electro-elasto-visco-plastic constitutive law used, which involves three time-dependent nonlinear integral equations. We assume that the contact is modelled with the Signorini condition involving a slip-dependent friction law, which is described by a subdifferential of a nonconvex potential. These conditions lead to a non-standard mathematical model that we investigate using results of pseudomonotone operators and fixed point arguments within the framework of variational-hemivariational inequalities.

The paper is structured as follows. In Section 2 we introduce the notation and some preliminaries we shall use in our study. Section 3 is dedicated to describing the mechanical problem and deriving its variational formulation. Then, under certain assumptions, we discuss its weak solvability in Section 4.

2 PRELIMINARIES

We introduce some preliminary materials. For further details, we refer the reader to [17, 24]. Here and below, the indices i, j, k, l run between 1 and d (d=2, 3). We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, and we define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$w \cdot v = \sum_{i=1}^{d} w_i v_i, \;\; |v| = \sqrt[2]{v \cdot v} \text{ for all } w = (w_i)\,, \; v \; = (v_i) \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sum_{1 \le i,j \le d} \sigma_{ij} \tau_{ij}, \;\; |\sigma| = \sqrt[2]{\sigma \cdot \sigma} \text{ for all } \sigma = (\sigma_{ij})\,, \; \tau = (\tau_{ij}) \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ, let $x = (x_i) \in \Omega \cup \Gamma$ be the spatial variable and let ν be the unit outer normal on Γ. We denote by $H = L^2(\Omega)$ the standard Lebesgue space for scalar functions, $\mathbf{H} = (L^2(\Omega))^d$ the standard Lebesgue space for vector functions, $H^1(\Omega)$ the standard Sobolev space for scalar functions, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$ the standard Sobolev space for vector functions, and we introduce the following spaces

$$\begin{aligned} \mathcal{Q} &= \{\sigma = (\sigma_{ij}) \mid \ \sigma_{ij} = \sigma_{ji} \in H\}, \\ \mathcal{H} &= \{\sigma \in \mathcal{Q} \mid \ Div\sigma \in \mathbf{H}\}, \\ \mathcal{W} &= \{D \in \mathbf{H} \mid \ \operatorname{div} D \in H\}. \end{aligned}$$

Here and below, $Div : \mathcal{H} \to \mathbf{H}$ is the divergence operator for tensor functions

$$Div\sigma = (\varsigma_i), \ \varsigma_i = \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} \text{ for all } \sigma \in \mathcal{H},$$

$\operatorname{div} : \mathcal{W} \to H$ denotes the divergence operator for vector functions

$$\operatorname{div} D = \sum_{i=1}^{d} \frac{\partial D_i}{\partial x_i} \text{ for all } D = (D_i) \in \mathcal{W},$$

$\varepsilon : \mathbf{H}^1(\Omega) \to \mathcal{Q}$ is the linearized strain tensor

$$\varepsilon(w) = (\varepsilon_{ij}(w)), \ \varepsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \text{ for all } w \in \mathbf{H}^1(\Omega).$$

Note that $\mathbf{H}$, $\mathcal{Q}$, $\mathbf{H}^1(\Omega)$, $\mathcal{H}$ and $\mathcal{W}$ are real Hilbert spaces endowed with the following inner products

$$\begin{aligned} (w,v)_{\mathbf{H}} &= \int_\Omega w \cdot v dx \text{ for all } w, \ v \in \mathbf{H}, \\ (\sigma,\tau)_{\mathcal{Q}} &= \int_\Omega \sigma \cdot \tau dx \text{ for all } \sigma, \ \tau \in \mathcal{Q}, \\ (w,v)_{\mathbf{H}^1(\Omega)} &= (w,v)_{\mathbf{H}} + (\varepsilon(w),\varepsilon(v))_{\mathcal{Q}} \text{ for all } w, \ v \in \mathbf{H}^1(\Omega), \\ (\sigma,\tau)_{\mathcal{H}} &= (\sigma,\tau)_{\mathcal{Q}} + (Div\sigma, Div\tau)_{\mathbf{H}} \text{ for all } \sigma, \ \tau \in \mathcal{H}, \\ (D,B)_{\mathcal{W}} &= (D,B)_{\mathbf{H}} + (\operatorname{div} D, \operatorname{div} B)_H \text{ for all } D, \ B \in \mathcal{W}. \end{aligned}$$

The associated norms on the spaces $\mathbf{H}$, $\mathcal{Q}$, $\mathbf{H}^1(\Omega)$, $\mathcal{H}$ and $\mathcal{W}$ are denoted by $\|.\|_{\mathbf{H}}$, $\|.\|_{\mathcal{Q}}$, $\|.\|_{\mathbf{H}^1(\Omega)}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{W}}$. Let w be a vector field defined on Γ.

Then, w_ν denotes the normal component of w and w_τ denotes the projection of w on the tangent plane of Γ, and they are given by

$$w_\nu = w \cdot \nu, \ w_\tau = w - w_\nu \nu \text{ on } \Gamma.$$

Similarly, the normal component and the tangential components of a tensor σ are denoted by σ_ν and σ_τ, and they are given by

$$\sigma_\nu = \sigma\nu \cdot \nu, \ \sigma_\tau = \sigma\nu - \sigma_\nu \nu \text{ on } \Gamma.$$

For a regular (say C^1) tensor field σ the following Green formula holds

$$(\sigma, \varepsilon(w))_{\mathcal{Q}} + (Div\sigma, w)_{\mathbf{H}} = \int_\Gamma \sigma\nu \cdot w da \text{ for all } w \in \mathbf{H}^1(\Omega),$$

where da is the surface measure element. Also, when $D \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(D, \nabla\phi)_{\mathbf{H}} + (\operatorname{div} D, \phi)_H = \int_\Gamma D \cdot \nu\phi da \text{ for all } \phi \in H^1(\Omega).$$

Throughout this section, X stands for a real Banach space. We denote by $\|.\|_X$ its norm, by X^* its topological dual, and we denote by $\langle , \rangle_{X^* \times X}$ the duality pairing between X^* and X. For the particular case when $X = \mathbb{R}^n$ with $n \in \mathbb{N}^*$, we take $X^* = \mathbb{R}^n$ and $\langle w, v\rangle_{X^* \times X} = w \cdot v$ for all w, $v \in \mathbb{R}^n$. The symbol 2^{X^*} is the set of all subsets of X^*. Let $\mathbb{R}_+ = [0, \infty)$, we denote by $C(\mathbb{R}_+; X)$ the space of continuous functions from $\mathbb{R}_+$ to X. For a multivalued operator $\Lambda : X \to 2^{X^*}$, its domain $Dom(\Lambda)$, range $Rang(\Lambda)$ and graph $Gra(\Lambda)$ are defined, respectively, by

$$Dom(\Lambda) = \{w \in X, \ \Lambda w \neq \emptyset\},$$

$$Rang(\Lambda) = \underset{w \in X}{\cup} \Lambda w,$$

$$Gra(\Lambda) = \{(w, w^*) \in X \times X^* \mid w^* \in \Lambda w\}.$$

Definitions of maximal monotone, pseudomonotone and generalized pseudomonotone multivalued operators, as well as their properties, can be found in the books [5, 8, 20]. In particular, the following proposition follows from [20, Theorem 2.12].

Proposition 1 *Let X be a real reflexive Banach space with the dual space X^*. Let $\Psi : X \to 2^{X^*}$ be a maximal monotone operator with $0_X \in Dom(\Psi)$. Let $\Lambda : X \to 2^{X^*}$ be a bounded pseudomonotone operator, which is coercive in the sense that a real-valued function $\kappa : \mathbb{R}_+ \to \mathbb{R}$ exists such that*

$$(i) \ \lim_{s \to +\infty} \kappa(s) = +\infty,$$

$$(ii) \ \langle w^*, w \rangle_{X^* \times X} \geq \kappa(\|w\|_X) \|w\|_X \ \text{for all} \ (w, w^*) \in Gra(\Lambda).$$

Then, $Rang(\Lambda + \Psi) = X^*$.

Let us conclude this Section with the notions of the generalized (Clarke) directional derivative and the generalized gradient. For more details see [5, 7]. For a locally Lipschitz continuous function $\beta : X \to \mathbb{R}$, the generalized (Clarke) directional derivative of β at $w \in X$ in the direction $v \in X$ is

$$\beta^\circ(w; v) = \limsup_{y \to w, \lambda \downarrow 0} \frac{\beta(y + \lambda v) - \beta(y)}{\lambda}.$$

The generalized gradient (subdifferential) of β at w is a non-empty subset of the dual space X^* given by

$$\partial\beta(w) = \left\{ \xi \in X^* \mid \langle \xi, v \rangle_{X^* \times X} \leq \beta^\circ(w; v) \ \text{for all} \ v \in X \right\}.$$

We will use the following result (cf. Section 2.1 of [7]).

