

FINE SHAPE OF METRIZABLE SPACES AS A LEFT FRACTION LOCALIZATION

V.K. ZEMLYANOV 

Abstract: The strong shape category of compact metrizable spaces (compacta) is very well-studied; extending it to noncompact spaces, however, introduces computational complexity that makes it hard to work with. The fine shape category, as defined by Melikhov, seems to hold promise in terms of both applicability and simplicity: it is a different extension of compact strong shape to a generalized homotopy theory of metrizable spaces that is compatible with both Čech cohomology and Steenrod-Sitnikov homology, and its definition lends itself to straightforward proofs. Further research seems to be in order. One goal to have in mind is to show the fine shape category to be a homotopy category in Quillen's sense, which implies representation as a localization. But the strong shape of compacta was shown to be a left fraction localization in several ways; we extend the representation given by Cathey to fine shape, introducing the notion of FDR-embeddings to extend Cathey's SSDR-maps. In the process, we also introduce what we call the mapping cylinder of an approaching map; such a construction has been defined by Ferry and elaborated on by Mrozek in the compact case, yet it seems the direct extension on noncompact spaces is not possible. Thus we resort to a somewhat different definition.

Keywords: Metrizable topological spaces; shape theory; fine shape; localization of categories; calculus of fractions.

MSC: 18E35, 54C56, 54E35, 55P60

УДК: 515.124.32, 515.142.275

1 Introduction

Strong shape theory has been introduced as early as 1944 by Christie [6]. Many definitions of the same have been later given independently for compacta, that is, compact metrizable spaces [2, 10, 4, 5, 7], all equivalent on these, and only differing in representation; these mostly arose as modifications of Borsuk's shape theory after its publication [3]. The sole resulting category over compacta is well-established, having been extensively studied and used

The work was supported by the Advanced Doctoral Programme at Higher School of Economics.

to solve various problems. There is also a known strong shape category of all topological spaces [1, 11], which is quite complex in both definition and computations; thus it is much less researched or applied.

A different notion has been suggested by Melikhov [12], called fine shape. This name originally was used by Kodama and Ono [10] for compacta only, whereas [12] extends it to all metrizable spaces. This fine shape still coincides with strong shape on compacta. However, fine shape has a far simpler definition than the noncompact strong shape; thus, it should be easier to prove meaningful results for it. In particular, invariants of fine shape include (see [12, Corollary 1.3]) both Čech cohomology and Steenrod-Sitnikov homology (the latter being defined as the direct limit of Steenrod homology of compacta; see [12, footnote 3]). These two theories seem to be a good choice of universal homology and cohomology pair for metrizable spaces (a survey to that effect can be found in [18]). Thus this fine shape theory promises to be a working (generalized) homotopy theory for metrizable spaces complementing Čech cohomology and Steenrod-Sitnikov homology.

The construction of this fine shape that we shall use relies on the notion of an approaching map, originally going as far back as Quigley's work [15]. The version we use is somewhat more general than the one often used in the past for compacta:

Definition 1. *Let X be a closed subspace of a metrizable topological space M . We say that X is homotopy negligible in M to mean that there is a homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = id_M$ (the identity map of M) and $H(M \times (0, 1]) \subseteq M \setminus X$ (so the image of M under H_t does not cross X for $t > 0$).*

Now assume also Y is closed and homotopy negligible in N in the same way, and that $\phi: M \setminus X \rightarrow N \setminus Y$ is a continuous map. We say that ϕ is an $X - Y$ -approaching map to mean that for every sequence $\{m_i\} \subset M \setminus X$ converging to a point of X , the sequence $\{\phi(m_i)\}$ has a subsequence converging to a point of Y (in other terms, $\{\phi(m_i)\}$ has an accumulation point in Y).

There are several other equivalent definitions [12, Theorem 1]. It should be also noted that if we restrict ourselves to compact metrizable spaces, this is equivalent to saying that ϕ is proper (i.e., inverse images of compact subsets of $N \setminus Y$ are compact). This provides a way of showing that fine shape restricts to strong shape on compacta.

Now fine shape itself can be explained as follows:

Definition 2. *Given two metrizable spaces X and Y and any absolute retracts (ARs) M and N containing X and Y respectively as closed homotopy negligible subsets, the set of fine shape classes $[X, Y]_{fSh}$ is the set of $X - Y$ -approaching maps $\phi: M \setminus X \rightarrow N \setminus Y$ up to an $X - Y$ -approaching homotopy (such a homotopy is an $(X \times [0, 1]) - Y$ -approaching map). It turns out that the set $[X, Y]_{fSh}$ is independent of the choice of M and N .*

The fine shape category $fSh(M)$ has all metrizable spaces as objects, and fine shape classes as morphisms.

It is not hard to prove that every homotopy class of ordinary continuous maps $[f] \in [X, Y]$ determines a unique fine shape class $[f]_{fSh} \in [X, Y]_{fSh}$. Fine shape is in general weaker; in particular, a map f that is not a homotopy equivalence can induce an isomorphism in fine shape (such a map is called a *fine shape equivalence*).

One can hope to extract further use and applications out of this version of shape theory for noncompact metrizable spaces where it is hard to do so for noncompact strong shape. A simple test of this is to take a result that has been proven for compact strong shape, and extend it to fine shape of all metrizable spaces. Yet it should also be possible to prove results for fine shape that do not restrict to compacta at all. In particular, we expect to be able to prove that the fine shape category is in fact a homotopy category in the sense of Quillen [16, Chapter I, section 1, Definition 6] by constructing a corresponding model structure on the category M of metrizable spaces and continuous maps. An equivalent of this has never been constructed for the strong shape of compacta; we do not expect it to be possible either, because some of the spaces arising there (the path spaces in particular) are practically bound to be noncompact even when starting with compacta (in fact, Cathey runs into this problem in [5, Theorem (2.5)]; the space $|X|$ there is in general not compact). This search for a model structure also raises the question of representing fine shape morphisms by usual maps of spaces via the localization used in Quillen's definition.

Among others, Cathey [5] defines the strong shape category of compacta as a left fraction localization (of hCM , the category of compacta and homotopy classes of continuous maps); it is thus known that this description of compact strong shape is equivalent to the others. Also works by Calder and Hastings [4] and by Mrozek [14] construct localizations by different morphism classes (of the same category), and show those still to be equivalent to the same strong shape category of compacta.

The goal of the present work is to extend Cathey's definition to fine shape, representing the fine shape category of [12] as a left fraction localization. This gives us a simple representation of fine shape by (homotopy classes of) usual maps:

Theorem 1. *(Corollary 8) For any fine shape class $[\phi] \in [X, Y]_{fSh}$, there exist a space Z along with maps (which can be chosen to be closed embeddings) $u: X \rightarrow Z$ and $i: Y \rightarrow Z$ such that i is a fine shape equivalence and $[i]_{fSh}^{-1} \circ [u]_{fSh} = [\phi]$.*

Moreover, if fine shape will be proven to be a homotopy category in Quillen's sense (which we expect to do), we will be able to represent its morphisms with left fractions of homotopy classes alone, as opposed to the general case (which uses equivalence classes of finite chains of morphisms and inversions of morphisms of the original category); this will provide a

convenient way of working with $\text{fSh}(\mathcal{M})$ as a homotopy category. In fact, due to the universal property of left fraction localization (stated for reference in Proposition 5), the homotopy category functor $\mathcal{M} \rightarrow \text{fSh}(\mathcal{M})$, if it exists, must factorize through hM . We structure our work as follows.

Section 2 contains all previously established results we shall make use of; it is by necessity quite large, to ensure that all inobvious definitions and statements are explained and supported by references. In section 3, we introduce and explore the notion of an FDR-embedding, extending Cathey's SSDR-maps to noncompact metrizable spaces. Same as for these, we have a simple description (Theorem 4): an FDR-embedding is precisely a closed embedding that is a fine shape equivalence. In spirit of [5], in section 4 we consider the class $[FDR]$ of homotopy classes of FDR-embeddings in hM , the category of metrizable spaces and homotopy classes of continuous maps; we show (Corollary 7) that there exists a category $[FDR] \backslash \text{hM}$, the left fraction localization of hM at $[FDR]$ (the same cannot be done without resorting to homotopy classes). Section 5 defines the functor S from this localization to the fine shape category $\text{fSh}(\mathcal{M})$. Construction of the inverse functor T , as well as the proof of the inversion, are postponed until section 7; there we show (Theorem 8) the two categories to be isomorphic. Thus the final result of the present work can be condensed to the following

Theorem 2. *(Corollary 7, Theorem 8) There exists a well-defined category of left fractions $[FDR] \backslash \text{hM}$, isomorphic to the fine shape category $\text{fSh}(\mathcal{M})$.*

One additional notable fact arises during the proof:

Theorem 3. *(Corollary 9) Given two maps $f, g: X \rightarrow Y$ such that $[f]_{\text{fSh}} = [g]_{\text{fSh}}$, there exist a space Z along with a map $h: Y \rightarrow Z$ such that h is a closed embedding and a fine shape equivalence, and hf is homotopic to hg .*

Section 6 introduces a construction we use in the process. Specifically, we extend the concept of mapping cylinder to approaching maps between metrizable spaces; for compacta, this has been done explicitly by Mrozek [14], referring an earlier construction of Ferry [8], yet it seems that a modification is required for the noncompact case. This modification is what section 6 is devoted to; the appendix at the end elaborates on why the modification is needed at all. Our description is unfortunately quite cumbersome (if not complex), though we do offer some explanation of what it actually represents (see Remark 9). What we prove for this cylinder construction suffices for our goals, yet there are some unresolved questions there; in particular (see Remark 11 and the appendix), the question of the universal property echoing that of the usual mapping cylinder. We do not dwell on this question here, but it may still be of some interest.

2 Preliminaries

In this section, various notions and results are collated for reference. All of these are either previously established or trivial (often both), though not all may be widely known.

2.1. Spaces. We work exclusively with metrizable topological spaces and (topologically) continuous maps between them, so we adopt the following

Convention. *By a space, we shall always mean a metrizable topological space, unless specified otherwise. By a map between spaces, we shall always mean a continuous map.*

Definition 3. *By \mathbf{M} we denote the category of metrizable topological spaces and continuous maps. By \mathbf{hM} we denote the category of metrizable topological spaces and homotopy classes of continuous maps.*

Notation. *We denote the homotopy class of a map f by $[f]$, and we write $f \simeq g$ to mean that maps f and g are homotopic. We denote the set of all homotopy classes from X to Y by $[X, Y]$. For every space X , we denote the identity map of X by id_X . Restriction of a map $f: X \rightarrow Y$ to a subspace $A \subseteq X$, we denote by $f|_A$.*

2.2. Metrizable joins and metrizable mapping cylinders. As we restrict ourselves to metrizable spaces, we will need a number of constructions that provide metrizable analogues to well-known topological objects. This subsection is fully based on [13, Chapter II].

Definition 4. *Given two spaces M and N whose topologies are generated by some metrics d_M and d_N , both bounded by unity, the metrizable join $M \star N$ is the space that has the same underlying set as the usual (topological) join $-(M \times [-1, 1] \times N)/\sim$, where $(m, 1, n) \sim (m, 1, n')$ and $(m, -1, n) \sim (m', -1, n)$ for all $m, m' \in M$ and $n, n' \in N$ — with the topology given by the metric [13, before Remark 7.26]*

$$d((m, s, n), (m', s', n')) := \min \left\{ \begin{array}{l} d_M(m, m') + |s - s'| + d_N(n, n'), \\ d_M(m, m') + |1 - s| + |1 - s'|, \\ |s + 1| + |s' + 1| + d_N(n, n'), \\ 4 - |s - s'| \end{array} \right\}.$$

Remark 1. (1) *As per the reference, the topology of $M \star N$ is independent of the choice of d_M and d_N .*

(2) *Often (but not always) this space simply coincides with the topological join — for example, whenever M and N are compact.*

(3) *Since we work with metrizable spaces, we reiterate that the notation $M \star N$ shall always refer to the metrizable join, rather than the topological one.*

(4) *We note explicitly that in $M \star N$, $(m_k, s_k, n_k) \rightarrow (m, 1) = m$ if and only if $m_k \rightarrow m$ and $s_k \rightarrow 1$, $(m_k, s_k, n_k) \rightarrow (-1, n) = n$ if and only if $n_k \rightarrow n$ and $s_k \rightarrow -1$, and for $s \in (-1, 1)$, $(m_k, s_k, n_k) \rightarrow (m, s, n)$ if and only if*

$m_k \rightarrow m$, $s_k \rightarrow s$, and $n_k \rightarrow n$. Thus metrizable join has not only the same underlying set, but also the same sequence convergence as the topological one.

(5) It is easy to see that for a given topology on a set, there can be at most one metrizable topology on the same set with the same sequence convergence. Therefore, metrizable join is uniquely defined as the metrizable space with the same underlying set and sequence convergence as the topological join, and the explicit metric shows that it does exist for any two metrizable spaces.

(6) As usual, we have standard embeddings of M and N into their join.

Notation. A point $m \in M$ in the join $M \star N$ shall be denoted by $(m, 1)$, or by $(m, 1, n)$ for any $n \in N$, or simply by m . A point $n \in N$, similarly, by $(-1, n)$, $(m, -1, n)$, or simply n . Any other point of $M \star N$ shall be denoted by $(m, s, n) \in M \times (-1, 1) \times N$.

From this join, we can construct the metrizable mapping cylinder. We do this by embedding the latter in the former (the same can be done in the usual topological case):

Definition 5. Given spaces M and N , and a map $f: M \rightarrow N$, we define the metrizable mapping cylinder $MC(f)$ to be the unique metrizable space having the same underlying set (which we take to be $M \times (0, 1] \cup N$) and sequence convergence as the topological cylinder of f . $MC(f)$ can be constructed by inserting the underlying set into $M \star N$ with $N \ni n \mapsto n$ and $M \times (0, 1] \ni (m, s) \mapsto (m, 2s - 1, f(m))$, and taking the subspace topology.

Remark 2. (1) This explicit construction induces the product topology on $M \times (0, 1]$, and the original topology on N . In particular, we have two standard embeddings, which we denote, within this remark, by $u: M = M \times \{1\} \rightarrow MC(f)$ and $i: N \rightarrow MC(f)$.

(2) $MC(f)$ strong deformation retracts onto its base N , same as in the topological case. What is more, this provides a homotopy (from $M \times [0, 1]$ to $MC(f)$) between u and i .

