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# $D$-FINITE POWER SERIES AND $P$-RECURSIVE SEQUENCES IN A SIMPLICIAL CONE

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**Abstract:** We study the relationship between multidimensional P-recursive sequences supported on simplicial lattice cones and D-finite generating functions. For a simplicial rational cone generated by linearly independent integer vectors, we introduce difference operators acting on formal power series and define a notion of D-finiteness. We prove that a function on the lattice points of such a cone is P-recursive if and only if its generating series satisfies a finite system of linear difference equations with polynomial coefficients. The proposed framework extends the classical correspondence between P-recursive sequences and D-finite power series to simplicial cones and provides a convenient operator formalism for studying multidimensional difference equations in conic domains.

**Keywords:** D-finite power series; P-recursive sequences; simplicial lattice cones; multidimensional difference equations; linear recurrences; generating functions.

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# 1 Introduction

The correspondence between linear recurrences and differential equations is a classical theme in enumerative combinatorics and symbolic computation ([4, 17, 18, 19, 15]). In the one–dimensional case, P-recursive sequences are known to be equivalent to D-finite generating functions [13, 9]. This duality was subsequently extended to multivariate power series and lattice-supported functions, with important contributions by Lipshitz, Stanley, and Bousquet-Mélou and Petkovšek [13, 9, 2].

From a geometric viewpoint, multidimensional recurrences naturally arise on affine semigroups generated by finite sets of integer vectors, while differential operators act in directions belonging to the dual space. In the simplicial setting considered here, this relationship admits a particularly transparent interpretation: shifts of the coefficients are performed along the generators $\alpha_j$ of the cone $K$, whereas differential operators correspond to the generators $j$ of the dual cone $K^*$. Thus, the analytic structure of D-finite generating functions is intrinsically linked to the geometry of the pair $(K, K^*)$.

This perspective places the present work within the general framework of affine semigroup methods in combinatorics and commutative algebra [23, 24], and provides a natural geometric interpretation of multidimensional P-recursiveness on conic domains.

# 2 Difference operators and generating series on simplicial cones

Let $\mathbb{Z}, \mathbb{R}$ denote the integer and real numbers, $\mathbb{Z}^n = \mathbb{Z} \times \ldots \times \mathbb{Z}$, $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$, then $\mathbb{Z}_{\geqslant}, \mathbb{R}_{\geqslant}$, $\mathbb{Z}^n_{\geqslant}$, $\mathbb{R}^n_{\geqslant}$ denote corresponding subsets with non-negative coordinates. Let $A = \{\alpha^1, \alpha^2, \ldots, \alpha^N\} \subset \mathbb{Z}^n$ be a set of $N$ vector-columns $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_n)^\top$, $j = 1, \ldots, N$, with integer coordinates.

Let $K$ be a rational polyhedral cone

$$K = \{\lambda \colon \lambda = \lambda_1 \alpha^1 + \cdots + \lambda_N \alpha^N, \lambda_1, \ldots, \lambda_N \in \mathbb{R}_{\geqslant}\} \subset \mathbb{R}^n.$$

A cone $K$ is called *simplicial* if its generating vectors $\{\alpha^1, \alpha^2, \ldots, \alpha^N\}$ are linearly independent. A cone $K$ is called *pointed* if $\lambda, -\lambda \in K$ if and only if $\lambda = 0$. In this paper we consider only simplicial lattice cones? Nope.

Let $\pi_j \colon K \longrightarrow K$, $j = 1, \ldots, N$, be the projection operator

$$\pi_j \alpha^i = \begin{cases} 0, & i = j; \\ \alpha^i, & i \neq j \end{cases}$$

for $1 \leqslant i \leqslant N$. For example, if $\lambda = x_1 \alpha^1 + \ldots + x_N \alpha^N$, then $\pi_j \lambda = \lambda - x_j \alpha^j$, $j = 1, \ldots, N$. Applying $\pi_j$ to cone $K$ yields its $(N-1)$-facet $\pi_j K$ spanned by vectors from $\Delta \setminus \{\alpha^j\}$, see [12]. We also define subsets of $K$ as

$$K_{j,k} = k\alpha_j + \pi_j K, j = 1, \ldots, N, k \in \mathbb{Z}_{\geqslant}.$$

The corresponding monioid of integer points

$$K_{\mathbb{Z}} = K \cap \mathbb{Z}^n$$

coincides with its submonoid

$$M = \{\lambda : \lambda = \lambda_1\alpha^1 + \cdots + \lambda_N\alpha^N, \lambda_1, \dots, \lambda_N \in \mathbb{Z}_{\geqslant}\} \subset \mathbb{Z}^n.$$

if cone $K$ is unimodular. Throughout the paper we consider functions supported on the affine submoniod $M$. The cone $K$ is used only as the real hull of $M$ ([13, 21]). Affine semigroups and their associated semigroup algebras provide a natural framework for functions supported on rational cones [23].