Proposition 2 *Let $\beta : X \to \mathbb{R}$ be a locally Lipschitz function. Then*

$$\beta^\circ(w; v) = \max\left\{ \langle \xi, v \rangle_{X^* \times X} \mid \xi \in \partial\beta(w) \right\} \text{ for all } w, v \in X.$$

3 Problem formulation

The physical setting is as follows. A deformable body occupies the reference configuration $\Omega \subset \mathbb{R}^{d=2,3}$ which is a bounded open set with a Lipschitz boundary Γ. The body is assumed to have a magneto-electro-elastic-visco-plastic behaviour and the process is quasistatic in the time interval $\mathbb{R}_+$. We consider three partitions of Γ into three open disjoint parts $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma_1^{el} \cup \Gamma_2^{el} \cup \Gamma_3 = \Gamma_1^{ma} \cup \Gamma_2^{ma} \cup \Gamma_3$ such that meas$(\Gamma_1) > 0$, meas$(\Gamma_1^{el}) > 0$ and meas$(\Gamma_1^{ma}) > 0$. We assume that body forces of density f_0, electric charges of density q_0 and electric current of density p_0 act in Ω. Homogeneous Dirichlet conditions of displacement, electrical potential and magnetic potential fields are considered on Γ_1, Γ_1^{el} and Γ_1^{ma}, respectively. We assume that surface tractions of density f_2, surface free electric charges of density q_2 and a surface magnetic induction of density p_2 are prescribed on Γ_2, Γ_2^{el} and Γ_2^{ma}, respectively. The body is supposed to be in unilateral contact over Γ_3 with an insulated foundation where a subdifferential contact condition with slip-dependent friction are included. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 3 *Find a displacement field* $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$*, a stress field* $\sigma : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$*, an electric potential field* $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$*, an electric displacement field* $D : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$*, a magnetic potential field* $\psi : \Omega \times \mathbb{R}_+$

$\rightarrow \mathbb{R}$ *and a magnetic induction field* $B : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ *such that*

$$\begin{aligned}
\dot{\sigma} &= \mathcal{A}\varepsilon(\dot{u}) + \mathcal{E}^*\nabla\dot{\varphi} + \mathcal{M}^*\nabla\dot{\psi} \\
&\quad +\mathcal{G}_1\left(\sigma, D, B, \varepsilon(u), \nabla\varphi, \nabla\psi\right) \;\; in\ \Omega \times (0, \infty)\,, \qquad (1)\\
\dot{D} &= \mathcal{E}\varepsilon(\dot{u}) - \mathcal{C}\nabla\dot{\varphi} - \mathcal{S}\nabla\dot{\psi} \\
&\quad +\mathcal{G}_2\left(\sigma, D, B, \varepsilon(u), \nabla\varphi, \nabla\psi\right) \;\; in\ \Omega \times (0, \infty)\,, \qquad (2)\\
\dot{B} &= \mathcal{M}\varepsilon(\dot{u}) - \mathcal{S}\nabla\dot{\varphi} - \mathcal{Z}\nabla\dot{\psi} \\
&\quad +\mathcal{G}_3\left(\sigma, D, B, \varepsilon(u), \nabla\varphi, \nabla\psi\right) \;\; in\ \Omega \times (0, \infty)\,, \qquad (3)\\
Div\sigma + f_0 &= 0\ in\ \Omega \times (0, \infty)\,, \qquad (4)\\
divD - q_0 &= 0\ in\ \Omega \times (0, \infty)\,, \qquad (5)\\
divB - p_0 &= 0\ in\ \Omega \times (0, \infty)\,, \qquad (6)\\
u &= 0\ on\ \Gamma_1 \times (0, \infty)\,, \qquad (7)\\
\sigma\nu &= f_2\ on\ \Gamma_2 \times (0, \infty)\,, \qquad (8)\\
\varphi &= 0\ on\ \Gamma_1^{el} \times (0, \infty)\,, \qquad (9)\\
D \cdot \nu &= q_2\ on\ \Gamma_2^{el} \times (0, \infty)\,, \qquad (10)\\
\psi &= 0\ on\ \Gamma_1^{ma} \times (0, \infty)\,, \qquad (11)\\
B \cdot \nu &= p_2\ on\ \Gamma_2^{ma} \times (0, \infty)\,, \qquad (12)\\
D \cdot \nu &= 0\ on\ \Gamma_3 \times (0, \infty)\,, \qquad (13)\\
B \cdot \nu &= 0\ on\ \Gamma_3 \times (0, \infty)\,, \qquad (14)\\
& u_\nu \leq 0,\;\; \sigma_\nu \leq 0,\; u_\nu\sigma_\nu = 0\ on\ \Gamma_3 \times (0, \infty)\,, \qquad (15)\\
-\sigma_\tau &\in \mu\left(|u_\tau|\right)\partial h_\tau\left(u_\tau\right) \;\; on\ \Gamma_3 \times (0, \infty)\,, \qquad (16)\\
(i)\;\; u\left(0\right) &= u_0,\;\; (ii)\;\; \varphi\left(0\right) = \varphi_0,\;\; (iii)\;\; \psi\left(0\right) = \psi_0\ in\ \Omega, \qquad (17)\\
(i)\;\; \sigma\left(0\right) &= \sigma_0,\;\; (ii)\;\; D\left(0\right) = D_0,\;\; (iii)\;\; B\left(0\right) = B_0\ in\ \Omega. \qquad (18)
\end{aligned}$$

The time-dependent integral equations (1)-(3) represent the magneto-electro-elastic-visco-plastic constitutive law in which the dot above a variable represents its derivative with respect to the time variable, $\varepsilon\left(u\right)$ denotes the linearized strain tensor, $-\nabla\varphi$ is the electric field, $-\nabla\psi$ stands for the magnetic field, $\mathcal{A}$ is the elasticity operator, $\mathcal{E}$ denotes the piezoelectric operator, $\mathcal{E}^*$ is the transposed of $\mathcal{E}$, $\mathcal{M}$ denotes the piezomagnetic operator, $\mathcal{M}^*$ is the transposed of $\mathcal{M}$, $\mathcal{C}$ stands for the electric permittivity operator, $\mathcal{Z}$ represents the magnetic permeability operator and $\mathcal{S}$ denotes the electromagnetic coupling operator. Here, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are given nonlinear constitutive functions. When $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3 = 0$, equations (1)-(3) can be reduced to the classical magneto-electro-elastic constitutive law, i.e.

$$\begin{aligned}\sigma &= \mathcal{A}\varepsilon(u) + \mathcal{E}^*\nabla\varphi + \mathcal{M}^*\nabla\psi,\\ D &= \mathcal{E}\varepsilon(u) - \mathcal{C}\nabla\varphi - \mathcal{S}\nabla\psi,\\ B &= \mathcal{M}\varepsilon(u) - \mathcal{S}\nabla\varphi - \mathcal{Z}\nabla\psi.\end{aligned}$$

Equations (4)-(6) represent the balance equations for the stress, electric displacement and magnetic induction fields, respectively. Equations (7)-(8) are the displacement-traction boundary conditions where ν stands for the unit outward normal vector to Γ and $\sigma\nu$ represents the Cauchy stress vector. Conditions (9)-(10) characterize the electric boundary conditions over $\Gamma_1^{el} \cup \Gamma_2^{el}$, whereas (11)-(12) represent the magnetic boundary conditions over $\Gamma_1^{ma} \cup \Gamma_2^{ma}$. In (13)-(14), the foundation is assumed to be non-conductive, so that the normal components of the electrical displacement and the magnetic induction fields vanish over Γ_3. Relations (15)-(16) describe the contact of the body with a foundation on Γ_3. Relation (15) is the Signorini contact condition, which assumes that the foundation is perfectly rigid. The condition (16) describes the friction law in which σ_τ represents the tangential stress, h_τ is a given locally Lipschitz function where ∂h_τ is its generalized subdifferential, and μ is a given function which may depend on the tangential displacement u_τ. As an example of the function μ, we can take

$$\mu(r) = (\delta_1 - \delta_2)\, e^{-\delta_3 r} + \delta_2 \text{ for all } r \in \mathbb{R}_+, \tag{19}$$

where δ_1, δ_2, $\delta_3 > 0$ and $\delta_1 \geq \delta_2$ (see, e.g., [2]). In particular, for $a \in (0,1)$, the following form of the function h_τ was considered for instance in [2, 17]

$$h_\tau(w) = \begin{cases} \dfrac{a-1}{2a}\,|w|^2 + |w| \;\text{ if }\; |w| \leq a, \\ \\ a\,|w| + \dfrac{a\,(1-a)}{2} \;\text{ if }\; |w| \geq a \end{cases} \tag{20}$$

for all $w \in \mathbb{R}^d$.

Finally, (17)-(18) represent the initial conditions.

In the study of Problem (1)-(18), we introduce the spaces $\mathbf{L}^2(\Gamma_i) = L^2(\Gamma_i)^d$ $(i = 1, 2)$, $\mathbf{Y} = L^2(\Gamma_3)^d$, the space V defined by

$$V = \{w \in \mathbf{H}^1(\Omega) \mid w = 0 \text{ on } \Gamma_1\},$$

and the set of admissible displacements U defined by

$$U = \{w \in V \mid w_\nu \le 0 \text{ on } \Gamma_3\}.$$

Since $meas\,(\Gamma_1) > 0$, Korn's inequality holds (see [21, p. 79]):

$$C_K \|w\|_{\mathbf{H}^1(\Omega)} \le \|\varepsilon(w)\|_{\mathcal{Q}} \text{ for all } w \in V,$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1. Over the space V, we consider the inner product given by

$$(w, v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}} \text{ for all } w, v \in V, \tag{21}$$

and let $\|.\|_V$ be the associated norm. We introduce the functional spaces

$$W_{el} = \{\phi \in H^1(\Omega) \mid \phi = 0 \ on \ \Gamma_1^{el}\},$$

$$W_{ma} = \{\vartheta \in H^1(\Omega) \mid \vartheta = 0 \ on \ \Gamma_1^{ma}\},$$

which are closed subspaces of $H^1(\Omega)$. Since $meas(\Gamma_1^{el}) > 0$ and $meas(\Gamma_1^{ma}) > 0$, the following Friedrichs–Poincare's inequalities hold

$$C_1^{el}\|\phi\|_{H^1(\Omega)} \le \|\nabla\phi\|_{\mathbf{H}} \text{ for all } \phi \in W_{el},$$

$$C_1^{ma}\|\vartheta\|_{H^1(\Omega)} \le \|\nabla\vartheta\|_{\mathbf{H}} \text{ for all } \vartheta \in W_{ma},$$

here $C_1^{el} > 0$ is a positive constant depending only on Ω and Γ_1^{el}, whereas $C_1^{ma} > 0$ is a positive constant depending only on Ω and Γ_1^{ma}. Over the spaces W_{el} and W_{ma}, we consider the inner products given by

$$(\varphi, \phi)_{W_{el}} = (\nabla\varphi, \nabla\phi)_{\mathbf{H}} \text{ for all } \varphi, \phi \in W_{el}, \tag{22}$$

$$(\psi, \vartheta)_{W_{ma}} = (\nabla\psi, \nabla\vartheta)_{\mathbf{H}} \text{ for all } \psi, \vartheta \in W_{ma}, \tag{23}$$

where the associated norms are denoted by $\|.\|_{W_{el}}$ and $\|.\|_{W_{ma}}$. Next, we use the functional space

$$W = W_{el} \times W_{ma}.$$

W is a real Hilbert space endowed with the inner product given by

$$((\varphi, \psi), (\phi, \vartheta))_W = (\nabla\varphi, \nabla\phi)_{\mathbf{H}} + (\nabla\psi, \nabla\vartheta)_{\mathbf{H}} \text{ for all } (\varphi, \psi), (\phi, \vartheta) \in W.$$