(3) Yet further, there is a strong deformation retraction of $MC(f) \times [0, 1]$ onto $u(M) \times [0, 1] \cup MC(f) \times \{0\}$: first take a strong deformation retraction of $[0, 1] \times [0, 1]$ onto $\{1\} \times [0, 1] \cup [0, 1] \times \{0\}$ in which $\{0\} \times [0, 1]$ retracts along itself onto $\{0\} \times \{0\}$; then, for each $m \in M$, apply this strong deformation retraction to the embedding $r_m \times id_{[0, 1]}: [0, 1] \times [0, 1] \rightarrow MC(f) \times [0, 1]$, where $r_m(s) := (m, s)$ for $s > 0$ and $r_m(0) := f(m)$.

We shall also extend the cylinder construction a little:

Definition 6. Assume that L is a subset of M , and that we have a map $f: L \rightarrow N$. Then we shall say that f is a partial map from M to N , and define the metrizable mapping cylinder of f as a partial map, denoted $MC_M(f)$, by taking the set $M \times \{1\} \cup L \times (0, 1] \cup N$ and inserting it into $M \star N$ such that M is embedded in the standard way, and $L \times (0, 1] \cup N$ is embedded as $MC(f)$ into $L \star N$, which is then embedded into $M \star N$. In other terms, $MC_M(f) := M \cup_{L \times \{1\}} MC(f)$.

Finally we shall make use of one more construction [13, Subsection 7.J]:

Definition 7. *Assume a map $f: M \rightarrow N$ and a closed subset X of M such that the restriction $f|_X$ is perfect (i.e., closed and has compact inverse image of every point). Then we can take $MC(f)$ and a continuous function $h: M \rightarrow [0, 1]$ such that $h^{-1}(0) = X$. By the relative metrizable mapping cylinder of f by X we shall mean the subspace $MC(f|X) := \{(m, s) \in MC(f) \mid s \leq h(m)\} \cup N$.*

Remark 3. (1) *It is easy to see that $MC(f|X)$ is a strong deformation retract of $MC(f)$, and that its homeomorphism class does not depend on the choice of the function h (this uses the fact that $f|_X$ is perfect).*

(2) *Moreover, $MC(f|X)$ contains N (as a closed subset) and $M \setminus X = \{(m, s) \mid s = h(m) > 0\}$. If f is injective on X , then $MC(f|X)$ even contains M as a closed subset.*

2.3. Approaching maps. Following [12], we construct fine shape using approaching maps, which we define here.

Definition 8. *Let X be a closed subset of a space M . We say that X is homotopy negligible in M to mean that there exists a homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$. In other words, there is a deformation of M into itself that never crosses X except at the initial (identity) map.*

Definition 9. *Let X and Y be closed homotopy negligible in spaces M and N respectively, and let $\phi: M \setminus X \rightarrow N \setminus Y$ be a map (continuous on its domain, as per our convention). We say that ϕ is $X - Y$ -approaching to mean that for any sequence $\{m_i\} \subset M \setminus X$ converging (in M) to a point of X , the sequence $\{\phi(m_i)\} \subset N \setminus Y$ contains a subsequence that converges (in N) to a point of Y . Equivalently, ϕ being $X - Y$ -approaching means that whenever a sequence in $M \setminus X$ has an accumulation point in X , the sequence's image in $N \setminus Y$ has an accumulation point in Y .*

It is clear that a composition of approaching maps is an approaching map, and that the identity map $id_{M \setminus X}$ is $X - X$ -approaching. Therefore we have no trouble with the following

Definition 10. *The approaching category \mathbf{MAppr} is defined as follows: its objects are pairs of spaces (M, X) with X closed homotopy negligible in M , and a morphism from (M, X) to (N, Y) is an $X - Y$ -approaching map $\phi: M \setminus X \rightarrow N \setminus Y$.*

For the rest of the present work, we adopt the following

Convention. *Whenever we speak of an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$, we mean that (M, X) and (N, Y) are objects of \mathbf{MAppr} , and ϕ is a morphism between those — that is, ϕ is both continuous on $M \setminus X$ and $X - Y$ -approaching.*

Homotopies of approaching maps are readily defined:

Definition 11. *Given two approaching maps $\phi, \psi: M \setminus X \rightarrow N \setminus Y$, by an approaching homotopy between ϕ and ψ we mean an approaching map $\Phi: (M \times [0, 1]) \setminus (X \times [0, 1]) \rightarrow N \setminus Y$ such that $\Phi_0 = \phi$ and $\Phi_1 = \psi$. Whenever such approaching homotopy exists, we say that ϕ and ψ are approaching homotopic, which is an equivalence relation. Finally, we define the approaching homotopy category hMAppr to consist of the same objects as MAppr and approaching homotopy classes of approaching maps.*

In the present work, we make use of two concepts having to do with passing between an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ and a map defined on X , or a subset of X , taking it to Y :

Definition 12. *In some cases, an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ can be obtained by extending a map $f: X \rightarrow Y$, so that the two combine into a continuous map $\bar{f}: M \rightarrow N$ with $\bar{f}^{-1}(Y) = X$. In such case we say that ϕ extends f .*

Conversely, for an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$, define the extension set of ϕ , denoted $X(\phi)$, as follows: a point $x \in X$ is in $X(\phi)$ if and only if there is a point $y \in Y$ such that for every sequence in $M \setminus X$ converging to x , the ϕ -image of the sequence converges to y .

The following fact is clear:

Proposition 1. *$X(\phi)$ is the largest subset of X for which there is a continuous map $f: X(\phi) \rightarrow Y$ combining with ϕ into a continuous map; f is defined uniquely; for any subset A of X , ϕ extends on A by a map $f': A \rightarrow Y$ if and only if $A \subseteq X(\phi)$, and in that case $f' = f|_A$.*

Remark 4. *Note that homotopy negligibility of X in M implies that every point of X is the limit of some sequence in $M \setminus X$.*

Finally, we shall also make use of a specific kind of homotopy:

Definition 13. *Given any homotopy $H: X \times [0, 1] \rightarrow Y$, we say H is additive whenever $H_s \circ H_t = H_{\min\{s+t, 1\}}$ for all $s, t \in [0, 1]$. For a space M and a closed subset X of M , we say that X is additive homotopy negligible in M whenever there is an additive homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = \text{id}_M$ and $H(M \times (0, 1]) \subset M \setminus X$.*

2.4. Absolute retracts. The second notion we use to construct fine shape, along with approaching maps, is that of an absolute retract. This, of course, goes back to the Borsuk's definition of shape using absolute neighborhood retracts; here, though, we will not need the latter.

Definition 14. *By an absolute retract (AR), we shall mean a space M that is, in fact, an absolute extensor for all metrizable spaces in the following sense: given any space X and a closed subset A of X , any map $f: A \rightarrow M$*

can be extended to a map $\bar{f}: X \rightarrow M$ such that $\bar{f}|_A = f$:

$$\begin{array}{ccc} A & \xrightarrow{f} & M \\ \downarrow & \nearrow \bar{f} & \\ X & & \end{array}$$

It is known [13, Corollary 18.3] that (at least with respect to metrizable spaces) absolute retracts and absolute extensors (AEs) are exactly the same spaces; by convention, we call them all ARs.

Aside from the definition, we make use of the following facts about ARs:

Proposition 2. [13, Chapter 18] (1) *Every contractible polyhedron is an AR, including, in particular, the unit interval $[0, 1]$;*

(2) *A direct product of any countable set of ARs is an AR;*

(3) *If M and N are ARs, then so is $M \star N$, and so is $MC(f)$ for any $f: M \rightarrow N$;*

(4) *If $M = M_1 \cup M_2$ is such that M_1 and M_2 are ARs, both are closed in M , and $M_1 \cap M_2$ is an AR, then M is an AR (therefore, $MC_M(f)$ is an AR for any $f: A \rightarrow N$ such that M, N , and A are ARs, and A is a closed subset of M);*

(5) *A retract of an AR is an AR (therefore, $MC(f|_X)$ is an AR for any $f: M \rightarrow N$ such that M and N are ARs, X is closed in M , and $f|_X$ is perfect);*

(6) [13, Theorem 19.3] *If X is closed and homotopy negligible in M , then M is an AR if and only if $M \setminus X$ is an AR;*

(7) *For any space X , there exists an AR M containing X as a closed subset (therefore, $M \times [0, 1]$ contains $X = X \times \{0\}$ as a closed additive homotopy negligible subset, homotopy given by $H_t(m, s) := (m, \min\{s + t, 1\})$).*

2.5. Fine shape. The notions of an approaching map and of an absolute retract are combined into the notion of fine shape based on the following fact [12, Lemma 2.13]:

Lemma 1. *Let X be a closed subset of any space M , and Y be closed homotopy negligible in an AR N . Then*

(1) *Every map $f: X \rightarrow Y$ extends to a map $\bar{f}: M \rightarrow N$ such that $\bar{f}^{-1}(Y) = X$ (and therefore $\bar{f}|_{M \setminus X}$ is $X - Y$ -approaching);*

(2) *For any homotopy $F: X \times [0, 1] \rightarrow Y$ and any extensions $\bar{F}_0, \bar{F}_1: M \rightarrow N$ of F_0 and F_1 such that $\bar{F}_0^{-1}(Y) = \bar{F}_1^{-1}(Y) = X$, there is an extension $\bar{F}: M \times [0, 1] \rightarrow N$ such that $\bar{F}(-, 0) = \bar{F}_0$, $\bar{F}(-, 1) = \bar{F}_1$, and $\bar{F}^{-1}(Y) = X$.*

Proof. (1) First we can extend f to any map $f': M \rightarrow N$, since N is an AR. Then choose any homotopy $H: N \times [0, 1] \rightarrow N$ such that $H_0 = id_N$ and $H(N \times (0, 1]) \subseteq N \setminus Y$, and any continuous function $h: M \rightarrow [0, 1]$ such that $h^{-1}(0) = X$. From those we define $\bar{f}(m) := H_{h(m)} \circ f'(m)$. Then $\bar{f}(M \setminus X) \subseteq N \setminus Y$, and $\bar{f}|_{M \setminus X}$ is $X - Y$ -approaching (as it actually extends on X by a map into Y), as needed.

(2) $M \times \{0, 1\} \cup X \times [0, 1]$ is a closed subset of $M \times [0, 1]$; we combine the maps \bar{F}_0, \bar{F}_1 , and F into a map of $M \times \{0, 1\} \cup X \times [0, 1]$ into N , then extend it to a map $\bar{F}' : M \times [0, 1] \rightarrow N$. Now same as in 1), choose any homotopy $H : N \times [0, 1] \rightarrow N$ such that $H_0 = id_N$ and $H(N \times (0, 1]) \subseteq N \setminus Y$, and any continuous function $h : M \times [0, 1] \rightarrow [0, 1]$ such that $h^{-1}(0) = M \times \{0, 1\} \cup X \times [0, 1]$, and define $\bar{F}(m, t) := H_{h(m)} \circ \bar{F}'(m, t)$. \square

Now fine shape is constructed from the following, which is clearly an equivalence relation:

Definition 15. *Let X and Y be any two spaces. Let M and M' be ARs, each containing X as a closed homotopy negligible subset, whereas N and N' are ARs each containing Y as such. For any approaching maps $\phi : M \setminus X \rightarrow N \setminus Y$ and $\psi : M' \setminus X \rightarrow N' \setminus Y$, we say that ϕ and ψ are of the same fine shape class (from X to Y) whenever there are some maps $\bar{id}_X : M \rightarrow M'$ and $\bar{id}_Y : N \rightarrow N'$, extending id_X and id_Y , such that $\bar{id}_X^{-1}(X) = X$, $\bar{id}_Y^{-1}(Y) = Y$, and that there is an approaching homotopy (from $(M \setminus X) \times [0, 1]$ to $N' \setminus Y$) between $\bar{id}_Y \circ \phi$ and $\psi \circ \bar{id}_X$, so that the following diagram commutes in approaching homotopy:*

$$\begin{array}{ccc} M \setminus X & \xrightarrow{\phi} & N \setminus Y \\ \bar{id}_X|_{M \setminus X} \downarrow & & \downarrow \bar{id}_Y|_{N \setminus Y} \\ M' \setminus X & \xrightarrow{\psi} & N' \setminus Y \end{array}$$

By using Lemma 1(1), we can construct fine shape classes from ordinary maps:

Definition 16. *Given a map $f : X \rightarrow Y$, the fine shape class (from X to Y) induced by f , denoted $[f]_{fSh}$, is defined as follows: take any ARs M and N containing X and Y respectively as closed homotopy negligible subsets, extend f to a map $\bar{f} : M \setminus X \rightarrow N \setminus Y$ such that $\bar{f}^{-1}(Y) = X$, and take the fine shape class of $\bar{f}|_{M \setminus X}$.*

There are, however, fine shape classes that are not induced by any ordinary maps. This is well-known already for strong shape of compacta; we mention a class of examples in Remark 6.

By definition, every fine shape from X to Y is represented by some approaching map $\phi : M \setminus X \rightarrow N \setminus Y$ for some ARs M and N (containing X and Y respectively as closed homotopy negligible subsets). In fact, however, it can be represented for *any* such choice of ARs:

Lemma 2. *Let X and Y be any two spaces, M and N be ARs containing X and Y respectively as closed homotopy negligible subsets, and $\phi : M \setminus X \rightarrow N \setminus Y$ be an approaching map. For any two other ARs M' and N' containing X and Y respectively as closed homotopy negligible subsets, there is an approaching map $\psi : M' \setminus X \rightarrow N' \setminus Y$ that is of the same fine shape class as ϕ .*

Proof. Take some maps $\bar{id}_X: M' \rightarrow M$ and $\bar{id}_Y: N \rightarrow N'$ extending id_X and id_Y respectively and such that $\bar{id}_X^{-1}(X) = X$ and $\bar{id}_Y^{-1}(Y) = Y$. Then the map $\bar{id}_Y \circ \phi \circ \bar{id}_X: M' \setminus X \rightarrow N' \setminus Y$ is of the same fine shape class as ϕ . \square

Corollary 1. *Fine shape classes are composable: a fine shape class from X to Y , represented by $\phi: M \setminus X \rightarrow N \setminus Y$, and a fine shape class from Y to Z , represented by $\psi: N' \setminus Y \rightarrow L' \setminus Z$, compose through taking any $\phi': M \setminus X \rightarrow N' \setminus Y$ of the same fine shape class as ϕ and taking the fine shape class of $\psi \circ \phi'$, or by taking any map $\psi': N \setminus Y \rightarrow L' \setminus Z$ of the same fine shape class as ψ and taking the fine shape class of $\psi' \circ \phi$ — the two compositions are of the same fine shape class from X to Z .*

Thus fine shape can be defined from X to Y without relying on any specific choice of spaces containing them; this is what differentiates fine shape from approaching maps.