We define $f, g\colon \mathbb{Z}^n \longrightarrow \mathbb{C}$, let $f(\lambda)$ vanish if $\lambda \notin M$ and $B$ is a finite subset of $M$. The difference equation with polynomial coefficients $p_\beta(\lambda) \in \mathbb{C}_{K_\mathbb{Z}}[\lambda], \beta \in B$, is

$$\sum_{\beta\in B} p_\beta(\lambda) f(\lambda+\beta) = g(\lambda), \quad \lambda \in M. \tag{1}$$ cauchy1

Let $m \in M$, then

$$X_m = M \setminus (m + M)$$

is a set of initial data such that for a given function $\varphi\colon X_m \longrightarrow \mathbb{C}$ we get

$$f(\lambda) = \varphi(\lambda) \tag{2}$$ cauchy2

for $\lambda \in X_m$. The problem (1)–(2) is said to be the Cauchy problem for multidimensional difference equation with polynomial coefficients in $M$. The question about correctness and solvability of a linear difference equation with constant coefficients was considered in [1, 7, 5, 11].

We consider a ring $\mathbb{C}_M[[z]]$ of Laurent series in $M$ as

$$F(z) = \sum_{\lambda\in M} f(\lambda) z^\lambda. \tag{3}$$ gf

Since vectors $A = \{\alpha^1, \alpha^2, \dots, \alpha^N\}$ are linearly independent, they form a basis of $K$. Let $\{a^1, \dots, a^N\}$ be a biorthogonal (dual) basis of the dual cone $K^*$, which means that $(\alpha^i, a^j) = \delta_{ij}$, where $\delta_{ij}, i, j = 1, \dots, N$, is the Kronecker symbol. We define a linear function $\mu_j(\lambda) = \langle A^{-1}\lambda, e^j\rangle = \langle \lambda, a^j\rangle$, that represents $j$th coordinate of $\lambda$ in the basis $\alpha_1, \dots, \alpha_N$ for $j = 1, \dots, N$. We also use a linear difference operator

$$\mu_j\left(z\frac{\partial}{\partial z}\right) = \left\langle z\frac{\partial}{\partial z}, a^j\right\rangle = a_1^j z_1 \frac{\partial}{\partial z_1} + \cdots + a_n^j z_n \frac{\partial}{\partial z_n}.$$

**Definition 1.** *For each $j = 1, \dots, N$, the differential operator $D_j\colon M \longrightarrow M$ is defined as the composition of an Euler-type differential operator with a monomial shift by*

$$D_j := z^{-\alpha^j}\left\langle z\frac{\partial}{\partial z}, a^j\right\rangle.$$

In what follows, the vectors $j$ naturally determine the differential directions acting on generating functions, while the vectors $\alpha_j$ specify the shifts on the affine semigroup $M$ generated by $A = \{\alpha_1, \dots, \alpha_n\}$. Thus, the algebraic operators in our approach are intrinsically related to the geometry of the primal–dual pair of cones $(K, K^*)$.

The operator $D_j$ is an analog of the differential operator $\frac{\partial}{\partial x_j}$ in $\mathbb{C}_M[[z]]$, for example:

$$D_j z^{\lambda} = \mu_j(\lambda) z^{\lambda - \alpha^j}, \tag{4} \quad \boxed{\texttt{Dj}}$$

and was considered in [5, 8].