We consider the following assumptions. The elasticity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(i) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \mathcal{A}\varepsilon \cdot \varepsilon \geq m_{\mathcal{A}} \left|\varepsilon\right|^2 \text{ for all } \varepsilon \in \mathbb{S}^d, \ a.e. \ x \in \Omega; \\ \\ \text{(ii) } \mathcal{A} \in \mathbf{Q}_\infty, \end{cases} \tag{24}$$

where $\mathbf{Q}_\infty$ is the space of fourth-order tensor fields defined by

$$\mathbf{Q}_\infty = \left\{ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega) \right\},$$

which is a real Banach space with the norm

$$\left\|\mathcal{E}\right\|_{\mathbf{Q}_\infty} = \max_{1 \leq i,j,k,l \leq d} \left\|\mathcal{E}_{jikl}\right\|_{L^\infty(\Omega)}.$$

The electric permittivity operator $\mathcal{C} = (\mathcal{C}_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(i) } \left(\mathcal{C}w\right)_i = \sum\limits_{1 \leq j \leq d} \mathcal{C}_{ij} w_j \\ a.e. \ x \in \Omega, \ \text{for all } w = (w_j) \in \mathbb{R}^d; \\ \\ \text{(ii) } \mathcal{C}_{ij} = \mathcal{C}_{ji} \ \in L^\infty\left(\Omega\right). \end{cases} \tag{25}$$

The operator $\mathcal{E} = (\mathcal{E}_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(i) } \left(\mathcal{E}\varepsilon\right)_i = \sum\limits_{1 \leq j,k \leq d} \mathcal{E}_{ijk} \varepsilon_{jk} \\ a.e. \ x \in \Omega, \ \text{for all } \varepsilon \in \mathbb{S}^d; \\ \\ \text{(ii) } \mathcal{E}_{ijk} = \mathcal{E}_{ikj} \ \in L^\infty\left(\Omega\right). \end{cases} \tag{26}$$

The operator $\mathcal{E}^* : \Omega \times \mathbb{R}^d \to \mathbb{S}^d$ is defined by

$$\varepsilon \cdot \mathcal{E}^* w = \mathcal{E}\varepsilon \cdot w \text{ for all } w \in \mathbb{R}^d, \ \text{for all } \varepsilon \in \mathbb{S}^d, \ a.e. \ x \in \Omega. \tag{27}$$

The magnetic permeability operator $\mathcal{Z} = (\mathcal{Z}_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(i) } \left(\mathcal{Z}w\right)_i = \sum\limits_{1 \leq j \leq d} \mathcal{Z}_{ij} w_j, \\ a.e. \ x \in \Omega, \ \text{for all } w = \left(w_j\right)_{1 \leq j \leq d} \in \mathbb{R}^d; \\ \\ \text{(ii) } \mathcal{Z}_{ij} = \mathcal{Z}_{ji} \ \in L^\infty\left(\Omega\right). \end{cases} \tag{28}$$

The operator $\mathcal{M}$ =$(\mathcal{M}_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i)}\ \ (\mathcal{M}\varepsilon)_i = \displaystyle\sum_{1 \le j,k \le d} \mathcal{M}_{ijk}\varepsilon_{jk}, \\ a.e.\ x \in \Omega,\ \text{for all } \varepsilon \in \mathbb{S}^d; \\ \\ \text{(ii)}\ \mathcal{M}_{ijk} = \mathcal{M}_{ikj}\ \ \in L^\infty(\Omega). \end{array} \right. \tag{29}$$

The operator $\mathcal{M}^* : \Omega \times \mathbb{R}^d \to \mathbb{S}^d$ is defined by

$$\varepsilon \cdot \mathcal{M}^* w = \mathcal{M}\varepsilon \cdot w \text{ for all } w \in \mathbb{R}^d,\ \text{for all } \varepsilon \in \mathbb{S}^d,\ a.e.\ x \in \Omega. \tag{30}$$

The operator $\mathcal{S}$ =$(\mathcal{S}_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(i)}\ \ (\mathcal{S}w)_i = \displaystyle\sum_{1 \le j \le d} \mathcal{S}_{ij} w_j\ a.e.\ x \in \Omega, \\ \ \text{for all } w = (w_j) \in \mathbb{R}^d; \\ \\ \text{(ii)}\ \mathcal{S}_{ij} = \mathcal{S}_{ji}\ \ \in L^\infty(\Omega). \end{array} \right. \tag{31}$$

We assume that there exists $m > 0$ such that

$$\left\{ \begin{array}{c} \mathcal{C}w_1 \cdot w_1 + 2\mathcal{S}w_1 \cdot w_2 + \mathcal{Z}w_2 \cdot w_2 \ge m\left(|w_1|^2 + |w_2|^2\right) \\ \\ a.e.\ x \in \Omega,\ \text{for all } w_1, w_2 \in \mathbb{R}^d. \end{array} \right. \tag{32}$$

The nonlinear constitutive function $\mathcal{G}_1 : \Omega \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{S}^d$ is assumed to satisfy the following

$$\left\{ \begin{array}{l} \text{(i) There exists } L_{\mathcal{G}_1} > 0 \text{ such that} \\ |\mathcal{G}_1(x, \sigma_1, D_1, B_1, \varepsilon_1, w_1, v_1) - \mathcal{G}_1(x, \sigma_2, D_2, B_2, \varepsilon_2, w_2, v_2)| \le \\ \ L_{\mathcal{G}_1}(|\sigma_1 - \sigma_2| + |D_1 - D_2| + |B_1 - B_2|) \\ +\ L_{\mathcal{G}_1}(|\varepsilon_1 - \varepsilon_2| + |w_1 - w_2| + |v_1 - v_2|)\ \ a.e.\ x \in \Omega, \\ \text{for all } \sigma_1,\ \varepsilon_1,\ \sigma_2,\ \varepsilon_2 \in \mathbb{S}^d \\ \text{and for all } D_1,\ B_1,\ w_1,\ v_1,\ D_2,\ B_2,\ w_2,\ v_2 \in \mathbb{R}^d; \\ \\ \text{(ii) The mapping } x \mapsto \mathcal{G}_1(x, \sigma, D, B, \varepsilon, w, v)\ \text{ is Lebesgue} \\ \text{measurable on } \Omega \text{ for all } \sigma,\ \varepsilon \in \mathbb{S}^d \\ \text{and for all } D,\ B,\ w,\ v \in \mathbb{R}^d; \\ \\ \text{(iii) The mapping } x \mapsto \mathcal{G}_1(x, 0, 0, 0, 0, 0, 0) \in \mathcal{Q}. \end{array} \right. \tag{33}$$

The nonlinear constitutive function $\mathcal{G}_i : \Omega \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ $(i = 2,\, 3)$ is assumed to satisfy the following

$$\begin{cases} \text{(i) There exists } L_{\mathcal{G}_i} > 0 \text{ such that} \\ |\mathcal{G}_i\left(x, \sigma_1, D_1, B_1, \varepsilon_1, w_1, v_1\right) - \mathcal{G}_i\left(x, \sigma_2, D_2, B_2, \varepsilon_2, w_2, v_2\right)| \leq \\ \ L_{\mathcal{G}_i}\left(|\sigma_1 - \sigma_2| + |D_1 - D_2| + |B_1 - B_2|\right) \\ +\, L_{\mathcal{G}_i}\left(|\varepsilon_1 - \varepsilon_2| + |w_1 - w_2| + |v_1 - v_2|\right) \ \ a.e.\ x \in \Omega, \\ \text{for all } \sigma_1,\ \varepsilon_1,\ \sigma_2,\ \varepsilon_2 \in \mathbb{S}^d \\ \text{and for all } D_1,\ B_1,\ w_1,\ v_1,\ D_2,\ B_2,\ w_2,\ v_2 \in \mathbb{R}^d; \\ \\ \text{(ii) The mapping } x \mapsto \mathcal{G}_i\left(x, \sigma, D, B, \varepsilon, w, v\right) \text{ is Lebesgue} \\ \text{measurable on } \Omega \text{ for all } \sigma,\ \varepsilon \in \mathbb{S}^d \\ \text{and for all } D,\ B,\ w,\ v \in \mathbb{R}^d; \\ \\ \text{(iii) The mapping } x \mapsto \mathcal{G}_i\left(x, 0, 0, 0, 0, 0, 0\right) \in \mathbf{H}. \end{cases} \tag{34}$$

We assume that the function $h_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(i) The mapping } x \mapsto h_\tau\left(x, w\right) \text{ is Lebesgue measurable on } \Gamma_3, \\ \forall w \in \mathbb{R}^d; \\ \\ \text{(ii) } \exists\, v_0 \in \mathbf{Y} \text{ such that } x \mapsto h_\tau\left(x, v_0\left(x\right)\right) \ \in L^1\left(\Gamma_3\right); \\ \\ \text{(iii) } h_\tau\left(x, .\right) \text{ is locally Lipschitz on } \mathbb{R}^d \ \ a.e.\ x \in \Gamma_3; \\ \\ \text{(iv) There exists } l_\tau > 0 \text{ such that} \\ |\partial h_\tau\left(x, w\right)| \leq l_\tau \ a.e.\ x \in \Gamma_3,\ \text{for all } w \in \mathbb{R}^d; \\ \\ \text{(v) There exists } L_\tau \geq 0 \text{ such that} \\ h_\tau^\circ(x, w_1; w_2 - w_1) + h_\tau^\circ(x, w_2; w_1 - w_2) \leq L_\tau\, |w_1 - w_2|^2 \\ a.e.\ x \in \Gamma_3,\ \text{for all } w_1, w_2 \in \mathbb{R}^d. \end{cases} \tag{35}$$