By calling onto Lemma 1 again, we easily see that fine shape is weaker than homotopy:

Proposition 3. *The fine shape class $[f]_{fSh}$ depends only on the homotopy class $[f]$. Composition of maps, or of homotopy classes, induces composition of the corresponding fine shape classes.*

With all the pieces in place, we introduce

Definition 17. *The fine shape category $fSh(\mathbf{M})$ is defined as follows:*

- (1) *Its objects are the objects of \mathbf{M} (metrizable topological spaces);*
- (2) *Morphisms from X to Y are the fine shape classes from X to Y ;*
- (3) *Composition is given by Corollary 1;*
- (4) *The identity morphism for a space X is given by $[id_X]_{fSh}$, the fine shape class of id_X .*

Now Proposition 3 has the following

Corollary 2. *There is a functor from $h\mathbf{M}$ to $fSh(\mathbf{M})$ that is constant on objects and sends every homotopy class to the sole fine shape class induced by it.*

Notation. *For an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$, we shall use $[\phi]$ to denote the fine shape class from X to Y defined by ϕ . For an actual map $f: X \rightarrow Y$, we shall use $[f]_{fSh}$ to denote the fine shape class (again from X to Y) defined by f , or by its homotopy class $[f]$. The set of all fine shape classes from X to Y , we denote by $[X, Y]_{fSh}$.*

We finish this subsection by discussing maps of \mathbf{M} that induce isomorphisms in $fSh(\mathbf{M})$:

Definition 18. *A map $f: X \rightarrow Y$ is called a fine shape equivalence whenever $[f]_{fSh}$ is an isomorphism.*

In regards to our search for a model structure on \mathbf{M} for which $\text{fSh}(\mathbf{M})$ is the homotopy category, fine shape equivalences must clearly be the weak equivalences. To that end we state the obvious

Proposition 4. (1) *Fine shape equivalences satisfy the 2-out-of-3-property: for any maps $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, if any two of the maps f , g , and gf are fine shape equivalences, then so is the third;*

(2) *Fine shape equivalences are closed under retracts: for any commutative diagram*

$$\begin{array}{ccccc} & & id_A & & \\ & & \curvearrowright & & \\ A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ g \downarrow & & f \downarrow & & \downarrow g \\ B & \xrightarrow{j} & Y & \xrightarrow{s} & B, \\ & & id_B & & \end{array}$$

if f is a fine shape equivalence, then so is g .

2.6. Left fraction localization. In general, a category can be localized at a selected class of its morphisms by allowing to invert each morphism in this class. Under some conditions on the selected class, the localization may have a simple representation. In the following, we make use of a specific case of this, the left fraction localization, as used by Cathey [5, Theorem (1.10)], which references [9] and [17]:

Definition 19. *Assume \mathbf{C} is a category, and Σ is a class of morphisms of \mathbf{C} . We say that Σ satisfies the left invertibility conditions (sometimes called the left Ore conditions) if:*

- (1) Σ contains all isomorphisms and is closed under morphism composition;
- (2) For any morphisms $i: A \rightarrow X$ and $u: A \rightarrow Y$ with $i \in \Sigma$, there exists an object Z along with morphisms $j: Y \rightarrow Z$ and $v: X \rightarrow Z$ such that $ju = vi$ and $j \in \Sigma$:

$$\begin{array}{ccc} A & \xrightarrow{u} & Y \\ i \downarrow & & \downarrow j \\ X & \xrightarrow{v} & Z \end{array}$$

- (3) For any morphisms $u, v: X \rightarrow Y$ such that $ui = vi$ for some $i: A \rightarrow X$ with $i \in \Sigma$, there exist an object Z and a morphism $j: Y \rightarrow Z$ such that $ju = jv$ and $j \in \Sigma$:

$$A \xrightarrow{i} X \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} Y \xrightarrow{j} Z$$

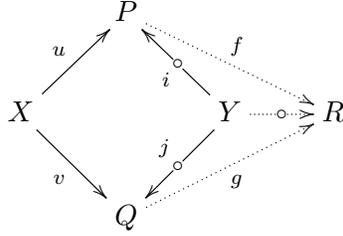
Here and below in the commutative diagrams we adopt the following

Convention. *All diagrams we include are to be either assumed or proved commutative. In any diagram used to illustrate a property, solid arrows are*

assumed to exist, dotted arrows must be shown to exist, and arrows with circles on them belong to the select class of morphisms satisfying the left invertibility conditions; an object is assumed to exist if it has at least one solid arrow entering or exiting it, and must be shown to exist otherwise.

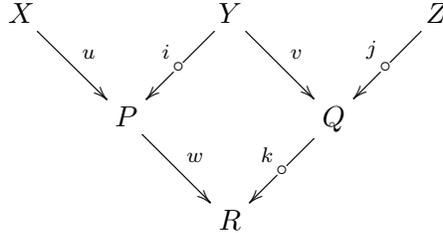
Definition 20. Given a category \mathbf{C} and a class of its morphisms Σ satisfying the left invertibility conditions, the category of left fractions $\Sigma \backslash \mathbf{C}$ is defined as follows:

- (1) Its objects are the objects of \mathbf{C} ;
- (2) A morphism from X to Y is an equivalence class of cospans $X \xrightarrow{u} P \xleftarrow{i} Y$ (also denoted by $i \backslash u$), where u (the “numerator”) and i (the “denominator”) are morphisms of \mathbf{C} with $i \in \Sigma$;
- (3) Two cospans $X \xrightarrow{u} P \xleftarrow{i} Y$ and $X \xrightarrow{v} Q \xleftarrow{j} Y$ belong to the same equivalence class precisely when there is a cospan $P \xrightarrow{f} R \xleftarrow{g} Q$ of morphisms of \mathbf{C} such that $fu = gv$, $fi = gj$, and the latter morphism is in Σ :



(in other words, $i \backslash u = fi \backslash fu = gj \backslash gv = j \backslash v$ whenever $fu = gv$ and $fi = gj$; note that neither f nor g has to be in Σ here);

- (4) The composition of two classes represented by cospans $X \xrightarrow{u} P \xleftarrow{i} Y$ and $Y \xrightarrow{v} Q \xleftarrow{j} Z$ can be obtained by choosing any cospan $P \xrightarrow{w} R \xleftarrow{k} Q$ with $wi = kv$ and $k \in \Sigma$ (which exists by the left invertibility condition (2)), and taking the class represented by the cospan $X \xrightarrow{wu} R \xleftarrow{kj} Z$:



(in other words, $(j \backslash v) \circ (i \backslash u) = (kj \backslash kv) \circ (wi \backslash wu) = kj \backslash wu$; any suitable k and w will give the same composition class by condition (3), and $kj \in \Sigma$ by condition (1));

- (5) The identity isomorphism for an object X is the class of the fraction $X \xrightarrow{id_X} X \xleftarrow{id_X} X$ (where id_X is the identity in \mathbf{C} , and $id_X \in \Sigma$ by condition (1)).

Convention. Here and later we write a specific fraction to mean its equivalence class (similar to how fractions are used in arithmetic to represent rational numbers).

We also state the universal property of the left fraction category:

Proposition 5. *The left fraction category $\Sigma \backslash \mathcal{C}$, whenever it exists, is uniquely characterized by the following:*

(1) *There is a functor $P: \mathcal{C} \rightarrow \Sigma \backslash \mathcal{C}$ acting on a morphism $f: X \rightarrow Y$ by $P(f) = id_Y \backslash f$;*

(2) *P sends Σ into the class of isomorphisms: if $i: X \rightarrow Y$ is in Σ , then $id_Y \backslash i$ has an inverse $i \backslash id_X$;*

(3) *For any category \mathcal{D} and any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ sending Σ into the class of isomorphisms in \mathcal{D} , there is a unique functor $Q: \Sigma \backslash \mathcal{C} \rightarrow \mathcal{D}$ (acting by $Q(i \backslash u) = (Fi)^{-1} \circ (Fu)$) such that $F = QP$.*

In short, P is the universal functor out of \mathcal{C} that sends all morphisms of Σ into invertible morphisms.

3 FDR-embeddings

As Cathey [5] starts by defining what he called SSDR-maps, so do we follow by defining the fine shape version of those.

Definition 21. *Let A be a closed subset of a space X , and assume that:*

- *there exists an AR M containing X as a closed homotopy negligible subset;*
- *there exists a closed subset L of M such that L is an AR, $L \cap X = A$, and A is homotopy negligible in L ;*
- *there exists an $(X \times [0, 1]) - X$ -approaching map $\Phi: (M \backslash X) \times [0, 1] \rightarrow M \backslash X$ such that $\Phi_0 = id_{M \backslash X}$, $\Phi_1(M \backslash X) = L \backslash A$, and $\Phi_t|_{L \backslash A} = id_{L \backslash A}$ for all $t \in [0, 1]$.*

In particular, considering Φ_1 as an approaching map from $M \backslash X$ to $L \backslash A$, the fine shape class $[\Phi_1]$ (from X to A) is the inverse to the fine shape class given by the embedding $A \subseteq X$ (which readily extends to the embedding $L \subseteq M$). In this case we shall say that the fine shape class $[\Phi]$ is a fine shape strong deformation retraction of X on A . We shall also say that A is a fine shape strong deformation retract of X , the inclusion $A \subseteq X$ is an FDR-embedding (which is simply a shorthand for “embedding of a fine shape strong deformation retract”), and Φ is an approaching strong deformation retraction representing the fine shape strong deformation retraction $[\Phi]$.

Example 1. *An inclusion $X \times \{0\} \subset X \times [0, 1]$ is clearly an FDR-embedding for any X . Lemma 7 below provides a larger class of examples.*

The rest of this section is concerned with the properties of FDR-embeddings and other statements that we will need to use them. Some of these simply extend results for SSDR-maps given by Cathey.

Proposition 6. *Let M be an AR, L and X closed subsets of M with $L \cap X = A$. Assume there are homotopies $H: M \times [0, 1] \rightarrow M$ and $G: L \times [0, 1] \rightarrow L$ such that $H_0 = id_M$, $G_0 = id_L$, $H(M \times (0, 1]) \subseteq M \setminus X$, and $G(L \times (0, 1]) \subseteq L \setminus A$. Then there is a homotopy $F: M \times [0, 1] \rightarrow M$ with $F_0 = id_M$, $F(M \times (0, 1]) \subseteq M \setminus X$, and $F|_{L \times [0, 1]} = G$ (so in particular $F(L \times (0, 1]) \subseteq L \setminus A$).*

Proof. First define F on $M \times \{0\} \cup L \times [0, 1]$, a closed subset of $M \times [0, 1]$, by $F|_{M \times \{0\}} = id_M$, $F|_{L \times [0, 1]} = G$. As M is an AR, this extends to some $F': M \times [0, 1] \rightarrow M$. Now to ensure that $F'(M \times (0, 1]) \subseteq M \setminus X$, take a continuous function $h: M \times [0, 1] \rightarrow [0, 1]$ with $h^{-1}(0) = M \times \{0\} \cup L \times [0, 1]$, and define $F_t(m) := H_{h(m,t)} \circ F'_t(m)$. \square

Corollary 3. *Under the conditions of Definition 21, there exists a single homotopy $H: M \times [0, 1] \rightarrow M$ with $H_0 = id_M$, $H(M \times (0, 1]) \subseteq M \setminus X$, and $H(L \times [0, 1]) \subseteq L$. In other words, (X, A) is homotopy negligible in (M, L) .*

The next two lemmas are needed to extend homotopy negligibility property to mapping cylinders. These results are already practically contained in [13, Proposition 19.8]; we simply confirm the specific formulations we need.

Lemma 3. *Assume Y is a closed homotopy negligible subset of a space N , X and L are closed subsets of a space M , $A = L \cap X$. Assume further that there is a homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = id_M$, $H(M \times (0, 1]) \subseteq M \setminus X$, and $H(L \times [0, 1]) \subseteq L$. Then for any map $f: L \rightarrow N$ such that $f^{-1}(Y) = A$, $MC_X(f|_A)$ is homotopy negligible in $MC_M(f)$. In particular, with $L = M$, $A = X$, and $f: M \rightarrow N$, $MC(f|_X)$ is homotopy negligible in $MC(f)$.*

Proof. In addition to H , choose some homotopy $G: N \times [0, 1] \rightarrow N$ with $G_0 = id_N$ and $G(N \times (0, 1]) \subseteq N \setminus Y$. Moreover, let d_N be some metric on N that provides its topology and is bounded by unity.

First, we will want to “slow down” H if necessary to obtain a new homotopy $H': M \times [0, 1] \rightarrow M$ such that the three assumed properties of H still apply, but in addition $d_N(f \circ H'_t(m), f(m)) \leq t$ for all $m \in M$ and $t \in [0, 1]$. To achieve that, in $M \times [0, 1] \times [0, 1]$ consider the closed subset $C := \{(m, t, p) \mid d_N(f \circ H_p(m), f(m)) \geq t\}$ containing $M \times \{0\} \times \{0\}$ and define a function $\delta(m, t) := \frac{1}{2}d((m, t, 0), C)$, taking any metric d giving the product topology and bounded by unity. Now take $H'_t(m) := H_{\delta(m,t)}(m)$. Then $H'_0 = id_M$ (as $(m, 0, 0) \in C$ for all $m \in M$), and for $(x, t) \in X \times (0, 1]$, $H'_t(x) \in M \setminus X$ as $(x, t, 0)$ is not in C . But also for all $(m, t) \in M \times [0, 1]$, we have $(m, t, \delta(m, t)) \in C$ if and only if $\delta(m, t) = 0$ and so $f \circ H'_t(m) = f(m)$. Thus $d_N(f \circ H'_t(m), f(m)) \leq t$ for all m and all t , as needed.