We define $D_j^s = \underbrace{D_j \circ \ldots \circ D_j}_{s \text{ times}}$, then

$$D_j^s z^{\lambda} := \mu_j^{\underline{s}}(\lambda) z^{\lambda - s\alpha^j}, \tag{5} \quad \boxed{\texttt{Djs}}$$

where $\mu_j^{\underline{s}}(\lambda) = \mu_j(\lambda)(\mu_j(\lambda) - 1) \cdots (\mu_j(\lambda) - s + 1)$ is a falling power. For example,

$$\begin{aligned} D_1 z^{3\alpha^1 + 2\alpha^2} &= 3 z^{2\alpha^1 + 2\alpha^2}, \\ D_1^2 z^{3\alpha^1 + 2\alpha^2} &= 3 \cdot 2 z^{\alpha^1 + 2\alpha^2}, \\ D_1^3 z^{3\alpha^1 + 2\alpha^2} &= 3 \cdot 2 \cdot 1 z^{2\alpha^2}, \\ D_1^4 z^{3\alpha^1 + 2\alpha^2} &= 0. \end{aligned}$$

**Lemma 1** (The Leibnitz rule)**.** *For $\lambda_1, \lambda_2 \in M, j = 1, \ldots, N$ and $F, G \in \mathbb{C}_M[[z]]$ we get*

$$D_j(z^{\lambda_1} z^{\lambda_2}) = D_j(z^{\lambda_1}) z^{\lambda_2} + z^{\lambda_1} D_j(z^{\lambda_2}), \tag{6} \quad \boxed{\texttt{simpleprod}}$$

$$D_j(F \cdot G) = D_j F \cdot G + F \cdot D_j G. \tag{7} \quad \boxed{\texttt{coneprod}}$$

*Proof.* Let us prove formula (6). Using (4) yields

$$\begin{aligned} D_j(z^{\lambda_1} z^{\lambda_2}) &= D_j(z^{\lambda_1 + \lambda_2}) = \\ &\mu_j(\lambda_1 + \lambda_2) z^{\lambda_1 + \lambda_2 - \alpha^j} = \mu_j(\lambda_1) z^{\lambda_1 + \lambda_2 - \alpha^j} + \mu_j(\lambda_2) z^{\lambda_1 + \lambda_2 - \alpha^j} \\ &= \mu_j(\lambda_1) z^{\lambda_1 - \alpha^j} z^{\lambda_2} + z^{\lambda_1} \mu_j(\lambda_2) z^{\lambda_2 - \alpha^j} = D_j(z^{\lambda_1}) z^{\lambda_2} + z^{\lambda_1} D_j(z^{\lambda_2}). \end{aligned}$$

The prove of (7) is identical. □

Let $\delta_j \colon M \longrightarrow M, j = 1, \ldots, N$, be a shift operator defined by

$$\delta_j \colon \lambda \longmapsto \lambda + \alpha^j.$$

Some properties of the shift operator were studied in [12].

**Lemma 2.** *Let $s \in \mathbb{Z}_{\geqslant}$. We can apply $D_j^s$ to the power series $F(z)$ as*

$$D_j^s F(z) = \sum_{\lambda \in M} \delta_j^s \left[ \mu_j^{\underline{s}}(\lambda) f(\lambda) \right] z^{\lambda}.$$

*Proof.* Using (5) and linearity of $D_j$ yields

$$D_j^s F(z) = \sum_{\lambda \in M} f(\lambda) D_j^s z^{\lambda} = \sum_{\lambda \in M} \mu_j^{\underline{s}}(\lambda) f(\lambda) z^{\lambda - s\alpha_j}.$$

Using a shift $\lambda \longmapsto \lambda + s\alpha_j$ and a fact that $f(\lambda) = 0$ for $\lambda \notin M$, we rewrite this sum as

$$\sum_{\lambda + s\alpha_j \in M} \mu_j^{\underline{s}}(\lambda + s\alpha_j) f(\lambda + s\alpha_j) z^{\lambda} = \sum_{\lambda \in M} \delta_j^s \big[\mu_j^{\underline{s}}(\lambda) f(\lambda)\big] z^{\lambda},$$

which completes the proof. □

⟨zalpha⟩ **Lemma 3.** *For $\beta \in M$*

$$z^{\beta} F(z) = \sum_{\lambda \in M} \delta^{-\beta} \big[f(\lambda)\big] z^{\lambda}.$$