We assume that the contact function $\mu : \Gamma_3 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies

$$\left\{ \begin{array}{l} \text{(i) The mapping } x \mapsto \mu(x,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \text{ for all } r \in \mathbb{R}_+; \\ \\ \text{(ii) There exists } L_\mu > 0 \text{ such that;} \\ |\mu(x,r_1) - \mu(x,r_2)| \leq L_\mu |r_1 - r_2| \text{ for all } r_1,\ r_2 \in \mathbb{R}_+,\ a.e.\ x \in \Gamma_3; \\ \\ \text{(iii) There exists } l_\mu > 0 \text{ such that} \\ \mu(x,r) \leq l_\mu \text{ for all } r \in \mathbb{R}_+,\ a.e\ x \in \Gamma_3\ . \end{array} \right. \tag{36}$$

We assume that the initial data satisfy

$$(i)\ u_0 \in V,\ (ii)\ \varphi_0 \in W_{el},\ (iii)\ \psi_0 \in W_{ma}, \tag{37}$$

$$(i)\ \sigma_0 \in \mathcal{Q},\ (ii)\ D_0 \in \mathbf{H},\ (iii)\ B_0 \in \mathbf{H}. \tag{38}$$

Finally, we assume that

$$(i)\ f_0 \in C(\mathbb{R}_+; \mathbf{H}),\ (ii)\ f_2 \in C\left(\mathbb{R}_+; \mathbf{L}^2(\Gamma_2)\right), \tag{39}$$

$$(i)\ q_0 \in C(\mathbb{R}_+; H),\ (ii)\ q_2 \in C\left(\mathbb{R}_+; L^2(\Gamma_2^{el})\right), \tag{40}$$

$$(i)\ p_0 \in C(\mathbb{R}_+; H),\ (ii)\ p_2 \in C\left(\mathbb{R}_+; L^2(\Gamma_2^{ma})\right). \tag{41}$$

Let $f : \mathbb{R}_+ \rightarrow V$, $q : \mathbb{R}_+ \rightarrow W_{el}$ and $p : \mathbb{R}_+ \rightarrow W_{ma}$ be functions defined by

$$(f(t), w)_V = \int_\Omega f_0(t) \cdot w dx + \int_{\Gamma_2} f_2(t) \cdot w da \text{ for all } w \in V, \tag{42}$$

$$(q(t), \phi)_{W_{el}} = \int_\Omega q_0(t) \phi dx - \int_{\Gamma_2^{el}} q_2(t) \phi da \text{ for all } \phi \in W_{el}, \tag{43}$$

$$(p(t), \vartheta)_{W_{ma}} = \int_\Omega p_0(t) \vartheta dx - \int_{\Gamma_2^{ma}} p_2(t) \vartheta da \text{ for all } \vartheta \in W_{ma}. \tag{44}$$

Using (39)-(41), we get

$$\left\{ \begin{array}{c} (i)\ f \in C(\mathbb{R}_+; V),\ (ii)\ q \in C(\mathbb{R}_+; W_{el}), \\ (iii)\ p \in C(\mathbb{R}_+; W_{ma}). \end{array} \right. \tag{45}$$

Under the assumptions (24)-(32), there are $L_{\mathcal{A}} > 0$, $L_{\mathcal{C}} > 0$, $L_{\mathcal{Z}} > 0$, $L_{\mathcal{S}} > 0$, $L_{\mathcal{E}} > 0$ and $L_{\mathcal{M}} > 0$ such that

$$\|\mathcal{A}\varepsilon\|_{\mathcal{Q}} \leq L_{\mathcal{A}} \|\varepsilon\|_{\mathcal{Q}} \text{ for all } \varepsilon \in \mathcal{Q}, \tag{46}$$

$$\left\{ \begin{array}{c} (i) \quad \|\mathcal{C}w\|_{\mathbf{H}} \leq L_{\mathcal{C}} \|w\|_{\mathbf{H}}\,, \ (ii) \quad \|\mathcal{Z}w\|_{\mathbf{H}} \leq L_{\mathcal{Z}} \|w\|_{\mathbf{H}}\,, \\ \\ (iii) \, \|\mathcal{S}w\|_{\mathbf{H}} \leq L_{\mathcal{S}} \|w\|_{\mathbf{H}} \text{ for all } w \in \mathbf{H}, \end{array} \right. \tag{47}$$

$$(i) \quad \|\mathcal{E}\varepsilon\|_{\mathbf{H}} \leq L_{\mathcal{E}} \|\varepsilon\|_{\mathcal{Q}}\,, \ (ii) \quad \|\mathcal{M}\varepsilon\|_{\mathbf{H}} \leq L_{\mathcal{M}} \|\varepsilon\|_{\mathcal{Q}} \text{ for all } \varepsilon \in \mathcal{Q}, \tag{48}$$

$$(i) \ \|\mathcal{E}^* w\|_{\mathcal{Q}} \leq L_{\mathcal{E}} \|w\|_{\mathbf{H}}, \ (ii) \quad \|\mathcal{M}^* w\|_{\mathcal{Q}} \leq L_{\mathcal{M}} \|w\|_{\mathbf{H}} \text{ for all } w \in \mathbf{H}. \tag{49}$$

Moreover, we have

$$(\mathcal{A}\varepsilon, \varepsilon)_{\mathcal{Q}} \geq m_{\mathcal{A}} \|\varepsilon\|_{\mathcal{Q}}^2 \text{ for all } \varepsilon \in \mathcal{Q}, \tag{50}$$

$$\left\{ \begin{array}{l} (\mathcal{C}w_1, w_1)_{\mathbf{H}} + 2\,(\mathcal{S}w_1, w_2)_{\mathbf{H}} + (\mathcal{Z}w_2, w_2)_{\mathbf{H}} \\ \\ \geq m\left(\|w_1\|_{\mathbf{H}}^2 + \|w_2\|_{\mathbf{H}}^2\right) \text{ for all } w_1, w_2 \in \mathbf{H}, \end{array} \right. \tag{51}$$

$$\left\{ \begin{array}{c} (i) \quad (\mathcal{E}^* w, \varepsilon)_{\mathcal{Q}} = (\mathcal{E}\varepsilon, w)_{\mathbf{H}}\,, \ (ii) \quad (\mathcal{M}^* w, \varepsilon)_{\mathcal{Q}} = (\mathcal{M}\varepsilon, w)_{\mathbf{H}} \\ \text{for all } (w, \varepsilon) \in \mathbf{H} \times \mathcal{Q}. \end{array} \right. \tag{52}$$

Let $\gamma : V \to \mathbf{Y}$ be the trace map. Then, by the Sobolev trace theorem, there is a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\gamma w\|_{\mathbf{Y}} \leq c_0 \|w\|_V \text{ for all } w \in V. \tag{53}$$

Define the operator $\gamma^* : \mathbf{Y} \to V^*$ by

$$\left\langle \gamma^* \zeta, w \right\rangle_{V^* \times V} = (\zeta, \gamma w)_{\mathbf{Y}} \text{ for all } (\zeta, w) \in \mathbf{Y} \times V. \tag{54}$$

We note that γ and γ^* are continuous linear operators. In the sequel, we use the functional $\mathcal{R}$: $V \times V \longrightarrow \mathbb{R}$ defined by

$$\mathcal{R}(w, v) = \int_{\Gamma_3} \mu\left(|\gamma w_\tau|\right) h_\tau^\circ\left(\gamma w_\tau; \gamma v_\tau\right) da \text{ for all } w, v \in V. \tag{55}$$

It follows from Proposition 2, (35), (36), (53) and (55) that

$$\left\{ \begin{array}{c} \mathcal{R}\left(w, v - w\right) + \mathcal{R}\left(v, w - v\right) \leq c_0^2 \left(l_\mu L_\tau + l_\tau L_\mu\right) \|w - v\|_V^2 \\ \text{for all } w, \ v \in V. \end{array} \right. \tag{56}$$

Assume u, φ, ψ, σ, D and B are smooth functions satisfying (1)-(18) and use integration by parts to obtain the following variational formulation.

Problem 4 *Find a displacement field $u : \mathbb{R}_+ \to V$, a stress field $\sigma : \mathbb{R}_+ \to \mathcal{Q}$, an electric potential field $\varphi : \mathbb{R}_+ \to W_{el}$, an electric displacement field $D : \mathbb{R}_+ \to \mathbf{H}$, a magnetic potential field $\psi : \mathbb{R}_+ \to W_{ma}$ and a magnetic induction field $B : \mathbb{R}_+ \to \mathbf{H}$ such that for all $t \in \mathbb{R}_+$*

$$\begin{aligned}\sigma(t) &= \sigma_0 - \mathcal{A}\varepsilon(u_0) - \mathcal{E}^*\nabla\varphi_0 - \mathcal{M}^*\nabla\psi_0 + \mathcal{A}\varepsilon(u(t)) + \mathcal{E}^*\nabla\varphi(t) + \mathcal{M}^*\nabla\psi(t) \\ &\quad + \int_0^t \mathcal{G}_1(\sigma(s), D(s), B(s), \varepsilon(u(s)), \nabla\varphi(s), \nabla\psi(s))\, ds, \qquad (57)\end{aligned}$$

$$\begin{aligned}D(t) &= D_0 - \mathcal{E}\varepsilon(u_0) + \mathcal{C}\nabla\varphi_0 + \mathcal{S}\nabla\psi_0 + \mathcal{E}\varepsilon(u(t)) - \mathcal{C}\nabla\varphi(t) - \mathcal{S}\nabla\psi(t) \\ &\quad + \int_0^t \mathcal{G}_2(\sigma(s), D(s), B(s), \varepsilon(u(s)), \nabla\varphi(s), \nabla\psi(s))\, ds, \qquad (58)\end{aligned}$$