With this we shall define a homotopy $F: MC_M(f) \times [0, 1] \rightarrow MC_M(f)$. First, F restricts to H' and G on the standard embeddings of M and N

respectively. Second, for a point $(l, s) \in L \times (0, 1)$, we define

$$F_t(l, s) := \begin{cases} G_{t-2s} \circ f \circ H_{2s}(l), & s \leq \frac{t}{2} \\ (H'_t(m), \frac{2s-t}{2-t}), & s > \frac{t}{2} \end{cases}.$$

Then $F_0 = id_{MC_M(f)}$ and the image of F_t does not intersect $MC_X(f|_A)$ for $t > 0$. Moreover, F is continuous: in the only inobvious case is if a sequence (l_k, s_k) in $L \times (0, 1)$ converges to $n \in N$, we have $d_N(f(l_k), n) \rightarrow 0$ and $s_k \rightarrow 0$. Now assume also $t_k \rightarrow t$. For those k which satisfy $2s_k \leq t_k$, we have $d_N(f \circ H'_{2s_k}(l_k), f(l_k)) \rightarrow 0$ and therefore $f \circ H'_{2s_k}(l_k) \rightarrow n$, so in addition to $s_k \rightarrow 0$ we have $G_{t_k-2s_k} \circ f \circ H'_{2s_k}(l_k) \rightarrow G_t(n)$, which means that $F_{t_k}(l_k, s_k) \rightarrow F_t(n)$. And those k for which $2s_k > t_k$ are finite in number unless $t_k \rightarrow 0$, and in the latter case we obtain $d_N(f \circ H'_{t_k}(l_k), f(l_k)) \rightarrow 0$, so also $f \circ H_{t_k}(l_k) \rightarrow n = G_t(n)$, so again $F_{t_k}(l_k, s_k) \rightarrow G_t(n)$, as needed. \square

Lemma 4. *Let $f: M \rightarrow N$ be a map, X and Y closed subsets of M and N respectively such that $f^{-1}(Y) = X$, $f|_X$ is perfect, and Y is homotopy negligible in N . Then Y is also homotopy negligible in $MC(f|_X)$.*

Proof. We have a homotopy $H: N \times [0, 1] \rightarrow N$ with $H_0 = id_N$ and $H(N \times (0, 1]) \subseteq N \setminus Y$. Now on the usual cylinder $MC(f)$ define a homotopy $G: MC(f) \times [0, 1] \rightarrow MC(f)$ by $G_t|_N(n) := H_t(n)$,

$$G_t|_{M \times (0, 1]}(m, s) := \begin{cases} (m, s - t), & t < s \\ H_{t-s} \circ f(m), & t \geq s \end{cases}$$

Then in particular $G(MC(f|_X) \times [0, 1]) \subseteq MC(f|_X)$. Moreover, $G(MC(f|_X) \times (0, 1]) \cap Y = \emptyset$, thus the restriction of G gives a homotopy of $MC(f|_X)$ proving Y homotopy negligible in it. \square

Lemma 5. *Assume the inclusion $A \subseteq X$ is an FDR-embedding, with M , L , and Φ as in Definition 21. For any other AR L' containing A as a closed homotopy negligible subset, we can always construct an AR M' containing X and L' as closed subsets with $L' \cap X = A$ and X homotopy negligible in M' , along with an approaching strong deformation retraction Φ' of $M' \setminus X$ onto $L' \setminus A$.*

Corollary 4. *Assume that every space A can be embedded as a closed homotopy negligible subset in an AR L such that some property P holds for either L , or the embedding $A \subseteq L$. Then for every FDR-embedding $A \subseteq X$, we can have M , L , and Φ as in Definition 21 such that the property P holds for L or $A \subseteq L$ correspondingly.*

Remark 5. *Note that the same cannot, in general, be said of M or $X \subseteq M$ in this situation.*

Proof (of Lemma 5). The identity map id_A extends to some map $\bar{id}_A: L \rightarrow L'$ with $\bar{id}_A^{-1}(A) = A$. Now $MC(\bar{id}_A|_A)$ is an AR, and so is $M' := M \cup_L MC(\bar{id}_A|_A)$. By applying Φ to $M \setminus X$ while keeping every point of $MC(\bar{id}_A|_A) \setminus$

A constant (these agree on $L \setminus A$), we obtain an approaching strong deformation retraction of $M' \setminus X$ onto $MC(\bar{id}_A|A) \setminus A$, which then approaching strong deformation retracts onto $L' \setminus A$.

To prove X homotopy negligible in M' , first use Corollary 3 to construct a homotopy $F: M \times [0, 1] \rightarrow M$ with $F_0 = id_M$, $F(M \times (0, 1]) \subseteq M \setminus X$, and $F(L \times [0, 1]) \subseteq L$; further use Lemma 4 to construct a homotopy $G: MC(\bar{id}_A|A) \times [0, 1] \rightarrow MC(\bar{id}_A|A)$ with $G_0 = id_{MC(\bar{id}_A|A)}$ and $G(MC(\bar{id}_A|A) \times (0, 1]) \subseteq MC(\bar{id}_A|A) \setminus A$. Then by Proposition 6 there is a homotopy $F': MC(\bar{id}_A|A) \times [0, 1] \rightarrow MC(\bar{id}_A|A)$ with $F'_0 = id_{MC(\bar{id}_A|A)}$, $F'(MC(\bar{id}_A|A) \times (0, 1]) \subseteq MC(\bar{id}_A|A) \setminus A$, and $F'|_{L \times [0, 1]} = F|_{L \times [0, 1]}$. Now combine F and F' into a single homotopy $F'': M' \times [0, 1] \rightarrow M'$. \square

Proposition 7. *Every identity map is an FDR-embedding. If $i: X \rightarrow Y$ and $j: Y \rightarrow Z$ are FDR-embeddings, so is ji .*

Proof. Identity part is clear.

For composition, first ji is a closed embedding. Now assume i being an FDR-embedding is proven (as per Definition 21) with ARs $N \supset Y$, $M \supset X$, and an approaching strong deformation retraction $\Phi: (N \setminus Y) \times [0, 1] \rightarrow N \setminus Y$, and j with ARs $L' \supset Z$, $N' \supset Y$, and $\Psi': (L' \setminus Z) \times [0, 1] \rightarrow L' \setminus Z$; all inclusions are those of closed homotopy negligible subsets. By the previous lemma, construct an AR L'' containing N and Z as closed subsets with $N \cap Z = Y$, Z a Z -set in L'' , and $L'' \setminus Z$ approaching strong deformation retracting onto $N \setminus Y$ by some Ψ'' . Then the composition $\Phi \circ \Psi''$ is well-defined (as the image of Ψ''_1 is $N \setminus Y$), and is an approaching strong deformation retraction of $L'' \setminus Z$ onto $M \setminus X$. \square

Lemma 6. *Let $A \subseteq X$ be a closed embedding with A nonempty. Then there exist an AR M containing X as a closed subset, a closed subset $L \subseteq M$ such that $L \cap X = A$ and L is an AR (in fact, L can be chosen to be a strong deformation retract of M), and a homotopy $H: M \times [0, 1] \rightarrow M$ such that $H_0 = id_M$, $H(M \times (0, 1]) \subseteq M \setminus X$, and $H(L \times [0, 1]) \subseteq L$.*

Proof. Take M' to be any AR containing X as a closed subset, and take any continuous function $h: M' \rightarrow [0, 1]$ such that $h^{-1}(0) = A$. Now let $M := M' \times [0, 1]$ with $X = X \times \{0\} \subset M$, $L := \{(m, s) \in M \mid s \geq h(m)\}$, $d_{M'}$, and $H_t(m, s) := (m, \min\{s + t, 1\})$. \square

Lemma 7. *Let A be a closed subset of a space X . Then the inclusion of $A \times [0, 1] \cup X \times \{0\}$ in $X \times [0, 1]$ is an FDR-embedding. (cf. [12, Corollary 1.6(b)], [5, Corollary (1.6)])*

Remark 6. *This provides nontrivial examples of FDR-embeddings. After all, a closed embedding $A \subseteq X$ is a cofibration if and only if $A \times [0, 1] \cup X \times \{0\}$ is a retract of $X \times [0, 1]$, and there exist closed embeddings of metrizable spaces that are not cofibrations. Note also that the fine shape classes of the approaching retractions are therefore not induced, in general, by any ordinary maps.*

Proof (of Lemma 7). If A is empty, then $X \times [0, 1]$ has a strong deformation retraction onto $X \times \{0\}$, which readily extends to a strong deformation retraction of $M \times [0, 1]$ onto $M \times \{0\}$ for any AR M containing X as a closed homotopy negligible subset.

Now assume A nonempty. Take M , L , and H given by the previous lemma. Let N to be the relative metrizable mapping cylinder of the embedding $L \subseteq M$ by A . Then N is an AR, and there is a homotopy $G: N \times [0, 1] \rightarrow N$ with $G_0 = id_N$, $G(N \times (0, 1]) \subseteq N \setminus X$, and $G(L \times [0, 1]) \subseteq L$: first apply Lemma 4 using H to prove that X is closed and homotopy negligible in N , then apply Proposition 6 with $H|_{L \times [0, 1]}$ as the second homotopy to obtain G with $G|_{L \times [0, 1]} = H|_{L \times [0, 1]}$. Now $N \times [0, 1]$ is an AR containing $X \times [0, 1]$ as a closed homotopy negligible subset, and $L \times [0, 1] \cup N \times \{0\}$ an AR in it intersecting $X \times [0, 1]$ precisely by $A \times [0, 1] \cup X \times \{0\}$, which is closed and homotopy negligible in it too (this uses the fact that $G(L \times [0, 1]) \subseteq L$).

For the approaching strong deformation retraction, note that $N \setminus X$ is homeomorphic to the (nonrelative) metrizable mapping cylinder of the embedding of $L \setminus A$ into $M \setminus X$. Then, as per Remark 2(3), we have a strong deformation retraction of $(N \setminus X) \times [0, 1]$ onto $(L \setminus A) \times [0, 1] \cup (N \setminus X) \times \{0\}$; this retraction is correctly approaching, thus proving the lemma. \square

Corollary 5. *If $A \subseteq X$ is an FDR-embedding, then so is $A \times [0, 1] \cup X \times \{0, 1\} \subseteq X \times [0, 1]$ (cf. [5, Corollary (1.6)]).*

Proof. Take spaces $M \supset X$ and $L \supset A$ and map Φ as in the definition of an FDR-embedding, and take $L \times [0, 1] \cup M \times \{0, 1\} \subseteq M \times [0, 1]$ to be the ARs, clearly containing $A \times [0, 1] \cup X \times \{0, 1\}$ and $X \times [0, 1]$ respectively as closed homotopy negligible subsets. Now define $\tilde{\Phi}: (M \times [0, 1] \setminus X \times [0, 1]) \times [0, 1] \rightarrow M \times [0, 1] \setminus X \times [0, 1]$ as a composition of $\tilde{\Phi}^{(1)}$ followed by $\tilde{\Phi}^{(2)}$. Here, $\tilde{\Phi}_t^{(1)}(m, s) := (\Phi_{(1-|2s-1|)t})(m, s)$, and then $\tilde{\Phi}^{(2)}$ applies the previous lemma to $(M \setminus X) \times [0, \frac{1}{2}]$ and $(M \setminus X) \times [\frac{1}{2}, 1]$ separately. The two maps for halves can be glued together into a single $\tilde{\Phi}^{(2)}$ because the image of $\tilde{\Phi}_1^{(1)}$ does not contain any points of the intersection $(M \setminus X) \times \{\frac{1}{2}\}$ that do not belong to $(L \setminus A) \times \{\frac{1}{2}\}$, and those stay unchanged under both maps, which therefore coincide on the intersection. Then $\tilde{\Phi}$ is an approaching strong deformation retraction, as needed. \square

The final part of this section is not used further in the present work, yet it pertains directly to the search for the model structure, as explained in the introduction:

Theorem 4. *Let $i: A \rightarrow X$ be a closed embedding. Then i is an FDR-embedding if and only if i is a fine shape equivalence.*

Corollary 6. (1) *A closed embedding homotopic to an FDR-embedding is itself an FDR-embedding;*

(2) *Let $i: X \rightarrow Y$ and $j: Y \rightarrow Z$ be closed embeddings. If any two of i , j , and ji are FDR-embeddings, then so is the third;*

(3) *Retracts of FDR-embeddings are FDR-embeddings.*

Remark 7. (1) *The proof of the theorem uses the same idea as for the property of cofibrations briefly remarked upon in [19] (the first paragraph of section 3.), or proven explicitly in [13, Lemma A.2].*

(2) *Cathey proves the properties of the theorem and the corollary for SSDR-maps between compacta (where fine shape reduces to strong shape), albeit in different order and with different methods [5, (1.15), (1.7), (1.5)].*

(3) *Cathey already mentions (in the introduction) that embeddings seem to be the cofibrations for compact strong shape; the same can be said for closed embeddings and fine shape over all metrizable spaces (Cathey did not need to specify “closed” because continuous maps between compacta are always closed). Thus the theorem states that FDR-embeddings are the acyclic, or trivial, cofibrations.*

Proof (of Theorem 4). The “only if” part is automatic. We deal with the “if” part.

First take some ARs M and L containing X and A respectively as closed homotopy negligible subsets, as well as some map $\bar{i}: L \rightarrow M$ extending i and such that $\bar{i}^{-1}(X) = A$. We can assume that \bar{i} is a closed embedding, because we can replace M with the relative metrizable cylinder $MC(\bar{i}|A)$; for the same reason we can assume that there is an approaching strong deformation retraction of $(M \setminus X) \times [0, 1]$ onto $(L \setminus A) \times [0, 1] \cup (M \setminus X) \times \{0\}$ as in the proof of Lemma 7.

Since i is a fine shape equivalence, the inverse of the fine shape class $[i]_{fSh} = [\bar{i}]_{L \setminus A}$ exists and can be represented by some approaching map $\phi: M \setminus X \rightarrow L \setminus A$, along with approaching homotopies $\Phi: (M \setminus X) \times [0, 1] \rightarrow M \setminus X$ and $\Psi: (L \setminus A) \times [0, 1] \rightarrow L \setminus A$ such that $\Phi_0 = id_{M \setminus X}$, $\Phi_1 = \bar{i}\phi$, $\Psi_0 = id_{L \setminus A}$, and $\Psi_1 = \phi\bar{i}|_{L \setminus A}$. In particular, the image of Φ_1 lies in $L \setminus A$; so define an approaching map $\omega: (L \setminus A) \times [0, 1] \cup (M \setminus X) \times \{0\} \rightarrow L \setminus A$ by $\omega|_{M \setminus X} = \Phi_1$ and $\omega|_{(L \setminus A) \times [0, 1]}(l, t) = \Psi_{1-t}(l)$, so that $\omega(l, 1) = l$ for all $l \in L \setminus A$; this is possible because, again, for all $l \in L$, we have $\Phi_1(l) = \phi(l)$, as \bar{i} is a closed embedding.