*Proof.* Since $f(\lambda) = 0$ for $\lambda \notin M$, we get

$$z^{\beta} F(z) = z^{\beta} \sum_{\lambda \in M} f(\lambda) z^{\lambda} = \sum_{\lambda \in M} f(\lambda) z^{\lambda + \beta} = \sum_{\lambda \in \beta + M} f(\lambda - \beta) z^{\lambda} =$$

$$= \sum_{\lambda \in \beta + M} \delta^{-\beta} f(\lambda) z^{\lambda} = \sum_{\lambda \in M} \delta^{-\beta} f(\lambda) z^{\lambda},$$

which completes the proof. □

For finite sets $V_j \subset M, j = 1, \ldots, n$, we denote a collection of Laurent polynomials (finite sums) as

$$Q_j^s(z) = \sum_{\beta \in V_j} c_{j,\beta}^s z^{\beta} \in \mathbb{C}_M[z], \quad j = 1, \ldots, n, \quad s = 0, \ldots, p,$$

then we consider

**Lemma 4.** *The product*

$$Q_j^s(z) F(z) = \sum_{\lambda \in M} Q_j^s(\delta^{-1}) \big[f(\lambda)\big] z^{\lambda}.$$

*Proof.* Routine manipulations as in Lemma 3 yields the proof. □

## 3 $P$-recursive sequences and $D$-finite series

Since the definition is recursive, we need to mention a one-dimensional case.

We define the partial order $\underset{K}{\leqslant}$ on $\mathbb{Z}^n$: for any $\lambda_1, \lambda_2 \in \mathbb{Z}^n$ we set

$$\lambda_1 \underset{K}{\leqslant} \lambda_2 \iff \lambda_2 \in \lambda_1 + K.$$

**Definition 2.** *Let $m \in M, B_m = \{\lambda \in K : 0 \underset{K}{\leqslant} \lambda \underset{K}{\leqslant} m\} \cap M$ be a finite subset, $p_{\beta,j}(\xi) \in \mathbb{C}_M[\xi], \beta \in B_m, j = 1, \ldots, N$, be polynomials in one variable not all identically equal to zero. A multidimensional sequence $f \colon M \longrightarrow \mathbb{C}$ is called $P$-recursive if it satisfies the system of $n$ difference equations for $\lambda \in M$:*

$$\sum_{\beta \in B_m} p_{\beta,j}(\mu_j(\lambda)) f(\lambda + \beta) = 0, \quad j = 1, \ldots, n, \tag{8}$$ ?precurs?

*and its restrictions $f\big|_{K_{j,k} \cap M}$ are also $P$-recursive for $j = 1, \ldots, N, k = 0, \ldots, \mu_j(m)$.*

**Remark 1.** *Generating series of $f(\lambda)$ supported on the sets $K_{j,k} \cap M$ are called sections and were considered in* [10, 12].

**Lemma 5.** *Let $\alpha_1, \dots, \alpha_n$ be linearly independent vectors in $\mathbb{R}^n$ and let $\mu_1, \dots, \mu_n$ be the dual basis defined by*

$$\langle \mu_i, \alpha_j \rangle = \delta_{ij}.$$

*Denote by*

$$K = \operatorname{cone}(\alpha_1, \dots, \alpha_n)$$

*the simplicial cone generated by $\alpha_i$. Then its dual cone satisfies*

$$K^* = \operatorname{cone}(\mu_1, \dots, \mu_n).$$

*Proof.* For any $x = \sum_i s_i \mu_i$ with $s_i \geq 0$ and any $y = \sum_j t_j \alpha_j \in K$ we have

$$\langle x, y \rangle = \sum_i s_i t_i \geq 0,$$

hence $x \in K^*$. Conversely, for any $x \in K^*$ define $s_i = \langle x, \alpha_i \rangle \geq 0$; then

$$x = \sum_i s_i \mu_i.$$

□

Although this fact is elementary, we state it explicitly since it provides the geometric bridge between shifts in the primal cone and differential directions acting on generating functions.

We define operators $\Omega_j \colon \mathbb{C}_M[[z]] \longrightarrow \mathbb{C}_M[[z]], j = 1, \dots, N$, as

$$\Omega_j := \sum_{s=0}^{p} Q_j^s(z) D_j^s.$$

**Definition 3** ([5])**.** *The series $F(z)$, given by* (3)*, is called D-finite, if it satisfies a system of difference equations*

$$\Omega_j F(z) = 0, j = 1, \dots, n.$$

**Theorem 1.** *Function $f \colon M \longrightarrow \mathbb{C}$ is $P$-recursive if and only if the its generating series $F(z)$ is $D$-finite.*