$$\begin{aligned}B(t) &= B_0 - \mathcal{M}\varepsilon(u_0) + \mathcal{S}\nabla\varphi_0 + \mathcal{Z}\nabla\psi_0 + \mathcal{M}\varepsilon(u(t)) - \mathcal{S}\nabla\varphi(t) - \mathcal{Z}\nabla\psi(t) \\ &\quad + \int_0^t \mathcal{G}_3(\sigma(s), D(s), B(s), \varepsilon(u(s)), \nabla\varphi(s), \nabla\psi(s))\, ds, \qquad (59)\end{aligned}$$

$$\left\{\begin{aligned} & u(t) \in U,\ (\sigma(t), \varepsilon(w - u(t)))_{\mathcal{Q}} + \mathcal{R}(u(t), w - u(t)) \\ & \qquad \geq (f(t), w - u(t))_V \ \textit{for all } w \in U, \end{aligned}\right. \qquad (60)$$

$$\left\{\begin{aligned} & -(D(t), \nabla\phi)_{\mathbf{H}} - (B(t), \nabla\vartheta)_{\mathbf{H}} = (q(t), \phi)_{W_{el}} \\ & \qquad + (p(t), \vartheta)_{W_{ma}} \ \textit{for all } (\phi, \vartheta) \in W. \end{aligned}\right. \qquad (61)$$

$$(i)\ u(0) = u_0,\ (ii)\ \varphi(0) = \varphi_0,\ (iii)\ \psi(0) = \psi_0, \qquad (62)$$

$$(i)\ \sigma(0) = \sigma_0,\ (ii)\ D(0) = D_0,\ (iii)\ B(0) = B_0. \qquad (63)$$

We conclude this section with some comments on assumptions (24)-(41) and (64). First, we note that the function h_τ given by (20) satisfies (35) with $(l_\tau, L_\tau) = (1, 1)$ (see [2, 17]). Second, the functions μ defined by (19) satisfies (36) with $(L_\mu, l_\mu) = ((\delta_1 - \delta_2)\,\delta_3, \delta_1)$. Finally, the conditions (64) are formulated for mathematical reasons; otherwise, the results obtained in Theorem 5 might be incorrect. For example, if we consider the case in which the function h_τ is defined by (20), and if the parameters of the function μ are small enough to satisfy the following inequality

$$l_\mu + L_\mu < \frac{m_{\mathcal{A}}}{c_0^2},$$

then Theorem 5 holds.

4 Existence and uniqueness results

The following theorem is the main result of this paper.

Theorem 5 *Assume (24)-(41) and, in addition, assume the smallness condition*

$$l_\mu L_\tau + l_\tau L_\mu < \frac{m_{\mathcal{A}}}{c_0^2}. \tag{64}$$

Then, Problem (57)-(63) has a unique solution which satisfies

$$(i)\ u \ \in \ C\left(\mathbb{R}_+; V\right),\ (ii)\ \varphi \in C\left(\mathbb{R}_+; W_{el}\right),\ (iii)\ \psi \in C\left(\mathbb{R}_+; W_{ma}\right), \tag{65}$$

$$(i)\ \sigma \ \in \ C\left(\mathbb{R}_+; \mathcal{H}\right),\ (ii)\ D \in C\left(\mathbb{R}_+; \mathcal{W}\right),\ (iii)\ B \in C\left(\mathbb{R}_+; \mathcal{W}\right). \tag{66}$$

We will divide the proof into several steps.

Step 1. Let $\mathcal{K} : W \to W$, $\mathcal{F} : W \to V$ and $\mathcal{N} : V \to W$ be linear operators defined by

$$\begin{cases} \left(\mathcal{K}\left(\varphi,\psi\right),\left(\phi,\vartheta\right)\right)_W = \left(\mathcal{C}\nabla\varphi,\nabla\phi\right)_{\mathbf{H}} + \\ \left(\mathcal{Z}\nabla\psi,\nabla\vartheta\right)_{\mathbf{H}} + \left(\mathcal{S}\nabla\varphi,\nabla\vartheta\right)_{\mathbf{H}} + \left(\mathcal{S}\nabla\psi,\nabla\phi\right)_{\mathbf{H}}, \\ \text{for all } \left(\varphi,\psi\right),\left(\phi,\vartheta\right) \in W, \end{cases} \tag{67}$$

$$\begin{cases} \left(\mathcal{F}\left(\phi,\vartheta\right),w\right)_V = \left(\mathcal{E}^*\nabla\phi,\varepsilon\left(w\right)\right)_{\mathcal{Q}} + \left(\mathcal{M}^*\nabla\vartheta,\varepsilon\left(w\right)\right)_{\mathcal{Q}}, \\ \text{for all } w \in V, \text{ for all } \left(\phi,\vartheta\right) \in W, \end{cases} \tag{68}$$

$$\begin{cases} \left(\mathcal{N}w,\left(\phi,\vartheta\right)\right)_W = \left(\mathcal{E}\varepsilon(w),\nabla\phi\right)_{\mathbf{H}} + \left(\mathcal{M}\varepsilon(w),\nabla\vartheta\right)_{\mathbf{H}}, \\ \text{for all } w \in V, \text{ for all } \left(\phi,\vartheta\right) \in W. \end{cases} \tag{69}$$

It is easy to show that $\mathcal{K}$ is a bijection. Moreover, the operator $A : V \to V^*$ defined by

$$\langle Aw, v\rangle_{V^*\times V} = \left(\mathcal{A}\varepsilon\left(w\right),\varepsilon\left(v\right)\right)_{\mathcal{Q}} + \left(\mathcal{F}\mathcal{K}^{-1}\mathcal{N}w, v\right)_V \text{ for all } w,\, v \in V, \tag{70}$$

satisfies

$$\|Aw\|_{V^*} \le L_A \|w\|_V \text{ for all } w \in V, \tag{71}$$

with $L_A > 0$;

$$m_{\mathcal{A}} \|w\|_V^2 \leq \langle Aw, w \rangle_{V^* \times V} \text{ for all } w \in V. \tag{72}$$

Let $\Pi : V \to 2^{V^*}$ be the operator defined by

$$\Pi w = \{Aw\} \text{ for all } w \in V. \tag{73}$$

Lemma 6 *The operator* Π *is bounded and pseudomonotone.*

Proof. The proof can be obtained using arguments similar to those used in [11, proof of Lemma 17]. ■

Step 2. Let $j_\tau : \mathbf{Y} \longrightarrow \mathbb{R}$ be a functional defined by

$$j_\tau(w) = \int_{\Gamma_3} h_\tau(w)\, da \text{ for all } w \in \mathbf{Y}.$$

From (35), using Aubin–Clarke theorem (see [5, Theorem 2.181]), it follows that j_τ is a locally Lipschitz function satisfying

$$\partial j_\tau(w_\tau) \subset \{ \varrho \in \mathbf{Y},\ \varrho \in \partial h_\tau(w_\tau) \ for \ a.e\ x \in \Gamma_3\} \tag{74}$$

for all $w \in \mathbf{Y}$. Let $w \in \mathbf{Y}$ and let $\varrho \in \partial j_\tau(w_\tau)$. Due to (35)-(36), the following mapping $j_{(w,\varrho)} : \mathbf{Y} \to \mathbb{R}$ defined by

$$j_{(w,\varrho)}(v) = \int_{\Gamma_3} \mu(|w_\tau|)\, \varrho \cdot v_\tau da \text{ for all } v \in \mathbf{Y} \tag{75}$$

is a continuous linear functional. Let $J : \mathbf{Y} \longrightarrow 2^{\mathbf{Y}}$ and $\Theta : V \to 2^{V^*}$ be multivalued operators defined by

$$J(w) = \left\{ \begin{array}{c} \zeta \in \mathbf{Y} \mid \ \exists\ \varrho \in \partial j_\tau(w_\tau) : (\zeta, v)_{\mathbf{Y}} = j_{(w,\varrho)}(v) \\ \text{for all } v \in \mathbf{Y} \end{array} \right\} \tag{76}$$

for all $w \in \mathbf{Y}$.

$$\Theta(w) = \gamma^* J(\gamma w) \text{ for all } w \in V. \tag{77}$$

We have the following result.

Lemma 7 *The operator* Θ *is bounded and pseudomonotone.*

Proof. The proof can be obtained using arguments similar to those used in [11, proof of Lemma 18]. ■

Step 3. Let $\mathcal{X} = \mathcal{Q}\times\mathbf{H}\times\mathbf{H}\times V\times W_{el}\times W_{ma}$. Then, for all $\eta\in C(\mathbb{R}_+;\mathcal{X})$ there are $\eta_1\in C(\mathbb{R}_+;\mathcal{Q})$, $\eta_2\in C(\mathbb{R}_+;\mathbf{H})$, $\eta_3\in C(\mathbb{R}_+;\mathbf{H})$, $\eta_4\in C(\mathbb{R}_+;V)$, $\eta_5\in C(\mathbb{R}_+;W_{el})$ and $\eta_6\in C(\mathbb{R}_+;W_{ma})$ such that $\eta(t) = (\eta_i(t))_{1\le i\le 6}$ for all $t\in\mathbb{R}_+$. Let $b_\eta\in C(\mathbb{R}_+;W)$ and $f_\eta\in C(\mathbb{R}_+;V^*)$ be defined by

$$\left\{\begin{array}{l} (b_\eta(t),(\phi,\vartheta))_W = (q(t),\phi)_{W_{el}} + (p(t),\vartheta)_{W_{ma}} \\ +(I_2\eta(t),\nabla\phi)_{\mathbf{H}} + (I_3\eta(t),\nabla\vartheta)_{\mathbf{H}} \\ \text{ for all } (\phi,\vartheta)\in W, \text{ for all } t\in\mathbb{R}_+, \end{array}\right. \tag{78}$$

$$\left\{\begin{array}{l} \langle f_\eta(t),w\rangle_{V^*\times V} = \left(f(t) - \mathcal{F}\mathcal{K}^{-1}b_\eta(t),w\right)_V + \\ \\ -(I_1\eta(t),\varepsilon(w))_{\mathcal{Q}} \text{ for all } w\in V, \text{ for all } t\in\mathbb{R}_+, \end{array}\right. \tag{79}$$