By the approaching strong deformation retraction mentioned above, we can extend ω to an approaching map $\Omega: (M \setminus X) \times [0, 1] \rightarrow L \setminus A$; in particular, Ω_1 is an approaching retraction of $M \setminus X$ onto $L \setminus A$. Compose Φ_t and Ω_t into a map $\Omega': (M \setminus X) \times [0, 1] \rightarrow M \setminus X$ with

$$\Omega'_t = \begin{cases} \Phi_{2t}, & t < \frac{1}{2} \\ \Omega_{2t-1}, & t \geq \frac{1}{2} \end{cases}$$

Thus Ω' is an approaching deformation retraction joining $\Omega'_0 = \Phi_0 = id_{M \setminus X}$ and $\Omega'_1 = \Omega_1$.

Now from the proof of Lemma 7 again, there is a map $\theta: (M \setminus X) \times [0, 1] \times [0, 1] \rightarrow (M \setminus X) \times [0, 1]$ which is an approaching strong deformation retraction onto $(L \setminus A) \times [0, 1] \cup (M \setminus X) \times \{1\}$. Define $\Theta: (M \setminus X) \times [0, 1] \rightarrow M \setminus X$

by $\Theta(m, t) := \Omega' \circ \theta(m, 0, t)$. Then $\Theta_0 = \Omega'_0 = id_{M \setminus X}$, the image of Θ_1 is in $L \setminus A$, and for all $l \in L \setminus A$, $\Theta(l, t) = \Omega' \circ \theta_t(l, 0) = \Omega'(l, 0) = l$. Thus Θ is an approaching strong deformation retraction of $M \setminus X$ onto $L \setminus A$, proving i to be an FSSDR-embedding. \square

4 The left fraction localization

Proposition 8. *Let $i: A \rightarrow X$ be an FDR-embedding and $f: A \rightarrow Y$ be a map. Then there exists a space Z with maps $j: Y \rightarrow Z$ and $g: X \rightarrow Z$ such that $gi \simeq jf$ and j is an FDR-embedding.*

Proof. Take Z to be $MC_X(f)$. Take j and f to be the usual inclusions of Y and X in $MC_X(f)$. Clearly $gi \simeq jf$ as usual for a mapping cylinder.

To show j is an FDR-embedding, first for $A \subseteq X$ we have some M , L , and Φ as in Definition 21. Also let N be an AR containing Y as a closed homotopy negligible subset. The map f extends to a map $\bar{f}: L \rightarrow N$ with $\bar{f}^{-1}(Y) = X$. For an AR containing Z , take $MC_M(\bar{f})$, which contains $MC_X(f)$ as a closed homotopy negligible subset by Lemma 3. Now by applying Φ to $M \setminus X$ while keeping each other point of $MC_M(\bar{f})$ in place, $MC_M(\bar{f}) \setminus MC_X(f)$ approaching strong deformation retracts onto $MC(f) \setminus MC(f)$. The latter in turn approaching strong deformation retracts onto its base (this even extends to a strong deformation retraction of $MC(f)$ onto N , of course). Composing the two proves j to be an FDR-embedding, as required. \square

Remark 8. *Clearly Z here is the pushout of the diagram in hM. Thus the proposition also says that in hM, the pushout of a homotopy class represented by an FDR-embedding is itself such a class.*

Note also that M, unlike hM, does not even have all pushouts (the topological pushout — the adjunction space — of metrizable spaces may not be metrizable). This is why we need to work with the homotopy category, same as Cathey and others before.

Proposition 9. *Let $i: A \rightarrow X$ be an FDR-embedding and $u, v: X \rightarrow Y$ be two maps such that there is a homotopy $G: ui \simeq vi$. Then there exist a space Z and an FDR-embedding $j: Y \rightarrow Z$ such that $ju \simeq jv$.*

Proof. Maps u , v , and G combine into a map $f: A \times [0, 1] \cup X \times \{0, 1\} \rightarrow Y$ by $f|_{X \times \{0\}} := u$, $f|_{X \times \{1\}} := v$, and $f|_{A \times [0, 1]} := G$. The inclusion $A \times [0, 1] \cup X \times \{0, 1\} \subseteq X \times [0, 1]$ is an FDR-embedding by Corollary 5, so we can apply the previous proposition to this inclusion and f to obtain a space Z , an FDR-embedding $j: Y \rightarrow Z$, and a homotopy $F: X \times [0, 1] \rightarrow Z$ such that $F_0 \simeq ju$ and $F_1 \simeq jv$. \square

Now we can define the left fraction category we shall use.

Definition 22. *In hM, we denote by $[FDR]$ the class of morphisms consisting of all homotopy classes that can be represented by FDR-embeddings (cf. Corollary 6(1)).*

One important property here is that if a homotopy class $[i]$ is in $[FDR]$, then the fine shape class $[i]_{fSh}$ is an isomorphism, and thus has an inverse $[i]_{fSh}^{-1}$.

Theorem 5. *In \mathbf{hM} , the class $[FDR]$ satisfies the conditions for left invertibility of Definition 19.*

Proof. Condition (1) is by Proposition 7. Conditions (2) and (3) are by Propositions 8 and 9. \square

Corollary 7. *There exists a well-defined category of left fractions $[FDR]\backslash\mathbf{hM}$.*

5 The S functor

The primary goal of this work is to prove $\mathbf{fSh}(\mathbf{M})$ and $[FDR]\backslash\mathbf{hM}$ to be isomorphic categories, thus extending the result of Cathey. So we need to define two functors between these that will be inverse to each other. One is readily defined.

Theorem 6. *There is a functor $S: [FDR]\backslash\mathbf{hM} \rightarrow \mathbf{fSh}(\mathbf{M})$, uniquely defined by the following properties:*

- (1) S is constant on objects;
- (2) Whenever a left fraction class can be represented by the fraction $[i]\backslash[u]$, $S([i]\backslash[u])$ is the fine shape class $[i]_{fSh}^{-1} \circ [u]_{fSh}$.

Proof. The action of S is already defined by the properties, so we need only check that $S([i]\backslash[u])$ does not depend on the choice of fraction representing the given fraction class, and that S respects identity and composition.

First, equivalent fractions are sent to the same fine shape class. To wit, assume $X \xrightarrow{u} Z \xleftarrow{i} Y$ and $X \xrightarrow{v} W \xleftarrow{j} Y$ are cospans of specific maps such that the fractions of their homotopy classes are in the same equivalence class. This means there exist $Z \xrightarrow{f} P \xleftarrow{g} W$ with $fu \simeq gv$ and $fi \simeq gj$, and the latter class belonging to $[FDR]$. Now since i, j , and $fi \simeq gj$ are all fine shape equivalencies, so must be f and g by Proposition 4(1) (albeit the homotopy classes $[f]$ and $[g]$ might not be in $[FDR]$). Then we have $[i]_{fSh}^{-1} \circ [u]_{fSh} = [fi]_{fSh}^{-1} \circ [fu]_{fSh} = [gj]_{fSh}^{-1} \circ [gv]_{fSh} = [j]_{fSh}^{-1} \circ [v]_{fSh}$, as required.

Second, an identity left fraction class clearly maps to the corresponding identity fine shape class.

Third, consider the composition. If we have fraction classes represented by specific maps $X \xrightarrow{u} P \xleftarrow{i} Y$ and $Y \xrightarrow{v} Q \xleftarrow{j} Z$ respectively, then the composition $([j]\backslash[v]) \circ ([i]\backslash[u])$ can be represented by $X \xrightarrow{fu} W \xleftarrow{kv} Z$ for some $P \xrightarrow{f} W \xleftarrow{k} Q$, where $fi \simeq kv$, and k is an FDR-embedding. In that case we obtain $S(([j]\backslash[v]) \circ ([i]\backslash[u])) = [kv]_{fSh}^{-1} \circ [fu]_{fSh} = [j]_{fSh}^{-1} \circ [k]_{fSh}^{-1} \circ [fu]_{fSh} \circ [i]_{fSh}^{-1} \circ [u]_{fSh} = [j]_{fSh}^{-1} \circ [kv]_{fSh} \circ [i]_{fSh}^{-1} \circ [u]_{fSh} = [j]_{fSh}^{-1} \circ [v]_{fSh} \circ [i]_{fSh}^{-1} \circ [u]_{fSh}$, which is exactly the composition $S([j]\backslash[v]) \circ S([i]\backslash[u])$. \square

Now our goal is to define the inverse functor to S . For this we shall need an additional construction.

6 Approaching and fine shape cylinders

The basic idea is that given an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ with X additive homotopy negligible in M (see Definition 13), we shall construct an object $(AC(\phi), FC(\phi))$ of MAppr that will be somewhat similar to a metrizable mapping cylinder in \mathbf{M} , or to a mapping cylinder in the topological category.

The primary property of $(AC(\phi), FC(\phi))$ is to be this: we want to have inclusions of pairs $\bar{u}: (M, X) \rightarrow (AC(\phi), FC(\phi))$ and $\bar{v}: (N, Y) \rightarrow (AC(\phi), FC(\phi))$ that satisfy $\bar{u}^{-1}(AC(\phi)) = X$ and $\bar{v}^{-1}(FC(\phi)) = Y$; and for any approaching homotopy $\Psi: (M \times [0, 1]) \setminus (X \times [0, 1]) \rightarrow L \setminus Z$ between $\Psi_0 = \psi: M \setminus X \rightarrow L \setminus Z$ and $\Psi_1 = \chi \circ \phi$ for some $\chi: N \setminus Y \rightarrow L \setminus Z$, we want to have an approaching map $\omega: (AC(\phi), FC(\phi)) \rightarrow L \setminus Z$ such that $\omega \circ \psi = \bar{u}|_{M \setminus X}$ and $\omega \circ \chi = \bar{v}|_{N \setminus Y}$.

For compact spaces, it is possible to dispense with the requirement of additive homotopy, and define a metrizable topology on $M \times (0, 1] \cup N$ making the pair $(M \times (0, 1] \cup N, X \times (0, 1] \cup Y)$ into an object having the desired property. Explicit definition is given in particular by Mrozek [14] (the pair $(D(\phi), C(\phi))$ there, defined in the beginning of section 2.), who refers to an earlier equivalent construction of Ferry [8]. In fact, Cathey's work [5] already effectively makes use of the same construction (note the space Z in the proof of Corollary (2.7) there), though no name or notation is given.

Unfortunately, it seems that this construction cannot be directly applied to noncompact spaces; see the appendix for a discussion of the problem. Instead, we shall obtain the desired pair of spaces by taking larger — and more convoluted — underlying sets.

The rest of this section, along with the next one, shall use the notation from the following

Definition 23. *Consider an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ with X additive homotopy negligible in M , and choose some specific additive homotopy $H: M \times [0, 1] \rightarrow M$ with $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$.*

In the metrizable join $M \star N$, take the following four subsets:

$$C_+(\phi, H) := \{(m, s, \phi \circ H_s(m)) \mid m \in M \setminus X, s \in (0, 1)\}$$

$$C_{+,0}(\phi, H) := \{(x, s, \phi \circ H_s(x)) \mid x \in X, s \in (0, 1)\}$$

$$C_-(\phi) := \{(m, s, n) \mid m \in M \setminus X, s \in (-1, 0], n \in N \setminus Y\}$$

$$C_{-,0}(\phi) := \{(x, s, y) \mid x \in X, s \in (-1, 0], y \in Y\}$$

Of course, C_- and $C_{-,0}$ depend only on (M, X) and (N, Y) , and not on the action of ϕ or on H . In addition, $C_-(\phi)$ is the metrizable mapping cylinder of the projection

$$(M \setminus X) \times (N \setminus Y) \ni (m, n) \mapsto n \in (N \setminus Y),$$

whereas $C_{-,0}(\phi)$ is the metrizable mapping cylinder of the projection

$$X \times Y \ni (x, y) \mapsto y \in Y$$

Finally, we define the approaching cylinder of (ϕ, H) to be the subset $AC(\phi, H) := M \cup C_+ \cup C_{+,0} \cup C_- \cup C_{-,0} \cup N$ of $M \star N$, and the fine shape cylinder of (ϕ, H) to be $FC(\phi, H) := X \cup C_{+,0} \cup C_{-,0} \cup Y$.

Remark 9. The compact approaching cylinder $C(\phi)$ of Mrozek has $X \times (0, 1] \cup Y$ as its underlying set (same as the usual mapping cylinder). Our construction of $FC(\phi, H)$, on the other hand, inserts the “negative half” $C_{-,0}(\phi) = X \times (-1, 0] \times Y$. In essence, we take the cylinder of Mrozek, “blow up” Y into $X \times Y$ (replacing every point of Y by a whole copy of X), and then attach the mapping cylinder of the projection $X \times Y \rightarrow Y$ (as we wouldn’t have an embedding of Y otherwise).

Proposition 10. In the notation of previous definition,

(1) $M \cup C_+(\phi, H) \cup C_{+,0}(\phi, H)$ is homeomorphic to $M \times (0, 1]$, and $AC(\phi, H) \setminus FC(\phi, H)$ is homeomorphic to the union $MC(p) \cup MC(q)$, where $p: M \setminus X \rightarrow (M \setminus X) \times (N \setminus Y)$ and $q: (M \setminus X) \times (N \setminus Y) \rightarrow N \setminus Y$ are defined by $p(m) := (m, \phi(m))$ and $q(m, n) := n$, and the union is along the image of p (equivalently, the graph of ϕ in $(M \setminus X) \times (N \setminus Y)$);

(2) $FC(\phi, H)$ is a closed homotopy negligible subset of $AC(\phi, H)$ – additive homotopy negligible if Y is additive homotopy negligible in N ;

(3) The choice of H does not affect the isomorphism class of $(AC(\phi, H), FC(\phi, H))$ in \mathbf{MAppr} ;

(4) If M and N are ARs, then $AC(\phi, H)$ is also an AR, and the fine shape class of $FC(\phi, H)$ is also independent of the choice of H .