*Proof.* Let $F(z)$ be a $D$-finite function. Since all series are formal, differentiation and summation commute, which yields

$$\begin{aligned}
\Omega_j F(z) &= \sum_{s=0}^{p} Q_j^s(z) D_j^s \sum_{\lambda \in M} f(\lambda) z^\lambda = \\
&= \sum_{s=0}^{p} Q_j^s(z) \sum_{\lambda \in M} \delta_j^s \left[ \mu_j^{\underline{s}}(\lambda) f(\lambda) \right] z^\lambda = \\
&= \sum_{\lambda \in M} \sum_{s=0}^{p} Q_j^s(\delta^{-1}) \delta_j^s \left[ \mu_j^{\underline{s}}(\lambda) f(\lambda) \right] z^\lambda
\end{aligned}$$

for $j = 1, \dots, n$. Since $\Omega_j F(z) = 0, j = 1, \dots, n$, we get a system of difference equations

$$\sum_{s=0}^{p} Q_j^s(\delta^{-1})\delta_j^s\left[\mu_j^{\underline{s}}(\lambda)f(\lambda)\right] = 0.$$

$\square$

## 4 Examples

**Example 1.** Applying $D_1$ to

$$F(z_1, z_2) = \frac{1}{1 - z_1^2 z_2 - z_1 z_2^2}$$

yields

$$D_1 F(z_1, z_2) = D_1 \sum_{(\lambda_1,\lambda_2)\in K} f(\lambda_1, \lambda_2) z_1^{\lambda_1} z_2^{\lambda_2} =$$
$$= \sum_{(\lambda_1,\lambda_2)\in K} \mu_1(\lambda_1, \lambda_2) f(\lambda_1, \lambda_2) z_1^{\lambda_1 - 2} z_2^{\lambda_2 - 1} =$$
$$= z_1^{-2} z_2^{-1} \sum_{(\lambda_1,\lambda_2)\in K} \frac{2\lambda_1 - \lambda_2}{3} f(\lambda_1, \lambda_2) z_1^{\lambda_1} z_2^{\lambda_2} =$$
$$= \frac{1}{3} z_1^{-2} z_2^{-1} \left( 2z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right) F(z_1, z_2) = \frac{1}{(1 - z_1^2 z_2 - z_1 z_2^2)^2}.$$

**Example 2.** Applying

$$\Omega_1 = c_1 z^{\alpha^2} + c_2 z^{2\alpha^1 + 3\alpha^2} D_1 + c_3 z^{3\alpha^1 + 4\alpha^2} D_1^5$$

to $F(\lambda)$ yields

$$\Omega_1 F(\lambda) = \left(c_1 z^{\alpha^2} + c_2 z^{2\alpha^1+3\alpha^2} D_1 + c_3 z^{3\alpha^1+4\alpha^2} D_1^5\right) \sum_{\lambda\in M} f(\lambda) z^\lambda =$$

$$= c_1 \sum_{\lambda\in M} f(\lambda) z^{\lambda+\alpha^2} + c_2 \sum_{\lambda\in M} \mu_1(\lambda) f(\lambda) z^{\lambda+\alpha^1+3\alpha^2}$$

$$+ c_3 \sum_{\lambda\in M} \mu_1^{\underline{5}}(\lambda) f(\lambda) z^{\lambda-2\alpha^1+4\alpha^2} =$$

$$= \sum_{\lambda\in M} \Big( c_1 f(\lambda-\alpha_2) + c_2\mu_1(\lambda-\alpha^1-3\alpha^2) f(\lambda-\alpha^1-3\alpha^2) +$$

$$+ c_3 \mu_1^{\underline{5}}(\lambda+2\alpha^1-4\alpha^2) f(\lambda+2\alpha^1-4\alpha^2)\Big) z^\lambda =$$

$$= \sum_{\lambda\in M} \Big( c_1 f(\lambda-\alpha^2) + c_2(\mu_1(\lambda)-1) f(\lambda-\alpha^1-3\alpha^2) +$$

$$+ c_3(\mu_1(\lambda)+2)^{\underline{5}} f(\lambda+2\alpha^1-4\alpha^2)\Big) z^\lambda.$$

Eventually, a shift $\lambda \longmapsto \lambda+\alpha^2$ yields a difference equation

$$c_1 f(\lambda) + c_2(\mu_1(\lambda)-1) f(\lambda-\alpha^1-2\alpha^2) + c_3(\mu_1(\lambda)+2)^{\underline{5}} f(\lambda+2\alpha^1-3\alpha^2) = 0.$$

for $\lambda \in K$.

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