where

$$\left\{\begin{array}{c} I_1\eta(t) = \sigma_0 - \mathcal{A}\varepsilon(u_0) - \mathcal{E}^*\nabla\varphi_0 - \mathcal{M}^*\nabla\psi_0 \\ +\displaystyle\int_0^t \mathcal{G}_1(\eta_1(s),\eta_2(s),\eta_3(s),\varepsilon(\eta_4(s)),\nabla\eta_5(s),\nabla\eta_6(s))\,ds. \end{array}\right. \tag{80}$$

$$\left\{\begin{array}{c} I_2\eta(t) = D_0 - \mathcal{E}\varepsilon(u_0) + \mathcal{C}\nabla\varphi_0 + \mathcal{S}\nabla\psi_0 \\ +\displaystyle\int_0^t \mathcal{G}_2(\eta_1(s),\eta_2(s),\eta_3(s),\varepsilon(\eta_4(s)),\nabla\eta_5(s),\nabla\eta_6(s))\,ds, \end{array}\right. \tag{81}$$

$$\left\{\begin{array}{c} I_3\eta(t) = B_0 - \mathcal{M}\varepsilon(u_0) + \mathcal{S}\nabla\varphi_0 + \mathcal{Z}\nabla\psi_0 \\ +\displaystyle\int_0^t \mathcal{G}_3(\eta_1(s),\eta_2(s),\eta_3(s),\varepsilon(\eta_4(s)),\nabla\eta_5(s),\nabla\eta_6(s))\,ds. \end{array}\right. \tag{82}$$

Let $I_U : V\to\mathbb{R}\cup\{+\infty\}$ be the indicator function of the set U, that is

$$I_U(w) = \left\{\begin{array}{l} 0 \text{ if } w\in U, \\ \\ +\infty \text{ if } w\notin U, \end{array}\right.$$

and let ∂I_U be its subdifferential in the sense of convex analysis. Consider the following auxiliary problem.

Problem 8 *Let $\eta\in C(\mathbb{R}_+;\mathcal{X})$. Find a function $u_\eta:\mathbb{R}_+\to V$ such that*

$$\left\{\begin{array}{c} u_\eta(t)\in U, \\ \\ f_\eta(t)\in Au_\eta(t) + \gamma^* J(\gamma u_\eta(t)) + \partial I_U(u_\eta(t)) \ \textit{for all } t\in\mathbb{R}_+. \end{array}\right. \tag{83}$$

Lemma 9 *Assume that (24)-(41) and (64) are fulfilled. Then, Problem (83) has a solution.*

Proof. Since U is a nonempty, closed, and convex subset of V, it follows that I_U is a proper, convex and lower semicontinuous function on V. Therefore, the operator ∂I_U: $V \to 2^{V^*}$ is maximal monotone (see [8, Theorem 1.3.19]).

Moreover, we have $U = Dom(\partial I_U)$, so $0_V \in Dom(\partial I_U)$. On the other hand, let $\Lambda : V \to 2^{V^*}$ be a multivalued operator defined by

$$\Lambda(w) = \Pi\,(w) + \gamma^* J(\gamma w) \text{ for all } w \in V. \tag{84}$$

It follows from Lemma 6, Lemma 7 and [8, Proposition 1.3.68] that Λ is a bounded pseudomonotone operator. We now show the coercivity of Λ. Let $(w, w^*) \in Gra\,(\Lambda)$. So, there is an element $\zeta \in J(\gamma w)$ such that

$$w^* = Aw + \gamma^* \zeta. \tag{85}$$

Let $v \in V$, let $\xi \in J(\gamma v)$. From (55)-(56) and (74)-(76), we deduce that

$$\begin{aligned}(\zeta - \xi, \gamma v - \gamma w)_{\mathbf{Y}} &\leq \int_{\Gamma_3} \mu\left(|\gamma w_\tau|\right) h_\tau^\circ\left(\gamma w_\tau; \gamma v_\tau - \gamma w_\tau\right) da \\ &\quad + \int_{\Gamma_3} \mu\left(|\gamma v_\tau|\right) h_\tau^\circ\left(\gamma v_\tau; \gamma w_\tau - \gamma v_\tau\right) da \\ &\leq c_0^2\left(l_\mu L_\tau + l_\tau L_\mu\right) \|w - v\|_V^2,\end{aligned}$$

which gives

$$\langle \gamma^* \zeta - \gamma^* \xi, w - v\rangle_{V^* \times V} \geq -c_0^2\left(l_\mu L_\tau + l_\tau L_\mu\right) \|w - v\|_V^2. \tag{86}$$

Using (85), we obtain

$$\langle w^*, w\rangle_{V^* \times V} = \langle Aw, w\rangle_{V^* \times V} + \langle \gamma^* \zeta - \gamma^* \xi, w\rangle_{V^* \times V} + \langle \gamma^* \xi, w\rangle_{V^* \times V}. \tag{87}$$

Let $v = 0_V$. Taking into account (35)$-$(iv), (53), (72), (86) and (87), we get

$$\langle w^*, w\rangle_{V^* \times V} \geq \left(m_{\mathcal{A}} - c_0^2\left(l_\mu L_\tau + l_\tau L_\mu\right)\right) \|w\|_V^2 - \alpha_3 \|w\|_V,$$

with $\alpha_3 > 0$. Therefore, it is sufficient to use the function $\kappa : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\kappa(s) = \left(m_{\mathcal{A}} - c_0^2\left(l_\mu L_\tau + l_\tau L_\mu\right)\right) s - \alpha_3$$

to conclude that Λ is a coercive operator. We apply now Proposition 1 to obtain a solution to Problem (83). ■

Consider the following problem.

Problem 10 *Let $\eta \in C(\mathbb{R}_+;\mathcal{X})$. Find a function $u_\eta : \mathbb{R}_+ \to V$ such that*

$$\left\{\begin{array}{c} u_\eta(t) \in U, \\ \langle Au_\eta(t), w - u_\eta(t)\rangle_{V^*\times V} \\ +\displaystyle\int_{\Gamma_3} \mu(|\gamma u_{\eta\tau}(t)|)\, h_\tau^\circ(\gamma u_{\eta\tau}(t);\gamma w_\tau - \gamma u_{\eta\tau}(t))da \\ \geq \langle f_\eta(t), w - u_\eta(t)\rangle_{V^*\times V} \text{ for all } w\in U, \text{ for all } t\in\mathbb{R}_+. \end{array}\right. \tag{88}$$

Lemma 11 *Problem (88) has a unique solution $u_\eta \in C(\mathbb{R}_+;V)$.*

Proof. From Lemma 9, there exists two functions $u_\eta : \mathbb{R}_+ \to V$ and $\zeta_\eta : \mathbb{R}_+ \to \mathbf{Y}$ such that

$$\left\{\begin{array}{c} \big(u_\eta(t), \zeta_\eta(t)\big) \in U\times J(\gamma u_\eta(t)), \\ \langle Au_\eta(t), w - u_\eta(t)\rangle_{V^*\times V} + \big\langle \gamma^*\zeta_\eta(t), w - u_\eta(t)\big\rangle_{V^*\times V} \\ \geq \langle f_\eta(t), w - u_\eta(t)\rangle_{V^*\times V} \text{ for all } w\in U, \text{ for all } t\in\mathbb{R}_+. \end{array}\right. \tag{89}$$

Using (75)-(76), we obtain for all $w \in U$

$$\begin{aligned} \big\langle \gamma^*\zeta_\eta(t), w - u_\eta(t)\big\rangle_{V^*\times V} &= \big(\zeta_\eta(t), \gamma w - \gamma u_\eta(t)\big)_{\mathbf{Y}} \\ &= \int_{\Gamma_3} \mu(|\gamma u_{\eta\tau}(t)|)\, \varrho_\eta(t)\cdot(\gamma w_\tau - \gamma u_{\eta\tau}(t))\, da, \end{aligned}$$

with $\varrho_\eta(t) \in \partial j_\tau(\gamma u_{\eta\tau}(t))$. Therefore, it follows from (74) that

$$\big\langle \gamma^*\zeta_\eta(t), w - u_\eta(t)\big\rangle_{V^*\times V} \leq \int_{\Gamma_3} \mu(|\gamma u_{\eta\tau}(t)|)\, h_\tau^\circ(\gamma u_{\eta\tau}(t);\gamma w_\tau - \gamma u_{\eta\tau}(t))da,$$

which, together with (89), implies that Problem (88) has a solution. Let t_1, $t_2 \in \mathbb{R}_+$. Using (88) with $(u_\eta(t), w) = (u_\eta(t_1), u_\eta(t_2))$ and $(u_\eta(t), w) = (u_\eta(t_2), u_\eta(t_1))$, adding the resulting inequalities and taking into account (55) and (72), we get

$$\left\{\begin{array}{c} m_{\mathcal{A}}\|u_\eta(t_1) - u_\eta(t_2)\|_V^2 \leq \mathcal{R}(u_\eta(t_1), u_\eta(t_2) - u_\eta(t_1)) \\ +\mathcal{R}(u_\eta(t_2), u_\eta(t_1) - u_\eta(t_2)) + \langle f_\eta(t_1) - f_\eta(t_2), u_\eta(t_1) - u_\eta(t_2)\rangle_{V^*\times V}, \end{array}\right.$$

which with (56) and (64) gives

$$\|u_\eta(t_1) - u_\eta(t_2)\|_V \leq \frac{1}{m_{\mathcal{A}} - c_0^2(l_\mu L_\tau + l_\tau L_\mu)} \|f_\eta(t_1) - f_\eta(t_2)\|_{V^*}.$$