Proof. (1) First, we have the bijection

$$M \cup C_+(\phi, H) \cup C_{+,0}(\phi, H) \ni (m, s, \phi \circ H_s(m)) \leftrightarrow (m, s) \in M \times (0, 1].$$

As the spaces are metrizable, we can prove continuity in terms of sequence convergence, as per Remark 1(5). Thus we prove that $(m_k, s_k, \phi \circ H_{s_k}(m_k)) \rightarrow (m, s, \phi \circ H_s(m))$ in $M \cup C_+(\phi, H) \cup C_{-,0}(\phi, H)$ if and only if $(m_k, s_k) \rightarrow (m, s)$ in $M \times (0, 1]$. The “only if” part is automatic, and for the “if” part, $(m_k, s_k) \rightarrow (m, s)$ implies $H_{s_k}(m_k) \rightarrow H_s(m)$, thus $\phi \circ H_{s_k}(m_k) \rightarrow \phi \circ H_s(m)$, since $s > 0$ and so $H_s(m) \in M \setminus X$.

Thus in particular $(M \setminus X) \cup C_+(\phi, H)$ is homeomorphic to $MC(p) \setminus ((M \setminus X) \times (N \setminus Y))$. The second homeomorphism now becomes obvious, as $C_-(\phi) \cup (N \setminus Y)$ is already precisely $MC(q)$ (note that $m_k \rightarrow m \in M \setminus X$ and $s_k \rightarrow 0$ imply $\phi \circ H_{s_k}(m_k) \rightarrow \phi(m)$ for any choice of H).

(2) The closed part is clear. Now take some homotopy $G: N \times [0, 1] \rightarrow N$ with $G_0 = id_N$ and $G(N \times (0, 1]) \subseteq N \setminus Y$. We shall define the required

homotopy $F: AC(\phi, H) \times [0, 1] \rightarrow AC(\phi, H)$ first on $C_-(\phi) \cup C_{-,0}(\phi) \cup N$ by $F_t(m, s, n) := (H_t(m), s, G_t(n))$.

Now define F on $M \cup C_+(\phi, H) \cup C_{+,0}(\phi, H)$ by

$$F_t(m, s, \phi \circ H_s(m)) := \begin{cases} (H_t(m), s - t, \phi \circ H_{s-t} \circ H_t(m)), & t < s \\ (H_{t-s} \circ H_s(m), 0, G_{t-s} \circ \phi \circ H_s(m)), & t \geq s \end{cases}$$

Then $F_0 = id_{AC(\phi, H)}$ and $F(AC(\phi, H) \times (0, 1]) \cap FC(\phi, H) = \emptyset$, as required. Moreover, F is fully continuous: the only complicated case is when a sequence $\{(m_k, s_k, \phi \circ H_{s_k}(m_k))\}$ of points of $M \cup C_+(\phi, H) \cup C_{+,0}(\phi, H)$ converges to a point $(x, 0, y) \in C_{-,0}(\phi)$. In this case, we have $m_k \rightarrow x$, $s_k \rightarrow 0$, and $\phi \circ H_{s_k}(m_k) \rightarrow y$. Now for a sequence $t_k \rightarrow t$ in $[0, 1]$, consider two separate cases: $t_k < s_k$ and $t_k \geq s_k$; any sequence can clearly be split into two (at most) sequences with either satisfying one of these conditions for all indices.

In the case $t_k \geq s_k$, we have $H_{s_k}(m_k) \rightarrow x = H_0(x)$ (as $s_k \rightarrow 0$, $m_k \rightarrow x$, and H is continuous), so $H_{t_k-s_k} \circ H_{s_k}(m_k) = H_{t_k}(m_k) \rightarrow H_t(x)$, because H is additive; and also from $\phi \circ H_{s_k}(m_k) \rightarrow y$ we get $G_{t_k-s_k} \circ \phi \circ H_{s_k}(m_k) \rightarrow G_t(y)$ (again, as $s_k \rightarrow 0$), thus $F_{t_k}(m_k, s_k, \phi \circ H_{s_k}(m_k)) = (H_{t_k-s_k} \circ H_{s_k}(m_k), 0, G_{t_k-s_k} \circ \phi \circ H_{s_k}(m_k)) \rightarrow (H_t(x), 0, G_t(y))$.

In the case $t_k < s_k$, we must have $t_k \rightarrow 0$, and therefore $\phi \circ H_{s_k-t_k} \circ H_{t_k}(m_k) = \phi \circ H_{s_k}(m_k) \rightarrow y = G_0(y)$, since H is additive. Thus $F_{t_k}(m_k, s_k, \phi \circ H_{s_k}(m_k)) = (H_{t_k}(m_k), s_k - t_k, \phi \circ H_{s_k-t_k} \circ H_{t_k}(m_k)) \rightarrow (H_0(x), 0, G_0(y)) = (x, 0, y)$, as required.

Finally, if Y is additive homotopy negligible in N , then G can be chosen additive. Now F becomes additive, as H and G are.

(3) Let $H^{(1)}$ and $H^{(2)}$ be two possible choices of homotopy, and denote $AC_i := AC(\phi, H^{(i)})$ and $FC_i := FC(\phi, H^{(i)})$ for $i = 1, 2$. By (2), the pairs (AC_1, FC_1) and (AC_2, FC_2) are objects of MAppr ; by (1), there is a homeomorphism between $AC_1 \setminus FC_1$ and $AC_2 \setminus FC_2$. This homeomorphism is FC_1 - FC_2 -approaching: it extends by two separate homeomorphisms onto $X \cup C_{-,0}$ and onto $C_{+,0} \cup Y$, so the only case we need to check is that of a sequence $(m_k, s_k, H_{s_k}^{(1)}(m_k))$ in $C_+(\phi, H^{(1)})$ converging to a point $(x, 0, y) \in C_{-,0}(\phi)$. But in this case $m_k \rightarrow x$ and $s_k \rightarrow 0$, so also $H_{s_k}^{(2)}(m_k) \rightarrow x$, and then $\phi \circ H_{s_k}^{(2)}(m_k)$ must have some accumulation point $y' \in Y$. Thus $(x, 0, y')$ is an accumulation point of $(m_k, s_k, H_{s_k}^{(2)}(m_k))$, proving the homeomorphism approaching. The inverse homeomorphism is similarly FC_2 - FC_1 -approaching, so we have an isomorphism of (AC_1, FC_1) and (AC_2, FC_2) as objects of MAppr .

(4) As $FC(\phi, H)$ is a homotopy negligible subset by (2), it suffices to prove that $AC(\phi, H) \setminus FC(\phi, H)$ is an AR, as per Proposition 2(7). But by (1), $AC(\phi, H) \setminus FC(\phi, H)$ is an intersection of two metrizable mapping cylinders between ARs along an AR (the graph of ϕ in $(M \setminus X) \times \{0\} \times (N \setminus Y) \subset M \star N$, which is homeomorphic to $M \setminus X$) that is a closed subset of either. Also given

any choices of $H^{(1)}$ and $H^{(2)}$, the approaching homeomorphisms of (3) now provide a fine shape isomorphism between $FC(\phi, H^{(1)})$ and $FC(\phi, H^{(2)})$. \square

It follows from the proposition that all versions of $FC(\phi)$ for various H represent the same isomorphism class of objects of $\text{fSh}(M)$, and all versions of the pair $(AC(\phi), FC(\phi))$ represent the same isomorphism class of objects of MAppr . Moreover, the fine shape class of any specific approaching map from $AC(\phi) \setminus FC(\phi)$, or into it, is also independent of the choice of H . Thus we are justified in adopting the following

Convention. *From now on, we shall refer to the two cylinders as $AC(\phi)$ and $FC(\phi)$, understanding that H is chosen arbitrarily (as long as it is additive). Whenever needed, an arbitrary point of $C_+(\phi, H) \cup C_{+,0}(\phi, H)$ shall be denoted by $(m, s, \phi \circ H_s(m))$ for some $m \in M$ and $s \in (0, 1)$, and a point of $C_-(\phi) \cup C_{-,0}(\phi)$ by (m, s, n) , where $m \in M$, $s \in (-1, 0]$, and $n \in N$.*

The first indication that these cylinders allow us to invert the S functor of section 5 is given by

Proposition 11. *Given an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$, if M and N are ARs, and X is additive homotopy negligible in M , denote the standard embeddings as $u: X \rightarrow FC(\phi)$ and $i: Y \rightarrow FC(\phi)$. Then i is an FDR-embedding, and $[i]_{fSh}^{-1} \circ [u]_{fSh} = [\phi]$.*

Corollary 8. *For any fine shape class $[\phi] \in [X, Y]_{fSh}$, there exist a space Z along with maps (which can be chosen to be closed embeddings) $u: X \rightarrow Z$ and $i: Y \rightarrow Z$ such that i is a fine shape equivalence and $[i]_{fSh}^{-1} \circ [u]_{fSh} = [\phi]$.*

Proof (of Proposition 11). u and i readily extend to embeddings $\bar{u}: M \rightarrow AC(\phi)$ and $\bar{i}: N \subset AC(\phi)$, and $FC(\phi)$ is closed and homotopy negligible in $AC(\phi)$ by the point 2) of the previous proposition. The approaching strong deformation retraction of $AC(\phi) \setminus FC(\phi)$ onto $N \setminus Y$ can be constructed in two steps. First, strong deformation retract $AC(\phi) \setminus FC(\phi)$ onto $C_-(\phi) \cup (N \setminus Y)$ by retracting each interval $\{(m, s, \phi \circ H_s(m)) \mid s \in [0, 1]\}$ along itself; given the point 1) of the previous proposition, this is the same as retracting the metrizable cylinder of the map

$$M \setminus X \ni m \mapsto (m, \phi(m)) \in (M \setminus X) \times (N \setminus Y).$$

Second, strong deformation retract $C_-(\phi) \cup (N \setminus Y)$ onto $N \setminus Y$, treating it as the metrizable cylinder of the map

$$(M \setminus X) \times (N \setminus Y) \ni (m, n) \mapsto n \in (N \setminus Y).$$

Both these retractions are clearly correctly approaching, noting yet again that if $(m_k, s_k, \phi \circ H_{s_k}(m_k)) \rightarrow (x, 0, y) \in C_{-,0}(\phi)$, then $\phi \circ H_{t_k}(m_k)$ has an accumulation point in Y for any sequence $t_k \rightarrow 0$.

Now for any point $m \in M$, we have $\bar{u}(m) = (m, 1)$, whose image at the end of this approaching strong deformation retraction is $\phi(m)$. Thus the class $[i]_{fSh}^{-1} \circ [u]_{fSh}$ is represented by ϕ . \square

Remark 10. Consider a fine shape class induced by a map $f: X \rightarrow Y$. For any ARs M and N containing X and Y respectively as closed homotopy negligible subsets (additive or not), and any extension $\bar{f}: M \rightarrow N$ with $\bar{f}^{-1}(Y) = X$, the metrizable mapping cylinder $MC(f)$ is closed and homotopy negligible in $MC(\bar{f})$ (by [13, Proposition 19.8(a)]), which is an AR (by Proposition 2(3)); thus $\bar{f}|_{M \setminus X}$ represents $[f]_{fSh}$. Since $MC(\bar{f})$ strong deformation retracts onto N , restricting to a strong deformation retraction of $MC(f)$ onto Y , we conclude that $MC(f)$ is (in particular) fine shape isomorphic to Y , and therefore also to $FC(f|_{M \setminus X})$. Thus our cylinder is consistent, at least in fine shape, with the usual metrizable cylinder.

Now we state the first result related to the fine shape mapping cylinder and extension set of an approaching map (see Definition 12).

Proposition 12. Let X be closed additively homotopy negligible in M , and let $\phi: M \setminus X \rightarrow N \setminus Y$ be an approaching map that extends on $A \subseteq X$ by a map $f: A \rightarrow Y$. Then there is an embedding of the metrizable mapping cylinder $MC_X(f)$ into $FC(\phi)$; therefore, if $u: X \subset FC(\phi)$ and $i: Y \subset FC(\phi)$ are the standard embeddings, then $if \simeq u|_A$.

Proof. Given any choice of homotopy H , we embed $MC(f) = X \times \{1\} \cup A \times (0, 1) \cup N$ by sending a point $y \in Y$ or $x \in X$ to itself (through the standard embeddings), while sending a point $(a, s) \in A \times (0, 1)$ to $(a, 2s - 1, \phi \circ H_{2s-1}(a))$ for $s \in (\frac{1}{2}, 1]$ or to $(a, 2s - 1, f(a))$ for $s \in (0, \frac{1}{2}]$. Continuity of this mapping is assured by the fact that $A \subseteq X(\phi)$. Specifically, denote the combination of ϕ and f by $\bar{\phi}$. Whenever $m_k \rightarrow m$ in $(M \setminus X) \cup A$, we have $\bar{\phi}(m_k) \rightarrow \bar{\phi}(m)$, and if also $s_k \rightarrow s \geq 0$, then $H_{s_k}(m_k) \rightarrow H_s(m)$, so $\bar{\phi} \circ H_{s_k}(m_k) \rightarrow \bar{\phi} \circ H_s(m)$. \square

The last proposition of this section describes the primary property of the pair $(AC(\phi), FC(\phi))$ that makes it similar to the usual mapping cylinder. In addition, it also includes a result about extension sets. This, combined with the previous proposition, will be used to prove fraction equivalences in the next section.

Proposition 13. Let X be closed additively homotopy negligible in M . Assume an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$. Take $(AC(\phi), FC(\phi))$, and denote the embeddings of M and N by \bar{u} and \bar{i} respectively. Given approaching maps $\psi: M \setminus X \rightarrow L \setminus Z$ and $\chi: N \setminus Y \rightarrow L \setminus Z$, along with an approaching homotopy $\Psi: (M \times [0, 1]) \setminus (X \times [0, 1]) \rightarrow L \setminus Z$ with $\Psi_1 = \psi$ and $\Psi_0 = \chi \circ \phi$, there is an approaching map $\omega: AC(\phi) \setminus FC(\phi) \rightarrow L \setminus Z$ such that $\omega \circ \bar{u}|_{M \setminus X} = \psi$ and $\omega \circ \bar{i}|_{N \setminus Y} = \chi$. Moreover, ω can be chosen so that it extends on $X(\psi)$ and $Y(\chi)$ by the same maps that extend ψ and χ .

Proof. Given any choice of additive homotopy H , define ω by

$$\omega|_{(M \setminus X) \cup C_+(\phi, H)}(m, s, \phi \circ H_s(m)) := \begin{cases} \psi(m), & s \in [\frac{1}{2}, 1] \\ \Psi_{2s}(m), & s \in (0, \frac{1}{2}] \end{cases}$$

and $\omega|_{C_-(\phi) \cup (N \setminus Y)}(m, s, n) := \chi(n)$. All required properties are trivial to prove. \square

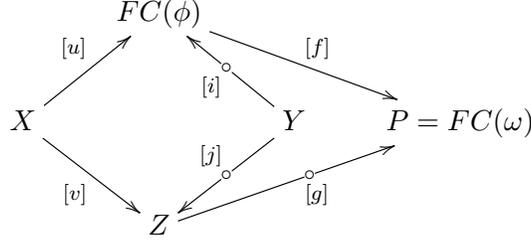
Remark 11. *In the topological category, the usual mapping cylinder of a map $f: X \rightarrow Y$ is the pushout of the diagram $X \times [0, 1] \xleftarrow{x=(x,0)} X \xrightarrow{f} Y$; the same is true for the metrizable mapping cylinder in \mathbf{M} (with respect to all metrizable spaces, but not to all topological ones), which had to be defined, to begin with, because the topological pushout of metrizable spaces may not be metrizable. The situation with \mathbf{MAppr} seems to be even worse and less studied too; our proposition, therefore, does not even show the existence part of the same universal property (and our cylinder pair, while definable even with H not additive, may not be an object of \mathbf{MAppr} in that case). It is not immediately clear whether \mathbf{MAppr} has pushouts even of this restricted type (see the appendix; the counterexample is applicable here too). What we do prove suffices for our purposes, but there may be some interest in studying the matter further.*

7 The T functor

Now we are in position to define the inverse to S . First we provide a lemma that uses Propositions 13 and 12 to prove fraction equivalence, and will be used in proving the next two theorems.

Lemma 8. *Let $\phi: M \setminus X \rightarrow N \setminus Y$ be an approaching map, $u: X \rightarrow FC(\phi)$ and $i: Y \rightarrow FC(\phi)$ be the standard embeddings, and $(M, X) \xrightarrow{\bar{v}} (L, Z) \xleftarrow{\bar{j}} (N, Y)$ be a cospan of maps of pairs with $\bar{v}^{-1}(Z) = X$ and $\bar{j}^{-1}(Z) = Y$; here M, N , and L are ARs, and X, Y , and Z are their closed homotopy negligible subsets, with X and Y additively so. Denote $v := \bar{v}|_X: X \rightarrow Z$ and $j := \bar{j}|_Y: Y \rightarrow Z$. If the homotopy class $[j]$ is in $[FDR]$, and if there is an approaching homotopy $\Omega: (M \times [0, 1]) \setminus (X \times [0, 1]) \rightarrow L \setminus Z$ between $\Omega_0 = \bar{v}|_{M \setminus X}$ and $\Omega_1 = \bar{j}|_{N \setminus Y} \circ \phi$, then the left fractions $X \xrightarrow{[v]} Z \xleftarrow{[j]} Y$ and $X \xrightarrow{[u]} FC(\phi) \xleftarrow{[i]} Y$ are equivalent.*

Proof. To prove the fractions equivalent, we need to exhibit a cospan $FC(\phi) \xrightarrow{[f]} P \xleftarrow{[g]} Z$ such that $[fu] = [gv]$, $[fi] = [gj]$, and the latter class is in $[FDR]$. So use Proposition 13 to obtain an approaching map $\omega: AC(\phi) \setminus FC(\phi) \rightarrow L \setminus Z$ that extends on X and Y by v and j respectively. Now take P to be $FC(\omega)$ (note that by Proposition 10(2), $FC(\phi)$ is additive homotopy negligible in $AC(\phi)$ since Y is so in N ; thus $(AC(\omega), FC(\omega))$ is defined), with f and g the standard embeddings:



Then by Proposition 12, P contains embeddings of $MC(v)$ and $MC(j)$ that prove $[fu] = [gv]$ and $[fi] = [gj]$, and $[gj]$ is in $[FDR]$ as $[j]$ and $[g]$ are. \square

Remark 12. *The only reason we require $[j] \in [FDR]$ is that otherwise the span $X \xrightarrow{[v]} Z \xleftarrow{[j]} Y$ does not define a fraction class in $[FDR] \setminus hM$. On the other hand, the approaching homotopy Ω may in some cases be provided by composing $\bar{v}|_{M \setminus X}$ with the approaching strong deformation retraction of $L \setminus Z$ onto $N \setminus Y$. The proof of Theorem 8 below uses exactly this.*

Also note that Z does not have to be additive homotopy negligible in L .

With this lemma we can prove

Theorem 7. *There is a functor $T: \text{fSh}(M) \rightarrow [FDR] \setminus hM$, uniquely defined by the following properties:*

(1) *T is constant on objects;*

(2) *Whenever a fine shape class $[\phi] \in [X, Y]_{fSh}$ can be represented by an $X - Y$ -approaching map $\phi: M \setminus X \rightarrow N \setminus X$, where M and N are ARs containing X and Y as closed additive homotopy negligible subsets, the left fraction class $T([\phi])$ can be represented by the fraction $X \xrightarrow{[u]} FC(\phi) \xleftarrow{[i]} Y$, where $[u]$ and $[i]$ are the homotopy classes of the standard embeddings.*

Proof. The properties define the action of T uniquely, and the left fraction specified in (2) is well-defined, as $i: Y \rightarrow FC(\phi)$ is an FDR-embedding by Proposition 11. Thus we only need to check that T is a functor.

First, we show that for any two approaching maps representing the same fine shape class, the left fractions specified in (2) are equivalent. To that end, let a given fine shape class from X to Y be represented by both approaching maps $\phi: M \setminus X \rightarrow N \setminus Y$ and $\psi: M' \setminus X \rightarrow N' \setminus Y'$, where M and M' are ARs each containing X as a closed additive homotopy negligible subset, and N and N' are ARs containing Y as such. This means (by Definition 15) that for some approaching maps $\bar{id}_X: M \setminus X \rightarrow M' \setminus X$ and $\bar{id}_Y: N \setminus Y \rightarrow N' \setminus Y$ extending id_X and id_Y we have an approaching homotopy $\Psi: (M \setminus X) \times [0, 1] \rightarrow N' \setminus Y$ between $\Psi_1 = \psi \circ \bar{id}_X$ and $\Psi_0 = \bar{id}_Y \circ \phi$. Let $X \xrightarrow{[u]} FC(\phi) \xleftarrow{[i]} Y$ and $X \xrightarrow{[v]} FC(\psi) \xleftarrow{[j]} Y$ be the fractions consisting of the homotopy classes of the embeddings (the embeddings themselves shall be denoted by u, v, i, j accordingly). We want to prove them equivalent.

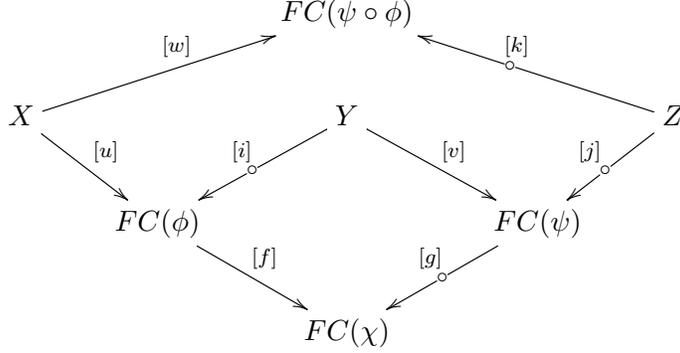
The maps u, v, i, j extend to embeddings $M \xrightarrow{\bar{u}} AC(\phi) \xleftarrow{\bar{i}} N$ and $M' \xrightarrow{\bar{v}} AC(\psi) \xleftarrow{\bar{j}} N'$. We have approaching maps $\bar{v} \circ \bar{id}_X: M \setminus X \rightarrow AC(\psi) \setminus FC(\psi)$ and $\bar{j} \circ \bar{id}_Y: N \setminus Y \rightarrow AC(\psi) \setminus FC(\psi)$. Assuming $AC(\psi) = AC(\psi, H)$ for some specific homotopy H , define $\Omega: (M \setminus X) \times [0, 1] \rightarrow AC(\psi) \setminus FC(\psi)$ by

$$\Omega(m, s) := \begin{cases} (\bar{v} \circ \bar{id}_X(m), 4s - 3, \psi \circ H_{4s-3} \circ \bar{v} \circ \bar{id}_X(m)), & s \in [\frac{3}{4}, 1] \\ (\bar{v} \circ \bar{id}_X(m), 4s - 3, \psi \circ \bar{v} \circ \bar{id}_X(m)), & s \in [\frac{1}{2}, \frac{3}{4}] \\ \Psi_{2s}(m) \in (N' \setminus Y) \subset AC(\psi) \setminus FC(\psi), & s \in [0, \frac{1}{2}] \end{cases}$$

As defined, Ω is an approaching homotopy between $\Omega_1 = \bar{v} \circ \bar{id}_X$ and $\Omega_0 = \bar{j} \circ \bar{id}_Y \circ \phi$. Then by Lemma 8, $[i] \setminus [u]$ and $[j] \setminus [v]$ are equivalent.

Second, it is clear that an identity fine shape class $[id_X]_{fsh}$ corresponds to the identity fraction class: represent $[id_X]_{fsh}$ by id_M for some AR M containing X as a closed additive homotopy negligible subset, and note that by Proposition 12, $FC(id_M)$, contains a copy of $X \times [0, 1] = MC(id_X)$, which makes the fraction $X \rightarrow FC(id_M) \leftarrow X$ equivalent to $X \rightarrow X \times [0, 1] \leftarrow X$, which belongs to the identity fraction class.

Third, consider the composition. Assume fine shape classes $[\phi] \in [X, Y]_{fsh}$ and $[\psi] \in [Y, Z]_{fsh}$ are represented by approaching maps $\phi: M \setminus X \rightarrow N \setminus Y$ and $\psi: N \setminus Y \rightarrow L \setminus Z$ (where M, N , and L are ARs containing X, Y , and Z as closed additive homotopy negligible subset), and consider the fractions. We have maps $X \xrightarrow{u} FC(\phi) \xleftarrow{i} Y$, $Y \xrightarrow{v} FC(\psi) \xleftarrow{j} Z$, and $X \xrightarrow{w} FC(\psi \circ \phi) \xleftarrow{k} Z$, which all extend to embeddings $M \xrightarrow{\bar{u}} AC(\phi) \xleftarrow{\bar{i}} N$, $N \xrightarrow{\bar{v}} AC(\psi) \xleftarrow{\bar{j}} L$, and $M \xrightarrow{\bar{w}} AC(\psi \circ \phi) \xleftarrow{\bar{k}} L$. For the composition of fraction classes $[j] \setminus [v] \circ [i] \setminus [u]$, we can take an approaching map $\chi: AC(\phi) \setminus FC(\phi) \rightarrow AC(\psi) \setminus FC(\psi)$ that simply collapses $AC(\phi) \setminus FC(\phi)$ onto $N \setminus Y$, which is then embedded into $AC(\psi) \setminus FC(\psi)$; clearly χ extends on Y by identity. Take $FC(\chi)$ ($FC(\phi)$ is additive homotopy negligible in $AC(\phi)$ since Y is so in N , per Proposition 10(2); thus $FC(\chi)$ is well-defined), and denote the standard embeddings by $FC(\phi) \xrightarrow{f} FC(\chi) \xleftarrow{g} FC(\psi)$, extending to $AC(\phi) \xrightarrow{\bar{f}} AC(\chi) \xleftarrow{\bar{g}} AC(\psi)$. Then $[gv] = [fi]$ (homotopy provided by χ extending on Y and Proposition 12), and g is an FDR-embedding, thus the fraction $[gj] \setminus [fu]$ belongs to the composition class $([j] \setminus [v]) \circ ([i] \setminus [u])$:



We have the embeddings $\bar{f} \circ \bar{u}: M \rightarrow AC(\chi)$ and $\bar{g} \circ \bar{j}: L \rightarrow AC(\chi)$, as well as an approaching homotopy $\Omega: (M \setminus X) \times [0, 1] \rightarrow AC(\chi) \setminus FC(\chi)$ with $\Omega_1 = \bar{f} \circ \bar{u}|_{M \setminus X}$ and $\Omega_0 = \bar{g} \circ \bar{j} \circ \psi \circ \phi$ (provided by approaching strong deformation retracting $AC(\chi) \setminus FC(\chi)$ onto $AC(\psi) \setminus FC(\psi)$ and then onto $L \setminus Z$). Thus by Lemma 8 we obtain the equivalence $T([\psi] \circ [\phi]) = [k] \setminus [w] = [gj] \setminus [fu] = ([j] \setminus [v]) \circ ([i] \setminus [u]) = T([\psi]) \circ T([\phi])$. \square

From this proof, we also derive

Corollary 9. *Given two maps $f, g: X \rightarrow Y$ such that $[f]_{fSh} = [g]_{fSh}$, there exist a space Z along with a map $h: Y \rightarrow Z$ such that h is a closed embedding and a fine shape equivalence, and $hf \simeq hg$. Moreover, there are closed embeddings of $MC(f)$ and $MC(g)$ into Z , at least one of which is a fine shape equivalence.*

Proof. Choose any ARs M and N containing X and Y respectively as closed homotopy negligible subsets, and choose respective extensions $\bar{f}, \bar{g}: M \rightarrow N$ with $\bar{f}^{-1}(Y) = \bar{g}^{-1}(Y) = X$ (of course, $\bar{f}|_X = f$ and $\bar{g}|_X = g$). As per Remark 10, we have well-defined left fractions $X \rightarrow MC(f) \leftarrow Y$ and $X \rightarrow MC(g) \leftarrow Y$ representing $T([f]_{fSh})$ and $T([g]_{fSh})$ (additive homotopy negligibility rendered irrelevant by the same remark); since $[f]_{fSh} = [g]_{fSh}$, these fractions must be equivalent. The corollary now follows from the definition of fraction equivalence (Definition 20(3)).