Therefore, $u_\eta \in C(\mathbb{R}_+; V)$. Let $t \in \mathbb{R}_+$, let u_1 and u_2 be two solutions to problem (88). Using (88) with $(u_\eta(t), w) = (u_1(t), u_2(t))$ and $(u_\eta(t), w) = (u_2(t), u_1(t))$, adding the resulting inequalities and taking into account (55) and (72), we get

$$m_{\mathcal{A}} \|u_1(t) - u_2(t)\|_V^2 \leq \mathcal{R}(u_1(t), u_2(t) - u_1(t)) + \mathcal{R}(u_2(t), u_1(t) - u_2(t)),$$

which with (56) and (64) gives

$$\|u_1(t) - u_2(t)\|_V \leq 0 \text{ for all } t \in \mathbb{R}_+.$$

It follows that $u_1 = u_2$. ■

Step 4. In the rest of this paper, the same letter c will be used to denote different positive constants which do not depend on $t \in \mathbb{R}_+$. Let $u_\eta \in C(\mathbb{R}_+; V)$ be the unique solution to Problem (88). Let $\varphi_\eta \in C(\mathbb{R}_+; W_{el})$, $\psi_\eta \in C(\mathbb{R}_+; W_{ma})$, $\sigma_\eta \in C(\mathbb{R}_+; \mathcal{Q})$, $D_\eta \in C(\mathbb{R}_+; \mathbf{H})$ and $B_\eta \in C(\mathbb{R}_+; \mathbf{H})$ be defined, for all $t \in \mathbb{R}_+$, by

$$\left(\varphi_\eta(t), \psi_\eta(t)\right) = \mathcal{K}^{-1}\mathcal{N}u_\eta(t) + \mathcal{K}^{-1}b_\eta(t), \tag{90}$$

$$\sigma_\eta(t) = \mathcal{A}\varepsilon(u_\eta(t)) + \mathcal{E}^*\nabla\varphi_\eta(t) + \mathcal{M}^*\nabla\psi_\eta(t) + I_1\eta(t), \tag{91}$$

$$D_\eta(t) = \mathcal{E}\varepsilon(u_\eta(t)) - \mathcal{C}\nabla\varphi_\eta(t) - \mathcal{S}\nabla\psi_\eta(t) + I_2\eta(t), \tag{92}$$

$$B_\eta(t) = \mathcal{M}\varepsilon(u_\eta(t)) - \mathcal{S}\nabla\varphi_\eta(t) - \mathcal{Z}\nabla\psi_\eta(t) + I_3\eta(t). \tag{93}$$

Using (68) and (90), we obtain

$$\left\{ \begin{array}{l} (\mathcal{F}(\mathcal{K}^{-1}\mathcal{N}u_\eta(t)) + \mathcal{F}(\mathcal{K}^{-1}b_\eta(t)), w)_V = \left(\mathcal{E}^*\nabla\varphi_\eta(t), \varepsilon(w)\right)_{\mathcal{Q}} \\ \quad + \left(\mathcal{M}^*\nabla\psi_\eta(t), \varepsilon(w)\right)_{\mathcal{Q}} \text{ for all } w \in V, \text{ for all } t \in \mathbb{R}_+. \end{array} \right. \tag{94}$$

Taking into account (55), (70), (79), (88) and (94), we infer that

$$\left\{ \begin{array}{l} (\mathcal{A}\varepsilon(u_\eta(t)), \varepsilon(w - u_\eta(t)))_{\mathcal{Q}} + \left(\mathcal{E}^*\nabla\varphi_\eta(t), \varepsilon(w - u_\eta(t))\right)_{\mathcal{Q}} \\ \quad + \left(\mathcal{M}^*\nabla\psi_\eta(t), \varepsilon(w - u_\eta(t))\right)_{\mathcal{Q}} \\ \quad + (I_1\eta(t), \varepsilon(w - u_\eta(t)))_{\mathcal{Q}} + \mathcal{R}(u_\eta(t), w - u_\eta(t)) \\ \quad \geq (f(t), w - u_\eta(t))_V \text{ for all } w \in U, \text{ for all } t \in \mathbb{R}_+. \end{array} \right. \tag{95}$$

On the other hand, using (67), (69), (78) and (90), we find that

$$\left\{\begin{array}{c}\left(\mathcal{C}\nabla\varphi_{\eta}(t),\nabla\phi\right)_{\mathbf{H}}+\left(\mathcal{Z}\nabla\psi_{\eta}(t),\nabla\vartheta\right)_{\mathbf{H}}+\left(\mathcal{S}\nabla\psi_{\eta}(t),\nabla\phi\right)_{\mathbf{H}}\\ +\left(\mathcal{S}\nabla\varphi_{\eta}(t),\nabla\vartheta\right)_{\mathbf{H}}=\left(\mathcal{E}\varepsilon(u_{\eta}(t)),\nabla\phi\right)_{\mathbf{H}}\\ +\left(\mathcal{M}\varepsilon(u_{\eta}(t)),\nabla\vartheta\right)_{\mathbf{H}}+\left(q(t),\phi\right)_{W_{el}}+\left(p(t),\vartheta\right)_{W_{ma}}\\ +\left(I_2\eta(t),\nabla\phi\right)_{\mathbf{H}}+\left(I_3\eta(t),\nabla\vartheta\right)_{\mathbf{H}}\\ \text{for all } (\phi,\vartheta)\in W,\text{ for all } t\in\mathbb{R}_+.\end{array}\right. \tag{96}$$

Define the operator $\Phi : C(\mathbb{R}_+;\mathcal{X}) \to C(\mathbb{R}_+;\mathcal{X})$ by

$$\Phi\eta(t)=\left(\sigma_{\eta}(t),D_{\eta}(t),B_{\eta}(t),u_{\eta}(t),\varphi_{\eta}(t),\psi_{\eta}(t)\right) \tag{97}$$

for all $t\in\mathbb{R}_+$, for all $\eta\in C(\mathbb{R}_+;\mathcal{X})$. We have the following result.

Lemma 12 *The operator Φ has a unique fixed point.*

Proof. Let η^1 and $\eta^2 \in C(\mathbb{R}_+;\mathcal{X})$. Let u_η be the corresponding solution to Problem (88) for $\eta\in\{\eta^1,\eta^2\}$. Let φ_η, ψ_η, σ_η, D_η and B_η be defined by (90)-(93) for $\eta\in\{\eta^1,\eta^2\}$. Using

$$\left\{\begin{array}{c}\left(u_{\eta}(t),\varphi_{\eta}(t),\psi_{\eta}(t),w\right)\\ =\left(u_{\eta^1}(t),\varphi_{\eta^1}(t),\psi_{\eta^1}(t),u_{\eta^2}(t)\right)\end{array}\right.$$

and

$$\left\{\begin{array}{c}\left(u_{\eta}(t),\varphi_{\eta}(t),\psi_{\eta}(t),w\right)\\ =\left(u_{\eta^2}(t),\varphi_{\eta^2}(t),\psi_{\eta^2}(t),u_{\eta^1}(t)\right)\end{array}\right.$$

in (95), consectively. Then, adding the resulting inequalities, we infer that

$$\left\{\begin{array}{c}\left(\mathcal{A}\varepsilon(u_{\eta^1}(t))-\mathcal{A}\varepsilon(u_{\eta^2}(t)),\varepsilon(u_{\eta^1}(t)-u_{\eta^2}(t))\right)_{\mathcal{Q}}\\ +\left(\mathcal{E}^*\nabla\varphi_{\eta^1}(t)-\mathcal{E}^*\nabla\varphi_{\eta^2}(t),\varepsilon(u_{\eta^1}(t)-u_{\eta^2}(t))\right)_{\mathcal{Q}}\\ +\left(\mathcal{M}^*\nabla\psi_{\eta^1}(t)-\mathcal{M}^*\nabla\psi_{\eta^2}(t),\varepsilon(u_{\eta^1}(t)-u_{\eta^2}(t))\right)_{\mathcal{Q}}\\ \leq\left(I_1\eta^2(t)-I_1\eta^1(t),\varepsilon(u_{\eta^1}(t)-u_{\eta^2}(t))\right)_{\mathcal{Q}}\\ +\mathcal{R}\left(u_{\eta^1}(t),u_{\eta^2}(t)-u_{\eta^1}(t)\right)\\ +\mathcal{R}\left(u_{\eta^2}(t),u_{\eta^1}(t)-u_{\eta^2}(t)\right)\text{ for all } t\in\mathbb{R}_+.\end{array}\right. \tag{98}$$

On the other hand, we set

$$\left\{ \begin{array}{c} \left(u_{\eta}(t), \varphi_{\eta}(t), \psi_{\eta}(t), \phi, \vartheta\right) = \\ \left(u_{\eta^1}(t), \varphi_{\eta^1}(t), \psi_{\eta^1}(t), \varphi_{\eta^1}(t) - \varphi_{\eta^2}(t), \psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right) \end{array} \right.$$

and

$$\left\{ \begin{array}{c} \left(u_{\eta}(t), \varphi_{\eta}(t), \psi_{\eta}(t), \phi, \vartheta\right) \\ = \left(u_{\eta^2}(t), \varphi_{\eta^2}(t), \psi_{\eta^2}(t), \varphi_{\eta^2}(t) - \varphi_{\eta^1}(t), \psi_{\eta^2}(t) - \psi_{\eta^1}(t)\right) \end{array} \right.$$

in (96), and add the resulting equations to obtain

$$\left\{ \begin{array}{c} \left(\mathcal{C}\nabla\varphi_{\eta^1}(t) - \mathcal{C}\nabla\varphi_{\eta^2}(t), \nabla\left(\varphi_{\eta^1}(t) - \varphi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \\ + \left(\mathcal{Z}\nabla\psi_{\eta^1}(t) - \mathcal{Z}\nabla\psi_{\eta^2}(t), \nabla\left(\psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \\ +2\left(\mathcal{S}\nabla\left(\varphi_{\eta^1}(t) - \varphi_{\eta^2}(t)\right), \nabla\left(\psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \\ = \left(\mathcal{E}\varepsilon(u_{\eta^1}(t)) - \mathcal{E}\varepsilon(u_{\eta^2}(t)), \nabla\left(\varphi_{\eta^1}(t) - \varphi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \\ + \left(\mathcal{M}\varepsilon(u_{\eta^1}(t)) - \mathcal{M}\varepsilon(u_{\eta^2}(t)), \nabla\left(\psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \\ + \left(I\eta^1(t) - I_2\eta^2(t), \nabla\left(\varphi_{\eta^1}(t) - \varphi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \\ + \left(I_3\eta^1(t) - I_3\eta^2(t), \nabla\left(\psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right)\right)_{\mathbf{H}} \text{ for all } t \in \mathbb{R}_+. \end{array} \right. \tag{99}$$

Adding now (98)-(99) and using (21), (22), (23), (33), (34), (50), (51), (52), (80), (81), and (82), we get

$$\left\{ \begin{array}{c} m_{\mathcal{A}}\left\|u_{\eta^1}(t) - u_{\eta^2}(t)\right\|_V^2 + m\left\|\varphi_{\eta^1}(t) - \varphi_{\eta^2}(t)\right\|_{W_{el}}^2 \\ +m\left\|\psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right\|_{W_{ma}}^2 \leq c\int_0^t \left\|\eta^1(s) - \eta^2(s)\right\|_{\mathcal{X}} ds \times \left\|u_{\eta^1}(t) - u_{\eta^2}(t)\right\|_V \\ +c\int_0^t \left\|\eta^1(s) - \eta^2(s)\right\|_{\mathcal{X}} ds \times \left\|\varphi_{\eta^1}(t) - \varphi_{\eta^2}(t)\right\|_{W_{el}} \\ +c\int_0^t \left\|\eta^1(s) - \eta^2(s)\right\|_{\mathcal{X}} ds \times \left\|\psi_{\eta^1}(t) - \psi_{\eta^2}(t)\right\|_{W_{ma}} \\ +\mathcal{R}\left(u_{\eta^1}(t), u_{\eta^2}(t) - u_{\eta^1}(t)\right) + \mathcal{R}\left(u_{\eta^2}(t), u_{\eta^1}(t) - u_{\eta^2}(t)\right), \end{array} \right.$$

which with (91)-(93), (56) and (64) gives

$$\begin{cases} \|\sigma_{\eta^1}(t)-\sigma_{\eta^2}(t)\|_{\mathcal{Q}}^2+\|D_{\eta^1}(t)-D_{\eta^2}(t)\|_{\mathbf{H}}^2 \\ +\|B_{\eta^1}(t)-B_{\eta^2}(t)\|_{\mathbf{H}}^2+\|u_{\eta^1}(t)-u_{\eta^2}(t)\|_V^2 \\ +\left\|\varphi_{\eta^1}(t)-\varphi_{\eta^2}(t)\right\|_{W_{el}}^2+\left\|\psi_{\eta^1}(t)-\psi_{\eta^2}(t)\right\|_{W_{ma}}^2 \\ \leq c\int_0^t\|\eta^1(s)-\eta^2(s)\|_{\mathcal{X}}^2\,ds. \end{cases} \tag{100}$$

It follows from (97) that

$$\begin{cases} \|\Phi\eta^1(t)-\Phi\eta^2(t)\|_{\mathcal{X}}^2=\|\sigma_{\eta^1}(t)-\sigma_{\eta^2}(t)\|_{\mathcal{Q}}^2+\|D_{\eta^1}(t)-D_{\eta^2}(t)\|_{\mathbf{H}}^2 \\ +\|B_{\eta^1}(t)-B_{\eta^2}(t)\|_{\mathbf{H}}^2+\|u_{\eta^1}(t)-u_{\eta^2}(t)\|_V^2+\left\|\varphi_{\eta^1}(t)-\varphi_{\eta^2}(t)\right\|_{W_{el}}^2 \\ +\left\|\psi_{\eta^1}(t)-\psi_{\eta^2}(t)\right\|_{W_{ma}}^2 \text{ for all } t\in\mathbb{R}_+ \end{cases}$$

which, due to (100), implies that

$$\left\|\Phi\eta^1(t)-\Phi\eta^2(t)\right\|_{\mathcal{X}}^2\leq c\int_0^t\left\|\eta^1(s)-\eta^2(s)\right\|_{\mathcal{X}}^2ds \text{ for all } t\in\mathbb{R}_+. \tag{101}$$

Let $\lambda>c$ be a positive constant, where c is the constant obtained in (101). Then, we have

$$\begin{aligned} \left\|\Phi\eta^1(t)-\Phi\eta^2(t)\right\|_{\mathcal{X}}^2 &\leq c\int_0^t\left\|\eta^1(s)-\eta^2(s)\right\|_{\mathcal{X}}^2ds \\ &\leq c\int_0^t\sup_{t\in\mathbb{R}_+}\left(e^{\lambda(s-t)}\left\|\eta^1(t)-\eta^2(t)\right\|_{\mathcal{X}}^2\right)ds \\ &\leq \left(c\int_0^t e^{\lambda s}ds\right)\sup_{t\in\mathbb{R}_+}\left(e^{-\lambda t}\left\|\eta^1(t)-\eta^2(t)\right\|_{\mathcal{X}}^2\right) \\ &\leq \frac{c\left(e^{\lambda t}-1\right)}{\lambda}\sup_{t\in\mathbb{R}_+}\left(e^{-\lambda t}\left\|\eta^1(t)-\eta^2(t)\right\|_{\mathcal{X}}^2\right). \end{aligned}$$

Therefore, we deduce that

$$e^{-\lambda t}\left\|\Phi\eta^1(t)-\Phi\eta^2(t)\right\|_{\mathcal{X}}^2\leq\frac{c}{\lambda}\sup_{t\in\mathbb{R}_+}\left(e^{-\lambda t}\left\|\eta^1(t)-\eta^2(t)\right\|_{\mathcal{X}}^2\right),$$

which gives

$$\sup_{t\in\mathbb{R}_+}\left(e^{-\lambda t}\left\|\Phi\eta^1(t)-\Phi\eta^2(t)\right\|^2_{\mathcal{X}}\right)\le\frac{c}{\lambda}\sup_{t\in\mathbb{R}_+}\left(e^{-\lambda t}\left\|\eta^1(t)-\eta^2(t)\right\|^2_{\mathcal{X}}\right).$$

Thus, we conclude that the operator Φ is a contraction on the real Banach space $C(\mathbb{R}_+;\mathcal{X})$ endowed with the Bielecki norm. This implies that Φ has a unique fixed point η^* satisfying

$$\eta^*(t)=\left(\sigma_{\eta^*}(t),D_{\eta^*}(t),B_{\eta^*}(t),u_{\eta^*}(t),\varphi_{\eta^*}(t),\psi_{\eta^*}(t)\right)$$

for all $t\in\mathbb{R}_+$. ■

Step 5. We have all the ingredients to prove Theorem 5. Let $\eta^*\in C(\mathbb{R}_+;\mathcal{X})$ be the fixed point of Φ defined by (97), let $u=u_{\eta^*}$ be the unique solution to Problem (88) for $\eta=\eta^*$. Let $\varphi=\varphi_{\eta^*}$, $\psi=\psi_{\eta^*}$, $\sigma=\sigma_{\eta^*}$, $D=D_{\eta^*}$ and $B=B_{\eta^*}$ be defined by (90)-(93). Taking into account (95), (96), (80), (81) and (82), we conclude that $(\sigma,D,B,u,\varphi,\psi)$ is a solution to Problem (57)-(63) such that (u,φ,ψ) satisfies (65) and (σ,D,B) satisfies

$$\left\{\begin{array}{c}(i)\ \ \sigma\in C(\mathbb{R}_+;\mathcal{Q}),\ \ (ii)\ \ D\in C(\mathbb{R}_+;\mathbf{H}),\\ (iii)\ \ B\in C(\mathbb{R}_+;\mathbf{H}).\end{array}\right.\tag{102}$$

Taking $w=u(t)\pm z$ with $z\in[D(\Omega)]^d$, $\phi\in D(\Omega)$, $\vartheta\in D(\Omega)$ in (60)-(63) and using (42)-(44), we get

$$\left\{\begin{array}{c}Div\sigma(t)=-f_0(t)\ \text{ in }\Omega,\\ \operatorname{div}D(t)=q_0(t)\ \text{ in }\Omega,\\ \operatorname{div}B(t)=p_0(t)\ \text{ in }\Omega\text{ for all }t\in\mathbb{R}_+.\end{array}\right.\tag{103}$$

Combining (103) with (39)-(i), (40)-(i), (41)-(i) and (102), we conclude that (σ,D,B) satisfies (66). To establish the uniqueness of the solution, we suppose that (24)-(41) and (64) are fulfilled. Let $(\sigma_1,D_1,B_1,u_1,\varphi_1,\psi_1)$ and $(\sigma_2,D_2,B_2,u_2,\varphi_2,\psi_2)$ be two solutions to Problem (57)-(63). Using arguments similar to those used in Lemma 12, we get

$$\begin{cases} \|\sigma_1(t)-\sigma_2(t)\|^2_{\mathcal{Q}}+\|D_1(t)-D_2(t)\|^2_{\mathbf{H}} \\ +\|B_1(t)-B_2(t)\|^2_{\mathbf{H}}+\|u_1(t)-u_2(t)\|^2_V \\ +\|\varphi_1(t)-\varphi_2(t)\|^2_{W_{el}}+\|\psi_1(t)-\psi_2(t)\|^2_{W_{ma}} \\ \\ \leq c\int_0^t\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathcal{Q}}\,ds+c\int_0^t\|D_1(s)-D_2(s)\|^2_{\mathbf{H}}\,ds \\ +c\int_0^t\|B_1(s)-B_2(s)\|^2_{\mathbf{H}}\,ds+c\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds \\ \\ +c\int_0^t\|\varphi_1(s)-\varphi_2(s)\|^2_{W_{el}}\,ds+c\int_0^t\|\psi_1(s)-\psi_2(s)\|^2_{W_{ma}}\,ds \end{cases}$$

for all $t\in\mathbb{R}_+$. Hence, using Gronwall's lemma in the last inequality, we deduce that $\sigma_1=\sigma_2$, $D_1=D_2$, $B_1=B_2$, $u_1=u_2$, $\varphi_1=\varphi_2$ and $\psi_1=\psi_2$. This completes the proof of Theorem 5.

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