The more or less explicit construction, as derived from our proofs, is as follows: first we take some approaching homotopy, say, $F: (M \setminus X) \times [0, 1] \rightarrow N \setminus Y$ between $F_0 = \bar{f}|_{M \setminus X}$ and $F_1 = \bar{g}|_{M \setminus X}$ (which affirms the equality $[f]_{fSh} = [g]_{fSh}$ as per Definition 15). Second, we construct $\omega: MC(\bar{f}) \setminus MC(f) \rightarrow MC(\bar{g}) \setminus MC(g)$ as in the proof of Proposition 13, using F , such that ω extends onto the standard embeddings of X and Y into $MC(f)$, and carries those to the standard embeddings of the same into $MC(g)$. Third, as in the proof of Lemma 8, we take $Z = FC(\omega)$; by the construction of $FC(\omega)$, Z contains embeddings of $MC(f)$ and $MC(g)$, the latter being an FDR-embedding (Proposition 11), but also Z contains (Proposition 12) embeddings of $X \times [0, 1]$ and $Y \times [0, 1]$ which are precisely the mapping cylinders of the extensions of ω onto X and Y respectively.

Fourth, consider the embedded thus “rectangular” subspace $MC(f) \cup (X \times [0, 1]) \cup MC(g) \cup (Y \times [0, 1])$ of Z ; here the union is such that $X \times \{0\}$ and $Y \times \{0\}$ are the standard embeddings into $MC(f)$, whereas $X \times \{1\}$ and $Y \times \{1\}$ are those into $MC(g)$. We see that the closed embeddings $h_0: Y = Y \times \{0\} \subset MC(f) \subset Z$ and $h_1: Y = Y \times \{1\} \subset MC(g) \subset Z$ are homotopic (along $Y \times [0, 1]$), and h_1 is an FDR-embedding as a composition of such; choose $h := h_1$. Now the embeddings $X = X \times \{0\} \subset MC(f) \subset Z$ and $X = X \times \{1\} \subset MC(g) \subset Z$ are homotopic to each other (along $X \times [0, 1]$), but also to h_0f and h_1g (along $MC(f)$ and $MC(g)$) respectively, thus $hf = h_1f \simeq h_0f \simeq h_1g = hg$. \square

Finally we obtain

Theorem 8. *The functors S and T are inverse to each other: $ST = id_{\text{fSh}(\mathcal{M})}$, $TS = id_{[FDR] \setminus \text{hM}}$, providing a category isomorphism $\text{fSh}(\mathcal{M}) \cong [FDR] \setminus \text{hM}$.*

Proof. First, both functors are constant on objects.

Second, for any fine shape class $[\phi]$ we have $ST([\phi]) = [\phi]$ from the definitions and Proposition 11.

Third, let a left fraction class from X to Y be represented by specific maps $X \xrightarrow{u} Z \xleftarrow{i} Y$ with i an FDR-map. The latter means there exist ARs $L \supset Z$ and $N \supset Y$, where inclusions are those of closed homotopy negligible subsets with Y additively so (use Proposition 2(7) and Corollary 4; see also Remark 12), the embedding $\bar{v}: N \subseteq L$ with $N \cap Z = Y$ extending i , and the $Z \times [0, 1] - Z$ -approaching strong deformation retraction $\Psi: (L \setminus Z) \times [0, 1] \rightarrow L \setminus Z$ such that Ψ_1 is a specific $Z - Y$ -approaching map from $L \setminus Z$ to $N \setminus Y$ representing the fine shape class $[i]_{\text{fSh}}^{-1}$. Now let M be any AR containing X as a closed additive homotopy negligible subset, and $\phi: M \setminus X \rightarrow L \setminus Z$ any specific approaching map extending u , and thus representing $[u]_{\text{fSh}}$. Then $S([i] \setminus [u])$ can be represented by $\Psi_1 \circ \phi$. Take $FC(\Psi_1 \circ \phi)$, and denote the embeddings by $X \xrightarrow{v} FC(\Psi_1 \circ \phi) \xleftarrow{j} Y$. The map

$$(M \setminus X) \times [0, 1] \ni (m, t) \mapsto \Psi_t \circ \phi(m) \in L \setminus Z$$

provides an approaching homotopy between $\phi = \Psi_0 \circ \phi$ and $\Psi_1 \circ \phi$. By Lemma 8, we obtain the desired equivalence $[j] \setminus [v] = [i] \setminus [u]$, and conclude that $TS = id_{[FDR] \setminus \text{hM}}$. \square

Appendix

Here we show why the approaching map cylinder, as defined by Mroziak [14] for compact spaces, does not seem to extend directly to the noncompact case.

Let an approaching map $\phi: M \setminus X \rightarrow N \setminus Y$ be given, and choose a homotopy $H: M \times [0, 1] \rightarrow M$ with $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$; here H being additive will not be relevant. We can easily define a metrizable topology on the set $M \times (0, 1] \cup N$ by inserting it into $M \star N$: take the standard embedding of N , and send $(m, s) \in M \times (0, 1]$ to $(m, 2s - 1, \phi \circ H_s(m))$. In

terms of Definition 23, this would extend $C_+ \cup C_{+,0}$ across the whole join and remove $C_- \cup C_{-,0}$ entirely (or collapse it onto N). Similarly to [14], denote the resulting space by $D(\phi)$, and write $C(\phi)$ for the subspace $X \times (0, 1] \cup Y$.

The problem is that the obvious retraction given by

$$D(\phi) \setminus C(\phi) \ni (m, s) \mapsto \phi(m) \in N \setminus Y$$

is not, in general, a $C(\phi) - Y$ -approaching map, and therefore there is no fine shape class inverse to the one given by the embedding $Y \subseteq C(\phi)$. By the definition of a metrizable join, a sequence $(m_k, 2s_k - 1, \phi \circ H_{s_k}(m_k))$ converges to a point $y \in Y$ if and only if $s_k \rightarrow 0$ and $\phi \circ H_{s_k}(m_k) \rightarrow y$ in N . For the retraction to be approaching, we would need $\phi(m_k)$ to have an accumulation point in Y , which does not follow from the premises in the noncompact case. This also means that $(D(\phi), C(\phi))$ does not have the universal property of the usual cylinder (as mentioned in Remark 11) outside of compacta.

For an actual counterexample, take the quotient set $M := \mathbb{N} \times [0, 1] / (\mathbb{N} \times \{1\})$ in the metrizable topology inherited from a piecewise linear insertion of M into \mathbb{R}^2 which inserts the apex 1 as $(0, 1)$, and $(k, 0)$ as $(k, 0)$ for all $k \in \mathbb{N}$. M is an AR because it is a contractible polyhedron. Take $X := \mathbb{N} \times \{0\} \subset M$, $N := \mathbb{R} \times [0, 1]$, and $Y := \mathbb{R} \times \{0\} \subset Y$. Define $\phi: M \setminus X \rightarrow N \setminus Y$ by $\phi(k, s) := (k \sin \frac{\pi}{s}, s)$; this is well-defined for $s = 1$. Then ϕ is $X - Y$ -approaching. Given a homotopy H on M (as usual, $H_0 = id_M$ and $H(M \times (0, 1]) \subseteq M \setminus X$; H can be chosen additive, but again it will not be relevant), we construct a sequence $((k, s_k), 2t_k - 1, \phi \circ H_{t_k}(k, s_k))$ in $M \star N$ such that $\phi \circ H_{t_k}(k, s_k) \rightarrow (0, 0)$ in N , and $t_k \rightarrow 0$, but $\phi(k, s_k)$ has no accumulation points in Y (or even in N).

We construct (s_k, t_k) separately for each k . First, assume U_k is the open set of $[0, 1] \times [0, 1]$ such that for all $(s, t) \in U$, $0 < s < \frac{1}{2^k}$, $0 \leq t < \frac{1}{2^k}$, and $H_t(k, s) \in \{k\} \times (0, 1)$ (U_k is nonempty since $H_0(k, 0) = (k, 0)$ and H is continuous). Then we can write $H_t(k, s) = (k, h_k(s, t))$ for some continuous function h_k and all $(s, t) \in U_k$.

Consider the set B_k of all pairs $(s, t) \in U_k$ such that $\sin \frac{\pi}{s} = 1$ and $\sin \frac{\pi}{h_k(s, t)} = 0$. We claim that B_k is nonempty: first, points $(s_l, t) = (\frac{2}{4l+1}, 0)$, satisfying $\sin \frac{\pi}{s_l} = 1$, are in U_k for all large natural l , and converge to $(0, 0)$ in $[0, 1] \times [0, 1]$. Second, if for every (large) l and for every $t \in [0, \frac{1}{2^k}]$ we were to have $\sin \frac{\pi}{h_k(s_l, t)} \neq 0$, then also for every t , $|s_l - h_k(s_l, t)| \rightarrow 0$. That implies $h_k(s_l, t) \rightarrow 0$ as $l \rightarrow +\infty$, and then $H_t(k, 0) = (k, 0)$ for $t > 0$, contradicting the choice of H . Therefore, for some l there is $t \in [0, \frac{1}{2^k}]$ such that $\sin \frac{\pi}{h_k(s_l, t)} = 0$. Thus we can choose $(s_k, t_k) \in B_k$. By construction, $s_k \rightarrow 0$, $t_k \rightarrow 0$, $\phi(k, s_k) = (k \sin \frac{\pi}{s_k}, s_k) = (k, s_k)$, and $\phi \circ H_{t_k}(k, s_k) = (0, h_k(s_k, t_k)) \rightarrow (0, 0)$. This shows that in general $\phi(m_k)$ having an accumulation point in Y does not follow from $\phi \circ H_{t_k}(m_k)$ having one.

This counterexample does not, of course, constitute definitive proof that there is no metrizable topology on $M \times (0, 1] \cup N$ that makes the pair $(M \times (0, 1] \cup N, X \times (0, 1] \cup Y)$ into an object of MAppr having all the properties we

need for Lemma 8. The question of defining a mapping cylinder in MAppr is still an open one in the noncompact case, in the sense of both that lemma and the pushout property. For compact M and N , it can be shown that $(D(\phi), C(\phi))$ defined above and $(AC(\phi), FC(\phi))$ are isomorphic objects of hMAppr by approaching strong deformation retracting the subset $C_-(\phi) \cup (N \setminus Y)$ of $AC(\phi) \setminus FC(\phi)$ onto its base $N \setminus Y$ — in short, this shows that the cylinder of Mroziak is sufficient in the compact case.

Acknowledgements The author expresses gratitude to S. Melikhov for the discussion of the subject and the results, and for remarks on theorem and proof formulations, as well as to the Advanced Doctoral Programme at Higher School of Economics for financial support.

References

- [1] M. Batanin. Categorical strong shape theory. *Cahiers Topol. Géom. Diff. Cat.*, volume 38:pages 3–66, 1997. ISSN 1245-530X. URL https://www.researchgate.net/publication/2659900_Categorical_Strong_Shape_Theory.
- [2] F. W. Bauer. Some relations between shape constructions. *Cahiers Topol. Géom. Diff. Cat.*, volume 19(4):pages 337–367, 1978. ISSN 1245-530X. URL <https://eudml.org/doc/91206>.
- [3] K. Borsuk. Concerning homotopy properties of compacta. *Fund. Math.*, volume 62(3):pages 223–254, 01 1968. ISSN 0016-2736. URL <http://dx.doi.org/10.4064/fm-62-3-223-254>.
- [4] A. Calder and H. M. Hastings. Realizing strong shape equivalences. *J. Pure and Appl. Alg.*, volume 20:pages 129–156, 02 1981. URL [http://dx.doi.org/10.1016/0022-4049\(81\)90088-8](http://dx.doi.org/10.1016/0022-4049(81)90088-8).
- [5] F. W. Cathey. Strong shape theory. In S. Mardesic and J. Segal (editors), *Shape theory and geometrical topology*, volume 870 of *Lecture Notes in Math.*, pages 215–238. Springer, 1981. URL <https://doi.org/10.1007/BFb0089717>.
- [6] D. Christie. Net homotopy for compacta. *Trans. Amer. Math. Soc.*, volume 56:pages 275–308, 01 1944. URL <http://dx.doi.org/10.1090/S0002-9947-1944-0010971-3>.
- [7] J. Dydak and J. Segal. *Strong shape theory*. Instytut Matematyczny Polskiej Akademii Nauk, 1981. URL <https://eudml.org/doc/268436>.
- [8] S. Ferry. A stable converse to the Vietoris-Smale theorem with applications to shape theory. *Trans. Amer. Math. Soc.*, volume 261(2):pages 369–386, 02 1980. URL <http://dx.doi.org/10.2307/1998371>.
- [9] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik 35. Springer, 1967. URL <http://dx.doi.org/10.1007/978-3-642-85844-4>.
- [10] Y. Kodama and J. Ono. On fine shape theory. *Fund. Math.*, volume 105(1):pages 29–39, 1979. ISSN 0016-2736. URL <https://eudml.org/doc/211079>.
- [11] S. Mardešić. *Strong Shape and Homology*. Springer-Verlag Berlin Heidelberg, 2000. ISBN 978-3-642-08546-8. URL <http://dx.doi.org/10.1007/978-3-662-13064-3>.
- [12] S. Melikhov. Fine shape I, 2022. URL <https://arxiv.org/abs/1808.10228v2>.
- [13] S. Melikhov. *Topology of Metric Spaces*. 11 2022. URL <https://www.researchgate.net/publication/365476532>.
- [14] P. Mroziak. Mapping cylinders of approaching maps and strong shape. *J. London Math. Soc.*, volume s2-41:pages 159–174, 02 1990. URL <http://dx.doi.org/10.1112/jlms/s2-41.1.159>.

- [15] J. B. Quigley. An exact sequence from the n th to the $(n - 1)$ -st fundamental group. *Fund. Math.*, volume 77:pages 195–210, 1973. URL <https://eudml.org/doc/214016>.
- [16] D. G. Quillen. *Homotopical Algebra*. Lecture Notes in Math. Springer, 1967. URL <http://dx.doi.org/10.1007/BFb0097438>.
- [17] H. Schubert. *Categories*. Springer, 1972. URL <http://dx.doi.org/10.1007/978-3-642-65364-3>.
- [18] E. Sklyarenko. On homology theory associated with the Aleksandrov-Čech cohomology. *Russ. Math. Surv.*, volume 34:pages 103–137, 1979. URL <http://dx.doi.org/https://doi.org/10.1070/RM1979v034n06ABEH003307>.
- [19] A. Strøm. Note on cofibrations II. *Math. Scand.*, volume 22:pages 130–142, 1968. URL <https://eudml.org/doc/166037>.

VLADISLAV KIRILLOVICH ZEMLYANOV
HIGHER SCHOOL OF ECONOMICS (HSE UNIVERSITY),
20 MYASNITSKAYA ST.
101000, MOSCOW, RUSSIA
Email address: tagambit@yandex.ru